Symbolic evaluation of determinants and rhombus tilings of holey hexagons

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(joint work with Thotsaporn Thanatipanonda)

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ÖAW RICAM

Beginning of the Story

Inventiones math. 53, 193 – 225 (1979)

Inventiones mathematicae

Plane Partitions (III): The Weak Macdonald Conjecture

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Dedicated to the memory of Alfred Young and F.J.W. Whipple

Determinant that counts descending plane partitions:

$$D_{0,0}(n) := \det_{1\leqslant i,j\leqslant n}\left(\delta_{i,j} + \binom{\mu+i+j-4}{j-1}\right),$$

where $\delta_{i,j}$ denotes the Kronecker delta function.

Andrews's Result

Theorem. We have

$$D_{0,0}(n) = 2 \prod_{i=1}^{n-1} R_{0,0}(i),$$

in other words $R_{0,0}(n) = D_{0,0}(n+1)/D_{0,0}(n)$, where

$$R_{0,0}(2n) = \frac{\left(\mu + 2n\right)_n \left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1}}{\left(n\right)_n \left(\frac{\mu}{2} + n + \frac{1}{2}\right)_{n-1}},$$

$$R_{0,0}(2n-1) = \frac{\left(\mu + 2n - 2\right)_{n-1} \left(\frac{\mu}{2} + 2n - \frac{1}{2}\right)_n}{\left(n\right)_n \left(\frac{\mu}{2} + n - \frac{1}{2}\right)_{n-1}},$$

and where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n := a \cdot (a+1) \cdots (a+n-1).$$

Another question is the possibility of other general determinants of this nature. At first glance

$$E_m(\mu) = \det\left(\delta_{ij} + \binom{\mu+i+j}{i+1}\right)_{0\le i,j\le m-1}$$

looks interesting. Indeed it turns out that

$$E_1(\mu) = \mu + 1,$$

$$E_2(\mu) = (\mu+2)(\mu+1),$$

$$E_3(\mu) = \frac{(\mu+14)(\mu+3)(\mu+2)(\mu+1)}{12},$$

$$E_4(\mu) = \frac{(\mu+14)(\mu+9)(\mu+4)(\mu+3)(\mu+2)(\mu+1)}{72},$$

$$E_5(\mu) = \frac{(\mu+9)(\mu+5)(\mu+4)(\mu+3)(\mu+2)(\mu+1)(\mu^3+45\mu^2+722\mu+3432)}{8640}.$$

Emprically it seems reasonable to guess that

$$\frac{E_{2m}(\mu)}{E_{2m-1}(\mu)} = f_{2m,2m}(\mu-2),$$

George Andrews (1980): Macdonald's conjecture and descending plane partitions

Andrews's Conjecture (1980)

Let $D_{1,1}(n)$ denote Andrews's interesting-looking determinant:

$$D_{1,1}(n) := \det_{1 \leqslant i,j \leqslant n} \left(\delta_{i,j} + \binom{\mu + i + j - 2}{j} \right).$$

Conjecture. The following holds:

$$\frac{D_{1,1}(2n)}{D_{1,1}(2n-1)} = (-1)^{(n-1)(n-2)/2}\, 2^n \, \frac{\left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1} \left(\frac{\mu}{2} + n\right)_{\lfloor (n+1)/2 \rfloor}}{\left(n\right)_n \left(-\frac{\mu}{2} - 2n + \frac{3}{2}\right)_{\lfloor (n-1)/2 \rfloor}}$$

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$$= 2^n \frac{\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}\left(\frac{\mu}{2}+n\right)_{\lfloor (n+1)/2\rfloor}}{\left(n\right)_n\left(\frac{\mu}{2}+\left\lfloor\frac{3n}{2}\right\rfloor+\frac{1}{2}\right)_{\lfloor (n-1)/2\rfloor}}$$

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$$= \frac{\left(\mu+2n\right)_n\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}}{\left(n\right)_n\left(\frac{\mu}{2}+n+\frac{1}{2}\right)_{n-1}}.$$

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$$D_{1,1}(n) := \det_{1\leqslant i,j\leqslant n}\left(\delta_{i,j} + \binom{\mu+i+j-2}{j}\right).$$

Theorem. The following holds:

$$\begin{aligned}\frac{D_{1,1}(2n)}{D_{1,1}(2n-1)} &= (-1)^{(n-1)(n-2)/2}\, 2^n \frac{\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}\left(\frac{\mu}{2}+n\right)_{\lfloor (n+1)/2\rfloor}}{\left(n\right)_n\left(-\frac{\mu}{2}-2n+\frac{3}{2}\right)_{\lfloor (n-1)/2\rfloor}}\\ &= 2^n \frac{\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}\left(\frac{\mu}{2}+n\right)_{\lfloor (n+1)/2\rfloor}}{\left(n\right)_n\left(\frac{\mu}{2}+\left\lfloor\frac{3n}{2}\right\rfloor+\frac{1}{2}\right)_{\lfloor (n-1)/2\rfloor}}\\ &= \frac{\left(\mu+2n\right)_n\left(\frac{\mu}{2}+2n+\frac{1}{2}\right)_{n-1}}{\left(n\right)_n\left(\frac{\mu}{2}+n+\frac{1}{2}\right)_{n-1}}.\end{aligned}$$

$\longrightarrow$ Proven by us in 2013.

$$
\begin{aligned}
D_{1,1}(1) &= \mu + 1 \\
D_{1,1}(2) &= (\mu+1)(\mu+2) \\
D_{1,1}(3) &= \tfrac{1}{12}(\mu+1)(\mu+2)(\mu+3)(\mu+14) \\
D_{1,1}(4) &= \tfrac{1}{72}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+9)(\mu+14) \\
D_{1,1}(5) &= \tfrac{1}{8640}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+9) \\
&\quad \times(\mu^3+45\mu^2+722\mu+3432) \\
D_{1,1}(6) &= \tfrac{1}{518400}(\mu+1)(\mu+2)(\mu+3)(\mu+4)(\mu+5)(\mu+6) \\
&\quad \times(\mu+8)(\mu+13)(\mu+15)(\mu^3+45\mu^2+722\mu+3432) \\
D_{1,1}(7) &= \tfrac{1}{870912000}(\mu+1)\circ\circ\circ(\mu+34)(\mu^3+47\mu^2+954\mu+5928) \\
D_{1,1}(8) &= \tfrac{1}{731566080000}(\mu+1)\circ\circ\circ(\mu+34)(\mu^3+47\mu^2+954\mu+5928) \\
D_{1,1}(9) &= \tfrac{1}{221225582592000000}(\mu+1)(\mu+2)\circ\circ\circ(\mu+21)^2 \\
&\quad \times(\mu^6+142\mu^5+8505\mu^4+277100\mu^3+5253404\mu^2+52937808\mu \\
D_{1,1}(10) &= \tfrac{1}{334493080879104000000}(\mu+1)(\mu+2)\circ\circ\circ(\mu+25)(\mu+27) \\
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Our Conjecture

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$$\times \left(\prod_{i=1}^{\left\lfloor \frac{3}{2} \left\lfloor \frac{n}{2} \right\rfloor - 2 \right\rfloor} \Big(\mu + 2i + 2\left\lfloor \frac{3}{2} \left\lfloor \frac{n}{2} + 1 \right\rfloor \right\rfloor - 1\Big)^{2\left\lfloor \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{3}(i-1) \right\rfloor - 1} \right),$$

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$$F_m(n) = \left(\prod_{i=1}^{\left\lfloor \frac{1}{4}(n-1) \right\rfloor} (\mu + 2i + n + m)^{1-2i-m} \right)$$

$$\times \left(\prod_{i=1}^{\left\lfloor \frac{n}{4} - 1 \right\rfloor} (\mu - 2i + 2n - 2m + 1)^{1-2i-m} \right),$$

Our Conjecture

... further let ...

$$F(n) = \begin{cases} E(n)F_0(n), & \text{if } n \text{ is even,} \\ E(n)F_1(n) \displaystyle\prod_{i=1}^{\frac{1}{2}(n-5)} (\mu + 2i + 2n - 1), & \text{if } n \text{ is odd,} \end{cases}$$

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$$\begin{aligned} T(k) = {} & 55296k^6 + 41472(\mu-1)k^5 + 384(30\mu^2 - 66\mu + 53)k^4 \\ & + 96(\mu-1)(15\mu^2 - 42\mu + 61)k^3 \\ & + 4(19\mu^4 - 122\mu^3 + 419\mu^2 - 544\mu + 72)k^2 \\ & + (\mu-1)(\mu^4 - 14\mu^3 + 101\mu^2 - 160\mu - 84)k \\ & + 2(\mu-3)(\mu-2)(\mu-1)(\mu+1), \end{aligned}$$

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. . . and let . . .

$$S_1(n) = \sum_{k=1}^{n-1} \Big(2^{6k}(\mu + 8k - 1) \left(\tfrac{1}{2}\right)_{2k-1}^2 \left(\tfrac{1}{2}(\mu+5)\right)_{2k-3}$$

$$\times \left(\tfrac{1}{2}(\mu + 4k + 2)\right)_{k-2} \left(\tfrac{1}{2}(\mu + 4k + 2)\right)_{2n-2k-2} T(k) \Big)$$

$$\Big/ \Big((2k)! \left(\tfrac{1}{2}(\mu + 6k - 3)\right)_{3k+4} \Big),$$

$$S_2(n) = \sum_{k=1}^{n-1} \Big(2^{6k}(\mu + 8k + 3) \left(\tfrac{1}{2}\right)_{2k}^2 \left(\tfrac{1}{2}(\mu+5)\right)_{2k-2}$$

$$\times \left(\tfrac{1}{2}(\mu + 4k + 4)\right)_{k-2} \left(\tfrac{1}{2}(\mu + 4k + 4)\right)_{2n-2k-2} T\big(k + \tfrac{1}{2}\big) \Big)$$

$$\Big/ \Big((2k+1)! \left(\tfrac{1}{2}(\mu + 6k + 1)\right)_{3k+5} \Big),$$

Our Conjecture

$$P_1(n) = 2^{3n-1}\frac{\left(\frac{1}{2}(\mu+6n-3)\right)_{3n-2}}{\left(\frac{1}{2}(\mu+5)\right)_{2n-3}} \times \left(\frac{\left(\frac{1}{2}(\mu+2)\right)_{2n-2}}{(\mu+3)^2} + \frac{\mu(\mu-1)}{2^{13}}S_1(n)\right),$$

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$$G(n) = \begin{cases} P_1\left(\frac{1}{2}(n+1)\right), & \text{if } n \text{ is odd,} \\ P_2\left(\frac{n}{2}\right), & \text{if } n \text{ is even.} \end{cases}$$

Then for every positive integer n we have

$$D_{1,1}(n) = C(n)\,F(n)\,G\left(\left\lfloor \tfrac{1}{2}(n+1)\right\rfloor\right).$$

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We found a beautiful formula for Andrews's determinant $D_{1,1}(n)$.
Let

$$C(n) = \frac{(-1)^n + 3}{2} \prod_{i=1}^{n} \frac{\lfloor \frac{i}{2} \rfloor !}{i!},$$

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$$P_1(n) = 2^{3n-1} \frac{\left(\frac{1}{2}(\mu + 6n - 3)\right)_{3n-2}}{\left(\frac{1}{2}(\mu+5)\right)_{2n-3}} \times \left(\frac{\left(\frac{1}{2}(\mu+2)\right)_{2n-2}}{(\mu+3)^2} + \frac{\mu(\mu-1)}{2^{13}} S_1(n) \right),$$

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Then for every positive integer n we have

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Desnanot-Jacobi-Carroll Identity (DJC)

Theorem. Let $\left(m_{i,j}\right)_{i,j\in\mathbb{Z}}$ be an infinite sequence and denote by $M_{s,t}(n)$ the determinant of the $(n\times n)$-matrix whose upper left entry is $m_{s,t}$, more precisely the matrix $\left(m_{i,j}\right)_{s\leqslant i<s+n,t\leqslant j<t+n}$. Then:

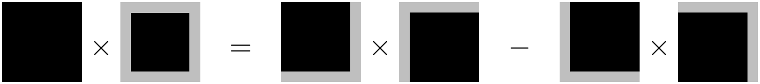
$$M_{s,t}(n)M_{s+1,t+1}(n-2) = \\ M_{s,t}(n-1)M_{s+1,t+1}(n-1) - M_{s+1,t}(n-1)M_{s,t+1}(n-1).$$

Desnanot-Jacobi-Carroll Identity (DJC)

Theorem. Let $\left(m_{i,j}\right)_{i,j\in\mathbb{Z}}$ be an infinite sequence and denote by $M_{s,t}(n)$ the determinant of the $(n \times n)$-matrix whose upper left entry is $m_{s,t}$, more precisely the matrix $\left(m_{i,j}\right)_{s\leqslant i<s+n,t\leqslant j<t+n}$. Then:

$$M_{s,t}(n)M_{s+1,t+1}(n-2) = \\ M_{s,t}(n-1)M_{s+1,t+1}(n-1) - M_{s+1,t}(n-1)M_{s,t+1}(n-1).$$

Schematically:



Generalization

Definition: For $n, s, t \in \mathbb{Z}$, $n \geqslant 1$, and μ an indeterminate, we define $D_{s,t}(n)$ to be the following $(n \times n)$-determinant:

$$D_{s,t}(n) := \det_{\substack{s \leqslant i < s+n \\ t \leqslant j < t+n}} \left(\delta_{i,j} + \binom{\mu + i + j - 2}{j} \right)$$

$$= \det_{1 \leqslant i,j \leqslant n} \left(\delta_{i+s,j+t} + \binom{\mu + i + j + s + t - 4}{j + t - 1} \right)$$

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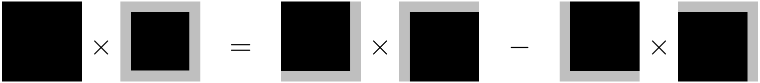
Known special cases:

- closed form for $D_{0,0}(n)$ (Andrews 1979)
- closed form for $D_{1,1}(2n)/D_{1,1}(2n-1)$ (Andrews 1980)
- monstrous conjecture for $D_{1,1}(n)$ (K-T 2013)

DJC for $D_{1,1}(n)$



By (DJC) we obtain a recurrence equation for $D_{1,1}(n)$:

$$D_{0,0}(n+1)D_{1,1}(n-1) = D_{0,0}(n)D_{1,1}(n) - D_{1,0}(n)D_{0,1}(n).$$

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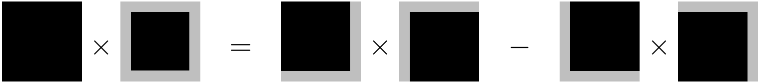

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We rewrite it slightly:

$$D_{1,1}(n) = \underbrace{\frac{D_{0,0}(n+1)}{D_{0,0}(n)}}_{=R_{0,0}(n)} D_{1,1}(n-1) + \frac{D_{1,0}(n)D_{0,1}(n)}{D_{0,0}(n)}.$$

$\longrightarrow$ Hence we need to know $D_{1,0}(n)$ and $D_{0,1}(n)$.

Zero Determinants

Task: We want to evaluate $D_{1,0}(n)$ and $D_{0,1}(n)$.

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- Use the holonomic systems approach (Zeilberger) to prove

$$M^{(2n)} \cdot c_n = 0, \text{ i.e., } \quad \sum_{j=1}^{2n} M_{i,j}^{(2n)} c_{n,j} = 0 \quad \text{for all } i \text{ and } n.$$

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Recipe for the Holonomic Ansatz

Problem: Given $a_{i,j}$ and $b_n \neq 0$. Show that $\det(a_{i,j})_{1 \leqslant i,j \leqslant n} = b_n$.

Method: "Pull out of the hat" a function $c_{n,j}$ and prove

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Then $\det(a_{i,j})_{1\leqslant i,j\leqslant n} = b_n$ holds.

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Example: For $D_{0,0}(2n)$ we obtain the following holonomic system of recurrence relations for $c_{n,j}$.

$$\{(j+\mu+2n-3)(2\mu j^6+8nj^6-2j^6+3\mu^2j^5-48n^2j^5-12\mu j^5-24nj^5+9j^5+\mu^3j^4+48n^3j^4-11\mu^2j^4-84\mu n^2j^4+204n^2j^4+21\mu j^4-20\mu^2nj^4+38\mu nj^4-10nj^4-11j^4+216n^4j^3-2\mu^3j^3+312\mu n^3j^3-408n^3j^3+7\mu^2j^3+28\mu^2n^2j^3+122\mu n^2j^3-198n^2j^3-2\mu j^3-9\mu^3nj^3+68\mu^2nj^3-113\mu nj^3+78nj^3-3j^3-864n^5j^2-756\mu n^4j^2+432n^4j^2-\mu^3j^2-112\mu^2n^3j^2-308\mu n^3j^2+600n^3j^2+11\mu^2j^2-3\mu^3n^2j^2-66\mu^2n^2j^2+189\mu n^2j^2-168n^2j^2-23\mu j^2-2\mu^4nj^2+15\mu^3nj^2-28\mu^2nj^2+33\mu nj^2-34nj^2+13j^2+864n^6j+432\mu n^5j+432n^5j-144\mu^2n^4j+1116\mu n^4j-1104n^4j+2\mu^3j-88\mu^3n^3j+384\mu^2n^3j-392\mu n^3j-36n^3j-10\mu^2j-14\mu^4n^2j+45\mu^3n^2j+40\mu^2n^2j-317\mu n^2j+270n^2j+14\mu j-\mu^5nj+3\mu^4nj+17\mu^3nj-89\mu^2nj+112\mu nj-42nj-6j+432\mu n^6-864n^6+432\mu^2n^5-1080\mu n^5+432n^5+144\mu^3n^4-324\mu^2n^4-156\mu n^4+456n^4+20\mu^4n^3-18\mu^3n^3-220\mu^2n^3+470\mu n^3-204n^3+\mu^5n^2+3\mu^4n^2-37\mu^3n^2+57\mu^2n^2+36\mu n^2-60n^2+2\mu^4n-18\mu^3n+54\mu^2n-62\mu n+24n)c_{n,j}-(j+\mu-3)(2j+\mu-3)(j-2n+1)(\mu+4n-1)(j^4+2\mu j^3-6j^3+\mu^2j^2-12n^2j^2-9\mu j^2-6\mu nj^2+6nj^2+13j^2-3\mu^2j-12\mu n^2j+36n^2j+13\mu j-6\mu^2nj+24\mu nj-18nj-12j+2\mu^2-2\mu^2n^2+20\mu n^2-24n^2-6\mu-\mu^3n+11\mu^2n-22\mu n+12n+4)c_{n,j+1}+2(2j+\mu-2)n(2n+1)(-j+2n+1)(-j+2n+2)(j+\mu+2n-1)(\mu+4n-3)(\mu+4n-1)c_{n+1,j},-(j+1)(2j+\mu)(j-2n)(j+\mu+2n-3)c_{n,j}+(4j^4+8\mu j^3-8j^3+5\mu^2j^2-8n^2j^2-5\mu j^2-4\mu nj^2+12nj^2-8j^2+\mu^3j+2\mu^2j-8\mu n^2j+8n^2j-15\mu j-4\mu^2nj+16\mu nj-12nj+12j+\mu^3-3\mu^2-2\mu^2n^2+16n^2-2\mu-\mu^3n+3\mu^2n+8\mu n-24n+8)c_{n,j+1}-(j+\mu-2)(2j+\mu-2)(j-2n+2)(j+\mu+2n-1)c_{n,j+2}\}$$

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Main Result

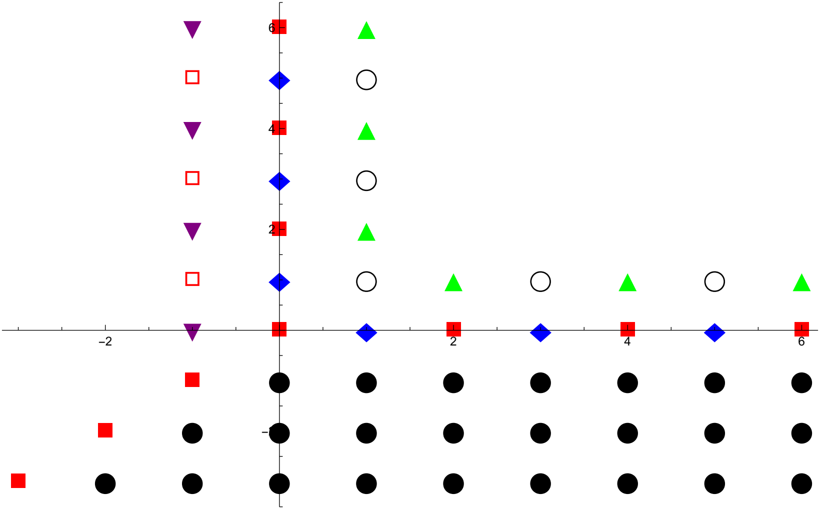
Theorem. Let μ be an indeterminate and let ρ_k be defined as $\rho_0(a,b) = a$ and $\rho_k(a,b) = b$ for $k > 0$.
If n is an odd positive integer then

$$D_{1,1}(n) = \sum_{k=0}^{(n+1)/2} \rho_k\left(4(\mu-2), \frac{1}{(2k-1)!}\right) \frac{(\mu-1)_{3k-2}}{2\left(\frac{\mu}{2}+k-\frac{1}{2}\right)_{k-1}} \times \left(\prod_{j=1}^{k-1} \frac{(\mu+2j+1)_{j-1}\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_{j-1}}{(j)_{j-1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}\right)^2 \times \left(\prod_{j=k}^{(n-1)/2} \frac{(\mu+2j)_j^2\left(\frac{\mu}{2}+2j-\frac{1}{2}\right)_j\left(\frac{\mu}{2}+2j+\frac{3}{2}\right)_{j+1}}{(j)_j\,(j+1)_{j+1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_j^2}\right)$$

If n is an even positive integer then. . . [similar formula]

More Results

We can give closed-form evaluations of some infinite 1-dimensional families of $D_{s,t}(n)$.



Lindström-Gessel-Viennot Lemma

Let G be a directed acyclic graph and consider base vertices $A = \{a_1, \ldots, a_n\}$ and destination vertices $B = \{b_1, \ldots, b_n\}$.

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$$e(a,b) = \sum_{P:a \to b} \omega(P) \qquad \text{and}$$

$$M = \begin{pmatrix} e(a_1,b_1) & e(a_1,b_2) & \cdots & e(a_1,b_n) \\ e(a_2,b_1) & e(a_2,b_2) & \cdots & e(a_2,b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n,b_1) & e(a_n,b_2) & \cdots & e(a_n,b_n) \end{pmatrix}.$$

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Then the determinant of M is the signed sum over all n-tuples $P = (P_1, \ldots, P_n)$ of non-intersecting paths from A to B:

$$\det(M) = \sum_{(P_1,\ldots,P_n)\colon\, A \to B} \operatorname{sign}(\sigma(P)) \prod_{i=1}^{n} \omega(P_i).$$

where σ denotes a permutation that is applied to B.

Lindström-Gessel-Viennot Lemma

Application: In our context, the lemma implies the following.

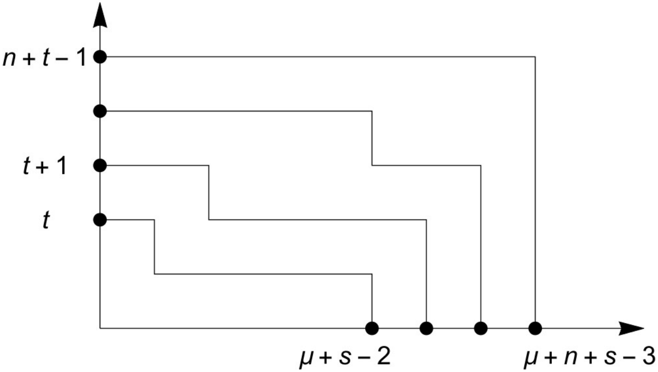
Look at the determinant without the Kronecker-Delta:

$$\det_{1\leqslant i,j\leqslant n} \binom{\mu+i+j+s+t-4}{j+t-1}.$$

It counts n-tuples of non-intersecting paths in the lattice $\mathbb{N}^2$:

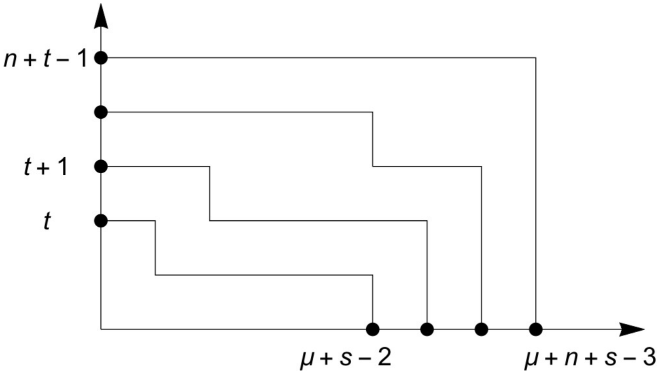
- The starting points are $(0,t),(0,t+1),\dots,(0,t+n-1)$.
- The end points are $(\mu+s-2,0),\dots,(\mu+s+n-3,0)$.
- The allowed steps are $(1,0)$ and $(0,-1)$.

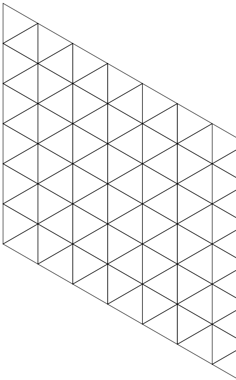
Non-intersecting Lattice Paths



For $1 \leqslant i, j \leqslant n$ the number of paths from $(0, t + j - 1)$ to $(\mu + s + i - 3, 0)$ is given by $\binom{\mu+i+j+s+t-4}{j+t-1}$, which is precisely the (i, j)-entry of our matrix.

Lattice Paths $\longrightarrow$ Rhombus Tilings





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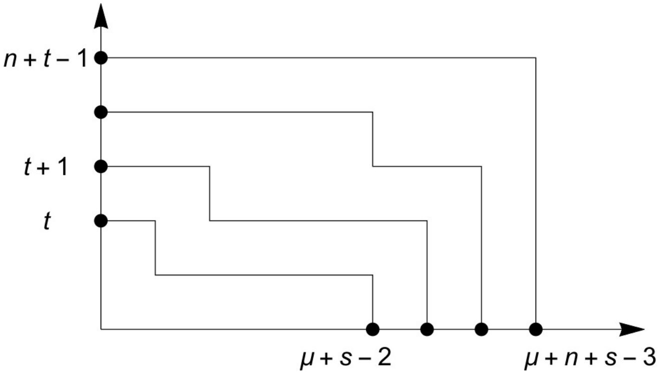
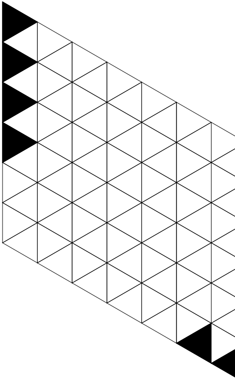
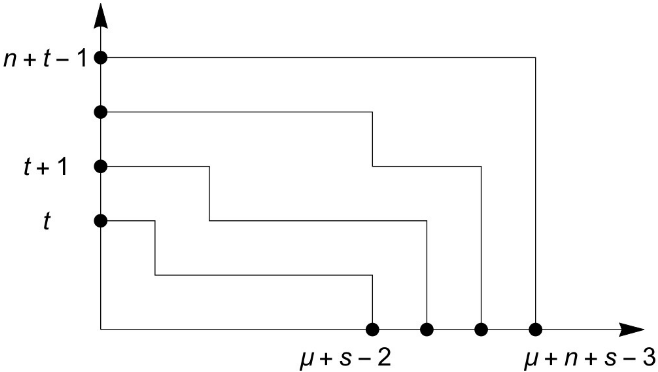
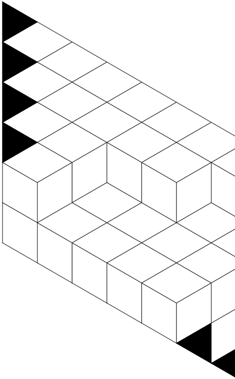




Lattice Paths ⟶ Rhombus Tilings





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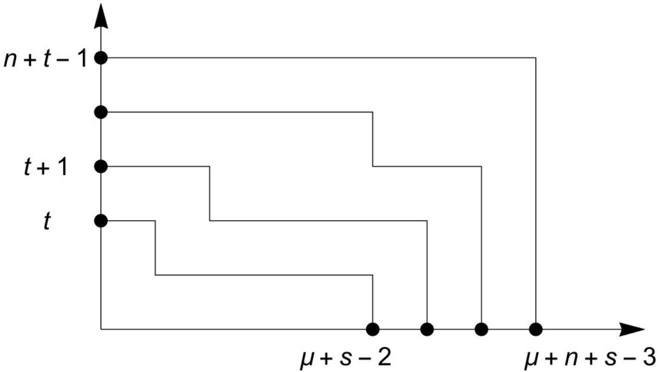

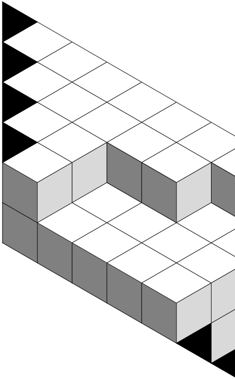

Determinant with Kronecker-Delta

From the Laplace expansion one immediately sees that

$$\begin{vmatrix} \cdots & b_{1,j}+1 & b_{1,j+1} & \cdots \\ \cdots & b_{2,j} & b_{2,j+1}+1 & \cdots \\ & \vdots & \vdots & \end{vmatrix} =$$

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By applying this procedure recursively, one obtains

$$D_{s,t}(n) = \sum_{I \subseteq \{1,\ldots,n-s+t\}} (-1)^{(s-t)\cdot|I|} \det\left(M_{I+s-t}^{I}\right) \qquad (s \geqslant t),$$

where M_J^I denotes the matrix that is obtained by deleting all rows with indices in I and all columns with indices in J from the matrix

$$\left(\binom{\mu+i+j+s+t-4}{j+t-1}\right)_{1 \leqslant i,j \leqslant n}.$$

Kronecker-Deltas on the Main Diagonal

General formula:

$$D_{s,t}(n) = \sum_{I \subseteq \{1,\dots,n-s+t\}} (-1)^{(s-t)\cdot|I|} \det\big(M^{I}_{I+s-t}\big) \qquad (s \geqslant t)$$

Special case: If $s = t$ we obtain

$$D_{s,s}(n) = \sum_{I \subseteq \{1,\dots,n\}} \det\big(M^{I}_{I}\big),$$

i.e., $D_{s,s}(n)$ is the sum of principal minors of the binomial matrix.

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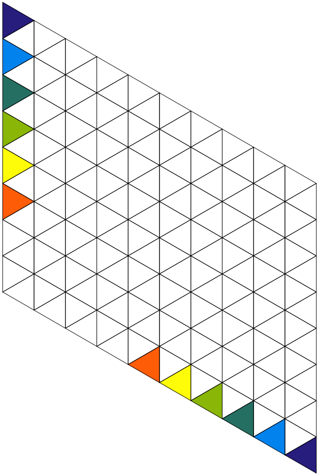
Hence: $D_{s,s}(n)$ counts all k-tuples of non-intersecting lattice paths, $k = 0, \ldots, n$, and where the start and end points are given by the same k-subset.

Kronecker-Deltas on the Main Diagonal

$s = 2$
$t = 2$
$n = 6$
$\mu = 4$



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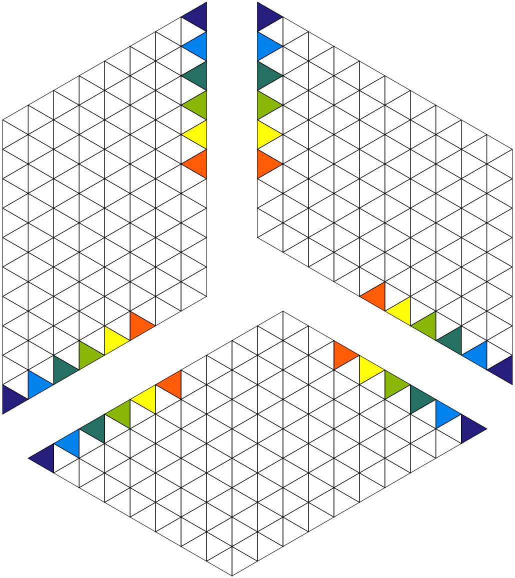

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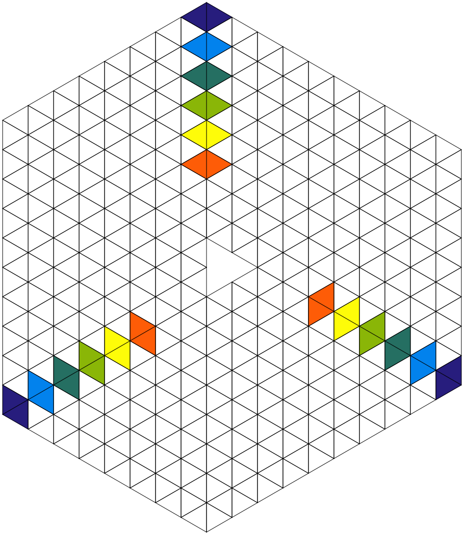

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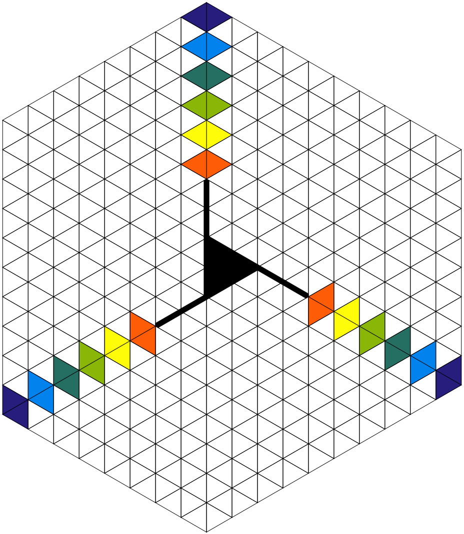

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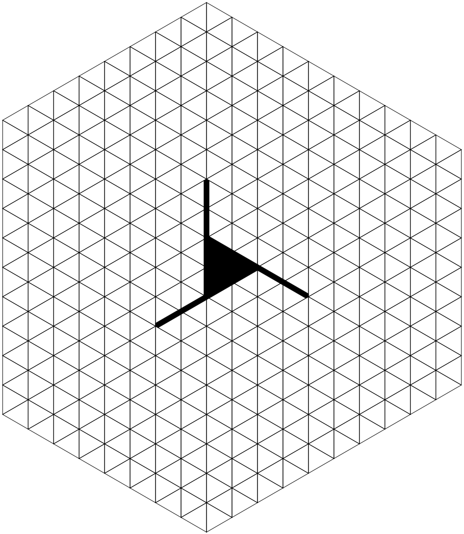

Rhombus Tilings

Finding: The determinant $D_{s,s}(n)$ counts

- rhombus tilings
- of a hexagon with a funny-shaped hole ("holey hexagon")
- that are cyclically symmetric.
- The hole has the shape of a triangle (of size $\mu - 2$) with "boundary lines" (of length s) sticking out of its corners.

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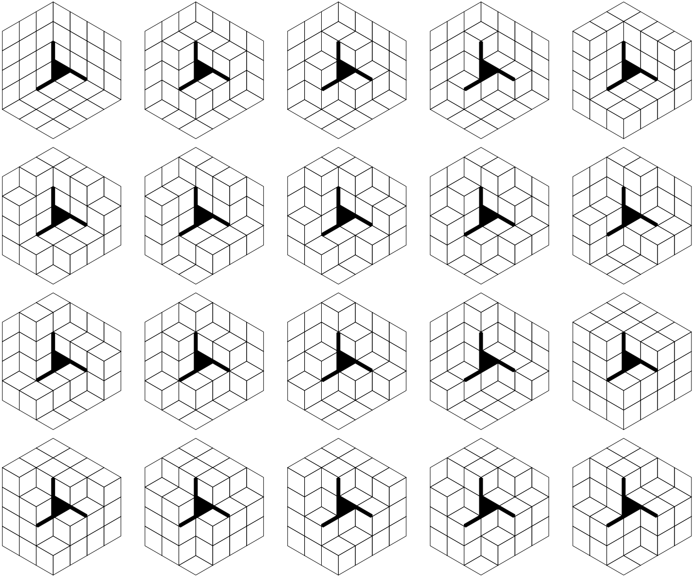
Example: For $s = t = 1$, $n = 2$, and $\mu = 3$ we obtain

$$D_{1,1}(2)\big|_{\mu \to 3} = \begin{vmatrix} 4 & 6 \\ 4 & 11 \end{vmatrix} = 20.$$

Cyclically Symmetric Rhombus Tilings of a Holey Hexagon



Off-Diagonal Kronecker-Deltas

Now let's look at the situation $s \neq t$.

General formula:

$$D_{s,t}(n) = \sum_{I \subseteq \{1,\ldots,n+s-t\}} (-1)^{(s-t)\cdot|I|} \, \det\bigl(M_I^{I+t-s}\bigr) \qquad (t \geqslant s)$$

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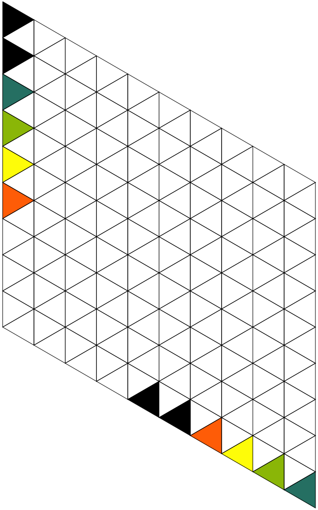
Remark: If $s-t$ is odd, we perform a weighted count with weights $+1$ and -1, according to the length of the tuples of paths.

Off-Diagonal Kronecker-Deltas

$s = 1$
$t = 3$
$n = 6$
$\mu = 5$



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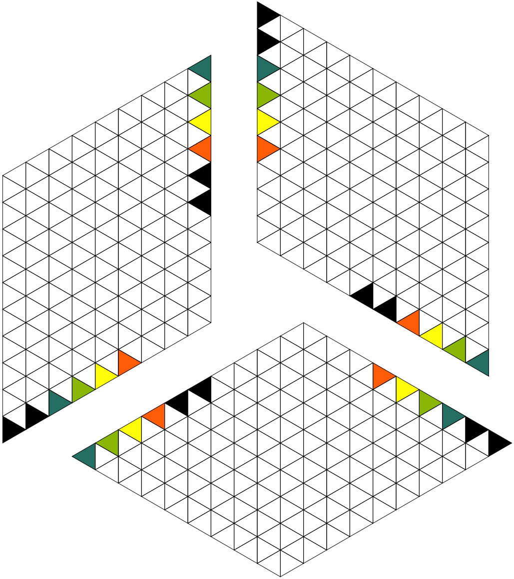

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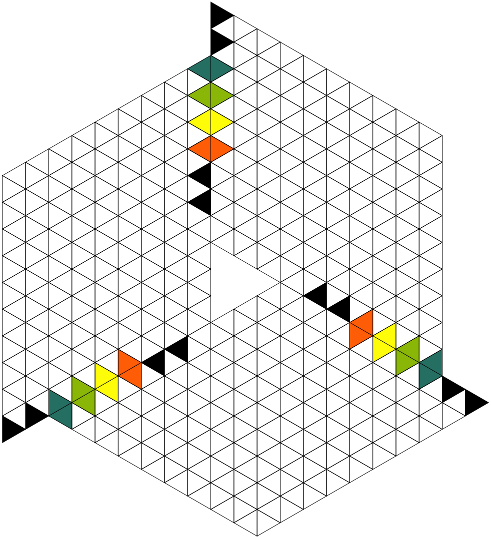

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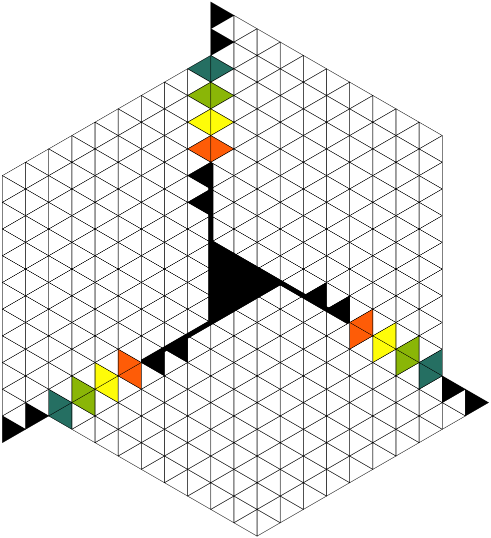

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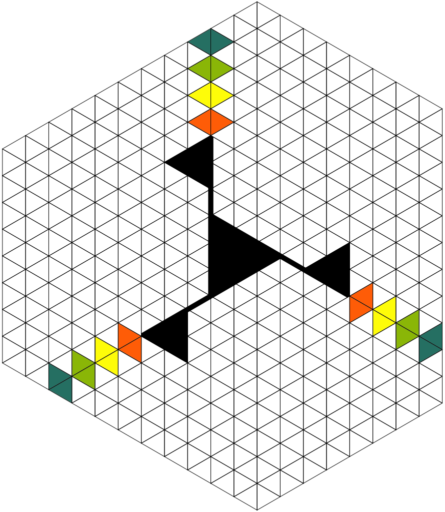

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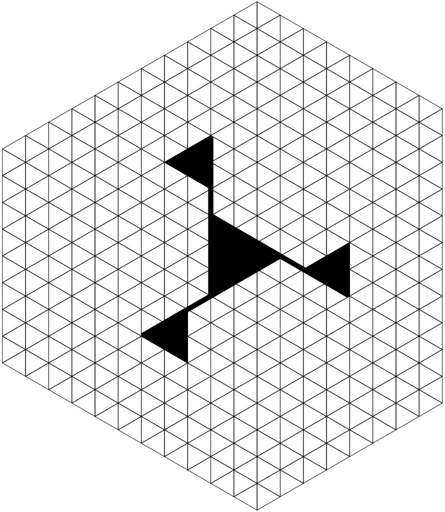
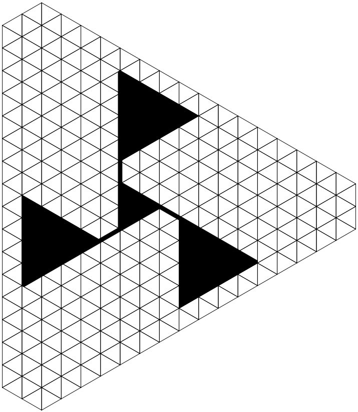
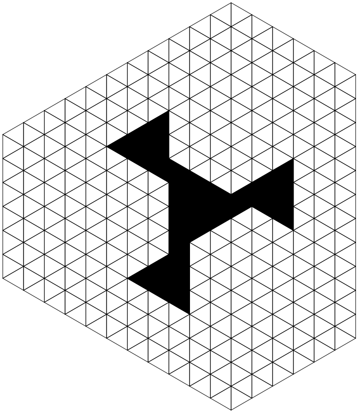


Off-Diagonal Kronecker-Deltas

Example: Shapes for different choices of the parameters



$s = 5,\ t = 1,\ n = 5,\ \mu = 4$



$s = -1,\ t = 2,\ n = 6,\ \mu = 6$

Example of an Infinite Family (A)

Family A: can be reduced to the base case $D_{0,0}(n)$:

$$D_{2r,0}(n) = D_{0,0}(n-2r)\big|_{\mu\to\mu+6r}$$

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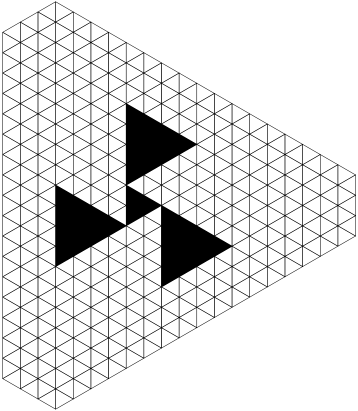

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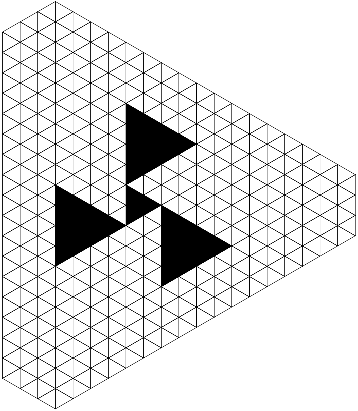

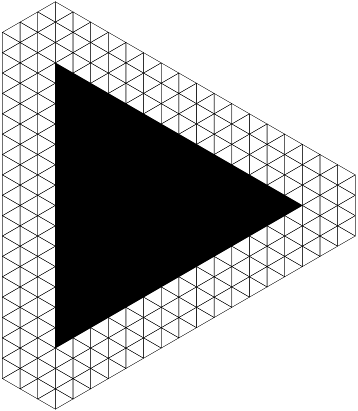

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Family B: If $n \geqslant 2r$ is an even number, then $D_{2r-1,0}(n) = 0$.

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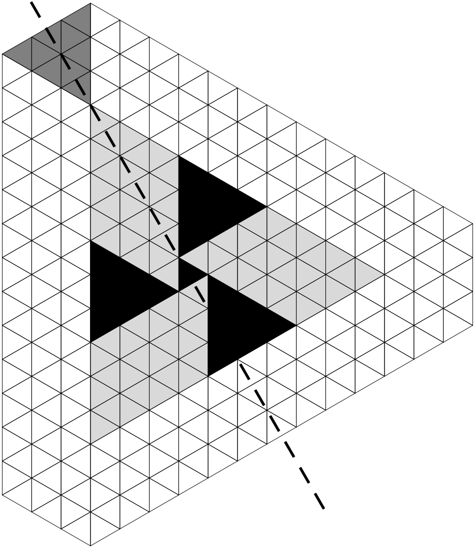

Example of an Infinite Family (B)

Theorem. Let μ be an indeterminate, and let r and n be positive integers. If n is an odd number, then

$$D_{2r-1,0}(n) = \prod_{i=r}^{(n-1)/2} \big(-R_{2r-1,0}(i)\big),$$

where $R_{2r-1,0}(n) =$

$$\frac{\big(\mu+2n+4r-4\big)_{n-r+1}\,\big(\mu+2n+4r-3\big)_{n-r}\,\big(\frac{\mu}{2}+2n+r-\frac{1}{2}\big)_{n-r}^2}{\big(n-r+1\big)_{n-r+1}\,\big(n-r+1\big)_{n-r}\,\big(\frac{\mu}{2}+n+2r-\frac{3}{2}\big)_{n-r}^2}$$

i.e., $R_{2r-1,0}(n) = D_{2r-1,0}(2n+1)/D_{2r-1,0}(2n-1)$ for $n \geqslant r$.
If $n \geqslant 2r$ is an even number, then $D_{2r-1,0}(n) = 0$. Moreover,

$$D_{0,2r-1}(n) = \left(\prod_{i=0}^{n-1} \frac{\big(\mu+i-1\big)_{2r-1}}{\big(i+1\big)_{2r-1}}\right) \cdot D_{2r-1,0}(n).$$

Reference

Christoph Koutschan and Thotsaporn Thanatipanonda:
A curious family of binomial determinants that count rhombus tilings of a holey hexagon

- Technical report no. 2017-30 in the RICAM Reports Series
- arxiv:1709.02616
- http://www.koutschan.de/data/det2/