In[1]:=
```
<< "/Users/georg/Documents/geo/conferences/2016/DARTVII/TenReS/TenReS.m"
```

```
Package TenReS version 0.2.3
written by Clemens G. Raab
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```

A few definitions need to be made by the user. Type ?CoeffQ, ?Specialization, and ?CyclicModules for more information.

Basic definitions

In[2]:=
```
Specialization = {F → {K, F̃}};
```

In[3]:=
```
CyclicModules = {{K, 1}, {"E", Eval}, {D, Diff}, {"I", Int}};
```

Membership checks

In[4]:=
```
CoeffQ = MemberQ_K;
```

Check for constants:

In[5]:=
```
MemberQ_K[f_] := (Diff[f] === 0)
```

Check for membership in the function algebra:

In[6]:=
```
MemberQ_F[f_?MemberQ_K] := True
MemberQ_F[F[_Integer]] := True
MemberQ_F[Diff[f_?MemberQ_F]] := True
MemberQ_F[Int[f_?MemberQ_F]] := True
MemberQ_F[f_mul] := True
```

Check for functionals:

In[11]:=
```
MemberQ_Φ̃[Eval] := True
MemberQ_Φ̃[Φ̃[_Integer]] := True
```

Check for membership in the other cyclic modules:

In[13]:=
```
MemberQ_D[Diff] := True
```

In[14]:=
```
MemberQ_"I"[Int] := True
```

In[15]:=
```
MemberQ_"E"[Eval] := True
```

Function algebra

Multiplication

In[16]:=
```
mul[a___, b_Plus, c___] := (mul[a, #, c] & /@ b)
mul[a___, b_Integer, c__] := b mul[a, c]
mul[a__, b_Integer, c___] := b mul[a, c]
mul[a___, d_Integer * b_, c___] := d mul[a, b, c]
mul[a___, mul[b__], c___] := mul[a, b, c]
mul[a_] := a
```

In[22]:=
```
mul[a___, c_?MemberQ_K, b___] := c mul[a, b]
mul[a___, (c_?MemberQ_K) f_, b___] := c mul[a, f, b]
```

Differentiation

In[24]:=
```
Diff[f_?NumericQ] := 0
Diff[(φ_?MemberQ𝔖)[f_]] := 0
Diff[Int[f_?MemberQF]] := f
Diff[a_mul] := Sum[MapAt[Diff, a, i], {i, Length[a]}]
Diff[(f_?MemberQK) g_] := f Diff[g]
Diff[Power[f_?MemberQK, _Integer]] := 0
```

Integration

In[30]:=
```
Int[f_?(MemberQK[#] && (# =!= 1) &)] := f Int[1]
Int[Diff[f_?MemberQF]] := f - Eval[f]
```

In[32]:=
```
Int /: mul[a___, Int[f_], Int[g_], b___] := mul[a, Int[mul[Int[f], g]], b] +
  mul[a, Int[mul[f, Int[g]]], b] + mul[a, Eval[Hold[mul[Int[f], Int[g]]]], b]
```

Induced evaluation

In[33]:=
```
Eval[f_?MemberQK] := f
Eval[Int[f_?MemberQF]] := 0
```

Basic reduction rules

In[35]:=
```
hFF[f_, g_] := Prod[mul[f, g]]
hDF[Diff, f_] := Prod[Diff[f]] + Prod[f, Diff]
hDI[Diff, Int] := Prod[]
hID[Int, Diff] := Prod[] - Prod[Eval]
hEFE[Eval, f_, Eval] := Prod[Eval[f], Eval]
```

In[40]:=
```
hK[1] := Prod[]
```

In[41]:=
```
hDE[Diff, Eval] := 0
hEI[Eval, Int] := 0
hEE[Eval, Eval] := Prod[Eval]
hIFD[Int, f_, Diff] := Prod[f] - Prod[Eval, f] - Prod[Int, Diff[f]]
```

In[45]:=
```
hIFE[Int, f_, Eval] := Prod[Int[f], Eval]
hIFI[Int, f_, Int] :=
 Prod[Int[f], Int] - Prod[Int, Int[f]] - Prod[Eval, Int[f], Int]
```

In[47]:=
```
hIE[Int, Eval] := Prod[Int[1], Eval]
hII[Int, Int] := Prod[Int[1], Int] - Prod[Int, Int[1]] - Prod[Eval, Int[1], Int]
```

In[49]:=
```
RedSys = {{{F, F}, hFF}, {{D, F}, hDF}, {{D, "I"}, hDI}, {{"I", D}, hID},
    {{"E", F, "E"}, hEFE}, {{K}, hK}, {{D, "E"}, hDE}, {{"E", "I"}, hEI},
    {{"E", "E"}, hEE}, {{"I", F, D}, hIFD}, {{"I", F, "E"}, hIFE},
    {{"I", F, "I"}, hIFI}, {{"I", "E"}, hIE}, {{"I", "I"}, hII}};
```

Check Resolvability

In[50]:=
```
CheckResolvability[RedSys, Count → True]
```

```
54 ambiguities in total
4 ambiguities have all S-polynomials equal to zero
54 ambiguities are resolvable
```

Out[50]= {}

In[51]:= **CheckResolvability[RedSys, Count → True, Print → True]**

```
54 ambiguities in total
4 ambiguities have all S-polynomials equal to zero
1: Overlap[{F, F, F}, {F}, {F}]
{{F, F}, h_FF}
{{F, F}, h_FF}
2: Overlap[{D, F, F}, {F}, {D}]
{{D, F}, h_DF}
{{F, F}, h_FF}
{{D, F}, h_DF}
{{F, F}, h_FF}
{{F, F}, h_FF}
3: Overlap[{D, I, D}, {D}, {D}]
{{D, E}, h_DE}
4: Overlap[{D, I, F, D}, {F, D}, {D}]
{{D, F}, h_DF}
{{D, E}, h_DE}
{{D, I}, h_DI}
5: Overlap[{D, I, F, E}, {F, E}, {D}]
{{D, F}, h_DF}
{{D, E}, h_DE}
6: Overlap[{D, I, F, I}, {F, I}, {D}]
{{D, I}, h_DI}
{{D, F}, h_DF}
{{D, E}, h_DE}
{{D, I}, h_DI}
7: Overlap[{D, I, E}, {E}, {D}]
{{D, F}, h_DF}
{{K}, h_K}
{{D, E}, h_DE}
8: Overlap[{D, I, I}, {I}, {D}]
{{D, I}, h_DI}
{{D, F}, h_DF}
{{D, E}, h_DE}
```

{{K}, h_K}

{{D, I}, h_{DI}}

9: Overlap[{I, D, F}, {F}, {I}]

{{I, F, D}, h_{IFD}}

10: Overlap[{I, D, I}, {I}, {I}]

{{E, I}, h_{EI}}

11: Overlap[{I, D, E}, {E}, {I}]

{{E, E}, h_{EE}}

12: Overlap[{E, F, E, F, E}, {F, E}, {E, F}]

{{E, F, E}, h_{EFE}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{E, F, E}, h_{EFE}}

13: Overlap[{E, F, E, I}, {I}, {E, F}]

{{K}, h_K}

{{E, I}, h_{EI}}

14: Overlap[{E, F, E, E}, {E}, {E, F}]

{{K}, h_K}

{{E, F, E}, h_{EFE}}

{{K}, h_K}

{{E, E}, h_{EE}}

15: Overlap[{D, E, F, E}, {F, E}, {D}]

{{D, F}, h_{DF}}

{{K}, h_K}

{{D, E}, h_{DE}}

16: Overlap[{D, E, I}, {I}, {D}]

17: Overlap[{D, E, E}, {E}, {D}]

{{D, E}, h_{DE}}

18: Overlap[{E, I, D}, {D}, {E}]

{{E, E}, h_{EE}}

19: Overlap[{E, I, F, D}, {F, D}, {E}]

{{E, E}, h_{EE}}

{{E, I}, h_{EI}}

20: Overlap[{E, I, F, E}, {F, E}, {E}]

{{E, F, E}, h_{EFE}}

21: Overlap[{E, I, F, I}, {F, I}, {E}]

{{E, I}, h_{EI}}

{{E, E}, h_{EE}}

22: Overlap[{E, I, E}, {E}, {E}]

{{E, F, E}, h_{EFE}}

23: Overlap[{E, I, I}, {I}, {E}]

{{E, I}, h_{EI}}

{{E, E}, h_{EE}}

24: Overlap[{E, E, F, E}, {F, E}, {E}]

{{E, F, E}, h_{EFE}}

{{E, F, E}, h_{EFE}}

25: Overlap[{E, E, I}, {I}, {E}]

{{E, I}, h_{EI}}

26: Overlap[{E, E, E}, {E}, {E}]

27: Overlap[{I, F, D, F}, {F}, {I, F}]

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{I, F, D}, h_{IFD}}

28: Overlap[{I, F, D, I}, {I}, {I, F}]

{{I, F, I}, h_{IFI}}

{{K}, h_{K}}

{{K}, h_{K}}

{{K}, h_{K}}

{{E, I}, h_{EI}}

29: Overlap[{I, F, D, E}, {E}, {I, F}]

{{E, F, E}, h_{EFE}}

{{I, F, E}, h_{IFE}}

30: Overlap[{I, F, E, F, E}, {F, E}, {I, F}]

{{F, F}, h_{FF}}

{{E, F, E}, h_{EFE}}

{{I, F, E}, h_{IFE}}

{{F, F}, h_{FF}}

31: Overlap[{I, F, E, I}, {I}, {I, F}]

{{E, I}, h_{EI}}

32: Overlap[{I, F, E, E}, {E}, {I, F}]

{{I, F, E}, h_{IFE}}

{{E, E}, h_{EE}}

33: Overlap[{I, F, I, D}, {D}, {I, F}]

{{I, F, E}, h_{IFE}}

{{I, F, D}, h_{IFD}}

{{I, D}, h_{ID}}

{{I, D}, h_{ID}}

{{E, F, E}, h_{EFE}}

34: Overlap[{I, F, I, F, D}, {F, D}, {I, F}]

{{F, F}, h_{FF}}

{{I, F, E}, h_{IFE}}

{{I, F, I}, h_{IFI}}

{{F, F}, h_{FF}}

{{I, F, D}, h_{IFD}}

{{I, F, D}, h_{IFD}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{I, F, D}, h_{IFD}}

{{E, F, E}, h_{EFE}}

35: Overlap[{I, F, I, F, E}, {F, E}, {I, F}]

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{I, F, E}, h_{IFE}}

{{I, F, E}, h_{IFE}}

{{I, F, E}, h_{IFE}}

{{I, F, E}, h_{IFE}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{K}, h_{K}}

{{E, F, E}, h_{EFE}}

{{E, F, E}, h_{EFE}}

{{E, F, E}, h_{EFE}}

{{K}, h_{K}}

36: Overlap[{I, F, I, F, I}, {F, I}, {I, F}]

{{I, F, I}, h_{IFI}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{I, F, I}, h_{IFI}}

{{I, F, I}, h_{IFI}}

{{I, F, E}, h_{IFE}}

{{F, F}, h_{FF}}

{{I, F, I}, h_{IFI}}

{{I, F, I}, h_{IFI}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{E, F, E}, h_{EFE}}

{{K}, h_{K}}

{{K}, h_{K}}

{{K}, h_{K}}

{{E, I}, h_{EI}}

37: Overlap[{I, F, I, E}, {E}, {I, F}]

{{I, F, E}, h_{IFE}}

{{I, E}, h_{IE}}

{{I, E}, h_{IE}}

{{F, F}, h_{FF}}

{{I, F, E}, h_{IFE}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{K}, h_{K}}

{{E, F, E}, h_{EFE}}

{{E, F, E}, h_{EFE}}

{{E, F, E}, h_{EFE}}

{{K}, h_{K}}

38: Overlap[{I, F, I, I}, {I}, {I, F}]

{{I, F, I}, h_{IFI}}

{{I, I}, h_{II}}

{{I, I}, h_{II}}

{{I, F, I}, h_{IFI}}

{{F, F}, h_{FF}}

{{I, F, E}, h_{IFE}}

{{F, F}, h_{FF}}

{{I, F, I}, h_{IFI}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{E, F, E}, h_{EFE}}

{{K}, h_{K}}

{{K}, h_{K}}

{{K}, h_{K}}

{{E, I}, h_{EI}}

39: Overlap[{I, E, F, E}, {F, E}, {I}]

{{K}, h_{K}}

{{E, F, E}, h_{EFE}}

{{I, E}, h_{IE}}

{{F, F}, h_{FF}}

40: Overlap[{I, E, I}, {I}, {I}]

{{E, I}, h_{EI}}

41: Overlap[{I, E, E}, {E}, {I}]

{{I, E}, h_{IE}}

{{E, E}, h_{EE}}

42: Overlap[{I, I, D}, {D}, {I}]

{{I, E}, h_{IE}}

{{I, F, D}, h_{IFD}}

{{I, D}, h_{ID}}

{{I, D}, h_{ID}}

{{K}, h_{K}}

{{E, F, E}, h_{EFE}}

43: Overlap[{I, I, F, D}, {F, D}, {I}]

{{I, E}, h_{IE}}

{{I, I}, h_{II}}

{{F, F}, h_{FF}}

{{I, F, D}, h_{IFD}}

{{I, F, D}, h_{IFD}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{I, F, D}, h_{IFD}}

{{E, F, E}, h_{EFE}}

44: Overlap[{I, I, F, E}, {F, E}, {I}]

{{I, F, E}, h_{IFE}}

{{F, F}, h_{FF}}

{{I, F, E}, h_{IFE}}

{{I, F, E}, h_{IFE}}

{{I, F, E}, h_{IFE}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{K}, h_{K}}

{{E, F, E}, h_{EFE}}

{{E, F, E}, h_{EFE}}

{{E, F, E}, h_{EFE}}

{{K}, h_{K}}

45: Overlap[{I, I, F, I}, {F, I}, {I}]

{{I, I}, h_{II}}

{{I, F, I}, h_{IFI}}

{{I, E}, h_{IE}}

{{F, F}, h_{FF}}

{{I, F, I}, h_{IFI}}

{{I, F, I}, h_{IFI}}

{{F, F}, h_{FF}}

{{I, F, I}, h_{IFI}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{E, F, E}, h_{EFE}}

{{K}, h_{K}}

{{K}, h_{K}}

{{K}, h_{K}}

{{E, I}, h_{EI}}

46: Overlap[{I, I, E}, {E}, {I}]

{{I, F, E}, h_{IFE}}

{{I, E}, h_{IE}}

{{I, E}, h_{IE}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{K}, h_{K}}

{{E, F, E}, h_{EFE}}

{{E, F, E}, h_{EFE}}

{{K}, h_{K}}

47: Overlap[{I, I, I}, {I}, {I}]

{{I, I}, h_{II}}

{{I, F, I}, h_{IFI}}

{{I, I}, h_{II}}

{{I, I}, h_{II}}

{{I, E}, h_{IE}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{F, F}, h_{FF}}

{{E, F, E}, h_{EFE}}

{{K}, h_{K}}

{{K}, h_{K}}

{{K}, h_{K}}

{{E, I}, h_{EI}}

```
48: SpecialInclusion[{K, F}, {}, {F}]
49: SpecialInclusion[{F, K}, {F}, {}]
50: SpecialInclusion[{D, K}, {D}, {}]
{{K}, h_K}
51: SpecialInclusion[{E, K, E}, {E}, {E}]
{{K}, h_K}
{{E, E}, h_EE}
52: SpecialInclusion[{I, K, D}, {I}, {D}]
{{K}, h_K}
{{K}, h_K}
{{I, D}, h_ID}
53: SpecialInclusion[{I, K, E}, {I}, {E}]
{{I, E}, h_IE}
54: SpecialInclusion[{I, K, I}, {I}, {I}]
{{I, I}, h_II}
54 ambiguities are resolvable
```

Out[51]= {}

Irreducible Words

In[68]:=
```
IrreducibleWords[ExtractReducibleWords[RedSys], 5]
```

Out[68]=
$$\{\{\tilde{F}\}, \{D\}, \{I\}, \{E\}, \{\tilde{F}, D\}, \{\tilde{F}, I\}, \{\tilde{F}, E\}, \{D, D\}, \{I, \tilde{F}\}, \{E, \tilde{F}\}, \{E, D\}, \{\tilde{F}, D, D\}, \{\tilde{F}, I, \tilde{F}\}, \{\tilde{F}, E, \tilde{F}\}, \{\tilde{F}, E, D\}, \{D, D, D\}, \{E, \tilde{F}, D\}, \{E, \tilde{F}, I\}, \{E, D, D\}, \{\tilde{F}, D, D, D\}, \{\tilde{F}, E, \tilde{F}, D\}, \{\tilde{F}, E, \tilde{F}, I\}, \{\tilde{F}, E, D, D\}, \{D, D, D, D\}, \{E, \tilde{F}, D, D\}, \{E, \tilde{F}, I, \tilde{F}\}, \{E, D, D, D\}, \{\tilde{F}, D, D, D, D\}, \{\tilde{F}, E, \tilde{F}, D, D\}, \{\tilde{F}, E, \tilde{F}, I, \tilde{F}\}, \{\tilde{F}, E, D, D, D\}, \{D, D, D, D, D\}, \{E, \tilde{F}, D, D, D\}, \{E, D, D, D, D\}\}$$