Some Progress on The Sweep Map

Guoce Xin
Capital Normal University (CNU)

Joint work with Adriano Garsia (UCSD)
Joint work with Yingrui Zhang (CNU)

Computer Algebra in Combinatorics
Nov. 13-17, 2017

Outline

- Human linear inverting algorithm for the sweep map on Fuss Dyck paths.
  - Standard Tableaux

- About the sweep map on (m,n)-Dyck paths.

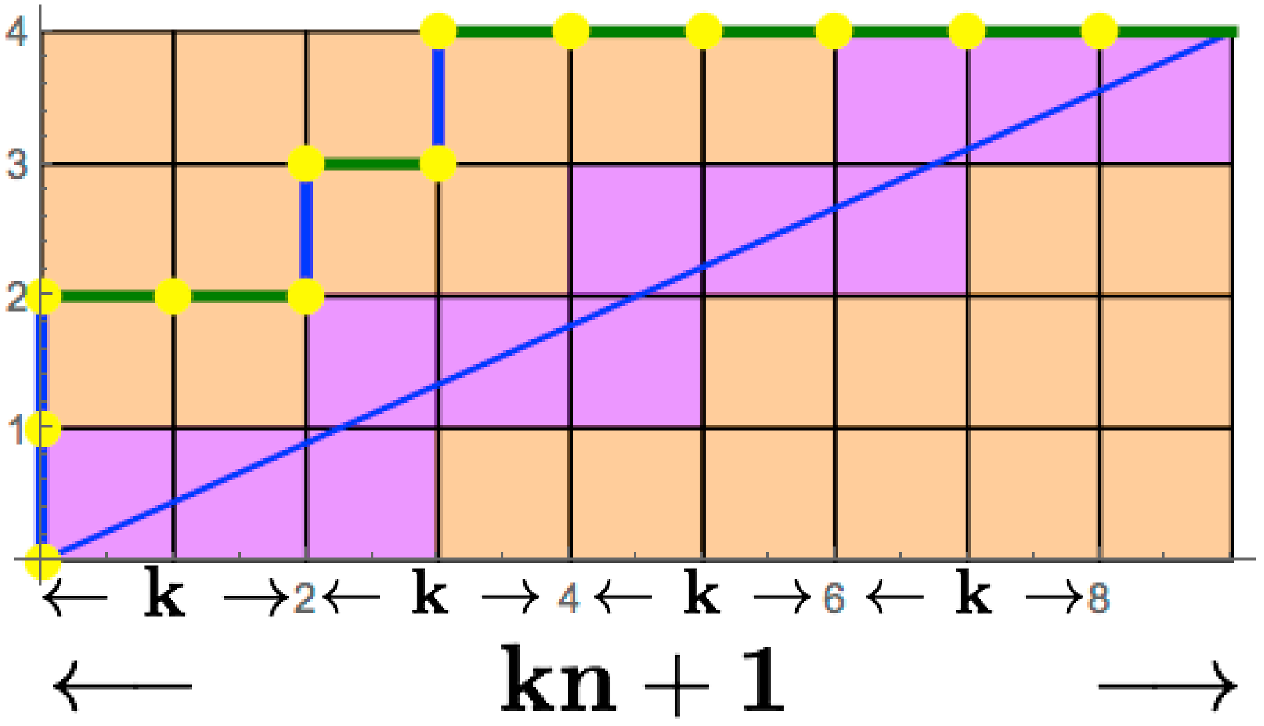
- An extension of the Human algorithm.
  - An example using Tableaux

A Fuss k, n Dyck path

For $k = 2$ and $n = 4$

$$D = \{0, 0, 2, 3\} =$$

$$\mathbf{m} = \mathbf{kn} + \mathbf{1}$$

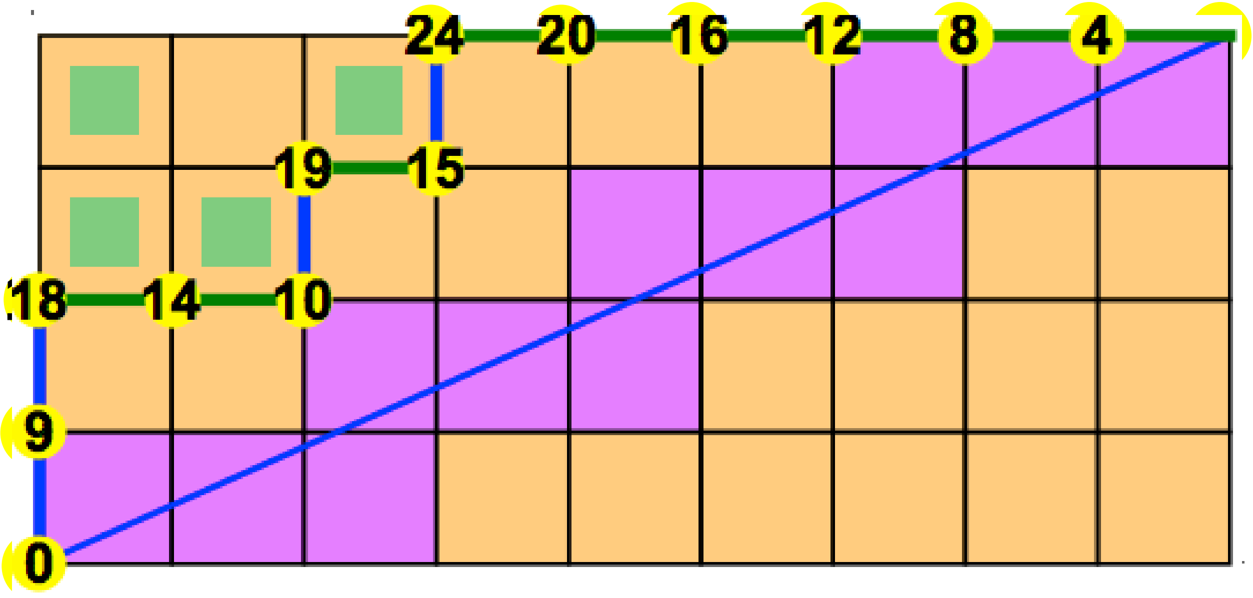


The Ranking Algorithm

(1) Start with rank 0

(2) Add **m** after a North step

(3) Subtract **n** after a East step



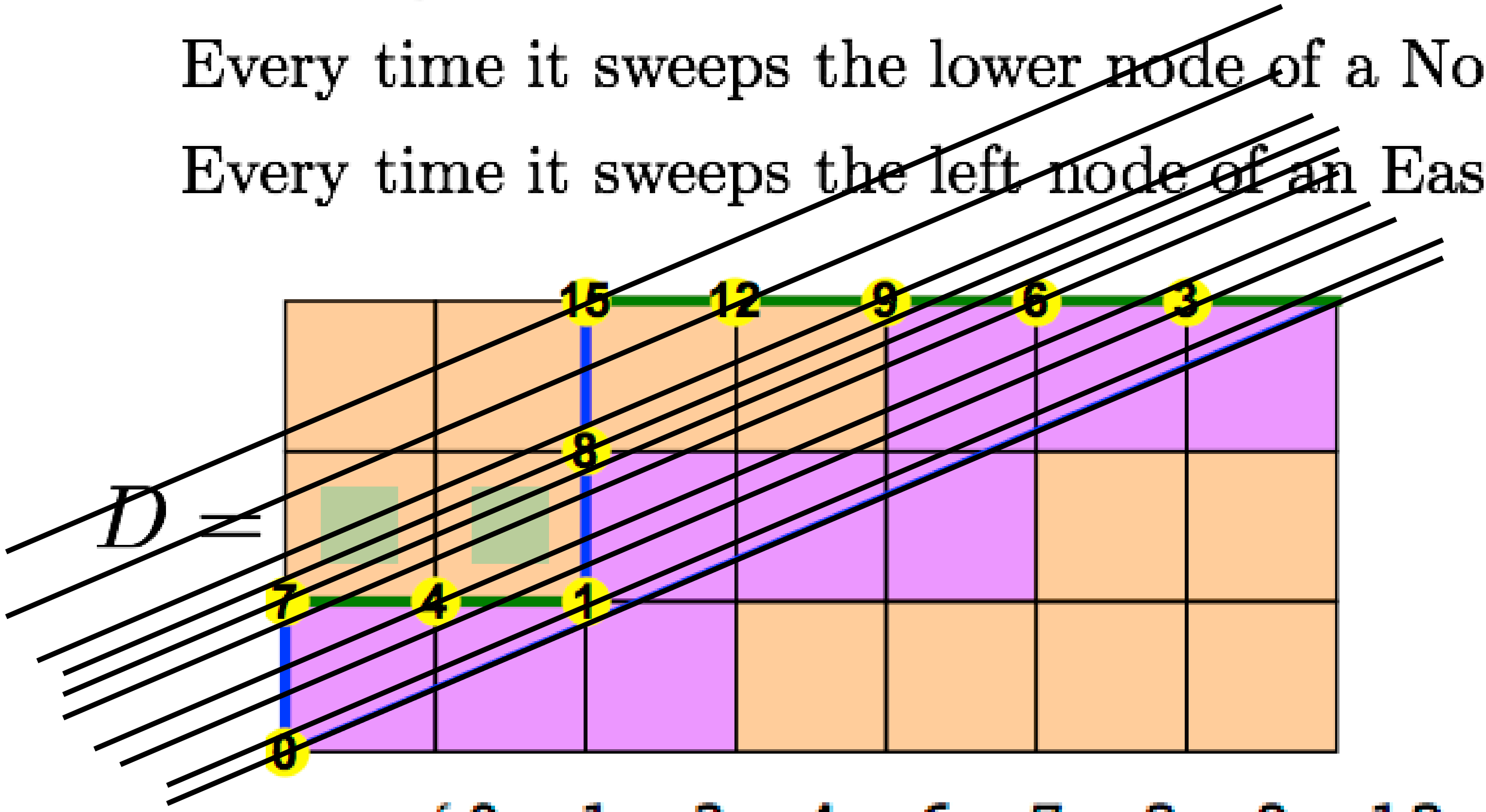
The Sweep Map

The diagonal sweeps the nodes of the path

Every time it sweeps the lower node of a North step **Draw** |

Every time it sweeps the left node of an East step **Draw** —

$D =$



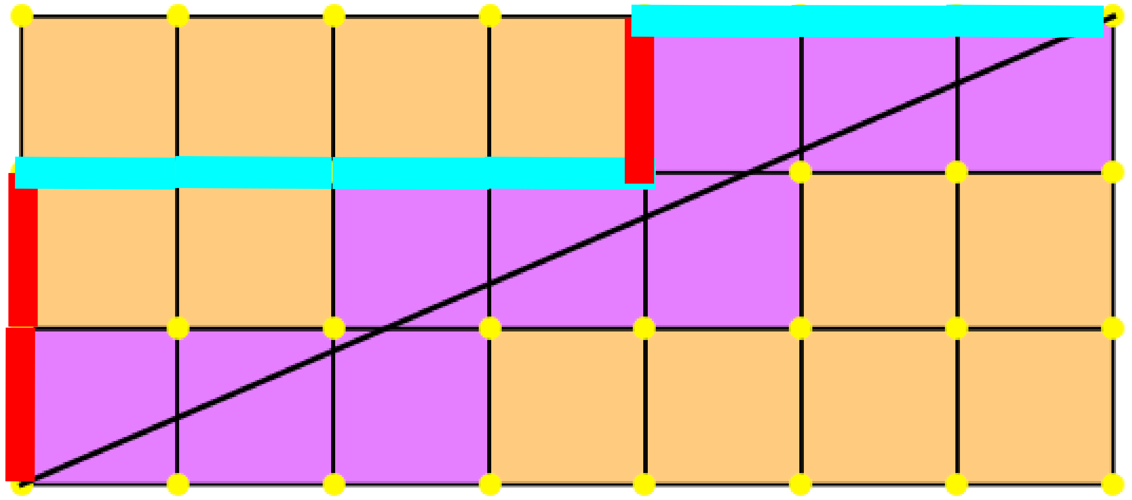
{0, 1, 3, 4, 6, 7, 8, 9, 12, 15}

Theorem

$\mathbf{area(D) = bounce(\Phi(D))}$

$\mathbf{dinv(D) = area(\Phi(D))}$

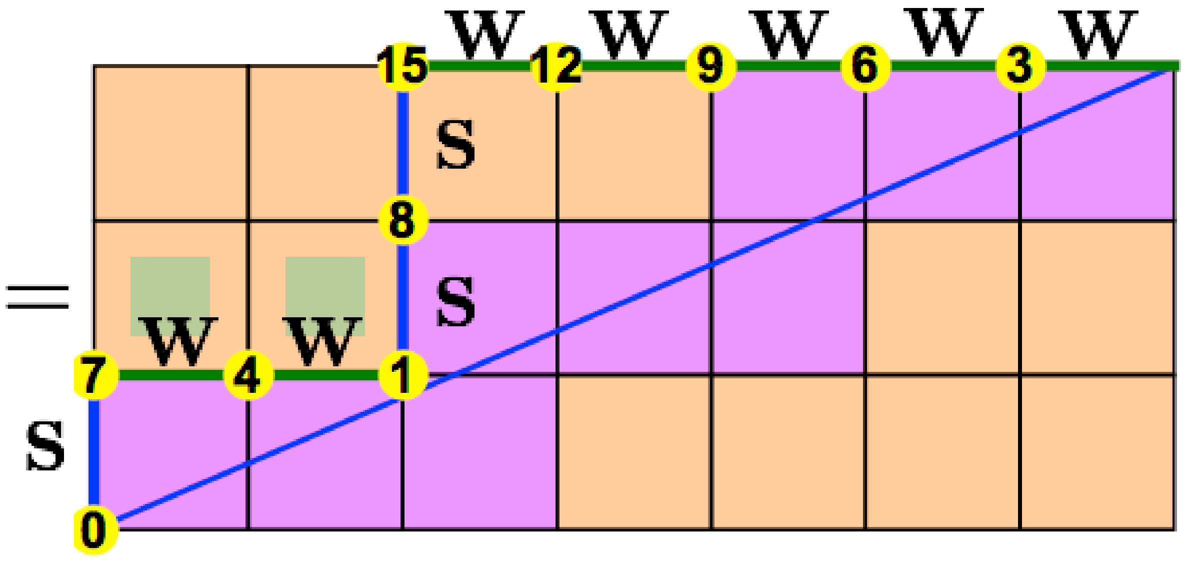
$\Phi[D] =$



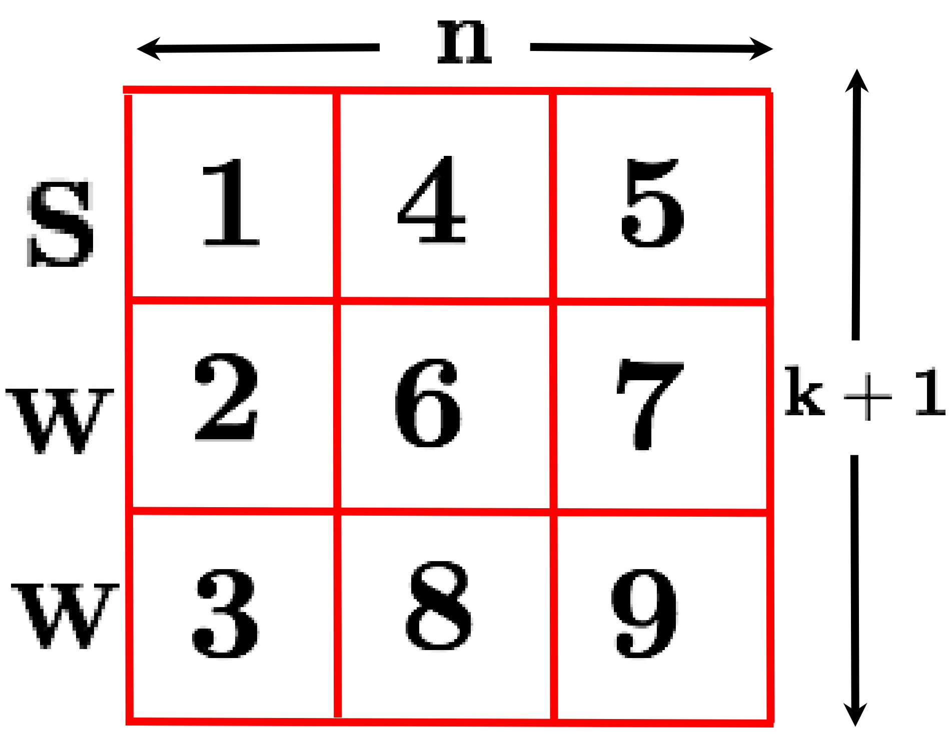
$$\sum_{\mathbf{D \in F_{k,n}}} \mathbf{t^{area(D)} q^{dinv(D)}} = \sum_{\mathbf{D \in F_{k,n}}} \mathbf{t^{bounce(\Phi(D))} q^{area(\Phi(D))}}$$

Fuss Dyck paths $\Longleftrightarrow$ Standard Tableaux

$D =$



Call North steps **S** and East steps **W**



S	**1**	**4**	**5**
W	**2**	**6**	**7**
W	**3**	**8**	**9**

10

Algorithm:

S $\longrightarrow$ start a new column

Labels not in row $k+1$ at bottom of columns are active

W $\longrightarrow$ place next label under smallest active label

Theorem

$\mathbf{k}, \mathbf{n}$-Fuss Dyck paths are in bijection with $(\mathbf{k}+\mathbf{1}) \times \mathbf{n}$ Tableaux

Characteristic properties:

(i) Row increasing left to right

(ii) Column increasing top to bottom

(iii) $\mathbf{a} < \mathbf{b} < \mathbf{c} < \mathbf{d}$ with $\mathbf{a}$ immediately above $\mathbf{d}$

then $\mathbf{b}$ and $\mathbf{c}$ are not in the same column

Magics of $\mathbf{T_{k,n}}$ Tableaux

The "Flip" Algorithm

(1) Reverse order of columns

(2) Reverse order of rows

(3) Complement the entries

1	**2**	**6**	**9**	**12**
3	4	8	11	14
5	7	10	13	15

16

⇔

1	**3**	**6**	**9**	**11**
2	5	8	12	13
4	7	10	14	15

16

Theorem

Flip is an involution of $\mathbf{T_{k,n}}$

Easy Fact:

The first row determines the rest

Corollary

The bottom row determines the rest

1	**3**	**6**	**9**	**11**
2	5	8	12	13
4	7	10	14	15

⇔

1	**2**	**6**	**9**	**12**
3	4	8	11	14
5	7	10	13	15

Characteristic properties:

$\mathbf{t_1, t_2, \ldots, t_n}$ are entries of the top row of a $T \in \mathcal{T}_{k,n}$ if and only if

$$\mathbf{t_i \leq i + k(i-1)} \qquad \forall \;\; \mathbf{1 \leq i \leq n}$$

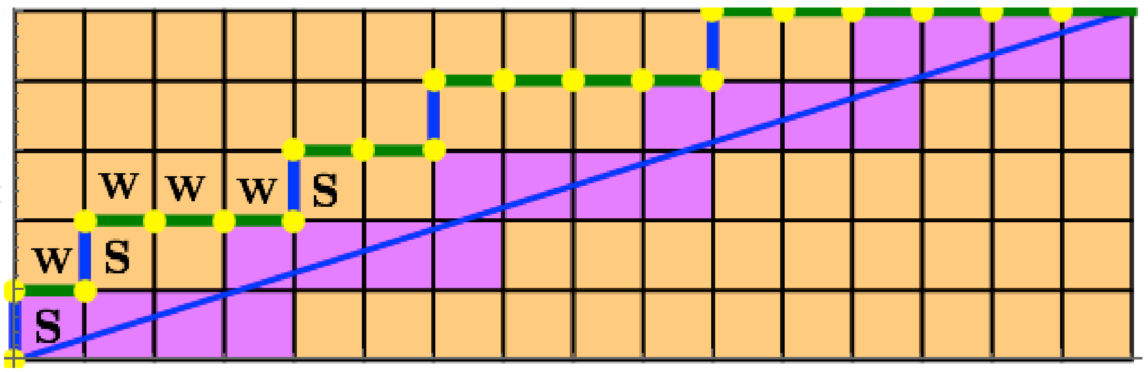
$\mathbf{b_1, b_2, \ldots, b_n}$ are entries of the bottom row of a $T \in \mathcal{T}_{k,n}$ if and only if

$$\mathbf{b_i \geq (k+1)i} \qquad \forall \;\; \mathbf{1 \leq i \leq n}$$

Inverting the Sweep Map

$\mathbf{k = 3 \quad n = 5 \quad m = 16}$

$D =$



$\mathbf{T(D)} =$

1	**3**	**⑦**	**10**	**⑮**
2	5	9	13	17
4	8	⑫	16	⑲
6	11	14	18	20
				㉑

A walk through $T(D)$ yields $\Phi^{-1}(D)$

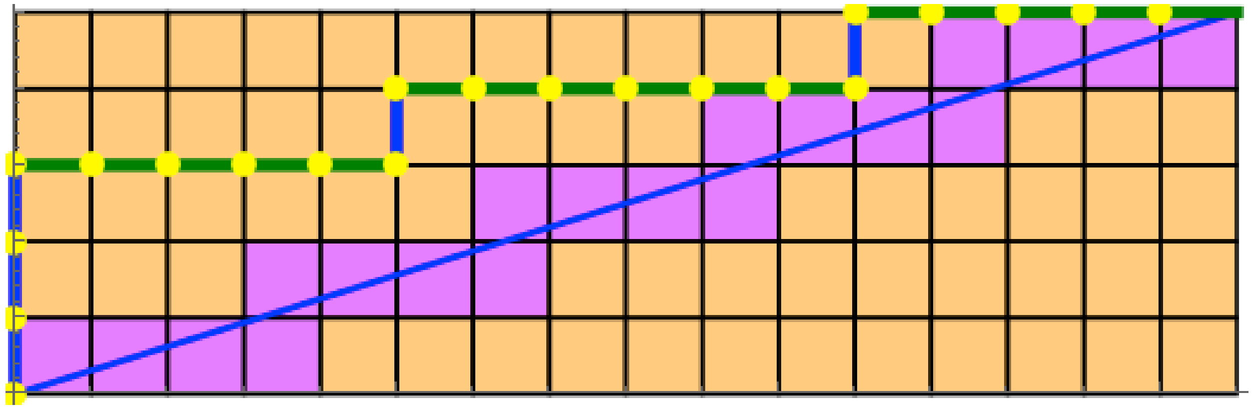
Algorithm:

- Circle entries 1 more than bottom row entries
- Write down all the entries you touch
- Touch 1
- From row one drop to bottom row and touch the circled letter
- If you are not in row one and the next entry up is not circled touch it
- If r is circled visit $r-1$ and continue... Touch only an uncircled letter

Here are all the letters you touch in this walk

1	7	15	21	20	18	16	13	10	19	17	14	11	8	5	3	12	9	6	4	2
S	**S**	**S**						**S**							**S**					
1	**2**	**3**						**9**							**16**					

$\mathbf{\Phi^{-1}(D) = \{0,0,0,5,11\}} =$



Reducing a k, n-Tableau

$$\begin{pmatrix} 1 & 3 & 7 & 10 & 15 \\ 2 & 5 & 9 & 13 & 17 \\ 4 & 8 & 12 & 16 & 19 \\ 6 & 11 & 14 & 18 & 20 \end{pmatrix}$$

- Remove first column
- Reduce the entries

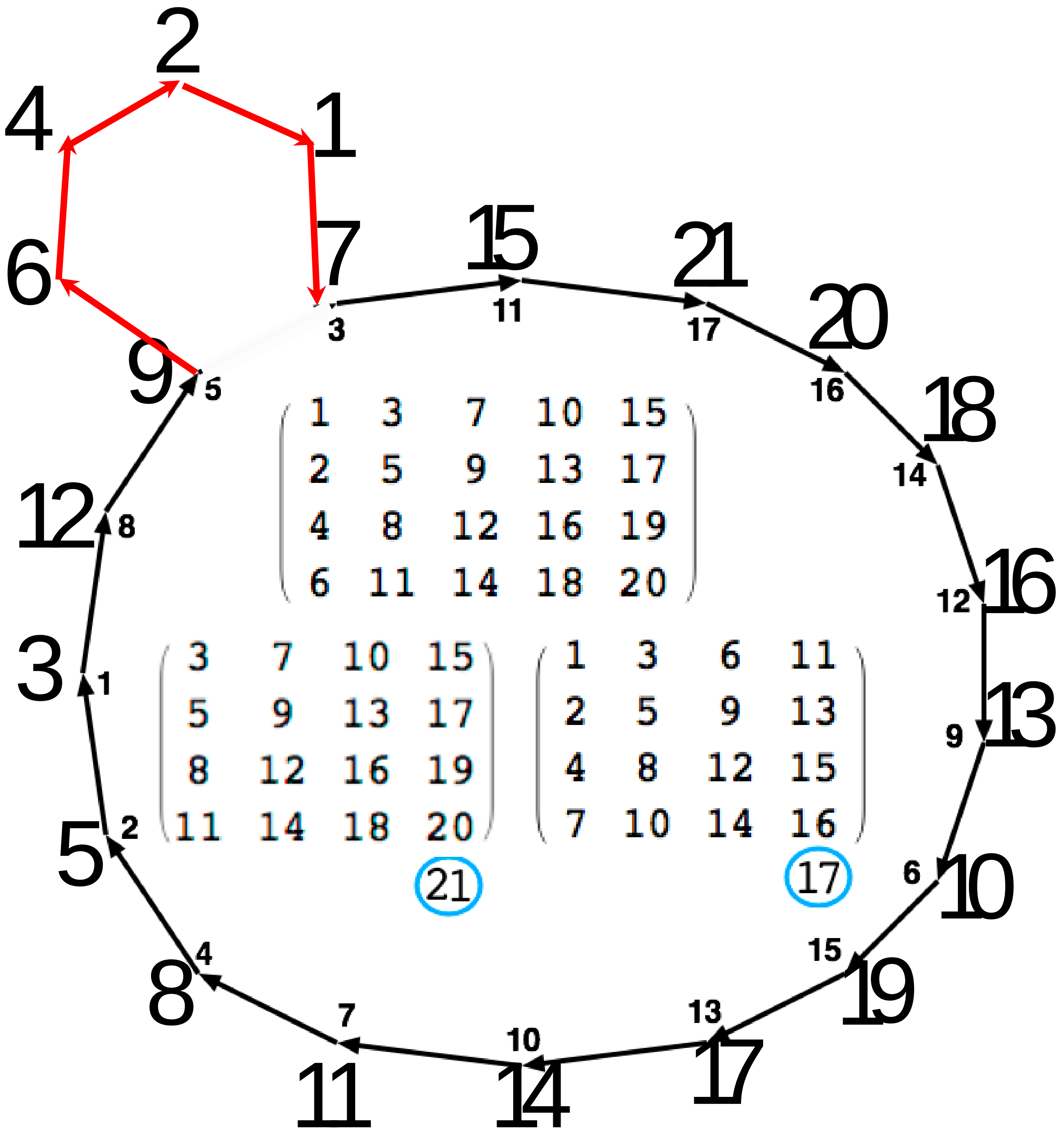
$$\begin{matrix} \begin{pmatrix} 3 & 7 & 10 & \enclose{circle}{15} \\ 5 & 9 & 13 & 17 \\ 8 & \enclose{circle}{12} & 16 & \enclose{circle}{19} \\ 11 & 14 & 18 & 20 \end{pmatrix} \\ \qquad\qquad\quad \enclose{circle}{21} \end{matrix} \Rightarrow \begin{matrix} \begin{pmatrix} 1 & 3 & 6 & \enclose{circle}{11} \\ 2 & 5 & 9 & 13 \\ 4 & \enclose{circle}{8} & 12 & \enclose{circle}{15} \\ 7 & 10 & 14 & 16 \end{pmatrix} \\ \qquad\qquad\quad \enclose{circle}{17} \end{matrix}$$

$$\mathbf{T_{k,n}} \Longrightarrow \mathbf{T_{k,n-1}}$$

1	8	5	3	11	17	16	14	12	9	6	15	13	10	7	4	2
3	12	9	7	15	21	20	18	16	13	10	19	17	14	11	8	5

Cycle Proof by Animation

3	12	9	7	15	21	20	18	16	13	10	19	17	14	11	8	5
1	8	5	3	11	17	16	14	12	9	6	15	13	10	7	4	2



$$\begin{pmatrix} 1 & 3 & 7 & 10 & 15 \\ 2 & 5 & 9 & 13 & 17 \\ 4 & 8 & 12 & 16 & 19 \\ 6 & 11 & 14 & 18 & 20 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 7 & 10 & 15 \\ 5 & 9 & 13 & 17 \\ 8 & 12 & 16 & 19 \\ 11 & 14 & 18 & 20 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 6 & 11 \\ 2 & 5 & 9 & 13 \\ 4 & 8 & 12 & 15 \\ 7 & 10 & 14 & 16 \end{pmatrix}$$

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1	7	15	21	20	18	16	13	10	19	17	14	11	8	5	3	12	9	6	4	2

Inversion Proof by Animation

0	16	32	48	43	38	33	28	23	39	34	29	24	19	14	9	25	20	15	10	5
1	7	15	21	20	18	16	13	10	19	17	14	11	8	5	3	12	9	6	4	2
S	**S**	**S**	**W**	**W**	**W**	**W**	**W**	**S**	**W**	**W**	**W**	**W**	**W**	**W**	**S**	**W**	**W**	**W**	**W**	**W**

Why this rearrangement of the SW sequence of D gives $\Phi^{-1}(D)$?

For a moment, call this path $\widetilde{D}$ and compute its rank sequence

Let us now read these ranks in the order given by the large labels

0	5	9	10	14	15	16	19	20	23	24	25	28	29	32	33	34	38	39	43	48

Their increasing rearrangement!!!

Recall that the large labels came from

$$\mathbf{T(D)} = \begin{array}{|c|c|c|c|c|}\hline \mathbf{1} & \mathbf{3} & \textcircled{\mathbf{7}} & \mathbf{10} & \textcircled{\mathbf{15}} \\ \hline 2 & 5 & 9 & 13 & 17 \\ \hline 4 & 8 & \textcircled{12} & 16 & \textcircled{19} \\ \hline 6 & 11 & 14 & 18 & 20 \\ \hline \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\textcircled{21}} \end{array}$$

This Tableau was constructed

by reading the **SW** sequence of **D**

Thus the **SW** sequence of **D**

sorts the ranks of $\widetilde{\mathbf{D}}$

This proves that $\widetilde{\mathbf{D}} = \Phi^{-1}(\mathbf{D})$

For $\mathbf{T'} \in \mathcal{T}_{\mathbf{k,n-1}}$, solve the equation $\mathbf{redT(D) = T'}$

Theorem

The number of solution is given by the entry at the bottom of the first column of T'

Algorithm

1) Construct the bottom rows of all solutions $\mathbf{T}$ of

$$\boxed{\mathbf{red(T) = T'}}$$

$$\mathbf{T'} = \begin{pmatrix} 1 & 3 & 6 & 11 \\ 2 & 5 & 9 & 13 \\ 4 & 8 & 12 & 15 \\ \textcircled{7} & 10 & 14 & 16 \end{pmatrix}$$

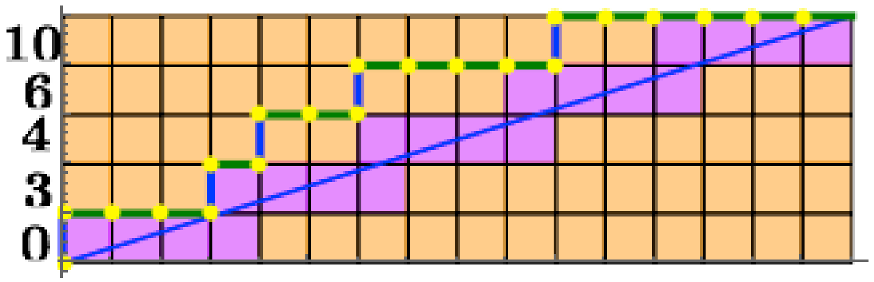
2) Use the "Flip" Algorithm to construct all their first rows

3) Use the resulting first rows to obtain all the corresponding paths $D \in \mathcal{T}_{k,n}$

Example :

$$red \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ \mathbf{x} & \mathbf{11} & \mathbf{14} & \mathbf{18} & \mathbf{20} \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 & 11 \\ 2 & 5 & 9 & 13 \\ 4 & 8 & 12 & 15 \\ 7 & 10 & 14 & 16 \end{pmatrix} \Rightarrow \boxed{\mathbf{4 \le x < 11}}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{7} & \mathbf{10} & \mathbf{17} \\ 2 & 5 & 9 & 13 & 18 \\ 4 & 8 & 12 & 15 & 19 \\ 6 & 11 & 14 & 16 & 20 \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{1} & 5 & \mathbf{7} & \mathbf{10} & 15 \\ 2 & 6 & 9 & 13 & 17 \\ 3 & 8 & 12 & 16 & 19 \\ \mathbf{4} & \mathbf{11} & \mathbf{14} & \mathbf{18} & \mathbf{20} \end{pmatrix} \Rightarrow \mathbf{D} =$$



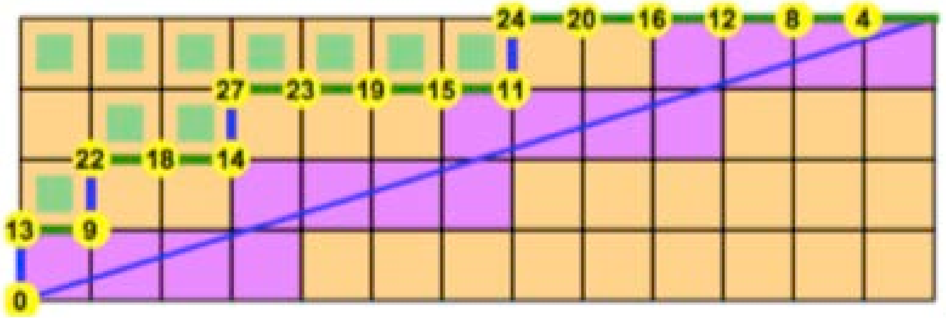
Solution based on our recursive algorithm

Algorithm

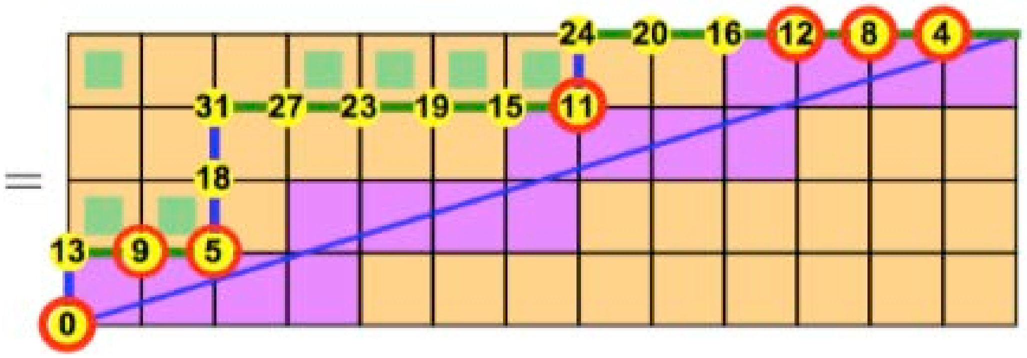
1. Construct the the path $\mathrm{D}' \in \mathcal{D}_{\mathrm{k(n-1)+1,n-1}}$ that corresponds to T'.
2. Construct the pree-image $\Phi^{-1}(\mathrm{D}')$ and its successive ranks.
3. Circle all the ranks of $\Phi^{-1}(\mathrm{D}')$ that are less than $\mathrm{k(n-1)+1}$.
4. Cut $\Phi^{-1}(\mathrm{D}')$ at the circled ranks and reverse the order of the two pieces.
5. Prepend to each of these paths a North step and append k East steps.
6. To obtain the desired D take the Φ images of the resulting Dyck paths.

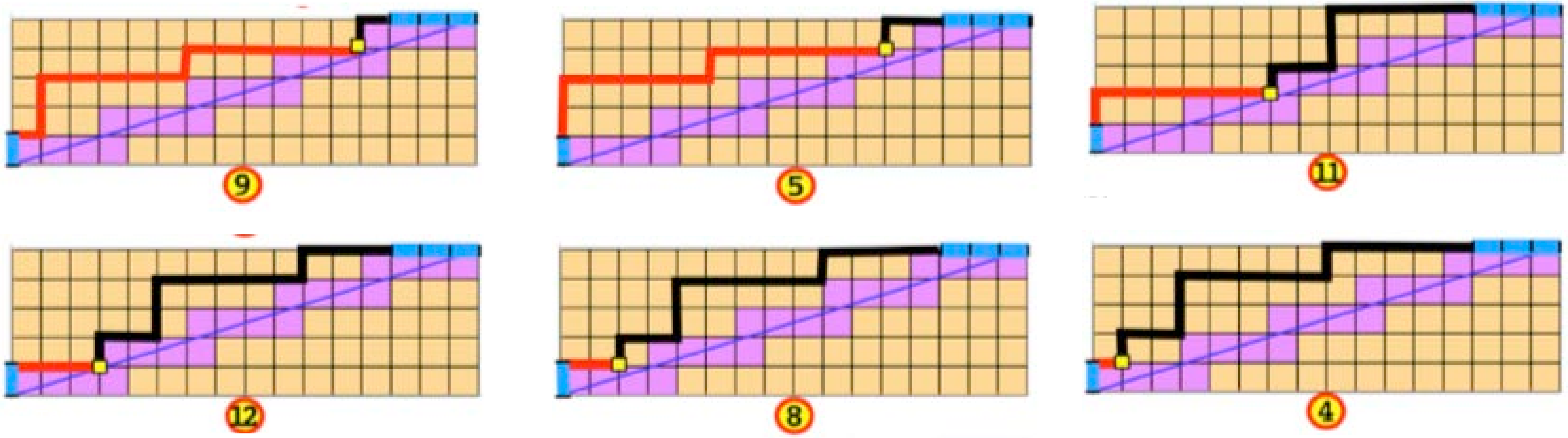
$$T' = \begin{pmatrix} 1 & 3 & 6 & 11 \\ 2 & 5 & 9 & 13 \\ 4 & 8 & 12 & 15 \\ 7 & 10 & 14 & 16 \end{pmatrix}$$

$D' =$

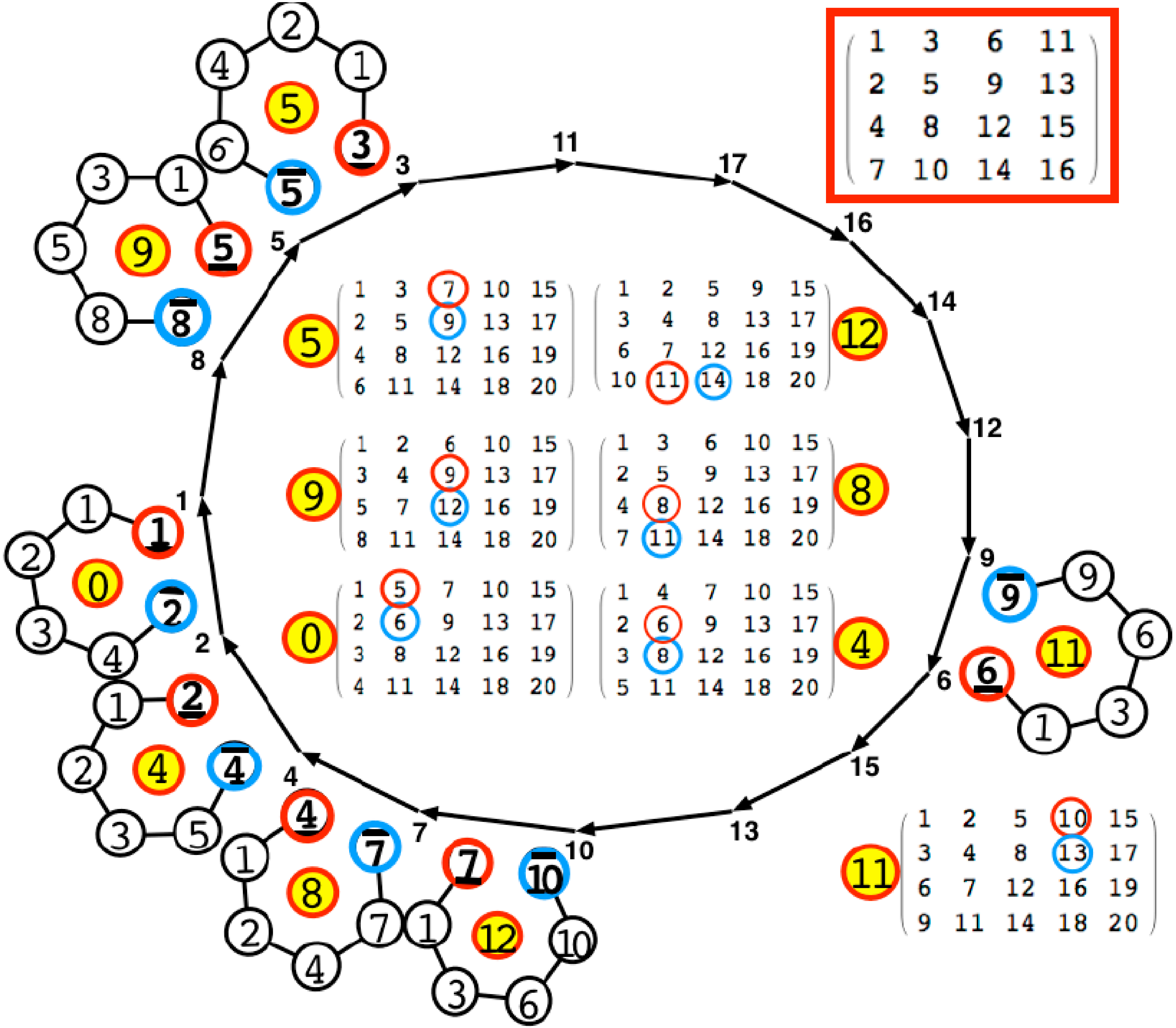


$\Phi^{-1}(D') =$





The Final Product



$$\boxed{\begin{pmatrix} 1 & 3 & 6 & 11 \\ 2 & 5 & 9 & 13 \\ 4 & 8 & 12 & 15 \\ 7 & 10 & 14 & 16 \end{pmatrix}}$$

5 $\begin{pmatrix} 1 & 3 & 7 & 10 & 15 \\ 2 & 5 & 9 & 13 & 17 \\ 4 & 8 & 12 & 16 & 19 \\ 6 & 11 & 14 & 18 & 20 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 5 & 9 & 15 \\ 3 & 4 & 8 & 13 & 17 \\ 6 & 7 & 12 & 16 & 19 \\ 10 & 11 & 14 & 18 & 20 \end{pmatrix}$ 12

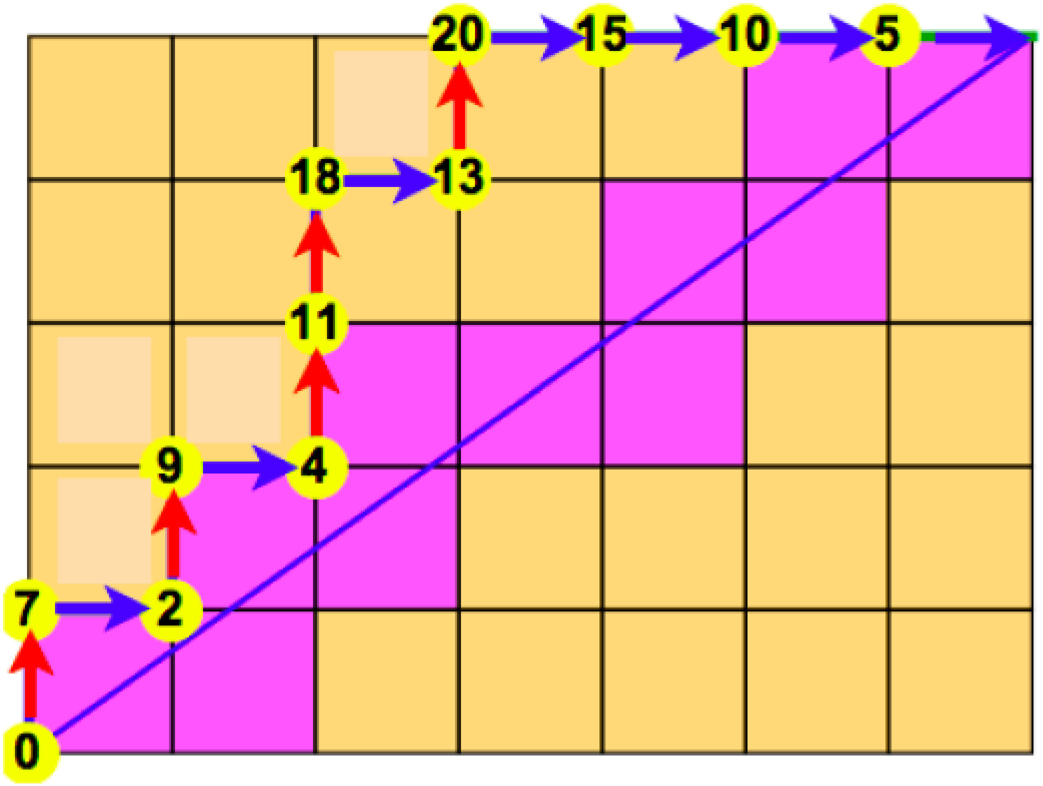
9 $\begin{pmatrix} 1 & 2 & 6 & 10 & 15 \\ 3 & 4 & 9 & 13 & 17 \\ 5 & 7 & 12 & 16 & 19 \\ 8 & 11 & 14 & 18 & 20 \end{pmatrix}$ $\begin{pmatrix} 1 & 3 & 6 & 10 & 15 \\ 2 & 5 & 9 & 13 & 17 \\ 4 & 8 & 12 & 16 & 19 \\ 7 & 11 & 14 & 18 & 20 \end{pmatrix}$ 8

0 $\begin{pmatrix} 1 & 5 & 7 & 10 & 15 \\ 2 & 6 & 9 & 13 & 17 \\ 3 & 8 & 12 & 16 & 19 \\ 4 & 11 & 14 & 18 & 20 \end{pmatrix}$ $\begin{pmatrix} 1 & 4 & 7 & 10 & 15 \\ 2 & 6 & 9 & 13 & 17 \\ 3 & 8 & 12 & 16 & 19 \\ 5 & 11 & 14 & 18 & 20 \end{pmatrix}$ 4

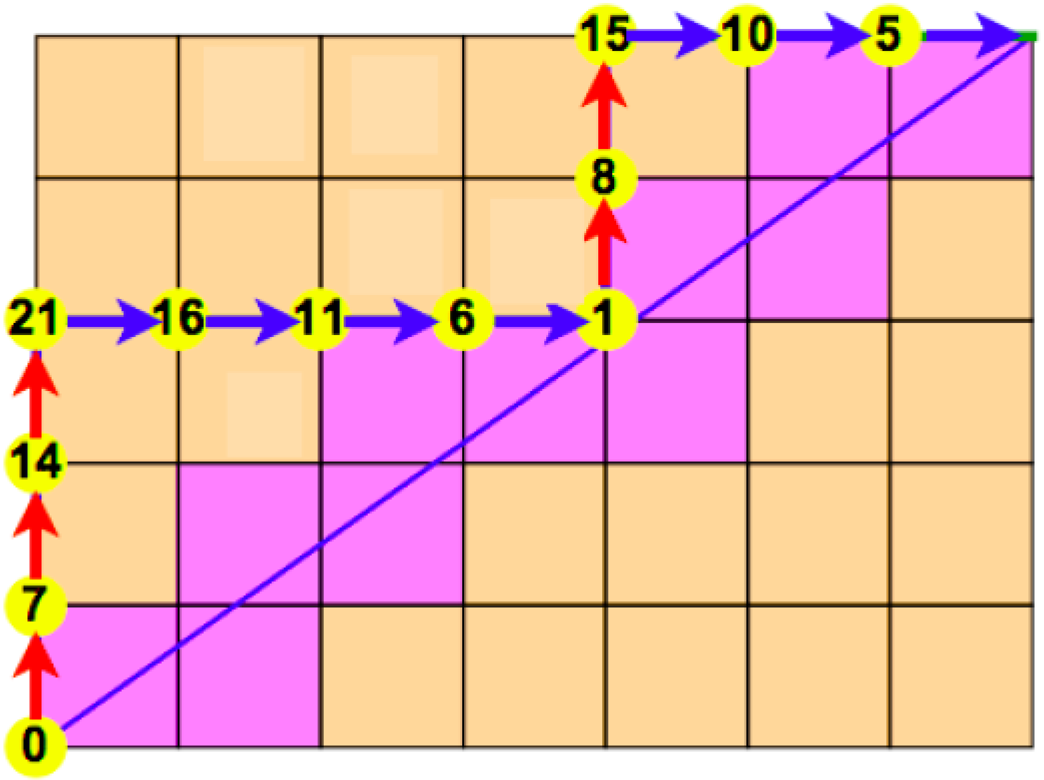
11 $\begin{pmatrix} 1 & 2 & 5 & 10 & 15 \\ 3 & 4 & 8 & 13 & 17 \\ 6 & 7 & 12 & 16 & 19 \\ 9 & 11 & 14 & 18 & 20 \end{pmatrix}$

Sweep map for (m,n)-Dyck paths: An example

D



$\Phi[D] =$



A (7,5)-Dyck path

Its sweep map image

S	W	S	W	S	S	W	S	W	W	W	W
0	7	2	9	4	11	18	13	20	15	10	5

S	S	S	W	W	W	W	S	S	W	W	W
0	2	4	5	7	9	10	11	13	15	18	20

Known Results about the Sweep Map

- The sweep map takes (m,n)-Dyck paths to (m,n)-Dyck paths.
- The sweep map takes dinv to area, and area to bounce*.
- Sweep map also appears in other context, such as core partitions.
- ex-conjecture (over 10 years): the sweep map is a bijection (known for Fuss case), and the conjecture is extended.
- Thomas-Williams proved the (general) conjecture for modulo sweep map.
- Garsia-Xin give a geometric proof for (m,n)-Dyck paths.

Problems on the sweep map

- What is the combinatorial meaning of the bounce statistic, i.e. area of the preimage.
- Is there a simple algorithm to invert the sweep map as for the Fuss case.
- The bi-statistic (dinv,area) is joint symmetric, i.e.,

$$\sum_D q^{dinv(D)} t^{area(D)} = \sum_D t^{dinv(D)} q^{area(D)}$$

Can we give a bijective proof? The joint symmetry property is only prove for the classical case. It is a consequence of the (m,n)-Shuffle conjecture, which is claimed to be proved.

- Can we generalize the joint symmetry property.

A puzzling (and frustrating) specialization

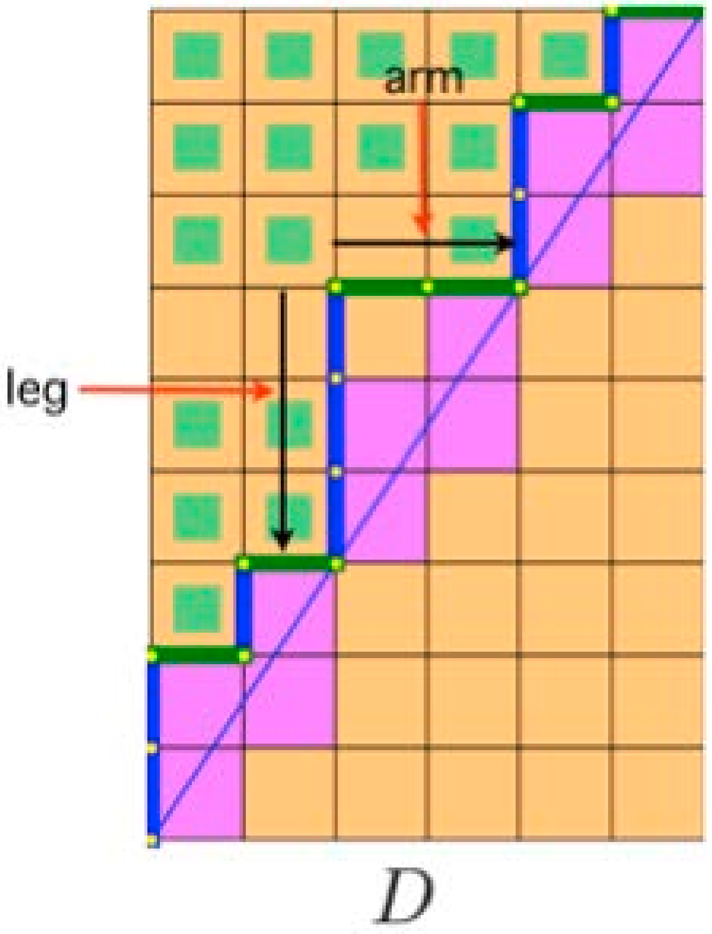
Theorem

On the validity of the Rational Compositional Shuffle Conjecture

$$\boxed{\sum_{D\in\mathcal{D}_{km,kn}} [ret(D)]_q \, q^{coarea(D)+dinv(D)} = \frac{[k]_q}{[km]_q} \begin{bmatrix} kn+km-1 \\ kn \end{bmatrix}_q}$$

Recall that a cell of the english Ferrers Diagram above the path contributes to the dinv if and only if

$$\boxed{\frac{arm(c)}{leg(c)+1} \le \frac{m}{n} < \frac{arm(c)+1}{leg(c)}}$$



The green squares show which cells contribute to the dinv

Cute exercise:

Prove the above identity for $q = 1$

Human linear algorithm extends for Dyck paths from (0,0) to (n,n) with steps $S^k = (0,k), W = (1,0)$

How to Sweep

Rank: add k after Sk, subtract 1 after W

Dyck---nonnegative ranks

Sort: small to large, right first for ties.

S^2	W	W	S^3	S^5	W	W	S^2	W	W	W	W	W	W	W	W
0	2	1	0	3	8	7	6	8	7	6	5	4	3	2	1
2	6	4	1	8	1	1	1	1	1	1	1	9	7	5	3
					6	4	2	5	3	1	0				

How to invert the sweep map

$$S^3 S^2 W W W W W S^5 W W W S^2 W W W W$$

$$1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16$$

Filling rule:
similar
abcd condition

$$S^3 \ S^2 \ S^5 \ S^2$$

$$1 \ 2 \ 8 \ 12$$

Ranking rule
Column: +1
First row:
r(i)=r(i-1)

1	2	8	12
3	4	9	14
5	6	10	16
7		11	
		13	
		15	

0	0	3	6
1	1	4	7
2	2	5	8
3		6	
		7	
		8	

Filling rule:
similar
abcd condition

S^3	S^2	S^5	S^2
1	2	8	12

1	*r* **2**	**8**	**12**
3	4	9	14
5	6	10	16
7		11	
		13	
		15	

0	0	3	6
1	1	4	7
2	2	5	8
3		6	
		7	
		8	

Ranking rule
Column: +1
First row:
r(i)=r(i-1)

Walking rule: same rank use large label

2	6	4	**1**	**8**	16	14	**12**	15	13	11	10	9	7	5	3
S^2	W	W	S^3	S^5	W	W	S^2	W	W	W	W	W	W	W	W