ABSTRACT

We present contributions to the theory of symmetric functions in three different but closely related directions. The first of these concerns the action of certain operators, the Verschiebung operators, on various families of symmetric functions. In the Schur function case this dates back to work of Littlewood and Richardson, and is intimately related with the decomposition of an integer partition into its core and quotient. More recently, Lecouvey and, indepdendently, Ayyer and Kumari provided similar expressions for the characters of the symplectic and orthogonal groups. We lift these to the level of universal characters and give a uniform generalisation involving a very general symmetric function defined by Hamel and King. The second direction concerns generalisations of Littlewood-type identities involving sums over partitions with empty 2-core. These formulae were recently conjectured by Lee, Rains and Warnaar as bounded Littlewood identities for Macdonald polynomials. We prove their conjectures in the Schur case using the powerful technique of virtual Koornwinder integrals developed by Rains and Warnaar. Finally, we provide combinatorial proofs of determinantal formulae, both of Jacobi–Trudi- and Giambelli-type, for skew symplectic and orthogonal characters. These are based on tableaux models for these skew characters given by Koike and Terada. Key in the proofs are the Lindström–Gessel–Viennot lemma and a modified reflection principle.

ZUSAMMENFASSUNG

Diese Dissertation behandelt drei verwandte Themen innerhalb der Theorie der symmetrischen Funktionen. Das erste Thema befasst sich mit dem sogenannten "Verschiebungsoperator" und dessen Wirkung auf verschiedene Familien von symmetrischen Funktionen. Für Schur-Funktionen wurde dieses Problem zuerst in Arbeiten von Littlewood und Richardson betrachtet und ein enger Zusammenhang zur Zerlegung von Zahlpartitionen in ihren Kern und Quotienten bewiesen. Für Charaktere von symplektischen beziehungsweise orthogonalen Gruppen wurden ähnliche Resultate von Lecouvey und unabhängig von Ayyer and Kumari entdeckt. In dieser Dissertation verallgemeinern wir obige Resultate uniform für universale Charaktere, indem wir eine von Hamel und King definierte Familie von symmetrischen Funktionen betrachten. Der zweite Teil dieser Arbeit befasst sich mit Verallgemeinerungen von Littlewood-Identitäten für Summen über Partitionen mit leerem 2-Kern. Diese Formeln wurden kürzlich von Lee, Rains und Warnaar in Form von beschränkten Littlewood-Identitäten für Macdonald Polynome vermutet. Wir beweisen ihre Vermutung für Schur-Funktionen mittels virtuellen Koornwinder-Integralen, welche von Rains und Warnar entwickelt wurden. Der letzte Teil der Dissertation beinhaltet kombinatorische Beweise von Jacobi–Trudi und Giambelli Determinantenformeln für symplektische und orthogonale Charaktere, welche von schiefen Partitionen indiziert und durch die Tableaux-Modelle von Koike und Terada definiert sind. Die Beweise bauen auf das Lindström–Gessel–Viennot Lemma und dem modifiziertem Spiegelungsprinzip auf.

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CONTENTS

INTRODUCTION

In this cumulative dissertation we present four contributions which broadly fall within the realm of symmetric functions. Each component involves Schur functions and classical group characters or their universal analogues. The first three articles are also intimately related with the combinatorics of cores and quotients of integer partitions. Before describing these contributions in detail, we place our work in its historical context. This is not often done, and particularly for the first two articles of the dissertation, brings forth aspects of the history of the theory of symmetric functions which are not as well known as they should be.

Much of the work we present here, in particular the first two sections, has its origins in work of A. R. Richardson and D. E. Littlewood which began with their seminal paper on Schur functions and characters of the symmetric group [47]. Richardson was aware of MacMahon's extensive work on permanents and determinants, including their relationship with the algebra of symmetric functions; see [53]. Inspired by this, he introduced what he called the immanant of a matrix (a name suggested by A. R. Forsyth), defined by replacing the usual sign of a permutation w in the Leibniz formula for the determinant by the value of an irreducible character of the symmetric group evaluated at w. The cases of the trivial representation and the sign representation correspond to the permanent and determinant of the matrix, respectively. Richardson computed immanants of various matrices, including those considered by MacMahon in relation with symmetric functions, before handing his computations over to Littlewood. Together they realised that, when phrased in terms of a finite set of variables $x_1, \dots, x_n$, certain immanants could be expressed as a ratio of determinants in powers of the x_i involving the partition labelling the corresponding irreducible character of the symmetric group. After a thorough scouring of the literature, Littlewood and Richardson discovered Schur's thesis [67], in which the relationship between the irreducible characters of the symmetric group and the bialternants first appeared. They dubbed these symmetric functions "S-functions" (short for Schur functions) and studied their properties in detail, including the first statement of what is now called the Littlewood–Richardson rule for the product of Schur functions. In modern terminology, they had essentially rediscovered the Frobenius characteristic map and shown that the image of the irreducible character χ^λ is the Schur function s_λ. This phase of their collaboration is beautifully described by Turnbull, who was close to Richardson, in [76].

Their collaboration continued with the further papers [48, 49], in which they consider various specialisations and variations of Schur functions which can be obtained either by way of the immanant, or by Littlewood's concept of "S-functions of series" [43]. These include, for example, the principal specialisation of the Schur functions, but in a different form than Stanley's hook-content formula [72, Theorem 15.3], since the notion of hook-length had not yet been invented. A different flavour of formula is obtained by applying certain operators to the power sum symmetric functions p_k. Let Λ denote the algebra of symmetric functions over $\mathbb{Q}$ which is generated by the p_k for $k \geqslant 1$. For a positive integer t define an algebra homomorphism $\varphi_t : \Lambda \longrightarrow \Lambda$ by

$$\varphi_t p_k = \begin{cases} tp_{k/t} & \text{if } t \text{ divides } k, \\ 0 & \text{otherwise.} \end{cases}$$

While not phrased in this precise manner, Littlewood and Richardson consider the action of φ_t on the Schur functions. They characterise the vanishing precisely in terms of the indexing partition and, when the action is nonzero, show that it is, up to a sign, a product of t Schur functions. These results, together with more detailed proofs, were included in Littlewood's book first published in 1940 [44, §7].

The combinatorial underpinnings of Littlewood and Richardson's Schur function factorisation was only made clear in the following decades. In 1940 Nakayama published the pair of papers [54, 55] on the modular representation theory of the symmetric group. At the end of the second paper, he makes the striking conjecture that for t prime the t-blocks of the symmetric group are characterised by t-core partitions, those being partitions with no hook length equal to t. He also shows that to each partition one may associate a unique t-core obtained by removing ribbons of length t. Nakayama's conjecture was proved several years later by Brauer and Robinson [18, 65]. Robinson went on to introduce the notion of a star diagram associated to a partition which encodes its t-hook structure [66]. This notion was independently discovered by Nakayama and Osima [56], and expanded on by Staal [71].

In view of the above constructions, Littlewood synthesised the above objects into what he dubbed the t-residue and t-quotient [45]. In fact, the t-residue is just Nakayama's t-core and the t-quotient contains the same information as the star diagram, but is more simply constructed. Let $\mathscr{P}$ denote the set of partitions and $\mathscr{C}_t$ the set of all t-cores. What is now known as the Littlewood decomposition amounts to a bijection

$$\begin{aligned} \mathscr{P} &\longrightarrow \mathscr{C}_t \times \mathscr{P}^t \\ \lambda &\longmapsto \big(t\text{-core}(\lambda), (\lambda^{(0)}, \dots, \lambda^{(t-1)})\big), \end{aligned}$$

where $t\text{-core}(\lambda)$ is Nakayama's t-core and the t-tuple of partitions $(\lambda^{(0)}, \dots, \lambda^{(t-1)})$ is Littlewood's t-quotient. The bijection may be realised in several equivalent ways. Littlewood gives an arithmetic construction in terms of beta sets, but there is also the beautiful abacus model; see both [34, §2.7] and [51, p. 12]. This bijection, while simple and elegant, has found many deep applications in various fields of mathematics beyond modular representation theory, where it is still an important tool. These include cranks for partition congruences [25], modular analogues of Nekrasov–Okounkov formulae [31, 77] and the study of Hilbert schemes of points on simple surface singularities [15, 28], to name only a few.

Littlewood gives two "applications" of the core-quotient construction in [45]: one to character values of the symmetric group and one to a particular plethysm of symmetric functions. In fact, both of these are equivalent to the theorem regarding the evaluation of $\varphi_t s_\lambda$ mentioned previously. Writing $\varnothing$ for the empty partition, this theorem may be stated as follows.

Theorem (Littlewood and Richardson)**.** *Let λ be a partition and $t \geqslant 2$ an integer. Then $\varphi_t s_\lambda = 0$ unless $t\text{-core}(\lambda) = \varnothing$, in which case*

$$\varphi_t s_\lambda = \operatorname{sgn}_t(\lambda) s_{\lambda^{(0)}} \cdots s_{\lambda^{(t-1)}},$$

where $\operatorname{sgn}_t(\lambda)$ is a sign defined in terms of the Littlewood decomposition of λ.

This essentially appears in [48, §4], and a slight generalisation in [49]. The proof given by Littlewood in his book [44, §7.3] uses a twist of the Schur function by a primitive t-th root of unity, and this equivalent formulation is often used to state the theorem. The first two articles of this dissertation deal with generalisations of this result involving the universal characters of the classical groups [2, 5]. This includes a detailed discussion of the relationship with plethysm and characters of the symmetric group. For more details see items (1) and (2) below.

The third part of this dissertation is based on the so-called Littlewood identities, which are a trio of summation formulae for Schur functions first written down together by Littlewood in his book [44, p.238]. The sums are indexed respectively by all partitions, partitions with even parts and partitions whose conjugate has even parts. In fact, the first of these identities was already known to Schur [68]. The first edition of his treatise on symmetric functions [50, §1.5], Macdonald gives a bounded analogue of the Schur–Littlewood identity in which one restricts the sum-side by demanding that the largest part of each partition occurring in the sum is bounded by a fixed positive integer m. When the number of variables n is finite the evaluation of this sum may be expressed as a determinant. While not noted by Macdonald, this determinant may itself be expressed as a character of the odd orthogonal group indexed by a partition of rectangular shape with n parts equal to m. He applied this bounded Littlewood identity to give a new proof of MacMahon's conjecture for the number of symmetric plane partitions in a box [52], and further develops a partial fraction technique for proving such identities which extends naturally to the Hall–Littlewood polynomial case. Bounded analogues of the other two Littlewood identities were given by Désarménien [20], Proctor [60] and Stembridge [73] and Okada [58], respectively.

There has been continued interest in Littlewood-type identities since Littlewood and Macdonald's initial examples. This includes many applications to areas such as alternating sign matrices and plane partitions [16, 17, 23, 73], longest increasing subsequences [14], Rogers–Ramanujan-type identities [27, 73, 79], elliptic hypergeometric series [63] and much more. The interested reader should consult [32] and [64] for comprehensive references to the literature, as well as further remarks on the history of such identities. Recently, Lee, Rains and Warnaar conjectured several fascinating bounded Littlewood identities for Macdonald polynomials in which the sum is over partitions with empty 2-core [42]. Surprisingly, their conjectures did not reduce to known results in the Schur case, corresponding to $q = t$. Our contribution in [4] is the resolution of their conjectures in the Schur case; see item (3) below for more details.

Our final section deals with determinantal formulae for skew classical group characters. While we have not delved into the full history of the Schur functions, one of their earliest appearances is the (h-)Jacobi–Trudi formula which expresses the Schur function as a determinant of complete homogeneous symmetric functions. This was written down by Jacobi [33], but a rigorous proof was only provided many years later by his student Trudi [75]. The e-Jacobi–Trudi formula was subsequently discovered by Nägelsbach [57], and proved again by Kostka [38]. A direct proof that these two determinants are equal in the case of the skew Schur functions is due to Aitken [10].

The Schur functions may be realised as the characters of the irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$. The classical groups $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{O}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n+1}(\mathbb{C})$ also carry irreducible representations indexed by partitions. The characters of these representations are Laurent polynomials in n variables which are invariant under permutation and reciprocation. In his book on the classical groups, Weyl gives Jacobi–Trudi-type expressions for these characters in terms of complete homogeneous symmetric functions with alphabets $(x_1, 1/x_1, \dots, x_n, 1/x_n)$.

The skew Schur functions may be expressed as a multivariate generating function for skew semistandard Young tableaux. By interpreting such tableaux as intersection lattice paths, Gessel and Viennot gave a beautiful combinatorial proof of the Jacobi–Trudi formulae for skew Schur functions by way of the Lindström–Gessel–Viennot lemma [26]. The main contribution of our article [7] is to extend this approach to the skew characters of the classical groups, extending work of Fulmek and Krattenthaler in the straight shape case [24]. The key tool is expressions for these skew characters as multivariate generating functions for special tableaux due to Koike and Terada [37]. Again, more detail is given in item (4) below.

To conclude the introduction we give a list of the articles produced during the candidature, [2, 4, 5, 6, 7, 9], together with a summary of their contents. The remainder of the thesis comprises of the first four of these, which are included unchanged from their published or submitted versions. Details about the publication status of each article is also included.

1. **Universal characters twisted by roots of unity**,
   Algebraic Combinatorics **6** (2023), 1653–1676.

   In the pair of papers [48, 49], essentially sequels to [47], Littlewood and Richardson computed the value of the Schur function with variables twisted by a primitive t-th root of unity ζ. That is, for $X_n := (x_1, \dots, x_n)$ a finite set of variables, they evaluated the Schur function at the alphabet $(X_n, \zeta X_n, \dots, \zeta^{t-1} X_n)$ where for $a \in \mathbb{C}$ we define $aX_n := (ax_1, \dots, ax_n)$. The vanishing is characterised in terms of the indexing partition. When the evaluation is nonzero, they further show that it is, up to a sign, a product of t Schur functions with alphabets $(x_1^t, \dots, x_n^t)$. This was rewritten by Littlewood in his book [44, §7.3] in the more general setup of "Schur functions of a series". In more modern terminology, their theorem asserts that such specialised Schur functions vanish unless the t-core of the indexing partition λ is empty, and if so then the expression factors as a product of Schur functions indexed by the t-quotient of λ. Farahat, a student of Littlewood, has given a generalisation to skew Schur functions $s_{\lambda/\mu}$ for which the inner shape μ is equal to the t-core of λ. The full skew Schur case appears in the second edition of Macdonald's book as an example [51, p. 91].

   Inspired by a recent rediscovery of Littlewood and Richardson's result by Prasad [59], Ayyer and Kumari considered similar twists for the characters of the symplectic and orthogonal groups [13]. Note that these are now Laurent polynomials in n variables invariant under permutation and inversion of the x_i. Again the vanishing and factorisations of these twisted characters are governed by the t-core and t-quotient of the indexing partition, but in a different form. For example, in the case of $\mathrm{SO}(2n+1, \mathbb{C})$ they show that the twisted character vanishes unless t-core(λ) is self-conjugate. Their proofs are based on the expressions for these characters as ratios of alternants, which is the same approach taken by Littlewood and Richardson.

   The main results of [2] are lifts of Ayyer and Kumari's results to the level of universal characters as defined by Koike and Terada [36]. These are symmetric function lifts of the ordinary characters, and so the notion of twisting by a root of unity is replaced by the Verschiebung operator φ_t, something which is already done by Macdonald for the skew Schur case mentioned above. The proofs are also rather based on Jacobi–Trudi-type formulae for the universal characters, often taken as a definition, and derived by Weyl in the case of the ordinary characters [81, Theorems 7.8.E & 7.8.A].

   The end of the paper contains some discussions of variations on character factorisations for other symmetric functions as well as short universal character lifts of some factorisation results of a very different kind discovered by Ciucu and Krattenthaler [19] and subsequently generalised by Ayyer and Behrend [11].

2. Character factorisations, z-asymmetric partitions and plethysm, preprint, arXiv:2501.18520.

This paper greatly expands on the main results of the previous one. In particular, we embed all of the factorisations of classical group characters (including the Schur functions, corresponding to the GL_n case) under the Verschiebung operator in an infinite family involving an integer z and a parameter q. Moreover, the signs involved are much more simply expressed in terms of statistics on the indexing partitions. What facilitates this generalisation is a very general symmetric function defined by Hamel and King [29, 30] which contains the Schur functions and the universal characters as special cases. We also study z-asymmetric partitions under the Littlewood decomposition, and prove a characterisation result in this context. This generalises classifications for self-conjugate and doubled distinct partitions, corresponding to the cases $z = 0$ and $z = 1$, respectively, which are well known.

The last part of the paper deals with the adjoint problem of computing plethysms of symmetric functions by power sums. In the Schur case this dates back to Littlewood [45, p. 351], and is referred to as the SXP rule. A generalisation of this rule, manifesting as the Schur function expansion of the expression $s_\tau(s_{\lambda/\mu} \circ p_t)$, has been given a purely combinatorial proof by Wildon [82], which we point out is equivalent to the action of the Verschiebung operator on the skew Schur functions. In [41], Lecouvey used his expressions for the symplectic and orthogonal characters under φ_t to give analogues of the SXP rule for the universal characters. Remarkably, the coefficients in these expressions may all be expressed in terms of branching coefficients associated with $\mathrm{SO}(2n+1,\mathbb{C})$. We simplify the statement of these rules, and, in particular, show that the coefficients vanish unless the t-core of the indexing partition is empty in all three cases. The paper concludes by explaining the connection between these results, symmetric functions twisted by roots of unity and characters of the symmetric group.

A preliminary version of this work was published in the proceedings of FPSAC 2025 and presented as a poster [3].

3. Proof of some Littlewood identities conjectured by Lee, Rains and Warnaar,
Proceedings of the American Mathematical Society, Series B **11** (2024), 133–146.

The original Littlewood identities are a trio of summation formulae for Schur functions, first written down together by Littlewood in his book [44, p. 238], where sum is over either all partitions, all partitions with even parts, or all partitions whose conjugate has even parts. They have many generalisations, including bounded variants and analogues for other families of symmetric functions. A major breakthrough in approaching these types of identities was made by Rains and Warnaar [64] who prove many bounded Littlewood identities for Macdonald–Koornwinder polynomials attached to root systems. The key tools are various virtual Koornwinder integrals, originally introduced by Rains [61]. Together with Lee [42], they further conjectured a new pair of bounded Littlewood identities for Macdonald polynomials in which the sum is over all partitions with empty 2-core, generalising Macdonald's lifts of the original Littlewood identities [51, p. 349].

Setting $q = t$ in the Macdonald polynomial $P_\lambda(q,t)$ recovers the Schur function s_λ. Surprisingly, the conjectured identities of Lee, Rains and

Warnaar do not reduce to known identities for Schur functions, and, in particular, the 2-core restriction is still present. (The same is not true of the Hall–Littlewood case, $q = 0$, in which the 2-core restriction drops out.) The main result of [4] is the resolution of the Schur cases of these conjectures in the bounded case using the approach outlined by Lee, Rains and Warnaar based on turning certain virtual Koornwinder integrals into Pfaffians, which may be simply evaluated. The full cases of their conjectures remain open.

4. **Skew symplectic and orthogonal characters through lattice paths**, (joint with I. Fischer, H. Höngesberg and F. Schreier-Aigner)
European Journal of Combinatorics **122** (2024), Paper No. 104000, 26 pp.

The classical Jacobi–Trudi formulae are a fundamental pair of identities expressing the (skew) Schur functions as determinants of elementary or complete homogeneous symmetric functions. These formulae have beautiful combinatorial proofs by way of the Lindström–Gessel–Viennot Lemma, interpreting skew semistandard tableaux as families of nonintersecting lattice paths [26]. Weyl proved analogues of the Jacobi–Trudi formulae for characters of the symplectic and orthogonal groups, which again may be expressed as determinants of elementary or complete homogeneous functions with variables $(x_1, 1/x_1, \dots, x_n, 1/x_n)$ where n is the rank of the corresponding Lie group [81, Theorems 7.8.E & 7.8.A]. Lattice path proofs of these identities were provided by Fulmek and Krattenthaler [24], who utilised various tableaux models due to Proctor, Sundaram and King and Welsh, as well as a modified reflection principle.

Compared to their non-skew counterparts, the skew characters of the symplectic and orthogonal groups have received little attention from a combinatorial point of view. Indeed, from the perspective of combinatorics one only finds the tableaux models of Koike and Terada [37] and some factorisation theorems involving skew Schur functions [11, 12] in the literature. Our paper expands on this by providing the first e-Jacobi–Trudi formulae for these skew characters together with lattice path proofs, the h-Jacobi–Trudi formulae first appearing only recently in the work of Jing, Li and Wang [35]. We use the tableaux models of Koike and Terada, which, even in the non-skew case, lead to simpler proofs than those of Fulmek and Krattenthaler.

Another type of determinantal formula for the Schur functions is the Giambelli formula, which expresses the Schur function as a determinant of Schur functions indexed by hook-shaped partitions. An extension of the Giambelli formula for skew Schur functions was provided by Lascoux and Pragacz [39]. Another upshot of the Koike–Terada tableaux is that we are able to provide Lascoux–Pragacz-type identities for the skew classical characters, together with purely combinatorial proofs.

5. **Elliptic A_n Selberg integrals**
(joint with E. M. Rains and S. O. Warnaar)
Constructive Approximation, to appear (accepted February 11th, 2025), arXiv:2306.02442.

The Selberg integral is perhaps the most important example of a hypergeometric integral. A k-dimensional generalisation of Euler's beta integral, it was discovered by Selberg in 1941 and he published a proof of his evaluation in 1944 [69, 70]. After a period of obscurity it has since found applications in a broad array of mathematical fields such as random matrix theory, analytic number theory, enumerative combinatorics and conformal field theory. In this work we unify two generalisations of the Selberg integral for the first time. The first of these is the Selberg integral associated with the Lie algebra A_n, first evaluated by Warnaar [80], building on work of Tarasov and Varchenko in the case of A_2 [74]. The second is the elliptic Selberg integral, conjectured by van Diejen and Spiridonov [21, 22] and proved by Rains [62]. Our main result is an elliptic extension of the A_n Selberg integral, giving the aforementioned unification of two important integral evaluations. The evaluation is based on the very powerful elliptic interpolation kernel of Rains. Indeed, by way of this kernel, we are able to prove extensions of Selberg-type integrals involving symmetric functions in the integrand such as the Alba–Fateev–Litvinov–Tarnopolsky (AFLT) integral, which recently arose in the verification of the AGT conjecture for SU(2) [1]. This integral was the subject of our earlier paper [8], in which many variations of the AFLT integral are derived, including an elliptic analogue associated to A_1.

6. **A generalization of conjugation of integer partitions**,
(joint with T. Eisenkölbl, I. Fischer, M. Gangl, H. Höngesberg, C. Krattenthaler and M. Rubey)
submitted, arXiv:2407.16043.

The starting point for this project was the observation that two statistics on the set of partitions of n have symmetric joint distribution. For a positive integer s these statistics are (1) the number of parts divisible by s and (2) the number of cells in the Young diagram with hook length divisible by s and zero leg-length. The proof is by way of an involution on the set of partitions of s which interchanges the two statistics, and further leaves invariant the sequence of nonzero remainders obtained by dividing each part of the partition by s. For $s = 1$ the involution reduces to ordinary conjugation of integer partitions. The bivariate generating function is also derived, which was the inspiration for the construction of the involution, and through which the symmetry may also be seen.

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UNIVERSAL CHARACTERS TWISTED BY ROOTS OF UNITY

SEAMUS P. ALBION

Abstract. A classical result of Littlewood gives a factorisation for the Schur function at a set of variables "twisted" by a primitive t-th root of unity, characterised by the core and quotient of the indexing partition. While somewhat neglected, it has proved to be an important tool in the character theory of the symmetric group, the cyclic sieving phenomenon, plethysms of symmetric functions and more. Recently, similar factorisations for the characters of the groups $\mathrm{O}(2n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ were obtained by Ayyer and Kumari. We lift these results to the level of universal characters, which has the benefit of making the proofs simpler and the structure of the factorisations more transparent. Our approach also allows for universal character extensions of some factorisations of a different nature originally discovered by Ciucu and Krattenthaler, and generalised by Ayyer and Behrend.

Keywords: Schur functions, symplectic characters, orthogonal characters, universal characters, t-core, t-quotient.

Contents

1. Introduction

In his 1940 book *The Theory of Group Characters and Matrix Representations of Groups*, D. E. Littlewood devotes a section to the evaluation of the Schur function s_λ at a set of variables "twisted" (not his term) by a primitive t-th root of unity ζ

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[27, §7.3]. In modern terminology, Littlewood's theorem asserts that s_λ evaluated at the variables $\zeta^j x_i$ for $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant t-1$ is zero unless the t-core of λ is empty. Moreover, when it is nonzero, it factors as a product of Schur functions indexed by the elements of the t-quotient of λ, each with the variables $x_1^t, \dots, x_n^t$.

The Schur functions are characters of the irreducible polynomial representations of the general linear group $\mathrm{GL}(n, \mathbb{C})$. Ayyer and Kumari [5] have recently generalised Littlewood's theorem to the characters of the other classical groups $\mathrm{O}(2n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SO}(2n+1, \mathbb{C})$ indexed by partitions. While their factorisations are still indexed by the t-quotient of the corresponding partition, the vanishing is governed by the t-core having a particular form. More precisely, t-core(λ) is of the form $(a \mid a+z)$ in Frobenius notation, where $z = -1, 1, 0$, for $\mathrm{O}(2n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SO}(2n+1, \mathbb{C})$ respectively. Note that these are the same partitions occurring in Littlewood's Schur expansion of the Weyl denominators for types B_n, C_n and D_n [27, p. 238] (see also [31, p. 79]).

Littlewood's proof, and the proofs of Ayyer and Kumari, use the Weyl-type expressions for the characters as ratios of alternants. In the Schur case, Chen, Garsia and Remmel [7] and independently Lascoux [24, Theorem 5.8.2] have given an alternate proof based on the Jacobi–Trudi formula (2.3). This approach was already known to Farahat, who used it to extend Littlewood's theorem to skew Schur functions $s_{\lambda/\mu}$ where μ is the t-core of λ [12, Theorem 2]. The full skew Schur case was then given by Macdonald [31, p. 91], again proved using the Jacobi–Trudi formula; see Theorem 3.1 below.

In this article we lift the results of Ayyer and Kumari to the much more general universal characters of the groups $\mathrm{O}(2n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{SO}(2n+1, \mathbb{C})$ as defined by Koike and Terada [20]. These are symmetric functions indexed by partitions which, under appropriate specialisation of the variables, become actual characters of their respective groups. In fact, these generalise the Jacobi–Trudi-type formulas for the characters of these groups, which were first written down by Weyl [43, Theorems 7.8.E & 7.9.A]. For the universal characters we generalise the notion of "twisting" a set of variables by introducing operators $\varphi_t : \Lambda \longrightarrow \Lambda$ for each integer $t \geqslant 2$ which act on the complete homogeneous symmetric functions as

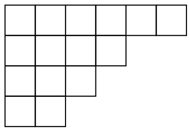
$$\varphi_t h_r = \begin{cases} h_{r/t} & \text{if } t \text{ divides } r, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

It is not at all hard to show that the image of φ_t acting on a symmetric function at the variables $x_1^t, \dots, x_n^t$ agrees with the result of twisting the variables $x_1, \dots, x_n$ by ζ. The advantages of this framework for such factorisations are that the proofs are much simpler, and the structure of the factorisations is made transparent. Moreover, we are able to discuss dualities between these objects which are only present at the universal level. A particularly important tool for our purposes is Koike's universal character $\mathrm{rs}_{\lambda,\mu}$ (2.10) associated with a rational representation of $\mathrm{GL}(n, \mathbb{C})$. This object, which is used later in Subsection 6.3 to prove other character factorisations, appears to be the correct universal character analogue of the Schur function with variables $(x_1, 1/x_1, \dots, x_n, 1/x_n)$.

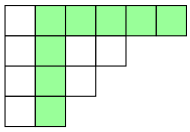
The remainder of the paper reads as follows. In the next section we outline the preliminaries on partitions and symmetric functions needed to state our main results, which follow in Section 3. In the following Section 4 we prepare for the proofs of these results by giving a series of lemmas regarding cores and quotients and their associated signs. The factorisations are then proved in Section 5, including a detailed proof of the Schur case, following Macdonald. The final Section 6 concerns other factorisation results relating to Schur functions and other characters. This includes universal extensions of factorisations very different from those already discussed originally due to Ciucu and Krattenthaler, later generalised by Ayyer and Behrend.

2. Preliminaries

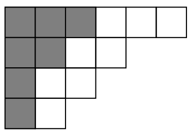
2.1. **Partitions.** A *partition* $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ is a weakly decreasing sequence of nonnegative integers such that only finitely many of the λ_i are nonzero. The nonzero λ_i are called *parts* and the number of parts the *length*, written $l(\lambda)$. We say λ is *a partition of* n if $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots = n$. Two partitions are regarded as the same if they agree up to trailing zeroes, and the set of all partitions is written $\mathscr{P}$. A partition is identified with its *Young diagram*, which is the left-justified array of squares consisting of λ_i squares in row i with i increasing downward. For example



is the Young diagram of $(6,4,3,2)$. We define the *conjugate* partition λ' by reflecting the diagram of λ in the main diagonal $x = y$, so that the conjugate of $(6,4,3,2)$ above is $(4,4,3,2,1,1)$. If $\lambda = \lambda'$ then λ is called *self-conjugate*. For a square at coordinate (i,j) where $1 \leqslant i \leqslant l(\lambda)$ and $1 \leqslant j \leqslant \lambda_i$ the *hook length* is $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$. For example the square $(1,2)$ in



has hook length 8, with its hook shaded. A partition λ *is a t-core* if it contains no squares of hook length t, the set of which is denoted $\mathscr{C}_t$. For a pair of partitions λ, μ we write $\mu \subseteq \lambda$ if the diagram of μ can be drawn inside the diagram of λ, i.e., if $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$. In this case we can form the *skew shape* λ/μ by removing the digram of μ from that of λ. For example $(3,2,1,1) \subseteq (6,4,3,2)$ and the diagram of $(6,4,3,2)/(3,2,1,1)$ is given by the non-shaded squares of



A skew shape is called a *ribbon* (or *border strip*, *rim hook*, *skew hook*) if its diagram is connected and contains no 2×2 square. A t-ribbon is a ribbon with t boxes. The *height* of a t-ribbon R, written $\mathrm{ht}(R)$, is one less than the number of rows it occupies. In our example above $R = (6,4,3,2)/(3,2,1,1)$ is an 8-ribbon with height $\mathrm{ht}(R) = 3$. We say a skew shape is *tileable by t-ribbons* or *t-tileable* if there exists a sequence of partitions

$$\mu = \nu^{(0)} \subseteq \nu^{(1)} \subseteq \cdots \subseteq \nu^{(k-1)} \subseteq \nu^{(k)} = \lambda \tag{2.1}$$

such that $\nu^{(i)}/\nu^{(i-1)}$ is a t-ribbon for $1 \leqslant i \leqslant k$. A sequence $D = (\nu^{(0)}, \dots, \nu^{(k)})$ (not to be confused with the t-quotient of ν below, for which we use the same notation) satisfying (2.1) is called a *ribbon decomposition* (or *border strip decomposition*) of λ/μ. We define the height of a ribbon decomposition to be the sum of the heights of the individual ribbons: $\mathrm{ht}(D) := \sum_{i=1}^{k} \mathrm{ht}(\nu^{(i)}/\nu^{(i-1)})$. As shown by van Leeuwen [26, Proposition 3.3.1] and Pak [35, Lemma 4.1] (also in [2, §6]), the quantity

$(-1)^{\mathrm{ht}(D)}$ is the same for every ribbon decomposition of λ/μ. We therefore define the *sign* of a t-tileable skew shape λ/μ as

$$\mathrm{sgn}_t(\lambda/\mu) := (-1)^{\mathrm{ht}(D)}. \tag{2.2}$$

Let $\mathrm{rk}(\lambda)$ be the greatest integer such that $\mathrm{rk}(\lambda) \geqslant \lambda_{\mathrm{rk}(\lambda)}$, usually called the *Frobenius rank* of λ. Equivalently, $\mathrm{rk}(\lambda)$ is the side length of the largest square which fits inside the diagram of λ (the *Durfee square*). A partition can alternatively be written in *Frobenius notation* as

$$\lambda = \big(\lambda_1 - 1, \dots, \lambda_{\mathrm{rk}(\lambda)} - \mathrm{rk}(\lambda) \mid \lambda'_1 - 1, \dots, \lambda'_{\mathrm{rk}(\lambda)} - \mathrm{rk}(\lambda)\big).$$

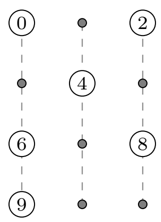
Any pair of integer sequences $a_1 > \cdots > a_k \geqslant 0$ and $b_1 > \cdots > b_k \geqslant 0$ thus determines a partition $\lambda = (a \mid b)$ with $\mathrm{rk}(\lambda) = k$. For $z \in \mathbb{Z}$ and an integer sequence of predetermined length $a = (a_1, \dots, a_k)$ we write $a + z := (a_1 + z, \dots, a_k + z)$. Following Ayyer and Kumari [5, Definition 2.9], λ is called *z-asymmetric* if it is of the form $\lambda = (a \mid a + z)$ for some integer sequence a and integer z. Clearly a 0-asymmetric partition is self-conjugate. Partitions which are -1- and 1-asymmetric are called *orthogonal* and *symplectic* respectively.

2.2. **Cores and quotients.** We now describe the t-core and t-quotient of λ arithmetically following [31, p. 12]. There are many equivalent descriptions, see for instance [13, 15, 16, 42]. We begin with the *beta set* of a partition, which is simply the set of n integers

$$\beta(\lambda; n) := \{\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_{n-1} + 1, \lambda_n\},$$

where $n \geqslant l(\lambda)$ is fixed. The number of elements in this set congruent to r modulo t is denoted by $m_r(\lambda; n) = m_r$. Each element which falls into residue class r for $0 \leqslant r \leqslant t-1$ can be written as $\xi_k^{(r)} t + r$ for some integers $\xi_1^{(r)} > \cdots > \xi_{m_r}^{(r)} \geqslant 0$. These integers are used to define a partition with parts $\lambda_k^{(r)} = \xi_k^{(r)} - m_r(\lambda; n) + k$ where $1 \leqslant k \leqslant m_r(\lambda; n)$, and the ordered sequence $(\lambda^{(0)}, \dots, \lambda^{(t-1)})$ of these partitions is called the *t-quotient*. The precise order of the constituents of the t-quotient depends on the residue class of n modulo t. However, the orders only differ by cyclic permutations, and Macdonald comments that it is best to think of the quotient as a sort-of "necklace" of partitions. To simplify things somewhat, we adopt the convention that the t-quotient is always computed with n a multiple of t, so that the order of its constituents is fixed. To define the t-core, one writes down the n distinct integers $kt + r$ where $0 \leqslant k \leqslant m_r(\lambda; n) - 1$ and $0 \leqslant r \leqslant t - 1$ in descending order, say as $\tilde{\xi}_1 > \cdots > \tilde{\xi}_n$. Then $t\text{-core}(\lambda)_i := \tilde{\xi}_i - n + i$. If t-core(λ) is empty then we say λ *has empty t-core*.

It will prove useful later on to work with the *bead configurations* (or *bead diagrams*, *abacus model*) of James and Kerber [16, §2.7], which give a different model for t-cores and t-quotients. The "board" for a bead configuration is the set of nonnegative integers arranged in t downward-increasing columns, called *runners*, according to their residues modulo t. A bead is then placed at the space corresponding to each element of $\beta(\lambda; n)$. For an example, let $\lambda = (4, 4, 3, 2, 1)$ so that $\beta(\lambda; 6) = \{9, 8, 6, 4, 2, 0\}$. Then the bead configuration for λ with $t = 3$ and $n = 6$ and the beads labelled by their position is



Moving a bead up one space is equivalent to reducing one of the elements of $\beta(\lambda; n)$ by t. This is, in turn, equivalent to removing a t-ribbon from λ such that what remains is still a Young diagram (see for instance [31, p. 12]). Pushing all beads to the top will give the bead configuration of $t\text{-core}(\lambda)$, and this is clearly independent of the order in which the beads are pushed. It follows that $t\text{-core}(\lambda)$ is the unique partition obtained by removing t-ribbons (in a valid way) from the diagram of λ until it is no longer possible to do so. We note that if removing a ribbon R corresponds to moving a bead from position b to $b-t$, then $\mathrm{ht}(R)$ is equal to the number of beads lying at the positions strictly between $b-t$ and b. The t-quotient can be obtained from the bead configuration by reading the r-th runner, bottom-to-top, as a bead configuration with $m_r(\lambda; n)$ beads. For our example, this means that $\beta(t\text{-core}(\lambda); 6) = \{6, 5, 3, 2, 1, 0\}$, so that $t\text{-core}(\lambda) = (1, 1)$, with the quotient $((1, 1), (1), (1))$ computed similarly.

The above procedure of computing the t-core and t-quotient actually encodes a bijection

$$\begin{aligned}\phi_t : \mathscr{P} &\longrightarrow \mathscr{C}_t \times \mathscr{P}^t \\ \lambda &\longmapsto \big(t\text{-core}(\lambda), (\lambda^{(0)}, \dots, \lambda^{(t-1)})\big),\end{aligned}$$

such that $|\lambda| = |t\text{-core}(\lambda)| + t(|\lambda^{(0)}| + \cdots + |\lambda^{(t-1)}|)$. The arithmetic description of this correspondence was first written down by Littlewood [29]. The idea of removing ribbons from a partition until a unique core is obtained goes back to Nakayama [33]. The t-quotient of a partition has its origin in the *star diagrams* of Nakayama, Osima, Robinson and Staal [34, 39, 40], which were shown to be equivalent to Littlewood's t-quotient by Farahat [10].

Let $w_t(\lambda; n)$ be the permutation of $\beta(\lambda; n)$ which sorts the elements of the beta set so that their residues modulo t are increasing, and the elements within each residue class decrease. The sign of $w_t(\lambda; n)$ will be denoted $\mathrm{sgn}(w_t(\lambda; n))$. The permutation $w_t(\lambda; n)$ can also be read off the bead configuration by first labelling the beads "backwards": label the bead with largest place 1, second-largest 2, and so on. Reading the labels column-wise from bottom-to-top gives $w_t(\lambda; n)$ in one-line notation. An inversion in this permutation corresponds to a pair of beads b_1, b_2 such that b_2 lies weakly below and strictly to the right of b_1. With the same example as before $w_3((4, 4, 3, 2, 1); 6) = 136425$ and the bead at position 0 generates three inversions, as it "sees" the beads $2, 4$ and 8.

As follows from the above, a partition λ has empty t-core if and only if it is t-tileable. In our results below we will need the following characterisation of when a skew shape is t-tileable, generalising the notion of "empty t-core" to this case. We briefly recall our convention that t-quotients are always computed with the number of beads in the bead configuration a multiple of t.

Lemma 2.1. *A skew shape λ/μ is tileable by t-ribbons if and only if* $t\text{-core}(\lambda) = t\text{-core}(\mu)$ *and* $\mu^{(r)} \subseteq \lambda^{(r)}$ *for each* $0 \leqslant r \leqslant t-1$.

Proof. The skew shape being t-tileable is equivalent to the diagram of μ being obtainable from the diagram of λ by removing t-ribbons. In other words, we can obtain the bead configuration of μ from that of λ, where both have nt beads, by moving beads upwards. Assume that this is the case. Then $m_r(\lambda; nt) = m_r(\mu; nt)$ for each $0 \leqslant r \leqslant t-1$, so that the r-th runner has the same number of beads in each diagram. This implies that $t\text{-core}(\lambda) = t\text{-core}(\mu)$. It also follows that the i-th bead in each runner of λ's bead configuration must lie weakly below the i-th bead in the same runner of μ's bead configuration. Equivalently, $\mu_i^{(r)} \leqslant \lambda_i^{(r)}$ for all $0 \leqslant r \leqslant t-1$ and $1 \leqslant i \leqslant m_r(\lambda; nt)$, which in turn is equivalent to $\mu^{(r)} \subseteq \lambda^{(r)}$. The reverse direction is now clear. □

Note that the lemma is also true when the t-quotients of λ and μ are computed using the same integer n of any residue class modulo t. If λ/μ is t-tileable, then we think of $\lambda^{(0)}/\mu^{(0)},\dots,\lambda^{(t-1)}/\mu^{(t-1)}$ as its t-quotient. When λ/μ is not t-tileable, it is not so clear how to define the t-quotient.

2.3. **Symmetric functions and universal characters.** Here we discuss some basics of the theory of symmetric functions, following [31]. Let Λ denote the *ring of symmetric functions* in an arbitrary countable set of variables $X=(x_1,x_2,x_3,\dots)$, called an *alphabet*. Where possible, we write elements of Λ without reference to an alphabet if the expression is independent of the chosen alphabet. If for a positive integer n one sets $x_i=0$ for all $i>n$ then the elements of Λ reduce to symmetric polynomials in the variables $(x_1,\dots,x_n)$. Another common specialisation sets $x_{n+i}=x_i^{-1}$ for $1\leqslant i\leqslant n$ and $x_i=0$ for $i>2n$. This gives Laurent polynomials in the x_i invariant under permutation and inversion of the variables (i.e., BC_n-symmetric functions). We will later write $(x_1^{\pm},\dots,x_n^{\pm})$ for this alphabet.

Two fundamental algebraic bases for Λ are the *complete homogeneous symmetric functions* and the *elementary symmetric functions*, defined for any positive integer r by

$$h_r(X):=\sum_{1\leqslant i_1\leqslant\cdots\leqslant i_r}x_{i_1}\cdots x_{i_r}\quad\text{and}\quad e_r(X):=\sum_{1\leqslant i_1<\cdots<i_r}x_{i_1}\cdots x_{i_r},$$

respectively. We further set $h_0=e_0:=1$ and $h_{-r}=e_{-r}=0$ for positive r. These admit the generating functions

$$\begin{aligned}H_z(X)&:=\sum_{r\geqslant 0}z^rh_r(X)=\prod_{i\geqslant 1}\frac{1}{1-zx_i}\\ E_z(X)&:=\sum_{r\geqslant 0}z^re_r(X)=\prod_{i\geqslant 1}(1+zx_i).\end{aligned}$$

The h_r and e_r for $r\geqslant 1$ are algebraically independent over $\mathbb{Z}$ and generate Λ. In view of this, we can define a homomorphism $\omega:\Lambda\longrightarrow\Lambda$ by $\omega h_r=e_r$. It then follows from the relation $H_z(X)E_{-z}(X)=1$ that $\omega e_r=h_r$, so that ω is an involution. We also define the power sums by

$$p_r(X):=\sum_{i\geqslant 1}x_i^r,$$

for $r\geqslant 1$ and $p_0:=1$. These satisfy $\omega p_r=(-1)^{r-1}p_r$.

The most important family of symmetric functions are the *Schur functions*. These have several definitions, but for our purposes it is best to define them, already for skew shapes, by the *Jacobi–Trudi formula*. If λ/μ is a skew shape and n an integer such that $n\geqslant l(\lambda)$ we define

$$s_{\lambda/\mu}:=\det_{1\leqslant i,j\leqslant n}(h_{\lambda_i-\mu_j-i+j}). \tag{2.3}$$

This is independent of n as long as $n\geqslant l(\lambda)$. If $\mu\not\subseteq\lambda$ then we set $s_{\lambda/\mu}:=0$. There is also an equivalent formula in terms of the e_r, called the *dual Jacobi–Trudi formula* (rarely also the *Nägelsbach–Kostka identity*)

$$s_{\lambda/\mu}=\det_{1\leqslant i,j\leqslant m}(e_{\lambda'_i-\mu'_j-i+j}).$$

Restricting to the μ empty case, we have $s_{(r)}=h_r$ and $s_{(1^r)}=e_r$. Moreover, it is clear that $\omega s_{\lambda/\mu}=s_{\lambda'/\mu'}$.

If the set of variables $(x_1,\dots,x_n)$ is finite then the Schur function for $\mu=0$ admits another definition as a ratio of alternants

$$s_\lambda(x_1,\dots,x_n)=\frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j})}. \tag{2.4}$$

The denominator is the Vandermonde determinant and has the product representation $\det_{1\leqslant i,j\leqslant n}(x_i^{n-j})=\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)$. In this case we also define $s_\lambda(x_1,\dots,x_n)=0$ if $l(\lambda)>n$. If λ is a partition of length at most n, then

$$s_{(\lambda_1+1,\dots,\lambda_n+1)}(x_1,\dots,x_n)=(x_1\cdots x_n)s_{(\lambda_1,\dots,\lambda_n)}(x_1,\dots,x_n). \tag{2.5}$$

This allows for Schur functions with a finite set of n variables to be extended to weakly decreasing sequences of integers of length exactly n.

Following Koike and Terada we define the *universal characters* for $\mathrm{O}(2n,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$ as the symmetric functions [20, Definition 2.1.1]

$$\mathrm{o}_\lambda:=\det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i-i+j}-h_{\lambda_i-i-j}\big) \tag{2.6}$$

$$\mathrm{sp}_\lambda:=\frac{1}{2}\det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+2}\big), \tag{2.7}$$

where $n\geqslant l(\lambda)$. Like the Schur functions, these determinants also have dual versions

$$\begin{aligned}\mathrm{o}_\lambda&=\frac{1}{2}\det_{1\leqslant i,j\leqslant m}\big(e_{\lambda'_i-i+j}+e_{\lambda'_i-i-j+2}\big)\\ \mathrm{sp}_\lambda&=\det_{1\leqslant i,j\leqslant m}\big(e_{\lambda'_i-i+j}-e_{\lambda'_i-i-j}\big),\end{aligned} \tag{2.8}$$

where $m\geqslant\lambda_1$. From this it is clear that $\omega\mathrm{o}_\lambda=\mathrm{sp}_{\lambda'}$. Koike alone added a third universal character for the group $\mathrm{SO}(2n+1,\mathbb{C})$ [19, Definition 6.4] (see also [25, Equation (3.8)])

$$\mathrm{so}_\lambda:=\det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+1}\big)=\det_{1\leqslant i,j\leqslant m}\big(e_{\lambda'_i-i+j}+e_{\lambda'_i-i-j+1}\big).$$

This universal character is self-dual under ω, so $\omega\mathrm{so}_\lambda=\mathrm{so}_{\lambda'}$. For later convenience we also define a variant of the above as

$$\mathrm{so}^-_\lambda:=\det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i-i+j}-h_{\lambda_i-i-j+1}\big)=\det_{1\leqslant i,j\leqslant m}\big(e_{\lambda'_i-i+j}-e_{\lambda'_i-i-j+1}\big).$$

If $X=(x_1,x_2,x_3,\dots)$ is a set of variables (which may be finite or countable) and $-X:=(-x_1,-x_2,-x_3,\dots)$, then

$$\mathrm{so}^-_\lambda(X)=(-1)^{|\lambda|}\mathrm{so}_\lambda(-X),$$

since the h_r and e_r are homogeneous of degree r.

For $l(\lambda)\leqslant n$, each of the three above universal characters become actual characters of irreducible representations of their associated groups when specialised to $(x_1^{\pm},\dots,x_n^{\pm})$ (hence the name universal characters).

The irreducible polynomial representations of $\mathrm{GL}(n,\mathbb{C})$ are indexed by partitions of length at most n. On the other hand, the irreducible rational representations are indexed by weakly decreasing sequences of integers of length n, which are called *staircases* by Stembridge [41]. Such sequences are equivalent to pairs of partitions λ,μ such that $l(\lambda)+l(\mu)\leqslant n$. Given such a pair, one defines the associated staircase $[\lambda,\mu]$ by $[\lambda,\mu]_i:=\lambda_i-\mu_{n-i+1}$ for $1\leqslant i\leqslant n$. The characters of the rational representations of $\mathrm{GL}(n,\mathbb{C})$ are then given by $s_{[\lambda,\mu]}(x_1,\dots,x_n)$ for all staircases with n entries. Note that (2.5) implies that this object is just a Schur function up to a power of $x_1\cdots x_n$. In [28], Littlewood gave the expansion

$$s_{[\lambda,\mu]}(x_1,\dots,x_n)=\sum_\nu(-1)^{|\nu|}s_{\lambda/\nu}(x_1,\dots,x_n)s_{\mu/\nu'}(1/x_1,\dots,1/x_n). \tag{2.9}$$

For a pair of partitions λ,μ and sets of indeterminates X, Y, this may be used to define the universal character associated to a rational representation of $\mathrm{GL}(n,\mathbb{C})$ as

$$\mathrm{rs}_{\lambda,\mu}(X;Y) := \sum_{\nu}(-1)^{|\nu|}s_{\lambda/\nu}(X)s_{\mu/\nu'}(Y). \tag{2.10}$$

Note that the only terms which contribute are those with $\nu \subseteq \lambda$ and $\nu' \subseteq \mu$. If we let ω_X and ω_Y denote the involution ω acting on the set of variables in its subscript, then

$$\begin{aligned}\omega_X\omega_Y\mathrm{rs}_{\lambda,\mu}(X;Y) &= \sum_{\nu}(-1)^{|\nu|}s_{\lambda'/\nu'}(X)s_{\mu'/\nu}(Y)\\ &= \sum_{\nu'}(-1)^{|\nu|}s_{\lambda'/\nu}(X)s_{\mu'/\nu'}(Y)\\ &= \mathrm{rs}_{\lambda',\mu'}(X;Y).\end{aligned}$$

As shown by Koike [18], this object has a Jacobi–Trudi-type expression as a block matrix

$$\mathrm{rs}_{\lambda,\mu}(X;Y) = \det_{1\leqslant i,j\leqslant n+m}\begin{pmatrix} \big(h_{\lambda_i-i+j}(X)\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}(X)\big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}} \\ \big(h_{\mu_i-i-j+1}(Y)\big)_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} & \big(h_{\mu_i-i+j}(Y)\big)_{1\leqslant i,j\leqslant m}\end{pmatrix}, \tag{2.11}$$

where $n \geqslant l(\lambda)$ and $m \geqslant l(\mu)$. As for the other determinants, this is independent of n and m as long as $n \geqslant l(\lambda)$ and $m \geqslant l(\mu)$. The relation under $\omega_X\omega_Y$ implies we have the dual form [18, Definition 2.1]

$$\mathrm{rs}_{\lambda,\mu}(X;Y) = \det_{1\leqslant i,j\leqslant n+m}\begin{pmatrix} \big(e_{\lambda'_i-i+j}(X)\big)_{1\leqslant i,j\leqslant n} & \big(e_{\lambda'_i-i-j+1}(X)\big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant m}} \\ \big(e_{\mu'_i-i-j+1}(Y)\big)_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} & \big(e_{\mu'_i-i+j}(Y)\big)_{1\leqslant i,j\leqslant m}\end{pmatrix}, \tag{2.12}$$

where $n \geqslant \lambda_1$ and $m \geqslant \mu_1$. The definition (2.10) and the determinants (2.11) and (2.12) are related by taking the Laplace expansion of each determinant according to its presented block structure; see [18, Equation (2.1)]. Also, by the definition (2.10) and (2.9) it is immediate that, for $l(\lambda)+l(\mu) \leqslant n$,

$$\mathrm{rs}_{\lambda,\mu}(x_1,\dots,x_n;1/x_1,\dots,1/x_n) = s_{[\lambda,\mu]}(x_1,\dots,x_n).$$

We will always take $X = Y$ in $\mathrm{rs}_{\lambda,\mu}(X;Y)$, which we write as $\mathrm{rs}_{\lambda,\mu}$ in the rest of the paper. In particular we note that $\mathrm{rs}_{\lambda,\mu} = \mathrm{rs}_{\mu,\lambda}$.

As mentioned in the introduction, the notion of twisting the set of variables $x_1,\dots,x_n$ by a primitive t-th root of unity ζ is replaced by the operator φ_t (1.1), which has been considered by Macdonald [31, p. 91] and, for $t=2$, by Baik and Rains [6, p. 25]. Let $X^t := (x_1^t, x_2^t, x_3^t \dots)$ and denote by ψ_t the homomorphism

$$\begin{aligned}\psi_t : \Lambda &\longrightarrow \Lambda_{X^t}\\ f &\longmapsto f(X,\zeta X,\dots,\zeta^{t-1}X).\end{aligned}$$

Since $\psi_t H_z(X) = H_{z^t}(X^t)$, both φ_t and ψ_t act on the h_r in the same way, i.e., the diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\varphi_t"] \arrow[dr, "\psi_t"'] & \Lambda \arrow[d, "X^t"] \\ & \Lambda_{X^t} \end{tikzcd} \tag{2.13}$$

commutes, where the arrow labelled X^t is the substitution map. This implies the claim of the introduction that the action of φ_t is equivalent to twisting the alphabet X by a primitive t-th root of unity ζ. If one wishes to think about this as a map $\Lambda_X \longrightarrow \Lambda_X$ where X is some concrete alphabet, then substitute each $x \in X$ by its set of t-th roots $x^{1/t}, \zeta x^{1/t}, \dots, \zeta^{t-1}x^{1/t}$ and evaluate this expression. By the action of this map on the h_r, such a map gives a symmetric function again in the variables X. Using the generating function $E_z(X)$ one may also show that [24, §5.8]

$$\varphi_t e_r = \begin{cases} (-1)^{r(t-1)/t} e_{r/t} & \text{if } t \text{ divides } r, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore ω and φ_t commute if t is odd, but not in general. Proposition 3.5 shows that, in some cases, the maps commute up to a computable sign. Different, but closely related operators are discussed at the end of this paper.

3. Summary of results

With the preliminary material of the previous section under our belts, we are now ready to state our main results regarding factorisations of universal characters under φ_t. The first of these is the action of the map on the skew Schur functions.

Theorem 3.1. *We have that $\varphi_t s_{\lambda/\mu} = 0$ unless λ/μ is tileable by t-ribbons, in which case*

$$\varphi_t s_{\lambda/\mu} = \operatorname{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}}.$$

For μ empty, this result is due to Littlewood [27, p. 131], who proves it by direct manipulation of the ratio of alternants (2.4). By (2.13) with $X = (x_1, \dots, x_n)$, one can recover Littlewood's result by simply evaluating the right-hand side of the equation at $(x_1^t, \dots, x_n^t)$. He also states the μ empty case of the theorem in the language of symmetric group characters: see both [27, p. 144] and [29, p. 340]. The generalisation to skew characters was discovered by Farahat [11] (see also [9, Theorem 3.3]). The form we state here is precisely that of Macdonald [31, p. 91]. Curiously, Prasad recently rediscovered the μ empty case independently, with a proof identical to Littlewood's, but in a more representation-theoretic context [37]. A version of the result for Schur's P-and Q-functions has been given by Mizukawa [32, Theorem 5.1].

Theorem 3.1 has been rediscovered many times for both skew and straight shapes, and often only in special cases. We make no attempt to give a complete history, but it appears to us that the theorem deserves to be better known. The interested reader can consult [44] for some exposition on the character theory side of this story. On the symmetric functions side, such an exposition is lacking in the literature.

We now state, in sequence, the three factorisations lifting [5, Theorems 2.11, 2.15 & 2.17] to the level of universal characters, beginning with the universal orthogonal character.

Theorem 3.2. *Let λ be a partition of length at most nt. Then $\varphi_t \mathrm{o}_\lambda = 0$ unless t-core(λ) is orthogonal, in which case*

$$\varphi_t \mathrm{o}_\lambda = (-1)^{\varepsilon^{\mathrm{o}}_{\lambda;nt}} \operatorname{sgn}(w_t(\lambda; nt)) \mathrm{o}_{\lambda^{(0)}} \prod_{r=1}^{\lfloor (t-1)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)}, \lambda^{(t-r)}} \times \begin{cases} \mathrm{so}^-_{\lambda^{(t/2)}} & t \text{ even,} \\ 1 & t \text{ odd,} \end{cases}$$

where

$$\varepsilon^{\mathrm{o}}_{\lambda;nt} = \sum_{r=\lfloor (t+2)/2 \rfloor}^{t-1} \binom{m_r(\lambda; nt) + 1}{2} + \operatorname{rk}(t\text{-core}(\lambda)) + \begin{cases} \binom{n+1}{2} + n \operatorname{rk}(t\text{-core}(\lambda)) & t \text{ even,} \\ 0 & t \text{ odd.} \end{cases}$$

Our next result is the same factorisation for the symplectic character.

Theorem 3.3. *Let λ be a partition of length at most nt. Then $\varphi_t \mathrm{sp}_\lambda = 0$ unless t-core(λ) is symplectic, in which case*

$$\varphi_t \mathrm{sp}_\lambda = (-1)^{\varepsilon^{\mathrm{sp}}_{\lambda;nt}} \operatorname{sgn}(w_t(\lambda;nt)) \mathrm{sp}_{\lambda^{(t-1)}} \prod_{r=0}^{\lfloor (t-3)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r-2)}} \times \begin{cases} \mathrm{so}_{\lambda^{((t-2)/2)}} & t \text{ even}, \\ 1 & t \text{ odd}, \end{cases}$$

where

$$\varepsilon^{\mathrm{sp}}_{\lambda;nt} = \sum_{r=\lfloor t/2 \rfloor}^{t-2} \binom{m_r(\lambda;nt)+1}{2} + \begin{cases} \binom{n+1}{2} + n\mathrm{rk}(t\text{-core}(\lambda)) & t \text{ even}, \\ 0 & t \text{ odd}. \end{cases}$$

Finally, we can claim a similar factorisation for so_λ.

Theorem 3.4. *Let λ be a partition of length at most nt. Then $\varphi_t \mathrm{so}_\lambda = 0$ unless t-core(λ) is self-conjugate, in which case*

$$\varphi_t \mathrm{so}_\lambda = (-1)^{\varepsilon^{\mathrm{so}}_{\lambda;nt}} \operatorname{sgn}(w_t(\lambda;nt)) \prod_{r=0}^{\lfloor (t-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r-1)}} \times \begin{cases} 1 & t \text{ even}, \\ \mathrm{so}_{\lambda^{((t-1)/2)}} & t \text{ odd}, \end{cases}$$

where[1]

$$\varepsilon^{\mathrm{so}}_{\lambda;nt} = \sum_{r=\lfloor (t+1)/2 \rfloor}^{t-1} \binom{m_r(\lambda;nt)+1}{2} + \begin{cases} 0 & t \text{ even}, \\ n\mathrm{rk}(t\text{-core}(\lambda)) & t \text{ odd}. \end{cases}$$

Some remarks are in order. Firstly, the three signs $\operatorname{sgn}(w_t(\lambda;nt))(-1)^{\varepsilon^{\bullet}_{\lambda;nt}}$ are actually independent of n as long as $nt \geqslant l(\lambda)$, a fact which we prove in Lemma 4.8 below. As remarked by Ayyer and Kumari [5, Remark 2.19], the order of the quotient is unchanged upon replacing $n \mapsto n+1$, so the product in the evaluation is independent of n. It is in principle possible to carry out our proof technique below under the assumption that $l(\lambda)$ is bounded by an arbitrary integer, say k, where k is not necessarily a multiple of t. In this case the evaluation is of course the same, however the sign will be expressed differently and the t-quotients in the evaluations will be a cyclic permutation of the ones presented. Since the proof is simplest when this k is a multiple of t, we stick to this case.

To obtain the theorems of Ayyer and Kumari one evaluates the right-hand side of each identity at the set of variables $(x_1^{\pm t}, \dots, x_n^{\pm t})$. Using (2.5) and the definition of $\mathrm{rs}_{\lambda,\mu}$ it follows that in this case the rational universal characters occurring in each evaluation agree with the Schur functions $s_{\mu_i^{(k)}}(x_1^{\pm t}, \dots, x_n^{\pm t})$ in the notation of [5].

As we have already seen the maps ω and φ_t do not commute in general. However, when acting on $s_{\lambda/\mu}$ and so_λ, they commute up to an explicitly computable sign.

Proposition 3.5. *We have the relations*

$$\omega\varphi_t s_{\lambda/\mu} = (-1)^{(t-1)(|\lambda^{(0)}/\mu^{(0)}|+\cdots+|\lambda^{(t-1)}/\mu^{(t-1)}|)} \varphi_t \omega s_{\lambda/\mu},$$

and

$$\omega\varphi_t \mathrm{so}_\lambda = (-1)^{(t-1)(|\lambda^{(0)}|+\cdots+|\lambda^{(t-1)}|)} \varphi_t \omega \mathrm{so}_\lambda.$$

We remark that the second relation does not hold with so_λ replaced by sp_λ or o_λ as written above since $\omega \mathrm{so}^-_\lambda = \mathrm{so}^-_{\lambda'}$.

[1]We have corrected the lower bound in the sum defining $\varepsilon^{\mathrm{so}}_{\lambda;nt}$ from $\lfloor t/2 \rfloor$ in [5, Theorem 2.17] (there denoted ϵ) to $\lfloor (t+1)/2 \rfloor$.

4. Auxiliary results

The purpose of this section is to collect all the small facts about beta sets and the signs (2.2) which we need to prove our main results. To begin, we relate the bead configurations of a partition and its conjugate.

Lemma 4.1. *Let λ be a partition of length at most nt such that $\lambda_1 \leqslant mt$. Then the bead configuration for $\beta(\lambda'; mt)$ can be obtained from the bead configuration for $\beta(\lambda; nt)$ with $n+m$ rows by rotating the picture by 180° and then interchanging beads and spaces.*

Proof. This is a consequence of the fact [31, p. 3] that for $l(\lambda) \leqslant n$ and $\lambda_1 \leqslant m$,

$$\{0,1,\dots,m+n-1\} = \{\lambda_i+n-i : 1 \leqslant i \leqslant n\} \sqcup \{m+n-1-(\lambda'_j+m-j) : 1 \leqslant j \leqslant m\},$$

where $\sqcup$ denotes a disjoint union. ☐

This lemma immediately implies the following relationship between the t-core and t-quotient of λ and λ'.

Corollary 4.2. *For a partition λ we have $t\text{-core}(\lambda') = t\text{-core}(\lambda)'$ and the t-quotient of λ' is $\big((\lambda^{(t-1)})', \dots, (\lambda^{(0)})'\big)$.*

The next pair of lemmas are due to Ayyer and Kumari, the first of which characterises partitions with z-asymmetric t-cores in terms of their beta sets [5, Lemma 3.6].

Lemma 4.3. *For a partition λ of length at most nt, $t\text{-core}(\lambda)$ is of the form $(a \mid a+z)$ for some integer $-1 \leqslant z \leqslant t-1$ if and only if*

$$m_r(\lambda, nt) + m_{t-r-z-1}(\lambda, nt) = 2n \quad \text{for } 0 \leqslant r \leqslant t-z-1, \tag{4.1a}$$

$$m_r(\lambda, nt) = n \quad \text{for } t-z \leqslant r \leqslant t-1, \tag{4.1b}$$

where the indices of the m_r are taken modulo t.

The second lemma of Ayyer and Kumari we need is [5, Lemma 3.13], which is used later on to simplify signs.

Lemma 4.4. *Let λ be a partition of length at most nt. If $t\text{-core}(\lambda)$ is orthogonal, then*

$$\operatorname{rk}(t\text{-core}(\lambda)) = \sum_{r=1}^{\lfloor (t-1)/2 \rfloor} |m_r(\lambda; nt) - n| = \sum_{r=\lfloor (t+2)/2 \rfloor}^{t-1} |m_r(\lambda; nt) - n| \tag{4.2}$$

If $t\text{-core}(\lambda)$ is symplectic, then

$$\operatorname{rk}(t\text{-core}(\lambda)) = \sum_{r=0}^{\lfloor (t-3)/2 \rfloor} |m_r(\lambda; nt) - n| = \sum_{r=\lfloor t/2 \rfloor}^{t-2} |m_r(\lambda; nt) - n| \tag{4.3}$$

If $t\text{-core}(\lambda)$ is self-conjugate, then

$$\operatorname{rk}(t\text{-core}(\lambda)) = \sum_{r=0}^{\lfloor (t-2)/2 \rfloor} |m_r(\lambda; nt) - n| = \sum_{r=\lfloor (t+1)/2 \rfloor}^{t-1} |m_r(\lambda; nt) - n| \tag{4.4}$$

Next, we show that the sign of a tileable skew shape can be expressed in terms of the signs of the permutations $w_t(\lambda; n)$.

Lemma 4.5. *For λ/μ t-tileable and any integer n such that $n \geqslant l(\lambda)$,*

$$\operatorname{sgn}_t(\lambda/\mu) = \operatorname{sgn}(w_t(\lambda; n)) \operatorname{sgn}(w_t(\mu; n)).$$

Proof. Since λ/μ is t-tileable, it has a ribbon decomposition $D = (\nu^{(0)}, \dots, \nu^{(k)})$ where $\nu^{(0)} = \mu$ and $\nu^{(k)} = \lambda$. Also, $\nu^{(k-1)}$ can be obtained from λ by moving one bead at some position upward one space. By our characterisation of the inversions in the permutation $w_t(\lambda; n)$, we see that moving a bead at position ℓ up one space changes the sign by $(-1)^{b_k}$ where b_k is the number of beads at positions between $\ell - t$ and ℓ. In other words, $\operatorname{sgn}(w_t(\lambda; n)) = (-1)^{b_k} \operatorname{sgn}(w_t(\nu^{(k-1)}; n))$. Moreover, $b_k = \operatorname{ht}(\nu^{(k)}/\nu^{(k-1)})$, so that

$$\operatorname{sgn}(w_t(\lambda; n)) \operatorname{sgn}(w_t(\mu; n)) = (-1)^{\sum_{i=1}^k b_i} = (-1)^{\operatorname{ht}(D)} = \operatorname{sgn}_t(\lambda/\mu). \qquad \square$$

We also have the following useful relationship between the sign of λ/μ and λ'/μ'.

Lemma 4.6. *For λ/μ t-tileable,*

$$\operatorname{sgn}_t(\lambda/\mu) \operatorname{sgn}_t(\lambda'/\mu') = (-1)^{(t-1)(|\lambda^{(0)}| + \cdots + |\lambda^{(t-1)}| - |\mu^{(0)}| - \cdots - |\mu^{(t-1)}|)}.$$

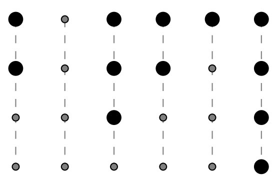
Proof. To prove the claim of the lemma we will proceed by induction on $|\lambda/\mu|$. If $|\lambda/\mu| = 0$ then $\lambda = \mu$ and the equation is trivial. Now fix μ and assume the result holds for λ/μ being t-tileable. Adding a t-ribbon to λ/μ moves one of the beads, say at position b, in the bead configuration for λ down a single space. The change in the number of inversions in $w_t(\lambda; nt)$ is the number of beads b' such that $b < b' < b + 1$. A consequence of Lemma 4.1 is that $w_t(\lambda'; mt)$ will change by the number of empty spaces between b and $b + 1$. There are $t - 1$ spaces and beads between b and $b + 1$, so the left-hand side changes by $(-1)^{t-1}$ when adding a t-ribbon. But adding a t-ribbon to λ/μ changes some element of the t-quotient of λ by a single box, also corresponding to a change in sign of $(-1)^{t-1}$. $\square$

There is another sign relation between orthogonal and symplectic t-cores, but this time using the permutations w_t.

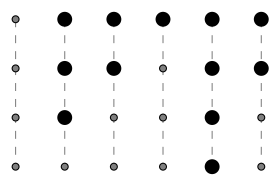
Lemma 4.7. *Let λ be an orthogonal or symplectic t-core whose diagram is contained in an $nt \times nt$ square. Then*

$$\operatorname{sgn}(w_t(\lambda; nt)) \operatorname{sgn}(w_t(\lambda'; nt)) = (-1)^{\operatorname{rk}(\lambda)}.$$

Proof. Assume that λ is a non-empty, orthogonal t-core (if λ is empty the result is trivial) and fix n so that the condition of the theorem holds. The key observation is that for an orthogonal t-core, the bead configuration of λ' with nt beads can be obtained from the bead configuration of λ with nt beads by reducing the labels by 1 modulo t. For example if $\lambda = (12, 7, 5, 3, 2, 2, 1, 1, 1, 1, 1)$ then $\lambda' = (11, 6, 4, 3, 3, 2, 2, 1, 1, 1, 1, 1)$ and their bead configurations for $t = 6$ and $n = 2$ are



and



respectively, where we have suppressed the labels. This is a consequence of Lemma 4.3 with $z = \pm 1$ and Lemma 4.1. When passing from λ to λ', the inversions contributed by the beads in the first runner are removed and replaced by additional inversions associated to the remaining beads in the first n rows. Modulo two, this is equivalent to each bead in the zeroth runner now seeing all of the beads in the same row twice, plus all other beads in the other runners once. Let b be the number of beads in the first n rows of the runners from 1 to $t - 1$ in the bead configuration of λ. Then the sign change is

$$\operatorname{sgn}(w_t(\lambda; nt)) = \operatorname{sgn}(w_t(\lambda'; nt))(-1)^{n^2(t-1)+b}.$$

Since λ is an orthogonal t-core, $n^2(t-1)+b \equiv \mathrm{rk}(\lambda) \pmod 2$ by (4.2) and Lemma 4.3 with $z=-1$. $\square$

The next lemma proves the claim made after Theorems 3.2–3.4 that the signs occurring in those factorisations are independent of n.

Lemma 4.8. *The signs* $(-1)^{\varepsilon^{\bullet}_{\lambda;nt}} \operatorname{sgn}(w_t(\lambda;nt))$ *for* $\bullet \in \{\mathrm{o}, \mathrm{sp}, \mathrm{so}\}$ *are independent of* n *as long as* $nt \geqslant l(\lambda)$.

Proof. Assume that $nt \geqslant l(\lambda)$. Incrementing n by one adds a row of beads to the top of the bead configuration of λ, and so $m_r(\lambda;(n+1)t) = m_r(\lambda;nt)+1$. In the inversion count, the rth bead in the new first row sees

$$\sum_{k=r+1}^{t-1} (m_k(\lambda;nt)+1)$$

other beads. Summing over $k=0,\dots,t-1$ we see that

$$\operatorname{sgn}(w_t(\lambda;(n+1)t)) = \operatorname{sgn}(w_t(\lambda;nt))(-1)^{\sum_{r=1}^{\lfloor t/2\rfloor}(m_{2r-1}(\lambda;nt)+1)}.$$

Now assume that λ has an orthogonal t-core. Then by Lemma 4.3 with $z=-1$ the above has the same parity as

$$\sum_{r=1}^{\lfloor t/2\rfloor}(m_{2r-1}(\lambda;nt)+1) \equiv \begin{cases} \frac{(n+1)t}{2} & t \text{ even},\\ \frac{t-1}{2} + \sum_{r=1}^{(t-1)/2} m_r(\lambda;nt) & t \text{ odd}, \end{cases} \pmod 2.$$

A short calculation shows that

$$\begin{aligned} \varepsilon^{\mathrm{o}}_{\lambda;(n+1)t} &= \varepsilon^{\mathrm{o}}_{\lambda;nt} + \sum_{r=1}^{\lfloor (t-1)/2\rfloor} m_r(\lambda;nt) + \begin{cases} n+\frac{t}{2}+\mathrm{rk}(t\text{-core}(\lambda)) & t \text{ even},\\ \frac{t-1}{2} & t \text{ odd}, \end{cases}\\ &\equiv \varepsilon^{\mathrm{o}}_{\lambda;nt} + \begin{cases} \frac{(n+1)t}{2} & t \text{ even},\\ \frac{t-1}{2}+\sum_{r=1}^{(t-1)/2} m_r(\lambda;nt) & t \text{ odd} \end{cases} \pmod 2. \end{aligned}$$

where the last equality uses (4.2). The remaining two cases follow similarly. $\square$

We conclude this section with a small lemma relating the indices in the Jacobi–Trudi determinants with partition quotients.

Lemma 4.9. *Let* λ, μ *be partitions of length at most* nt *and assume that for* $0 \leqslant r,s \leqslant t-1$ *we have* $\lambda_i - i \equiv r \pmod t$, $\mu_j - j \equiv s \pmod t$ *for* $1 \leqslant i,j \leqslant nt$. *If* $r-s+z \equiv 0 \pmod t$ *for some* $z \in \mathbb{Z}$, *then*

$$\frac{\lambda_i - \mu_j + j - i + z}{t} = \lambda_k^{(r)} - \mu_\ell^{(s)} - k + \ell + m_r(\lambda;nt) - m_s(\mu;nt) + (r-s+z)/t,$$

for some k,ℓ *such that* $1 \leqslant k \leqslant m_r(\lambda;nt)$ *and* $1 \leqslant \ell \leqslant m_s(\mu;nt)$. *Alternatively, if* $r+s+z \equiv 0 \pmod t$ *then*

$$\frac{\lambda_i + \mu_j - i - j + z}{t} = \lambda_k^{(r)} + \mu_\ell^{(s)} - k - \ell - 2n + 1 + m_r(\lambda;nt) + m_s(\mu;nt) + (r+s+z)/t$$

for some k,ℓ *such that* $1 \leqslant k \leqslant m_r(\lambda;nt)$ *and* $1 \leqslant \ell \leqslant m_s(\mu;nt)$.

Proof. We first write $\lambda_i + nt - i = \xi_k^{(r)} t + r$ and $\mu_j + nt - j = \pi_\ell^{(s)} t + s$ for $1 \leqslant k \leqslant m_r(\lambda;nt)$ and $1 \leqslant \ell \leqslant m_s(\mu;nt)$. Then

$$\begin{aligned} \frac{\lambda_i - \mu_j - i + j + z}{t} &= \xi_k^{(r)} + \pi_\ell^{(s)} + (r-s+z)/t\\ &= \lambda_k^{(r)} + \mu_\ell^{(s)} - k + \ell + m_r(\lambda;nt) - m_s(\mu;nt) + (r-s+z)/t, \end{aligned}$$

by the definition of the t-quotient. The second claim is analogous. $\square$

5. Proofs of theorems

In this section we provide proofs of Theorems 3.2, 3.3 and 3.4. Since our proof strategy follows that of Macdonald's proof of the skew Schur case [31, p. 91] (Theorem 3.1 above), we reproduce this proof in detail as preparation for what remains. We also give a detailed example in the orthogonal case in Section 5.2 to further elucidate the structure of the remaining proofs.

5.1. **Proof of Theorem 3.1.** Let n be a nonnegative integer and $\mu \subseteq \lambda$ be a pair of partitions such that $l(\lambda) \leqslant nt$. Consider the Jacobi–Trudi determinant

$$s_{\lambda/\mu} = \det_{1\leqslant i,j\leqslant nt}(h_{\lambda_i-\mu_j-i+j}).$$

Before applying the map φ_t, we rearrange the rows and columns of this determinant by the permutations $w_t(\lambda; nt)$ and $w_t(\mu; nt)$ respectively. By Lemma 4.5 this introduces a sign of $\mathrm{sgn}_t(\lambda/\mu)$. The rows and columns are now arranged in such a way that the residue classes of $\lambda_i - i$ and $\mu_j - j$ are grouped in ascending order, and the values within each class are decreasing. From this vantage point it is easy to apply the map φ_t since $\varphi_t h_{\lambda_i-\mu_j-i+j}$ vanishes unless $\lambda_i - i \equiv \mu_j - j \pmod t$. Therefore, $\varphi_t s_{\lambda/\mu}$ has a block-diagonal structure, with each block having size $m_r(\lambda; nt) \times m_r(\mu; nt)$ for $0 \leqslant r \leqslant t-1$. We conclude that $\varphi_t s_{\lambda/\mu} = 0$ unless $m_r(\lambda; nt) = m_r(\mu; nt)$ for all $0 \leqslant r \leqslant t-1$. Assuming this is the case, then the entries of the of the minor corresponding to the residue class r are given by Lemma 4.9, and are

$$h_{(\lambda_i-\mu_j-i+j)/t} = h_{\lambda_k^{(r)}-\mu_\ell^{(r)}-k+\ell}$$

for some k and ℓ with $1 \leqslant k, \ell \leqslant n$. Note that the rows and columns are in the desired order (i.e., in each $n \times n$ minor the indices increase from 1 to n) thanks to the permutations we applied at the beginning of the proof. We have therefore shown that if $m_r(\lambda; nt) = m_r(\mu; nt)$ for all $0 \leqslant r \leqslant t-1$, then

$$\varphi_t s_{\lambda/\mu} = \mathrm{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}}.$$

Now, if $\mu^{(r)} \not\subseteq \lambda^{(r)}$ for any r such that $0 \leqslant r \leqslant t-1$ this expression will give zero, from which we conclude, by Lemma 2.1, that $\varphi_t s_{\lambda/\mu} = 0$ unless λ/μ is t-tileable.

5.2. **An example.** The structure of the remaining proofs is best outlined through a detailed example. To this end, let $t = 4$ and $\lambda = (12, 12, 12, 8, 8, 8, 7, 7, 3, 3, 2)$. We therefore have that 4-core$(\lambda) = (4, 1, 1)$, which is clearly orthogonal, and

$$\big(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\big) = \big((2,2), (4,1), (3,2,1), (2,1,1)\big).$$

Now choose $n = 3$, so that $nt = 12 \geqslant l(\lambda)$. Using the definition of o_λ as a Jacobi–Trudi-type determinant (2.6) we immediately see that

$$\varphi_4 \mathrm{o}_\lambda = \begin{vmatrix}
h_3 & \cdot & -h_2 & \cdot & h_4 & \cdot & -h_1 & \cdot & h_5 & \cdot & -1 & \cdot \\
\cdot & h_3 - h_2 & \cdot & \cdot & \cdot & h_4 - h_1 & \cdot & \cdot & \cdot & h_5 - 1 & \cdot & \cdot \\
-h_2 & \cdot & h_3 & \cdot & -h_1 & \cdot & h_4 & \cdot & -1 & \cdot & h_5 & \cdot \\
\cdot & \cdot & \cdot & h_2 - 1 & \cdot & \cdot & \cdot & h_3 & \cdot & \cdot & \cdot & h_4 \\
h_1 & \cdot & -1 & \cdot & h_2 & \cdot & \cdot & \cdot & h_3 & \cdot & \cdot & \cdot \\
\cdot & h_1 - 1 & \cdot & \cdot & \cdot & h_2 & \cdot & \cdot & \cdot & h_3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & h_1 & \cdot & \cdot & \cdot & h_2 & \cdot & \cdot & \cdot & h_3 \\
1 & \cdot & \cdot & \cdot & h_1 & \cdot & \cdot & \cdot & h_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & h_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & h_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{vmatrix},$$

where we write $\cdot$ in place of 0 to avoid clutter. The next step is to permute the rows and columns of the matrix according to the permutations $w_4(\lambda;12)$ and $w_4(0;12)$, respectively. In this case, the first permutation is odd and the second even, so we are left with

$$\varphi_4\mathrm{o}_\lambda = -\begin{vmatrix}
h_2-1 & h_3 & h_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
h_1 & h_2 & h_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & h_3 & h_4 & h_5 & \cdot & \cdot & \cdot & -h_2 & -h_1 & -1 \\
\cdot & \cdot & \cdot & \cdot & 1 & h_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & h_3-h_2 & h_4-h_1 & h_5-1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & h_1-1 & h_2 & h_3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & h_1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -h_2 & -h_1 & -1 & \cdot & \cdot & \cdot & h_3 & h_4 & h_5 \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & h_1 & h_2 & h_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & h_1 & h_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{vmatrix}.$$

The top-left 3×3 minor and central 3×3 minor occupying rows 6–8 and columns 7–9 are clearly equal to $\mathrm{o}_{(2,2)}$ and $\mathrm{so}^-_{(3,2,1)}$, respectively. One way to isolate the copy of $\mathrm{so}^-_{(3,2,1)}$ is to push it so that it is the bottom-right 3×3 submatrix, while preserving the order of the other rows and columns. In this case such a procedure will introduce a sign of -1. Putting this together, we have shown that

$$\varphi_4\mathrm{o}_\lambda = \mathrm{o}_{(2,2)}\mathrm{so}^-_{(3,2,1)}\begin{vmatrix}
h_3 & h_4 & h_5 & -h_2 & -h_1 & -1 \\
\cdot & 1 & h_1 & \cdot & \cdot & \cdot \\
-h_2 & -h_1 & -1 & h_3 & h_4 & h_5 \\
-1 & \cdot & \cdot & h_1 & h_2 & h_3 \\
\cdot & \cdot & \cdot & 1 & h_1 & h_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{vmatrix}.$$

Our goal is to show that this final unidentified determinant is equal to $\mathrm{rs}_{(4,1),(2,1,1)}$. Clearly the extra signs can be cleared by multiplying the first two rows and first three columns by -1 each, generating an overall sign of -1. Then one need only push the first column past the second and third, which does not change the sign, and the resulting determinant is precisely a copy of $\mathrm{rs}_{(4,1),(2,1,1)}$. Thus,

$$\varphi_4\mathrm{o}_\lambda = -\mathrm{o}_{(2,2)}\mathrm{so}^-_{(3,2,1)}\mathrm{rs}_{(4,1),(2,1,1)}.$$

Note that $(-1)^{\varepsilon^{\mathrm{o}}_{\lambda;12}} = 1$ so the overall sign clearly agrees with Theorem 3.2. In the next sections we show that, with a little extra work, this argument also works in general for the universal characters $\mathrm{o}_\lambda, \mathrm{sp}_\lambda$ and so_λ.

5.3. **Proof of Theorem 3.2.** Let λ be a partition such that $l(\lambda) \leqslant nt$ and consider the definition (2.6) of o_λ

$$\mathrm{o}_\lambda = \det_{1\leqslant i,j\leqslant nt}\big(h_{\lambda_i-i+j} - h_{\lambda_i-i-j}\big).$$

We permute the rows and columns by $w_t(\lambda;nt)$ and $w_t(0;nt)$ respectively, which introduces a sign of

$$(-1)^{\binom{n+1}{2}\binom{t}{2}}\operatorname{sgn}(w_t(\lambda;nt)). \tag{5.1}$$

The modular behaviour of the indices of each row is now known. There are three possibilities for the entries of $\varphi_t\mathrm{o}_\lambda$: both h's may survive, one h may survive, or the entry is necessarily zero. For both to survive, we see that h_{λ_i-i+j} and h_{λ_i-i-j}

are nonzero under φ_t if and only if $\lambda_i - i \equiv -j \equiv 0 \pmod{t}$ or, if t is even, $\lambda_i - i \equiv -j \equiv t/2 \pmod{t}$. In the first instance, by Lemma 4.9,

$$\varphi_t\big(h_{\lambda_i-i+j} - h_{\lambda_i-i-j}\big) = h_{\lambda_k^{(0)}-k+\ell+m_0(\lambda;nt)-n} - h_{\lambda_k^{(0)}-k-\ell+m_0(\lambda;nt)-n},$$

where $1 \leqslant k \leqslant m_0(\lambda;nt)$ and $1 \leqslant \ell \leqslant n$. Moreover, all other entries in the first $m_0(\lambda;nt)$ rows and n columns are zero. If t is even then we also find a submatrix of size $m_{t/2}(\lambda;nt) \times n$ in the rows $1 + \sum_{r=0}^{(t-2)/2} m_r(\lambda;nt)$ to $\sum_{r=0}^{t/2} m_r(\lambda;nt)$ and columns $1 + nt/2$ to $n(t+2)/2$. The entries of this submatrix are

$$\varphi_t\big(h_{\lambda_i-i+j} - h_{\lambda_i-i-j}\big) = h_{\lambda_k^{(t/2)}-k+\ell+m_{t/2}(\lambda;nt)-n} - h_{\lambda_k^{(t/2)}-k-\ell+m_{t/2}(\lambda;nt)-n+1},$$

where $1 \leqslant k \leqslant m_{t/2}(\lambda;nt)$ and $1 \leqslant \ell \leqslant n$. Again, all other entries in these rows and columns are necessarily zero under φ_t. Given a row corresponding to the residue class r where $1 \leqslant r \leqslant \lfloor (t-1)/2 \rfloor$, there are two possibilities for the entry to potentially survive: the column corresponds to the residue class r or $t-r$. Again, by Lemma 4.9,

$$\varphi_t\big(h_{\lambda_i-i+j} - h_{\lambda_i-i-j}\big) = \begin{cases} h_{\lambda_k^{(r)}-k+\ell+m_r(\lambda;nt)-n} & \text{if } j \equiv -r \pmod{t}, \\ -h_{\lambda_k^{(r)}-k-\ell+m_r(\lambda;nt)-n+1} & \text{if } j \equiv r \pmod{t}. \end{cases}$$

The set of indices of the complete homogeneous symmetric functions in such a row are

(5.2a)
$$\begin{aligned}&\big\{\lambda_k^{(r)} - k - \ell + m_r(\lambda,nt) + 1 \mid 1 \leqslant \ell \leqslant 2n\big\} \\ &= \big\{\lambda_k^{(r)} - k + \ell \mid 1 \leqslant \ell \leqslant m_r(\lambda,nt)\big\} \sqcup \big\{\lambda_k^{(r)} - k - \ell + 1 \mid 1 \leqslant \ell \leqslant m_{t-r}(\lambda,nt)\big\}.\end{aligned}$$

If we look at the complementary row corresponding to $t-r$, then a similar computation shows that the indices are

(5.2b)
$$\begin{aligned}&\big\{\lambda_k^{(t-r)} - k - \ell + m_{t-r} + 1 \mid 1 \leqslant \ell \leqslant 2n\big\} \\ &= \big\{\lambda_k^{(t-r)} - k + \ell \mid 1 \leqslant \ell \leqslant m_{t-r}(\lambda,nt)\big\} \sqcup \big\{\lambda_k^{(t-r)} - k - \ell + 1 \mid 1 \leqslant \ell \leqslant m_r(\lambda,nt)\big\}.\end{aligned}$$

We have now identified the entries which do not necessarily vanish under φ_t. These can be rearranged into a block-diagonal matrix. If t is even, we move the submatrix corresponding to $t/2$ to the bottom-right $m_{t-1}(\lambda;nt)$ rows and n columns, which picks up a sign of

$$(-1)^{m_{t/2}(\lambda;nt)\sum_{r=(t+2)/2}^{t-1} m_r(\lambda;nt)+n^2(t-2)/2}.$$

We then group the rows and columns corresponding to the residue classes r and $t-r$ together with $0 \leqslant r \leqslant \lfloor (t-1)/2 \rfloor$ increasing. The determinant is now block-diagonal and the blocks have dimension $m_0(\lambda;nt) \times n$, $(m_r(\lambda;nt) + m_{t-r}(\lambda;nt)) \times 2n$ for $1 \leqslant r \leqslant \lfloor (t-1)/2 \rfloor$ and, if t is even, $m_{t/2}(\lambda;nt) \times n$. Since the determinant of a block-diagonal matrix vanishes if one of the blocks is not a square, we can therefore conclude that $\varphi_t \mathrm{o}_\lambda$ vanishes unless the conditions (4.1) with $z=-1$ hold in Lemma 4.3, i.e., unless t-core(λ) is orthogonal. In this case the top-left $n \times n$ minor is equal to $\mathrm{o}_{\lambda^{(0)}}$ and if t is even the bottom-right minor corresponds to $\mathrm{so}^-_{\lambda^{(t/2)}}$. Note that in this case the grouping of the $2n \times 2n$ minors does not change the sign of the determinant since each row and column is pushed past an even number of rows or columns. For each $1 \leqslant r \leqslant \lfloor (t-1)/2 \rfloor$ these final minors are of the form

$$\begin{pmatrix} h_{\lambda_1^{(r)}+m_r-n} & \cdots & h_{\lambda_1^{(r)}+m_r-1} & -h_{\lambda_1^{(r)}+m_r-n-1} & \cdots & -h_{\lambda_1^{(r)}+m_r-2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ h_{\lambda_{m_r}^{(r)}+1-n} & \cdots & h_{\lambda_{m_r}^{(r)}} & -h_{\lambda_{m_r}^{(r)}-n} & \cdots & -h_{\lambda_{m_r}^{(r)}-2n+1} \\ -h_{\lambda_1^{(t-r)}+m_{t-r}-n-1} & \cdots & -h_{\lambda_1^{(t-r)}+m_{t-r}-2n} & h_{\lambda_1^{(t-r)}+m_{t-r}-n} & \cdots & h_{\lambda_1^{(t-r)}+m_{t-r}-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ -h_{\lambda_{m_{t-r}}^{(t-r)}-n} & \cdots & -h_{\lambda_{m_{t-r}}^{(t-r)}-2n+1} & h_{\lambda_{m_{t-r}}^{(t-r)}+1-n} & \cdots & h_{\lambda_{m_{t-r}}^{(t-r)}} \end{pmatrix},$$

where we write $m_r = m_r(\lambda; nt)$. Clearing the negatives in this minor produces the sign $(-1)^{m_r(\lambda;nt)+n}$. If $m_r(\lambda; nt) = m_{t-r}(\lambda; nt) = n$ then we are done. If $m_r(\lambda; nt) > n$ then we need to move the columns $n+1$ to $m_r(\lambda; nt)$ so they are the first $m_r(\lambda; nt) - n$ columns, and then reverse the order. This gives a sign of

$$(-1)^{n(m_r(\lambda;nt)-n)+\binom{m_r(\lambda;nt)-n}{2}} = (-1)^{\binom{m_r(\lambda;nt)}{2}-\binom{n}{2}}.$$

If $m_r(\lambda; nt) < n$ then we need to push the $m_{t-r}(\lambda; nt) - n$ missing rows past the $n - m_r$ rows to their right and then reverse again, giving the same sign

$$(-1)^{(m_{t-r}(\lambda;nt)-n)m_r(\lambda;nt)+\binom{m_{t-r}(\lambda;nt)-n}{2}} = (-1)^{\binom{m_r(\lambda;nt)}{2}-\binom{n}{2}},$$

since $m_{t-r}(\lambda; nt) - n = n - m_r(\lambda; nt)$. In each of the three cases the resulting determinant is equal to $\mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r)}}$. Collecting all of the above determinant manipulations, the value of $\varepsilon^{\mathrm{o}}_{\lambda;nt}$ is

$$\frac{(n+1)nt(t-1)}{4} + \sum_{r=1}^{\lfloor (t-1)/2 \rfloor} \left(\binom{m_r+1}{2} + \binom{n+1}{2} \right) + \begin{cases} n\sum_{r=1}^{(t-2)/2} m_{t-r} + \frac{n^2(t-2)}{2} & t \text{ even}, \\ 0 & t \text{ odd}. \end{cases}$$

To see that this agrees with the sign of Ayyer and Kumari, we use (4.2) together with the fact that for odd t, $(t+1)(t-1)n(n+1)/4$ is even and for even t, $n(n+1)(t^2-2)/4$ has the same parity as $n(n+1)/2$. The above exponent therefore has the same parity as

$$\varepsilon^{\mathrm{o}}_{\lambda;nt} = \sum_{r=\lfloor (t+2)/2 \rfloor}^{t-1} \binom{m_r(\lambda; nt)+1}{2} + \mathrm{rk}(t\text{-core}(\lambda)) + \begin{cases} \binom{n+1}{2} + n\mathrm{rk}(t\text{-core}(\lambda)) & t \text{ even}, \\ 0 & t \text{ odd}. \end{cases}$$

This completes the proof.

5.4. **Proof of Theorem 3.3.** It is of course possible to prove Theorem 3.3 by direct manipulation of the h Jacobi–Trudi-type formula for sp_λ (2.7). However, it will be more insightful to begin with the e Jacobi–Trudi-type formula (2.8)

$$\mathrm{sp}_\lambda = \det_{1 \leqslant i,j \leqslant nt} \left(e_{\lambda'_i-i+j} - e_{\lambda'_i-i-j} \right),$$

where we assume that n is an integer such that $nt \geqslant \lambda_1$. We further assume that $nt \geqslant l(\lambda)$, since, in the end, our sign will be independent of n. The values of $\varphi_t h_r$ and $\varphi_t e_r$ differ by a sign of $(-1)^{(t-1)r/t}$, and the indices of the e's in this formula are the same as the h's in the formula for $\mathrm{o}_{\lambda'}$ ((2.6) with $\lambda \mapsto \lambda'$), so we can simply replace each h by a signed e in the previous proof. Moreover, by Corollary 4.2, we know that the t-quotient of λ' is simply the reverse of the t-quotient of λ. We can therefore already claim that $\varphi_t\mathrm{sp}_\lambda$ vanishes unless t-core(λ) is symplectic, in which case

$$\varphi_t\mathrm{sp}_\lambda=(-1)^\delta\operatorname{sgn}(w_t(\lambda';nt))\mathrm{sp}_{\lambda^{(t-1)}}\prod_{i=0}^{\lfloor (t-3)/2\rfloor}\mathrm{rs}_{\lambda^{(i)},\lambda^{(t-i-2)}}\times\begin{cases}\mathrm{so}_{\lambda^{((t-2)/2)}} & t \text{ even},\\ 1 & t\text{ odd},\end{cases}$$

where

$$\delta=(t-1)\sum_{r=0}^{t-1}|\lambda^{(r)}|+\sum_{r=\lfloor (t+1)/2\rfloor}^{t-1}\binom{m_r(\lambda';nt)}{2}\\ +\mathrm{rk}(t\text{-core}(\lambda))+\begin{cases}\binom{n+1}{2}+n\mathrm{rk}(t\text{-core}(\lambda)) & t\text{ even},\\ 0 & t\text{ odd}.\end{cases}$$

All that remains now is to show that this sign agrees with that of Theorem 3.3. By a combination of Lemmas 4.6 and 4.7 we may replace $\operatorname{sgn}(w_t(\lambda';nt))$ by $\operatorname{sgn}(w_t(\lambda;nt))$, which cancels $\mathrm{rk}(t\text{-core}(\lambda))+(t-1)\sum_{r=0}^{t-1}|\lambda^{(r)}|$ in δ. If we call this new exponent δ', then we also have by Lemma 4.1 that $m_r(\lambda';nt)=m_{r-1}(\lambda;nt)$ for $\lfloor (t+1)/2\rfloor\leqslant r\leqslant t-1$, which implies $\delta'=\varepsilon^{\mathrm{sp}}_{\lambda;nt}$.

5.5. **Proof of Theorem 3.4.** The final proof closely follows the first. Let λ be a partition of length at most nt and consider

$$\mathrm{so}_\lambda=\det_{1\leqslant i,j\leqslant nt}\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+1}\big).$$

As before we apply the permutations $w_t(\lambda;nt)$ and $w_t(0;nt)$ to the rows and columns of this determinant, introducing the sign (5.1). Unlike before, there is only one case in which both h's may survive. If t is odd and $\lambda_i-i\equiv -j\equiv (t-1)/2 \pmod t$ then we have

$$\varphi_t\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+1}\big)\\ =h_{\lambda_k^{((t-1)/2)}-k+\ell+m_{(t-1)/2}(\lambda;nt)-n}+h_{\lambda_k^{((t-1)/2)}-k-\ell+1+m_{(t-1)/2}(\lambda;nt)-n}.$$

where $1\leqslant k\leqslant m_{(t-1)/2}(\lambda;nt)$ and $1\leqslant \ell\leqslant n$. These entries lie in the rows $1+\sum_{r=0}^{(t-3)/2}m_r(\lambda;nt)$ to $\sum_{r=0}^{(t-1)/2}m_r(\lambda;nt)$ and columns $1+n(t-1)/2$ to $n(t+1)/2$, and outside of their intersection, all other entries in these rows and columns are zero. Now consider a row corresponding to the residue class r for $0\leqslant r\leqslant\lfloor (t-2)/2\rfloor$. Then the column must fall into the residue class r or $t-r-1$ in order for the entry to not necessarily vanish. In this case we now have

$$\varphi_t\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+1}\big)=\begin{cases}h_{\lambda_k^{(r)}-k+\ell_1+m_r(\lambda;nt)-n} & \text{if } j\equiv -r \pmod t,\\ h_{\lambda_k^{(r)}-k-\ell_1+m_r(\lambda;nt)-n+1} & \text{if } j\equiv r+1 \pmod t.\end{cases}$$

Again, a similar computation holds for the row corresponding to $t-r-1$, and the sets of indices agree with (5.2) but with $t-r\mapsto t-r-1$ in (5.2b). If t is odd we move the central submatrix corresponding to $(t-1)/2$ to the top-left, picking up a sign of

$$(-1)^{m_{(t-1)/2}(\lambda;nt)\sum_{r=0}^{(t-3)/2}m_r(\lambda;nt)+n^2(t-1)/2}.$$

The grouping and rearrangement of the remaining minors is the same as in the first proof above. We only remark that the result is the determinant of a block-diagonal matrix with blocks of dimensions $(m_r(\lambda;nt)+m_{t-r-1}(\lambda;nt))\times 2n$ for $0\leqslant r\leqslant\lfloor (t-2)/2\rfloor$ plus one of size $m_{(t-1)/2}(\lambda;nt)\times n$ if t is odd. Thus the determinant vanishes unless (4.1) holds with $z = 0$, i.e., unless t-core(λ) is self-conjugate. Accounting for the sign of $(-1)^{\binom{m_r(\lambda;nt)}{2}+\binom{n}{2}}$ from reordering the columns in the copies of $\mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r-1)}}$, the exponent $\varepsilon^{\mathrm{so}}_{\lambda;nt}$ has the value

$$\frac{(n+1)nt(t-1)}{4} + \sum_{r=0}^{\lfloor (t-2)/2\rfloor} \left(\binom{m_r}{2} + \binom{n}{2}\right) + \begin{cases} 0 & t \text{ even}, \\ n\sum_{r=0}^{(t-3)/2} m_r + n^2(t-1)/2 & t \text{ odd}. \end{cases}$$

By (4.4) this has the same parity as

$$\varepsilon^{\mathrm{so}}_{\lambda;nt} = \sum_{r=\lfloor (t+1)/2\rfloor}^{t-1} \binom{m_r(\lambda;nt)+1}{2} + \begin{cases} 0 & t \text{ even}, \\ n\mathrm{rk}(t\text{-core}(\lambda)) & t \text{ odd}. \end{cases}$$

5.6. **Proof of Proposition 3.5.** To close out this section, we sketch the proof of Proposition 3.5. In the Schur case, by Corollary 4.2 and the fact that λ/μ is t-tileable if and only if λ'/μ' is, we already have $\omega\varphi_t s_{\lambda/\mu} = \pm\varphi_t\omega s_{\lambda/\mu}$. The precise difference in sign is then provided by Lemma 4.6. Again using Corollary 4.2, we have

$$\begin{aligned} \omega\varphi_t \mathrm{so}_\lambda &= (-1)^{\varepsilon^{\mathrm{so}}_{\lambda;nt}} \operatorname{sgn}(w_t(\lambda;nt)) \prod_{r=0}^{\lfloor (t-2)/2\rfloor} \mathrm{rs}_{(\lambda^{(r)})',(\lambda^{(t-r-1)})'} \\ &\qquad\times \begin{cases} 1 & t \text{ even}, \\ \mathrm{so}_{(\lambda^{((t-1)/2)})'} & t \text{ odd}, \end{cases} \\ &= (-1)^{\varepsilon^{\mathrm{so}}_{\lambda;nt}+\varepsilon^{\mathrm{so}}_{\lambda';nt}} \operatorname{sgn}(w_t(\lambda;nt))\operatorname{sgn}(w_t(\lambda';nt))\varphi_t \mathrm{so}_{\lambda'}, \end{aligned}$$

where n should be large enough so that λ is contained in an $nt\times nt$ box. Combining Lemmas 4.5 and 4.6 shows that, in this case,

$$\operatorname{sgn}(w_t(\lambda;nt))\operatorname{sgn}(w_t(\lambda';nt)) = (-1)^{(t-1)(|\lambda^{(0)}|+\cdots+|\lambda^{(t-1)}|)}.$$

Moreover, $\varepsilon^{\mathrm{so}}_{\lambda;nt} = \varepsilon^{\mathrm{so}}_{\lambda';nt}$, so that the total sign agrees with the claim.

6. Other factorisations

6.1. **Littlewood-type factorisations.** In [27, §7.3], Littlewood proves a factorisation slightly more general than the one contained in Theorem 3.1 for μ empty; see also [5, Theorem 2.7]. Here, and below, we let $\lambda^{(r)} = \lambda^{(k)}$ if $k \equiv r \pmod t$.

Theorem 6.1. *Let λ be a partition of length at most $nt+1$ and $X = (x_1,\dots,x_n)$ a set of variables. Then for another variable q,*

$$s_\lambda(X,\zeta X,\dots,\zeta^{t-1}X,q) = 0$$

unless t-core$(\lambda) = (c)$ for some $0 \leqslant c \leqslant t-1$, in which case

$$s_\lambda(X,\zeta X,\dots,\zeta^{t-1}X,q) = \operatorname{sgn}_t(\lambda/(c))q^c s_{\lambda^{(c-1)}}(X^t,q^t) \prod_{\substack{r=0\\ r\neq c-1}}^{t-1} s_{\lambda^{(r)}}(X^t).$$

This theorem can also be placed in our framework, however in a somewhat less elegant manner than our other results. The operator $\varphi_t^q : \Lambda \longrightarrow \Lambda\otimes_{\mathbb{Z}}\mathbb{Z}[q]$ which gives the above may be defined by

$$\varphi_t^q h_{at+b} := q^b \sum_{k\geqslant 0} q^{kt} h_{a-k} = \sum_{k\geqslant 0} q^k \varphi_t h_{at+b-k}.$$

Note that the sums are finite since h_r vanishes for negative r, and that for $q=0$ this reduces to the operator φ_t from the earlier sections. Alternatively, since

$$h_r(X,q)=\sum_{k\geqslant 0}q^k h_{r-k}(X),$$

the image of φ_t^q acting on any symmetric function f is the same as the image of φ_t acting on $f(X,q)$, where φ_t acts only on the X variables. Using φ_t^q, Littlewood's above theorem may be phrased as follows. After the statement we provide a short proof which relies only on Theorem 3.1.

Proposition 6.2. *We have that* $\varphi_t^q s_\lambda=0$ *unless* t-core$(\lambda)=(c)$ *for some* c *such that* $0\leqslant c\leqslant t-1$*, in which case*

$$\varphi_t^q s_\lambda=\operatorname{sgn}_t(\lambda/(c))q^c\prod_{\substack{r=0\\r\neq c-1}}^{t-1}s_{\lambda^{(r)}}\sum_{k\geqslant 0}q^{kt}s_{\lambda^{(c-1)}/(k)}.$$

Proof. The first observation is that

$$\varphi_t^q s_\lambda=\sum_{k\geqslant 0}q^k\varphi_t s_{\lambda/(k)},$$

which is a simple consequence of the branching rule for Schur functions [31, p. 72]. In the case that $l(t\text{-core}(\lambda))>1$ then each term in the sum on the right-hand side vanishes by Theorem 3.1 as the t-cores of the inner and outer shape can never be equal. Now assume t-core$(\lambda)=(c)$ for some $0\leqslant c\leqslant t-1$, which is a complete set of t-cores with length one. Then the nonzero terms in the sum on the right-hand side are those for which k is of the form $\ell t+c$ with $\ell\geqslant 0$ and $\ell t+c\leqslant\lambda_1$. Therefore

$$\varphi_t^q s_\lambda=q^c\sum_{\ell\geqslant 0}q^{\ell t}\varphi_t s_{\lambda/(\ell t+c)}=q^c\sum_{\ell\geqslant 0}\operatorname{sgn}_t(\lambda/(\ell t+c))q^{\ell t}s_{\lambda^{(c-1)}/(\ell)}\prod_{\substack{r=0\\r\neq c-1}}^{t-1}s_{\lambda^{(r)}},$$

again by Theorem 3.1. By our convention we always compute the t-quotient using a beta set with number of elements a multiple of t. This means that the single row $(\ell t+c)$ has one non-empty element in its t-quotient, $\lambda^{(c-1)}$. Moreover, since the partitions $(\ell t+c)$ all differ by a ribbon of height zero, the sign of each term in the sum is the same and equal to $\operatorname{sgn}_t(\lambda/(c))$. Putting all of this together, we arrive at

$$\varphi_t^q s_\lambda=\operatorname{sgn}_t(\lambda/(c))q^c\prod_{\substack{r=0\\r\neq c-1}}^{t-1}s_{\lambda^{(r)}}\sum_{\ell\geqslant 0}q^{\ell t}s_{\lambda^{(c-1)}/(\ell)}. \qquad \square$$

We do not see, at this stage, whether it is possible to extend the previous result to skew Schur functions. If we expand $\varphi_t^q s_{\lambda/\mu}$ as in the proof above, we find that

$$\varphi_t^q s_{\lambda/\mu}=\sum_{\nu\succ\mu}q^{|\nu|-|\mu|}\varphi_t s_{\lambda/\nu},$$

where $\nu\succ\mu$ means that ν/μ is a *horizontal strip*, i.e., $\nu\supseteq\mu$ and ν/μ contains at most one box in each column of its Young diagram. Of course, this implies that $\varphi_t^q s_{\lambda/\mu}=0$ if there does not exist a ν such that $\nu\succ\mu$ and λ/ν is t-tileable. However, the sum may vanish even if such a ν exists. For example,

$$\begin{aligned}\varphi_2^q s_{(4,4)/(1)}&=q\varphi_t\big(s_{(4,4)/(2)}+s_{(4,4)/(1,1)}\big)+q^3\varphi_t\big(s_{(4,4)/(4)}+s_{(4,4)/(3,1)}\big)\\&=q\big(s_{(2)}s_{(2)/(1)}-s_{(2)}s_{(2)/(1)}\big)+q^3\big(s_{(2)}-s_{(2)}\big)\\&=0.\end{aligned}$$

In a similar direction Pfannerer [36, Theorem 4.4] has shown that, if λ has empty t-core and $m=\ell t+k$ is any integer, then the Schur function $s_\lambda(1,\zeta,\dots,\zeta^{m-1})$

always factors as a product of Schur functions with variables all one indexed by the t-quotient of λ. When m is a multiple of t this becomes a special case of Littlewood's theorem (Theorem 3.1 with μ empty) noted by Reiner, Stanton and White [38, Theorem 4.3]. Pfannerer's result has subsequently been generalised by Kumari [22, Theorem 2.2], in addition to analogues of Theorem 6.1 for other classical group characters. It is an open problem to see how these factorisations fit into our story.

6.2. **Factorisations of supersymmetric Schur functions.** Recently, Kumari has given a version of Theorem 3.1 for the so-called *skew hook Schur functions* (or *supersymmetric skew Schur functions*) [21, Theorem 3.2]. For two independent sets of variables (*alphabets*), we denote their plethystic difference by $X - Y$; see, e.g., [14, 23] for the necessary background on plethystic notation. We also note that for an alphabet X, we let εX be the alphabet with all variables negated. The *complete homogeneous supersymmetric function* used in [21] may be defined as

$$\sum_{j=0}^{r} h_j(X)e_{r-j}(Y) = h_r[X - \varepsilon Y].$$

The hook Schur function is then the Jacobi–Trudi determinant of these functions, so that

$$s_{\lambda/\mu}[X - \varepsilon Y] = \det_{1\leqslant i,j\leqslant n}\big(h_{\lambda_i-\mu_j-i+j}[X - \varepsilon Y]\big).$$

From this, it follows readily that Kumari's factorisation for the hook Schur functions is contained in Theorem 3.1 above at the alphabet $X - \varepsilon Y$.

6.3. **Factorisations of $\mathrm{rs}_{\lambda,\mu}$.** To close, we point out that the universal character $\mathrm{rs}_{\lambda,\mu}$ can be used to lift some factorisation results, discovered by Ciucu and Krattenthaler [8, Theorems 3.1–3.2] and subsequently generalised by Ayyer and Behrend [3, Theorems 1–2], to the universal character level. In the next result we write $\lambda + 1^n = (\lambda_1 + 1, \dots, \lambda_n + 1)$ where $n \geqslant l(\lambda)$.

Theorem 6.3. *For λ a partition of length at most n, there holds*

$$\mathrm{rs}_{\lambda,\lambda} = \mathrm{so}_\lambda \mathrm{so}^-_\lambda, \tag{6.1a}$$

and

$$\mathrm{rs}_{\lambda+1^n,\lambda} = \mathrm{o}_{\lambda+1^n}\mathrm{sp}_\lambda. \tag{6.1b}$$

Moreover, for λ a partition of length at most $n+1$,

$$\begin{aligned}\mathrm{rs}_{(\lambda_1,\dots,\lambda_n),(\lambda_2,\dots,\lambda_{n+1})} &+ \mathrm{rs}_{(\lambda_1-1,\dots,\lambda_{n+1}-1),(\lambda_2+1,\dots,\lambda_n+1)} \\ &= \mathrm{sp}_{(\lambda_1,\dots,\lambda_n)}\mathrm{o}_{(\lambda_2,\dots,\lambda_{n+1})},\end{aligned} \tag{6.2a}$$

and

$$\begin{aligned}\mathrm{rs}_{(\lambda_1+1,\dots,\lambda_n+1),(\lambda_2,\dots,\lambda_{n+1})} &+ \mathrm{rs}_{(\lambda_1,\dots,\lambda_{n+1}),(\lambda_2+1,\dots,\lambda_n+1)} \\ &= \mathrm{so}_{(\lambda_1+1,\dots,\lambda_n+1)}\mathrm{so}^-_{(\lambda_2,\dots,\lambda_{n+1})}.\end{aligned} \tag{6.2b}$$

To get back to the results of Ayyer and Behrend one simply evaluates both sides of each equation at the alphabet $(x_1^{\pm}, \dots, x_n^{\pm})$. The precise forms present in [3, Equations (18)–(21)] then follow from (2.5).[2] As identities for Laurent polynomials the pairs of identities (6.1) and (6.2) admit uniform statements. However no such uniform statement will exist for the above generalisation, since this requires characters indexed by half-partitions, which cannot be handled by the universal characters. Ayyer and Fischer [4] have also given skew analogues of the non-universal case of Theorem 6.3. Jacobi–Trudi formulae for the symplectic and orthogonal

[2] The factor of $(1 + \delta_{0,\lambda_{n+1}})$ in [3, Equation (20)] is not present in our generalisation (6.2a) since the second character vanishes if $\lambda_{n+1} = 0$.

characters have recently been derived in [1, 17], and so there are candidates for the universal characters for those objects. However, the main obstacle in lifting Ayyer and Fischer's results to the universal level is the lack of a skew analogue of $\mathrm{rs}_{\lambda,\mu}$.

Proof of Theorem 6.3. First up is (6.1a), which is the simplest of the four. In the determinant

$$\mathrm{rs}_{\lambda,\lambda}=\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n}\\ \big(h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i+j}\big)_{1\leqslant i,j\leqslant n}\end{pmatrix},$$

add the blocks on the right to the blocks on the left, and then subtract the blocks on the top from the blocks on the bottom, giving

$$\begin{aligned}\mathrm{rs}_{\lambda,\lambda}&=\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n}\\ 0 & \big(h_{\lambda_i-i+j}-h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n}\end{pmatrix}\\&=\mathrm{so}_\lambda\mathrm{so}^-_\lambda.\end{aligned}$$

For the second identity (6.1b),

$$\mathrm{rs}_{\lambda+1^n,\lambda}=\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j+1}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+2}\big)_{1\leqslant i,j\leqslant n}\\ \big(h_{\lambda-i-j+1}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i+j}\big)_{1\leqslant i,j\leqslant n}\end{pmatrix},$$

and we add columns $1,\dots,n-1$ to the columns $n+2,\dots,2n$ and then subtract the bottom two blocks from the top two, resulting in

$$\begin{aligned}&\mathrm{rs}_{\lambda+1^n,\lambda}\\&\quad=\frac{1}{2}\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j+1}-h_{\lambda_i-i-j+1}\big)_{1\leqslant i,j\leqslant n} & 0\\ \big(h_{\lambda-i-j+1}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+2}\big)_{1\leqslant i,j\leqslant n}\end{pmatrix}\\&\quad=\mathrm{o}_{\lambda+1^n}\mathrm{sp}_\lambda.\end{aligned}$$

In the third identity, we consider the second determinant in the sum in (6.2a)

$$\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j-1}\big)_{1\leqslant i,j\leqslant n+1} & \big(h_{\lambda_i-i-j}\big)_{\substack{1\leqslant i\leqslant n+1\\1\leqslant j\leqslant n-1}}\\ \big(h_{\lambda_{i+1}-i-j+2}\big)_{\substack{1\leqslant i\leqslant n-1\\1\leqslant j\leqslant n+1}} & \big(h_{\lambda_{i+1}-i+j+1}\big)_{1\leqslant i,j\leqslant n-1}\end{pmatrix}.$$

Push the first column so it becomes the $(n+1)$-st, and then push the $(n+1)$-st row to the final row, which picks up a minus sign. The resulting determinant differs from that of $\mathrm{rs}_{(\lambda_1,\dots,\lambda_n),(\lambda_2,\dots,\lambda_{n+1})}$ in only the last row, so we can take the sum of the two, giving

$$\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}\big)_{1\leqslant i\leqslant n}\\ \big(h_{\lambda_{i+1}-i-j+1}\big)_{\substack{1\leqslant i\leqslant n-1\\1\leqslant j\leqslant n}} & \big(h_{\lambda_{i+1}-i+j}\big)_{\substack{1\leqslant i\leqslant n-1\\1\leqslant j\leqslant n}}\\ (h_{\lambda_{n+1}-n-j+1}-h_{\lambda_{n+1}-n+j-1})_{1\leqslant j\leqslant n} & (h_{\lambda_{n+1}-n+j}-h_{\lambda_{n+1}-n-j})_{1\leqslant j\leqslant n}\end{pmatrix}.$$

In this new determinant, add columns $n+1,\dots,2n-1$ to columns $2,\dots,n$, and then subtract rows $2,\dots,n$ from rows $n+1,\dots,2n-1$, which gives

$$\frac{1}{2}\det_{1\leqslant i,j\leqslant 2n}\begin{pmatrix}\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+2}\big)_{1\leqslant i,j\leqslant n} & \big(h_{\lambda_i-i-j+1}\big)_{1\leqslant i\leqslant n}\\ 0 & \big(h_{\lambda_{i+1}-i+j}-h_{\lambda_{i+1}-i-j}\big)_{1\leqslant i,j\leqslant n}\end{pmatrix},$$

which equals $\mathrm{sp}_{(\lambda_1,\dots,\lambda_n)}\mathrm{o}_{(\lambda_2,\dots,\lambda_{n+1})}$. The final factorisation (6.2b) follows almost identically. $\square$

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CHARACTER FACTORISATIONS, z-ASYMMETRIC PARTITIONS AND PLETHYSM

SEAMUS ALBION

ABSTRACT. The Verschiebung operators φ_t are a family of endomorphisms on the ring of symmetric functions, one for each integer $t \geqslant 2$. Their action on the Schur basis has its origins in work of Littlewood and Richardson, and is intimately related with the decomposition of a partition into its t-core and t-quotient. Namely, they showed that the action on s_λ is zero if the t-core of the indexing partition is nonempty, and otherwise it factors as a product of Schur functions indexed by the t-quotient. Much more recently, Lecouvey and, independently, Ayyer and Kumari have provided similar formulae for the characters of the symplectic and orthogonal groups, where again the combinatorics of cores and quotients plays a fundamental role. We embed all of these character factorisations in an infinite family involving an integer z and parameter q using a very general symmetric function defined by Hamel and King. The proof hinges on a new characterisation of the t-cores and t-quotients of z-asymmetric partitions which generalise the well-known classifications for self-conjugate and doubled distinct partitions. We also explain the connection between these results, plethysms of symmetric functions and characters of the symmetric group.

1. INTRODUCTION

For each integer $t \geqslant 2$ the *Verschiebung operator*[1] φ_t is an endomorphism on the ring of symmetric functions defined by

$$\varphi_t h_k = \begin{cases} h_{k/t} & \text{if } t \text{ divides } k, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

where h_k denotes the k-th complete homogeneous symmetric function. The action of φ_t on the Schur basis was first computed by Littlewood and Richardson, but phrased in a different way [41, 42]. They classified the partitions for which $\varphi_t s_\lambda = 0$ and further show that when it is nonzero the result is a product of t Schur functions indexed by partitions depending only on λ. Almost two decades later, Littlewood realised that this action is intimately related with the decomposition of a partition into its t-core and t-quotient, concepts which were not yet known at the time of the work with Richardson. Much more recently, Lecouvey [33] and, independently, Ayyer and Kumari [3] computed the action of φ_t on the characters of the symplectic and orthogonal groups in a finite number of variables. In [2] we lifted these results to the universal characters of the associated groups. Again, the combinatorics of cores and quotients is at the heart of the evaluations. Our main result of the present paper, Theorem 4.3, embeds all of these "character factorisations" in an infinite family paramaterised by an integer z and involving a parameter q. This is achieved by computing the action of φ_t on a very general symmetric function of Hamel and King [19, 20]. For $q = 0$ we recover the Schur case and for $z \in \{-1, 0, 1\}$ the symplectic and orthogonal cases. What facilitates this generalisation is a characterisation of the t-cores and t-quotients of the z-asymmetric partitions of Ayyer and Kumari which are a z-deformation of self-conjugate partitions; see Theorem 2.3. Before explaining our contributions in detail, we survey the history of these results, since it appears that they are not so well-known. Moreover, it involves a rich interplay between (modular) representation theory, symmetric functions and the combinatorics of integer partitions.

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[1] The name *Verschiebung* (German for *shift*) comes from the theory of Witt vectors; see [17, §2.9] and [18, Exercise 2.9.10].

1.1. **Historical background.** The notion of a hook of an integer partition was introduced by Nakayama in the pair of papers [48, 49]. For an integer $t \geqslant 2$ he showed that one can associate to each partition a *t-core*, being a partition containing no hook of length t. His motivation came from the modular representation theory of the symmetric group, and in particular he conjectured that for t prime two partitions belong to the same t-block of the symmetric group if and only if they have the same t-core [49, §6]. This conjecture was proved several years later by Brauer and Robinson [6, 59].[2] Following the proof of Nakayama's conjecture, Robinson introduced the notion of a star diagram associated to a partition, which encodes its t-hook structure [60], work which was continued by Staal [62]. This was independently discovered by Nakayama and Osima, who gave a second, independent proof of Nakayama's conjecture [50].

Inspired by Robinson's work, Littlewood synthesised the aforementioned ideas into what he dubbed the t-residue and t-quotient of a partition [38]. In fact, the t-residue is just Nakayama's t-core, while the t-quotient contains the same information as the star diagram of Robinson, but is more simply constructed. Perhaps due to its more straightforward nature, Littlewood's construction is now the most well-known. To be a little more explicit, let $\mathscr{P}$ denote the set of partitions and $\mathscr{C}_t$ the set of all t-cores. What is now known as the *Littlewood decomposition* amounts to a bijection

$$\begin{aligned} \phi_t : \mathscr{P} &\longrightarrow \mathscr{C}_t \times \mathscr{P}^t \\ \lambda &\longmapsto \big(t\text{-core}(\lambda), (\lambda^{(0)}, \dots, \lambda^{(t-1)})\big), \end{aligned}$$

where t-core(λ) is Nakayama's t-core and the t-tuple of partitions $(\lambda^{(0)}, \dots, \lambda^{(t-1)})$ is Littlewood's t-quotient. The bijection may be realised in several equivalent ways. Below we will use the realisation in terms of Maya diagrams or, equivalently, the binary encoding of partitions. Littlewood's original construction was purely arithmetic, and his motivation was similar to the authors before him. In his paper he gives a short, independent proof of Nakayama's conjecture, and then uses the t-quotient as a tool to produce relationships between modular characters inside t-blocks. He gives two further applications of the construction: one to character values of the symmetric group and one to a particular plethysm of symmetric functions.

Let χ^λ denote the irreducible character of the symmetric group $\mathfrak{S}_n$ indexed by the partition λ of n. We use the usual notations for partitions; see Subsection 2.1 for the relevant definitions. Here we only note that $t\mu$ stands for the partition with all parts multiplied by t and for a partition with empty t-core $\operatorname{sgn}_t(\lambda)$ is equal to ± 1 and may be defined in terms of the heights of ribbons; see (2.1). Littlewood stated the following theorem.

Theorem 1.1 ([38, p. 340])**.** *Let λ be a partition of nt. Then $\chi^\lambda(t\mu) = 0$ unless the t-core of λ is empty, in which case*

$$\chi^\lambda(t\mu) = \operatorname{sgn}_t(\lambda) \operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{|\lambda^{(0)}|} \times \cdots \times \mathfrak{S}_{|\lambda^{(t-1)}|}} \big(\chi^{\lambda^{(0)}} \otimes \cdots \otimes \chi^{\lambda^{(t-1)}}\big)(\mu). \tag{1.2}$$

In fact, this result appears already in a paper of Littlewood and Richardson from seventeen years prior [41, Theorem IX].[3] There, however, the elegance of the theorem is almost completely obscured by the absence of the concepts of the t-core and t-quotient. An extension to skew characters $\chi^{\lambda/\mu}$ was given by Farahat, a student of Littlewood [12]. For more on this theorem and its generalisations see Subsection 6.2.

The second application is to a particular instance of plethysm of symmetric functions. Again, deferring precise definitions until later on, let $s_\lambda = s_\lambda(x_1, x_2, \dots)$ be the Schur function indexed by λ and $p_r(x_1, x_2, \dots) = x_1^r + x_2^r + \cdots$ the r-th power sum symmetric function. The plethysm $p_r \circ s_\lambda = s_\lambda \circ p_r$ is defined by

$$s_\lambda \circ p_r := s_\lambda(x_1^r, x_2^r, x_3^r, \dots). \tag{1.3}$$

Also, for a multiset of skew shapes $\mathcal{S}$ we let $c^\lambda_{\mathcal{S}}$ denote the coefficient of s_λ in the Schur expansion of $\prod_{\mu \in \mathcal{S}} s_\mu$. When $\mathcal{S}$ consists of only two straight shapes μ, ν then $c^\lambda_{\mu,\nu}$ are the Littlewood–Richardson coefficients famously characterised by Littlewood and Richardson in [40]. Thus we will refer to

[2]The proof is joint work but appears in separate papers published simultaneously in the Transactions of the Royal Society of Canada.

[3]Curiously, Littlewood's citation of this result points to his treatise [36], although it appears earlier in the work with Richardson.

the $c^{\lambda}_{\mathcal{S}}$ as *multi-Littlewood–Richardson coefficients*. Littlewood's second application is the Schur expansion of the plethysm (1.3).

Theorem 1.2 ([38, p. 351])**.** *For a partition λ and integer $t \geqslant 2$,*

$$s_\lambda \circ p_t = \sum_{\substack{\nu \\ t\text{-core}(\nu)=\varnothing}} \operatorname{sgn}_t(\nu) c^{\lambda}_{\nu^{(0)},\dots,\nu^{(t-1)}} s_\nu.$$

This formula has come to be known as the *SXP rule*. It has a generalisation as an expansion of the expression $s_\tau(s_{\lambda/\mu} \circ p_t)$ due to Wildon [68] which we will meet later on in Section 5.

A glance at the structure of the theorems suggests there must be a relation between them, and indeed the proof of Theorem 1.2 in [38] uses Theorem 1.1. Remarkably, Littlewood and Richardson's proof of the first theorem is based on a Schur function identity which is in a sense dual to the second theorem. (Littlewood provides a proof of a slightly more general result in [38], of which Theorem 1.1 is a special case, which is independent of the proof given earlier.) To explain this, recall that the *Hall inner product* is the inner product on the ring of symmetric functions Λ for which the Schur functions are orthonormal:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}, \tag{1.4}$$

where $\delta_{\lambda\mu}$ is the usual Kronecker delta. As an operator on the algebra of symmetric functions, the plethysm by p_t has an adjoint, which is denoted by φ_t. That is, for any $f, g \in \Lambda$,

$$\langle f \circ p_t, g \rangle = \langle f, \varphi_t g \rangle. \tag{1.5}$$

The operator φ_t turns out to be the Verschiebung operator defined above (1.1).

With the aid of Theorem 1.2 the evaluation of the action of φ_t on the Schur functions is a short exercise. Setting $(f, g) \mapsto (s_\mu, s_\lambda)$ in (1.5) gives $\langle s_\mu \circ p_t, s_\lambda \rangle = \langle s_\mu, \varphi_t s_\lambda \rangle$. Therefore

$$\langle s_\mu, \varphi_t s_\lambda \rangle = \begin{cases} \operatorname{sgn}_t(\lambda) c^{\mu}_{\lambda^{(0)},\dots,\lambda^{(t-1)}} & \text{if } t\text{-core}(\lambda) = \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of the multi-Littlewood–Richardson coefficients we obtain the following.

Theorem 1.3. *For λ a partition and $t \geqslant 2$ an integer we have that $\varphi_t s_\lambda = 0$ unless t-core$(\lambda) = \varnothing$, in which case*

$$\varphi_t s_\lambda = \operatorname{sgn}_t(\lambda) s_{\lambda^{(0)}} \cdots s_{\lambda^{(t-1)}}.$$

As already mentioned above, this result has its origins in the work of Littlewood and Richardson in the 1930's. At the generality of the above theorem the result first appears in Littlewood's book [36, §7.3]. There the language of cores and quotients is of course not used, nor is the Verschiebung operator. Rather he gives an equivalent formulation in terms of his notion of the "S-function of a series"; see [34] or [63, Exercise 7.91]. There is also an extension of Theorem 1.3 to skew Schur functions due to Farahat and Macdonald; see Theorem 3.2 below. How precisely Theorems 1.1, 1.2 and 1.3 are equivalent will be explained in Subsection 6.2.

1.2. **Generalisations to classical group characters.** In their paper on what are now called LLT polynomials, Lascoux, Leclerc and Thibon pointed out that the adjoint relationship (1.5) combined with a refinement of the Littlewood decomposition to ribbon tableaux due to Stanton and White [64] leads to a combinatorial proof of the identity of Theorem 1.3 [31, §IV]. Indeed, the operator φ_t and its plethysm adjoint are "q-deformed" and then used to define the LLT polynomials. In extending this construction to other types, Lecouvey proved beautiful variations of Theorem 1.3 for the characters of Sp_{2n} and O_{2n} in the case t is odd and SO_{2n+1} for general t [32, §3]. (Here and throughout all matrix groups are taken over $\mathbb{C}$.) Rather than expressing these results as products of characters, he gives the expansion of the evaluation in terms of Weyl characters where the coefficients are branching coefficients corresponding to the restriction of an irreducible polynomial representation to a subgroup of Levi type. The obstruction for t even in the first two cases is precisely that the coefficients cannot be interpreted as branching coefficients.

Recently Ayyer and Kumari rediscovered the factorisation results of Lecouvey, but in a slightly different form by "twisting" a finite set of n variables by a primitive t-th root of unity [3]. This point of view is explained in Section 6. By working with the explicit Laurent polynomial expressions for the symplectic and orthogonal characters they could show that for all $t \geqslant 2$ these twisted characters factor as a product of other characters. They also characterise the vanishing of these twisted characters in a much simpler manner. For example they show that the twisted character of SO_{2n+1} indexed by λ vanishes unless t-core(λ) is self-conjugate. The even orthogonal and symplectic cases admit similarly simple descriptions. For $t = 2$ these factorisations may be found already in the work of Mizukawa [46].

Lecouvey also proved striking extensions of Theorem 1.2 to the universal characters of the symplectic and orthogonal groups [33]. These are symmetric function lifts of the ordinary characters first defined by Koike and Terada [27] using the Jacobi–Trudi formulae of Weyl. (Lecouvey's extensions are anticipated by work of Littlewood [39] for the ordinary characters and Scharf and Thibon for the universal characters [61, §6], both only for $t = 2, 3$.) Inspired by the work of Ayyer and Kumari we lifted their factorisations to the level of universal characters [2].[4] Our proofs there are based on the Jacobi–Trudi formulae for these symmetric functions. In the present work we utilise a different approach based on expressions for the universal characters in terms of skew Schur functions. For example, let so_λ denote the universal odd orthogonal character. Then

$$\mathrm{so}_\lambda := \det_{1\leqslant i,j\leqslant l(\lambda)}\big(h_{\lambda_i-i+j} + h_{\lambda_i-i-j-1}\big) = \sum_{\substack{\mu\in\mathscr{P}_0\\ \mu\subseteq\lambda}} (-1)^{(|\mu|-\mathrm{rk}(\mu))/2} s_{\lambda/\mu}, \tag{1.6}$$

where $\mathscr{P}_0$ is the set of self-conjugate partitions, $\mu \subseteq \lambda$ means the Young diagram of μ is contained in that of λ and $\mathrm{rk}(\mu)$ denotes the Frobenius rank of μ. We will now state the expression for $\varphi_t\mathrm{so}_\lambda$, in which we will write $\tilde{\lambda} := t\text{-core}(\lambda)$, a short-hand also used below whenever it is convenient. We also note that for a pair of partitions λ, μ the symmetric function $\mathrm{rs}_{\lambda,\mu}$ is the universal character lift of the irreducible rational representation of GL_n indexed by the pair of partitions (λ, μ); see (3.14) and the surrounding discussion for a definition.

Theorem 1.4. *For λ a partition and $t \geqslant 2$ an integer we have that $\varphi_t\mathrm{so}_\lambda = 0$ unless t-core(λ) is self-conjugate, in which case*

$$\varphi_t\mathrm{so}_\lambda = (-1)^{(|\tilde{\lambda}|-\mathrm{rk}(\tilde{\lambda}))/2}\,\mathrm{sgn}_t(\lambda/\tilde{\lambda}) \prod_{r=0}^{\lfloor (t-2)/2\rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r-1)}} \times \begin{cases} 1 & t \text{ even},\\ \mathrm{so}_{\lambda^{((t-1)/2)}} & t \text{ odd}.\end{cases}$$

This may be found in various forms in [32, §3.2.4], [3, Theorem 2.17] and [2, Theorem 3.4]. The key difference between this theorem and all its previous iterations is that the overall sign is explicitly expressed in terms of statistics on λ and its t-core. While not visible from the above we are also able to show that in the symplectic and even orthogonal cases the sign is just as simple. The proof of the above theorem we present below uses the skew Schur expansion in (1.6), the skew Schur function case of Theorem 1.3 (Theorem 3.2 below) and properties of the Littlewood decomposition restricted to the set of self-conjugate partitions. More precisely, it was observed by Osima [52] that a partition is self-conjugate if and only if t-core(λ) is self-conjugate and

$$\lambda^{(r)} = (\lambda^{(t-r-1)})' \quad \text{for } 0 \leqslant r \leqslant t-1. \tag{1.7}$$

Note that the partitions paired by this condition are precisely the partitions paired in the factorisation of $\varphi_t\mathrm{so}_\lambda$.

In fact much more is true. Our main result, which we state as Theorem 4.3 below, embeds Theorem 1.4 as the $(z, q) = (0, 1)$ case of an infinite family of such factorisations where z is an arbitrary integer and q is a formal variable. The generalisation of the character so_λ, denoted $\mathcal{X}_\lambda(z; q)$, is a symmetric function defined by Hamel and King [19, 20], building on work of Bressoud and Wei [7]. It may be expressed as a Jacobi–Trudi-type determinant or as a sum of skew Schur functions à la (1.6). This sum is indexed by z-asymmetric partitions, a term coined by Ayyer and Kumari, which are a z-deformation of self-conjugate partitions. In fact, what facilitates the factorisation of this object under φ_t is that the Littlewood decomposition for z-asymmetric partitions has a nice structure, involving "conjugation conditions" such as (1.7). Indeed, this is our other main result, Theorem 2.3, which characterises z-asymmetric partitions in terms of their Littlewood decompositions. For $z = 0$ this is the self-conjugate case discussed prior, and for $z = 1$ this appears in the seminal work of Garvan, Kim and Stanton on cranks [14].

[4]At the time we were unfortunately not aware of the work of Lecouvey.

1.3. **Summary of the paper.** The paper reads as follows. In the next section we introduce the necessary definitions and conventions for integer partitions, including the Littlewood decomposition. This includes our first main result, Theorem 2.3, the characterisation of z-asymmetric partitions under the Littlewood decomposition. Section 3 then turns to symmetric functions and universal characters. We survey the action of the Verschiebung operators on the classical bases of the ring of symmetric functions, and introduce a new deformation of the rational universal characters which arise naturally in our main factorisation theorem. Section 4 then contains the companions of Theorem 1.4 for the symplectic and even orthogonal characters, our generalisation, stated as Theorem 4.3, and its proof. Then Section 5 is used to survey the known SXP rules for Schur functions and universal characters. This includes Wildon's generalisation of Theorem 1.2 which we show is equivalent to the skew case of Theorem 1.3 (Theorem 3.2 below). Using our combinatorial setup, we give reinterpretations of Lecouvey's SXP rules, and in particular show that for all types they may be expressed as sums over partitions with empty t-core. We close with some remarks about related results, including a discussion of the precise relationship between the first three theorems of the introduction.

2. Partitions and the Littlewood decomposition

This section contains the necessary preliminaries regarding integer partitions. We also describe the Littlewood decomposition in terms of Maya diagrams which is essentially the abacus model of James and Kerber [22]. Our main results in this section, Theorem 2.3 and its corollaries, give a characterisation of z-asymmetric partitions in terms of the Littlewood decomposition.

2.1. **Preliminaries.** A *partition* is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ such that the *size* $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$ is finite. The nonzero λ_i are called *parts* and the number of parts the *length*, denoted $l(\lambda)$. The set of all partitions is written $\mathscr{P}$ and the empty partition, the unique partition of 0, is denoted by $\varnothing$. We write (m^ℓ) for the partition with ℓ parts equal to m, and the sum $\lambda + (m^\ell)$ is then the partition obtained by adding m to the first ℓ parts of λ. We identify a partition with its *Young diagram*, which is the left-justified array of cells consisting of λ_i cells in row i with i increasing downward. An example is given in Figure 1. We define the *conjugate* partition λ' by reflecting the diagram of λ in the main diagonal, so that the conjugate of $(6, 5, 5, 1)$ is $(4, 3, 3, 3, 3, 1)$. A partition is *self-conjugate* if $\lambda = \lambda'$.

The *Frobenius rank* of a partition, $\mathrm{rk}(\lambda)$, is defined as the number of cells along the main diagonal of its Young diagram. We extend this by an integer $c \in \mathbb{Z}$ to a statistic $\mathrm{rk}_c(\lambda)$ which we call the *c-shifted Frobenius rank*. If $c \geqslant 0$ this is the Frobenius rank of the partition obtained by deleting the first c rows of λ, while for $c \leqslant 0$ it is the Frobenius rank of the partition with the first $-c$ columns of λ removed. Another way to notate partitions is with *Frobenius notation*, which records the number of cells to the right of and below each cell on this main diagonal. This is written

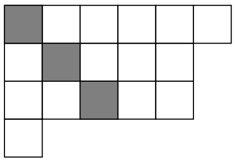
$$\lambda = \big(\lambda_1 - 1, \dots, \lambda_{\mathrm{rk}(\lambda)} - \mathrm{rk}(\lambda) \mid \lambda'_1 - 1, \dots, \lambda'_{\mathrm{rk}(\lambda)} - \mathrm{rk}(\lambda)\big);$$

again, see Figure 1 for an example. Any two strictly decreasing nonnegative integer sequences u, v with the same number of elements, say k, thus give a unique partition $\lambda = (u \mid v)$ of Frobenius rank k. Clearly self-conjugate partitions are those of the form $(u \mid u)$. Now let $u + z := (u_1 + z, \dots, u_k + z)$ for any $z \in \mathbb{Z}$. Ayyer and Kumari define *z-asymmetric partitions* to be those of the form $(u + z \mid u)$ for any sequence u (of any length) and fixed $z \in \mathbb{Z}$ [3, Definition 2.9]. The set of z-asymmetric partitions is denoted by $\mathscr{P}_z$ and $(6, 5, 5, 1)$ in Figure 1 is 2-asymmetric. The generating function for z-asymmetric partitions is given by

$$\sum_{\lambda \in \mathscr{P}_z} q^{|\lambda|} = (-q^{1+|z|}; q^2)_\infty.$$

This is easy to see by the fact that a z-asymmetric partition is uniquely determined by its set of hook lengths on the main diagonal. These are all distinct integers of the form "odd plus $|z|$", which gives the proof. Clearly the conjugate of a z-asymmetric partition is $-z$-asymmetric.

Given a cell s in the Young diagram of λ its *hook length* is one more than the sum of the number of cells below and to the right of s; see Figure 1. The *hook* of s is then the set of cells counted. A hook is a *principal hook* if it is the hook of a cell on the main diagonal. For an integer $t \geqslant 2$ we say a partition is a *t-core* if it contains no cell with hook length t (or, equivalently, no cell with hook length divisible by t). For a pair of partitions λ, μ we say μ is *contained* in λ, written $\mu \subseteq \lambda$, if its



9	7	6	5	4	1
7	5	4	3	2	
6	4	3	2	1	
1					

FIGURE 1. The partition $\lambda = (6,5,5,1) = (5,3,2 \mid 3,1,0)$ with its main diagonal shaded (left) and the same partition with hook length of each cell inscribed (right). We have $|\lambda| = 17$, $l(\lambda) = 4$, $\mathrm{rk}(\lambda) = 3$, $\mathrm{rk}_2(\lambda) = 1$ and $\mathrm{rk}_{-3}(\lambda) = 2$.

Young diagram may be drawn inside the Young diagram of λ. The corresponding *skew shape* is the arrangement of cells formed by removing μ's diagram from λ's. A skew shape is a *ribbon* if it is edge-connected and contains no 2×2 square of cells, and a *t-ribbon* is a ribbon containing t cells.[5] The *height* of a ribbon R, $\mathrm{ht}(R)$, is one less than the number of rows it occupies; see Figure 2.

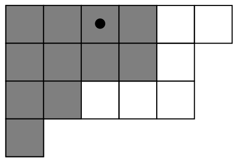


FIGURE 2. The pair of partitions $(4,4,2,1) \subseteq (6,5,5,1)$. The unshaded cells form a 6-ribbon of height 2 and the corresponding cell with hook length 6 is marked.

We say a skew shape λ/μ is *t-tileable* if there exists a sequence of partitions

$$\mu =: \nu^{(0)} \subseteq \nu^{(1)} \subseteq \cdots \subseteq \nu^{(m-1)} \subseteq \nu^{(m)} := \lambda$$

such that the skew shapes $\nu^{(r)}/\nu^{(r-1)}$ are each t-ribbons for $1 \leqslant r \leqslant m$. It is a non-trivial fact, see, e.g. [53, Lemma 4.1], that the sign

$$\mathrm{sgn}_t(\lambda/\mu) := (-1)^{\sum_{r=1}^{m} \mathrm{ht}(\nu^{(r)}/\nu^{(r-1)})} \tag{2.1}$$

is constant over the set of all t-ribbon decompositions of λ/μ (so, indeed, the above is well-defined). In the case $\mu = \varnothing$ and $t = 2$ the above sign is simply equal to

$$\mathrm{sgn}_2(\lambda) = (-1)^{\mathrm{odd}(\lambda)/2}$$

where $\mathrm{odd}(\lambda)$ is equal to the number of odd parts of λ; see, e.g., [5, Equation (5.15)].

2.2. **Littlewood's decomposition.** Here we describe the Littlewood decomposition through the lens of Maya diagrams, which is essentially the *abacus* of James and Kerber [22, §2.7] or the *Brettspiele* of Kerber, Sänger and Wagner [23]. Littlewood's original algebraic description may be found in [38] and [44, p. 12].

Given a partition λ its *beta set* is the subset of the half integers given by

$$\beta(\lambda) := \Big\{\lambda_i - i + \frac{1}{2} : i \geqslant 1\Big\}.$$

This is visualised as a configuration of "beads" on the real line placed at the positions indicated by $\beta(\lambda)$, and this visualization is the *Maya diagram*. Note that for any partition the configuration will eventually contain only beads to the left and only empty spaces to the right. The map from partitions to Maya diagrams is clearly a bijection, and one way to reconstruct λ from $\beta(\lambda)$ is to count the number of empty spaces to the left of each bead starting from the right. From the Maya diagram we extract t subdiagrams, called *runners*, formed by the beads at positions x such that $x - 1/2$ is equal to r modulo t for $0 \leqslant r \leqslant t-1$. Arranging the runners with r increasing upward we obtain the *t-Maya diagram*. An example of this procedure is given in Figure 3. The partitions corresponding to each runner are denoted by $\lambda^{(r)}$ according to the residues modulo t of the original positions, and these precisely form Littlewood's *t-quotient*.

The next important observation is that t-hooks in λ correspond to beads in its t-Maya diagram which contain no bead immediately to their left. For example, Figure 1 shows that $(6,5,5,1)$

[5] Elsewhere in the literature ribbons are variously called *border strips*, *rim hooks* or *skew hooks*.

contains two 3-hooks, and in Figure 3 one bead in runner 0 and one in runner 2 have free spaces to their left. Moving such a bead one space to its left removes the t-ribbon associated with that hook. Repeating this procedure until all beads are flush-left in the t-Maya diagram produces a unique partition t-core(λ) which, as the notation suggests, is a t-core. The uniqueness is clear from the t-Maya diagram picture. Furthermore, the height of the removed ribbon is equal to the number of beads between its initial and terminal position, i.e., to $|\beta(\lambda) \cap \{x-1, \ldots, x-t+1\}|$ if we move the bead at position x. Note that in the ordinary Maya diagram this corresponds to the number of beads "jumped over". Let us collect these observations into the following theorem.

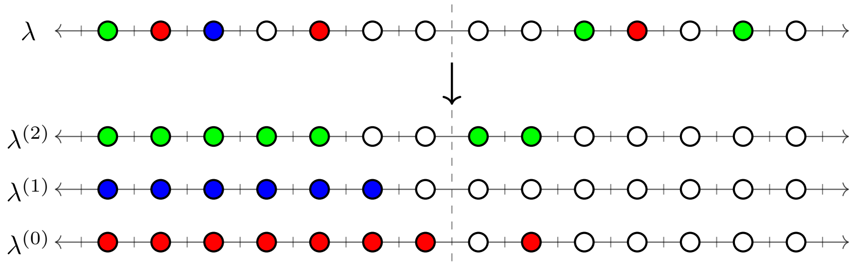


FIGURE 3. The Maya diagram of $\lambda = (6, 5, 5, 1)$ (top) and the 3-Maya diagram of the same partition (bottom). We have that 3-core$(\lambda) = (1, 1)$, $\kappa_3((1, 1)) = (1, -1, 0)$ and $(\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}) = ((1), \varnothing, (2, 2))$.

Theorem 2.1 (Littlewood's decomposition)**.** *For any integer $t \geqslant 2$ the above procedure encodes a bijection*

$$\begin{aligned} \mathscr{P} &\longrightarrow \mathscr{C}_t \times \mathscr{P}^t \\ \lambda &\longmapsto \big(t\text{-core}(\lambda), (\lambda^{(0)}, \ldots, \lambda^{(t-1)})\big) \end{aligned}$$

such that $|\lambda| = |t\text{-core}(\lambda)| + t(|\lambda^{(0)}| + \cdots + |\lambda^{(t-1)}|)$.

When a skew shape λ/μ is t-tileable can be characterised completely in terms of the Littlewood decomposition of λ and μ. Since λ/μ being t-tileable means that we may obtain the diagram of μ from that of λ by removing ribbons in any order, it follows that λ/μ is t-tileable if and only if t-core$(\lambda) = t$-core(μ) and $\mu^{(r)} \subseteq \lambda^{(r)}$ for each $0 \leqslant r \leqslant t-1$.

We will also need a different characterisation of t-cores. Call a Maya diagram *balanced* if it contains as many beads to the right of 0 as empty spaces to the left. The way we defined Maya diagrams ensures they are always balanced, but Figure 3 shows that the constituent diagrams of the quotient need not be. Let c_r^+ (resp. c_r^-) denote the number of beads to the right of 0 (resp. number of empty spaces to the left of 0) in row $\lambda^{(r)}$ of the t-Maya diagram. Now the sequence of integers $(c_0, \ldots, c_{t-1})$ defined by $c_r := c_r^+ - c_r^-$ has total sum zero, and is invariant under valid bead movements. As observed by Garvan, Kim and Stanton, this encodes a bijection [14, Bijection 2]

$$\kappa_t : \mathscr{C}_t \longrightarrow \{(c_0, \ldots, c_{t-1}) \in \mathbb{Z}^t : c_0 + \cdots + c_{t-1} = 0\} \tag{2.2}$$

such that for $\mu \in \mathscr{C}_t$

$$|\mu| = \sum_{r=0}^{t-1} \left(\frac{tc_r^2}{2} + rc_r \right).$$

In what follows we extend (2.2) to a map $\mathscr{P} \longrightarrow \mathbb{Z}^t$, the fibres of which are the sets of all partitions with a given core.

In the introduction we noted that self-conjugate partitions satisfy a nice symmetry with respect to the Littlewood decomposition. To explain where this comes from, note that the conjugate of a partition can be read off its (ordinary) Maya diagram by interchanging beads and empty spaces and then reflecting the picture about 0. In the t-Maya diagram this corresponds to conjugating each runner and reversing the order of the runners. This implies that the t-quotient of λ' is given by $((\lambda^{(t-1)})', \ldots, (\lambda^{(0)})')$ in terms of the t-quotient of λ. Furthermore, we have that t-core$(\lambda') = t$-core$(\lambda)'$ which, if $\kappa_t(\lambda) = (c_0, \ldots, c_{t-1})$, translates to $\kappa_t(\lambda') = (-c_{t-1}, \ldots, -c_0)$ in terms of (2.2). From these properties it immediately follows that the Littlewood decomposition of a self-conjugate partition much satisfy $c_r + c_{t-r-1} = 0$ for $0 \leqslant r \leqslant t-1$ and $\lambda^{(r)} = (\lambda^{(t-r-1)})'$ for r in the same range. This is equivalent to the conditions given in the introduction. Garvan, Kim and Stanton [14, §8] show that something similar holds for 1-asymmetric partitions.

Proposition 2.2. *If $\lambda \in \mathscr{P}_1$ then t-core$(\lambda), \lambda^{(0)} \in \mathscr{P}_1$ and the remaining entries in the quotient satisfy $\lambda^{(r)} = (\lambda^{(t-r)})'$ for $1 \leqslant r \leqslant t-1$.*

Our first main result is a generalisation of this proposition to z-asymmetric partitions. To fix some notation, let $\mathcal{C}_{z;t} \subset \mathbb{Z}^t$ consist of those t-tuples for which $c_r + c_{z-r-1} = 0$ for $0 \leqslant r \leqslant z-1$ and $c_s + c_{t+z-s-1} = 0$ for $z \leqslant s \leqslant t-1$. Also recall the c-shifted Frobenius rank $\mathrm{rk}_c(\lambda)$ from the previous Subsection 2.1.

Theorem 2.3. *Let $t \geqslant 2$ and z be integers and λ a partition such that $0 \leqslant z \leqslant t-1$ and $\lambda \in \mathscr{P}_z$. Then $\kappa_t(t\text{-core}(\lambda)) \in \mathcal{C}_{z;t}$ and the quotient $(\lambda^{(0)}, \dots, \lambda^{(t-1)})$ is such that for $0 \leqslant r \leqslant z-1$ with $c_r \geqslant 0$ there exist partitions $\nu^{(r)}$ with*

$$\lambda^{(r)} = \nu^{(r)} + (1^{c_r + \mathrm{rk}_{c_r}(\nu^{(r)})}) \quad \textit{and} \quad \lambda^{(z-r-1)} = (\nu^{(r)})' + (1^{\mathrm{rk}_{c_r}(\nu^{(r)})}) \tag{2.3a}$$

and for $z \leqslant s \leqslant t-1$,

$$\lambda^{(s)} = (\lambda^{(t+z-s-1)})'. \tag{2.3b}$$

Proof. The proof is by induction on z. For $z = 0$ the result is clear from the properties of self-conjugate partitions under the Littlewood decomposition. Now choose a strict partition v and let $\lambda = (v + z - 1 \mid v)$ for some fixed $z \geqslant 1$. Assume that $\kappa_t(t\text{-core}(\lambda)) \in \mathcal{C}_{z-1;t}$ and further that the conditions (2.3) are satisfied (with z replaced by $z-1$ in the latter). We wish to show that the partition $\mu = (v + z \mid v)$ has $\kappa_t(t\text{-core}(\mu)) \in \mathcal{C}_{z;t}$ and that the conditions (2.3) hold for μ. Also set $\kappa_t(t\text{-core}(\lambda)) = (c_1, \dots, c_{t-1})$ and $\kappa_t(t\text{-core}(\mu)) = (d_1, \dots, d_{t-1})$.

The key observation is that we may obtain the t-Maya diagram of μ from that of λ as follows: beads lying at positive positions are moved upwards cyclically one runner in the same column, except those passing from $\lambda^{(t-1)}$ to $\lambda^{(0)}$, which move an additional space to the right. An example of this is given in Figure 4. If we imagine that the t-Maya diagram is wrapped around a bi-infinite cylinder, then this corresponds to cutting the cylinder along 0, "twisting" so that beads passing from $r = t-1$ to $r = 0$ are also moved one space to the right, and then re-gluing. From this construction we observe that for $0 \leqslant r \leqslant z-1$

$$d_r + d_{z-r-1} = c^+_{r-1} - c^-_r + c^+_{z-r-2} - c^-_{z-r-1}. \tag{2.4}$$

We already have that $c_r + c_{z-r-2} = 0$ for $0 \leqslant r \leqslant z-2$ by our assumption. However, (2.3a) implies the slightly stronger condition that $c^-_r = c^+_{z-r-2}$ (or, equivalently, $c^+_r = c^-_{z-r-2}$). This is because conjugation of a runner interchanges c^+_r and c^-_r. Thus, in the range $1 \leqslant r \leqslant z-2$ we have that (2.4) vanishes. For $r = 0$ one needs to use that $\lambda^{(z)} = (\lambda^{(t-1)})'$ and $c_z + c_{t-1} = 0$. The same argument in the range $z \leqslant s \leqslant t-1$ completes the proof that $\kappa_t(t\text{-core}(\mu)) \in \mathcal{C}_{z;t}$.

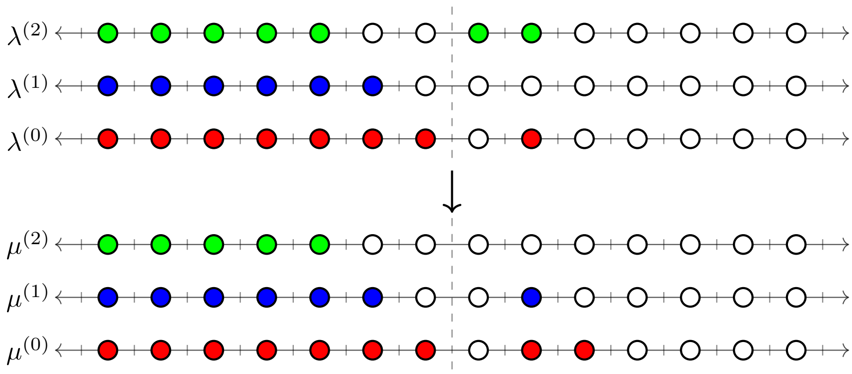


FIGURE 4. The 3-Maya diagram of $(6, 5, 5, 1)$ (top) and the 3-Maya diagram of $(7, 6, 6, 1)$ (bottom) corresponding to action the "cut and twist" map.

Let $\lambda^{(r)}_{<0}$ (resp. $\lambda^{(r)}_{>0}$) denote the negative (resp. positive) half of the runner corresponding to $\lambda^{(r)}$. It is clear that, when indices are read modulo t, $\mu^{(r)} = \lambda^{(r)}_{<0} \cup \lambda^{(r-1)}_{>0}$ and $\mu^{(z-r-1)} = \lambda^{(z-r-1)}_{<0} \cup \lambda^{(z-r-2)}_{>0}$ for $0 \leqslant r \leqslant z-1$. Since the $\lambda^{(r)}$ satisfy (2.3) in the range $1 \leqslant r \leqslant z-2$, then so will the $\mu^{(r)}$ with c_r replaced by d_r. The cases $r = 0$ and for $z \leqslant s \leqslant t-1$ follow by the same argument, the former using the fact that the positive beads in $\lambda^{(t-1)}_{>0}$ will move one space to the right. $\square$

While we have stated the above theorem only for $0 \leqslant z \leqslant t-1$, it may be extended to arbitrary $z \geqslant 0$. Since this is not as elegant as the above, we now state this separately as a corollary.

Corollary 2.4. *Let $t \geqslant 2$ and $z = at + b$ be integers and λ a partition such that $0 \leqslant b \leqslant t-1$ and $\lambda \in \mathscr{P}_z$. Then $\kappa_t(t\text{-core}(\lambda)) \in \mathcal{C}_{b;t}$ and the quotient $(\lambda^{(0)}, \dots, \lambda^{(t-1)})$ is such that for $0 \leqslant r \leqslant b-1$ with $c_r \geqslant 0$ there exist partitions $\nu^{(r)}$ with*

$$\lambda^{(r)} = \nu^{(r)} + ((a+1)^{c_r + \mathrm{rk}_{c_r}(\nu^{(r)})}) \quad \textit{and} \quad \lambda^{(b-r-1)} = (\nu^{(r)})' + ((a+1)^{\mathrm{rk}_{c_r}(\nu^{(r)})}), \tag{2.5a}$$

and for $b \leqslant s \leqslant t-1$ with $c_s \geqslant 0$ there exist partitions $\xi^{(s)}$ with

$$\lambda^{(s)} = \xi^{(s)} + (a^{c_s + \mathrm{rk}_{c_s}(\xi^{(s)})}) \quad \textit{and} \quad \lambda^{(t+b-s-1)} = (\xi^{(s)})' + (a^{\mathrm{rk}_{c_s}(\xi^{(s)})}). \tag{2.5b}$$

This corollary follows simply from the observation that t iterations of the "cut and twist" map used in the previous proof shift all beads at positive positions one place to the right. If b is odd then (2.5a) says that $\lambda^{((b-1)/2)} \in \mathscr{P}_{a+1}$ and if $t+b$ is odd then (2.5b) says that $\lambda^{((t+b-1)/2)} \in \mathscr{P}_a$. Since we may obtain negative z by conjugation, Corollary 2.4 gives a characterisation of z-asymmetric partitions under the Littlewood decomposition.

Our next corollary, which will prove useful in the statement of our main results, characterises when a t-core is z-asymmetric, and gives the minimal z-asymmetric partition with a given core. The first part of this is due to Ayyer and Kumari [3, Lemma 3.6] in a slightly different form.

Corollary 2.5. *A t-core μ is z-asymmetric if and only if $0 \leqslant z \leqslant t-2$ and $\kappa_t(\mu)$ satisfies $c_r = 0$ for $0 \leqslant r \leqslant z-1$. Moreover, for any sequence $\mathbf{c} \in \mathcal{C}_{z;t}$ the unique z-asymmetric partition λ with $\kappa_t(t\text{-core}(\lambda)) = \mathbf{c}$ and minimal $|\lambda|$ has quotient $\lambda^{(r)} = (1^{c_r})$ for those r with $0 \leqslant r \leqslant z-1$ and $c_r > 0$.*

Proof. By Theorem 2.3 a z-asymmetric partition μ must have $\kappa_t(\mu) \in \mathcal{C}_{z;t}$ and $\lambda^{(r)} = \varnothing$ for all $0 \leqslant r \leqslant t-1$. However, the restrictions (2.3a) admit the empty partition as a solution if and only if $c_r = 0$. The second part of the corollary is then immediate. $\square$

This shows that while the t-core of a 0- or 1-asymmetric partition is always itself 0- or 1-asymmetric, the same is not necessarily true for z-asymmetric partitions when $z \geqslant 2$. Indeed, our running example of $(6,5,5,1)$ is 2-asymmetric but has t-core $(1,1)$ which is clearly not 2-asymmetric.

A key tool we need below is an expression for the Frobenius rank of a partition in terms of the Frobenius ranks of its core and quotient. This is due to Brunat and Nath, however, we restate it in our terminology and provide a short proof. Some related results about the Frobenius ranks of -1-, 0- and 1-asymmetric partitions may be found in [3, Lemma 3.13].

Lemma 2.6 ([8, Corollary 3.29])**.** *For any partition λ and integer $t \geqslant 2$,*

$$\mathrm{rk}(\lambda) = \mathrm{rk}(t\text{-core}(\lambda)) + \sum_{r=0}^{t-1} \mathrm{rk}_{c_r}(\lambda^{(r)}).$$

Proof. Let $\kappa_t(\lambda) = (c_0, \dots, c_1)$. As we have already remarked, $\mathrm{rk}(\lambda)$ is equal to the number of beads at positive positions in the Maya or t-Maya diagram, i.e., $\mathrm{rk}(\lambda) = \sum_{r=0}^{t-1} c_r^+$. A simple rewriting of this expression gives

$$\sum_{r=0}^{t-1} c_r^+ = \sum_{\substack{r=0 \\ c_r > 0}}^{t-1} (c_r^+ - c_r^-) + \sum_{\substack{r=0 \\ c_r > 0}}^{t-1} c_r^- + \sum_{\substack{r=0 \\ c_r \leqslant 0}}^{t-1} c_r^+.$$

The first sum on the right is equal to $\mathrm{rk}(t\text{-core}(\lambda))$ since, after pushing all beads to the left, this will count the beads remaining at positive positions. If $c_r = 0$ then the beads on runner r do not contribute to $\mathrm{rk}(t\text{-core}(\lambda))$ and so $\mathrm{rk}(\lambda^{(r)}) = c_r^+$. Now consider the case $c_r > 0$. Counted from the right, the first c_r beads in this runner are already accounted for by $\mathrm{rk}(t\text{-core}(\lambda))$. The quantity $c_r^- = c_r^+ - c_r$ then counts the number of remaining beads at positive positions, which is equal to the Frobenius rank of λ with the first c_r rows removed, i.e., to $\mathrm{rk}_{c_r}(\lambda)$. By conjugation the same argument works in the case $c_r < 0$, completing the proof. □

Observe that if the t-core of λ is empty then

$$\mathrm{rk}(\lambda) = \sum_{r=0}^{t-1} \mathrm{rk}(\lambda^{(r)}),$$

since $\mathrm{rk}_0(\lambda^{(r)}) = \mathrm{rk}(\lambda^{(r)})$. An example of the computation of the Frobenius rank using the lemma is given in Figure 5.

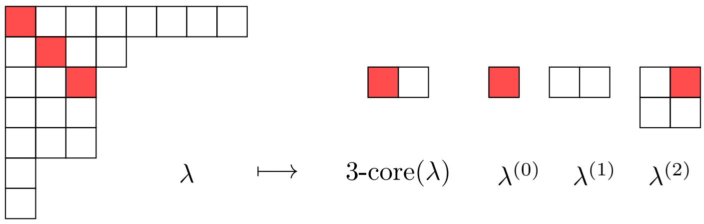


FIGURE 5. The Littlewood decomposition of $\lambda = (8, 4, 3, 3, 3, 1, 1)$ with $t = 3$ and $\kappa_3(\lambda) = (0, 1, -1)$. The marked cells explain the computation of the Frobenius rank: the left- and right-hand sides both contain three shaded cells since the first row of $\lambda^{(1)}$ and the first column of $\lambda^{(2)}$ are ignored.

To conclude this section, we give an alternate characterisation of the sign (2.1) in terms of certain permutations. For a partition λ and an integer n such that $n \geqslant l(\lambda)$ we write $\sigma_t(\lambda; n)$ for the permutation on n letters which sorts the list $(\lambda_1 - 1, \dots, \lambda_n - n)$ such that their residues modulo t are increasing, and the elements within each residue class are decreasing. For example if $t = 3$, $\lambda = (6, 5, 5, 1)$ and $n = 6$ then the list is $(5, 3, 2, -3, -5, -6)$. Our permutation is then $\sigma_3(\lambda; 6) = 246513$ in one-line notation. Inversions in this permutation may be read off the t-Maya diagram. They correspond to pairs of beads (b_1, b_2) such that b_2 lies weakly to the right of and strictly above b_1. (Note that we only consider the first n beads, read top-to-bottom and right-to-left.)

In the following lemma we write $\mathrm{sgn}(w)$ for the sign of the permutation w.

Lemma 2.7 ([2, Lemma 4.5]). *Let λ/μ be a t-tileable skew shape. Then for any integer $n \geqslant l(\lambda)$ we have*

$$\mathrm{sgn}_t(\lambda/\mu) = \mathrm{sgn}(\sigma_t(\lambda; n))\,\mathrm{sgn}(\sigma_t(\mu; n)). \tag{2.6}$$

Proof. Let λ/μ have ribbon decomposition

$$\mu =: \nu^{(0)} \subseteq \nu^{(1)} \subseteq \cdots \subseteq \nu^{(m-1)} \subseteq \nu^{(m)} := \lambda,$$

where $\nu^{(r)}/\nu^{(r-1)}$ is a t-ribbon for each $1 \leqslant r \leqslant m$. The contribution of the ribbon $\lambda/\nu^{(m-1)}$ to the sign on the left is $(-1)^{\mathrm{ht}(\lambda/\nu^{(m-1)})}$. If the removal of this ribbon corresponds to moving a bead from position x to $x - t$, then this sign is equal to $(-1)^b$ where $b = |\beta(\lambda) \cap \{x - 1, \dots, x - t + 1\}|$ counts the number of beads strictly between x and $x - t$. By the construction of the permutations $\sigma_t(\lambda; n)$ and $\sigma_t(\nu^{(m-1)}; n)$, we have $(-1)^b = \mathrm{sgn}(\sigma_t(\lambda; n))\,\mathrm{sgn}(\sigma_t(\nu^{(m-1)}; n))$. In other words, upon removing a single ribbon, both the left- and right-hand sides of (2.6) change by the same quantity. Iterating this completes the proof. □

3. Generalised universal characters

We now return to symmetric functions. The first part of this section is devoted to the Verschiebung operator, defined as the adjoint of the plethysm by a power sum symmetric function. After briefly surveying its action on various classes of symmetric functions we state our variants of this action on the universal characters. This is followed by the main theorems, which compute the image of the general symmetric function $\mathcal{X}_\lambda$ defined in the introduction.

3.1. **Symmetric functions and plethysm.** Here we give the basic facts relating to symmetric functions; see [44, Chapter 1] or [63, Chapter 7]. We work in the *algebra of symmetric functions* over $\mathbb{Q}$ and in a countable alphabet $X = (x_1, x_2, x_3, \ldots)$, denoted Λ. Important families of symmetric functions we require are the *elementary symmetric functions* and the *complete homogeneous symmetric functions*, defined for integers $k \geqslant 0$ by

$$e_k(X) := \sum_{1\leqslant i_1<\cdots<i_k} x_{i_1}\cdots x_{i_k} \quad \text{and} \quad h_k(X) := \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k} x_{i_1}\cdots x_{i_k},$$

respectively. As in the introduction we will drop the alphabet of variables and write e_k and h_k for the above. These are extended to partitions by $h_\lambda := h_{\lambda_1}h_{\lambda_2}h_{\lambda_3}\cdots$ and analogously for the e_λ and p_λ. The final "obvious" basis consists of the *monomial symmetric functions*

$$m_\lambda(X) := \sum_\alpha x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\cdots,$$

where the sum is over all distinct permutations of the partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$. Over $\mathbb{Q}$, all four of the above families form linear bases for Λ.

The most important family of symmetric functions are certainly the *Schur functions* s_λ. The simplest way to define them at the generality of skew shapes is by the Jacobi–Trudi determinant

$$s_{\lambda/\mu} := \det_{1\leqslant i,j\leqslant l(\lambda)}(h_{\lambda_i-\mu_j-i+j}), \tag{3.1}$$

where $h_{-k} := 0$ for $k \geqslant 1$. Similarly we have the dual Jacobi–Trudi formula

$$s_{\lambda/\mu} = \det_{1\leqslant i,j\leqslant \lambda_1}(e_{\lambda'_i-\mu'_j-i+j}),$$

and again $e_{-k} := 0$ for $k \geqslant 1$. The symmetric functions s_λ form a basis for Λ which is orthonormal with respect to the Hall inner product (1.4). Another way to define the skew Schur function is by the adjoint relation

$$\langle s_{\lambda/\mu}, f\rangle = \langle s_\lambda, s_\mu f\rangle. \tag{3.2}$$

for any $f \in \Lambda$. As already covered above, the *Littlewood–Richardson coefficients* $c^\lambda_{\mu\nu}$ are the structure constants of the Schur basis:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

Combining this with (3.2) in the case $f = s_\nu$ then gives

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu.$$

From these equations one sees that $c^\lambda_{\mu\nu}$ is symmetric in μ, ν and will vanish unless $\mu, \nu \subseteq \lambda$ and $|\lambda| = |\mu| + |\nu|$. These properties extend analogously to the multi-Littlewood–Richardson coefficients.

We also have the following orthogonality relations among other symmetric functions

$$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu} \quad \text{and} \quad \langle p_\lambda, p_\mu\rangle = z_\lambda\delta_{\lambda\mu}, \tag{3.3}$$

where $\delta_{\lambda\mu}$ is the usual Kronecker delta and $z_\lambda := \prod_{i\geqslant 1} m_i(\lambda)! i^{m_i(\lambda)}$. It is customary to define a homomorphism on symmetric functions by $\omega h_k = e_k$, which is in fact an involution. One may show using the Jacobi–Trudi formulae that $\omega s_{\lambda/\mu} = s_{\lambda'/\mu'}$. Moreover, $\omega m_\lambda = f_\lambda$ where the f_λ are the *forgotten symmetric functions*, and ω is an isometry.

Plethysm is a composition of symmetric functions first introduced by Littlewood [35]; see also [44, p. 135]. In the introduction we defined the plethysm by a power sum symmetric function p_t, which raises each variable to the power of t. Some properties of this plethysm are $f \circ p_t = p_t \circ f$ and $p_s \circ p_t = p_{st}$ for $s, t \in \mathbb{N}$. Moreover if f is homogeneous of degree n then [63, Exercise 7.8]

$$\omega(f \circ p_t) = (-1)^{n(t-1)}(\omega f) \circ p_t. \tag{3.4}$$

Recall from earlier that the adjoint of this plethysm with respect to the Hall scalar product is denoted φ_t, the t-th Verschiebung operator. We take the adjoint relation as defining this operator; the definition in (1.1) given at the very beginning will serve as a special case of the following. The next proposition gives the action of this operator on most of the families of symmetric functions we have seen so far. We also provide short proofs of these claims, writing λ/t as short-hand for the partition $(\lambda_1/t, \lambda_2/t, \lambda_3/t, \ldots)$ when all parts of λ are divisible by t.

Proposition 3.1. *Let $t \geqslant 2$ be an integer and λ a partition. If t does not divide each part of λ then $\varphi_t h_\lambda = \varphi_t e_\lambda = \varphi_t p_\lambda = 0$. If it does, then*

$$\varphi_t h_\lambda = h_{\lambda/t}, \qquad \varphi_t e_\lambda = (-1)^{|\lambda|(t-1)/t} e_{\lambda/t} \quad \text{and} \quad \varphi_t p_\lambda = t^{l(\lambda)} p_{\lambda/t}. \tag{3.5}$$

Proof. Beginning with the complete homogeneous symmetric function case, it is clear from the definition of the m_μ that $m_\mu \circ p_t = m_{t\mu}$. Therefore

$$\langle \varphi_t h_\lambda, m_\mu \rangle = \langle h_\lambda, m_\mu \circ p_t \rangle = \langle h_\lambda, m_{t\mu} \rangle = \delta_{\lambda, t\mu},$$

where the last equality is an application of (3.3). This implies that $\varphi_t h_\lambda = 0$ if t does not divide each part of λ. If it does, then the above is equal to $\delta_{\lambda/t,\mu}$, which implies that $\varphi_t h_\lambda = h_{\lambda/t}$.

For the second case, note that since ω is an isometry

$$\langle \varphi_t e_\lambda, f_\mu \rangle = \langle e_\lambda, f_\mu \circ p_t \rangle = \langle h_\lambda, \omega(f_\mu \circ p_t) \rangle.$$

By (3.4) we now have $\omega(f_\mu \circ p_t) = (-1)^{|\mu|(t-1)} m_{t\mu}$. Therefore

$$\langle \varphi_t e_\lambda, f_\mu \rangle = (-1)^{|\mu|(t-1)} \langle h_\lambda, m_{t\mu} \rangle = (-1)^{|\mu|(t-1)} \delta_{\lambda, t\mu},$$

again with the aid of (3.3). Exactly as before this implies that $\varphi_t e_\lambda = 0$ unless all parts of λ are divisible by t. If they are then $\varphi_t e_\lambda = (-1)^{|\lambda|(t-1)/t} e_{\lambda/t}$, completing the proof of this case.

The power sum case is almost identical. First we use (3.3) to obtain

$$\langle \varphi_t p_\lambda, p_\mu \rangle = \langle p_\lambda, p_\mu \circ p_t \rangle = \langle p_\lambda, p_{t\mu} \rangle = z_\lambda \delta_{\lambda, t\mu}.$$

This tells us that $\varphi_t p_\lambda$ vanishes unless all parts of λ are divisible by t. Thus the power sum expansion of $\varphi_t p_\lambda$ has a single term with coefficient $z_\lambda / z_{\lambda/t} = t^{l(\lambda)}$. $\square$

The actions of φ_t presented in the previous proposition are all rather simple, and follow the same pattern of dividing all parts of the partition by t if possible. A much richer structure underlies the action of the t-th Verschiebung operator on the (skew) Schur functions, utilising Littlewood's core and quotient construction.

Theorem 3.2. *For any integer $t \geqslant 2$ and skew shape λ/μ we have $\varphi_t s_{\lambda/\mu} = 0$ unless λ/μ is t-tileable, in which case*

$$\varphi_t s_{\lambda/\mu} = \operatorname{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}},$$

where the sign is defined in (2.1).

For $\mu = \varnothing$ this reduces to Theorem 1.3 of the introduction. As alluded to there, the skew case was first worked out by Farahat, but only when $\mu = t\text{-core}(\lambda)$ [13]. To our knowledge, the first statement of the full skew Schur case appears in the second edition of Macdonald's book as an example [44, p. 92]. It then makes a further appearance in the work of Lascoux, Leclerc and Thibon [31, p. 1049], which cites Kerber, Sänger and Wagner [23]. However, the latter does not use Schur functions, and rather gives a new proof of Farahat's skew generalisation of Theorem 1.1 using "Brettspiele", which are essentially our Maya diagrams. In none of these references is the vanishing described in terms of the tileability of the skew shape λ/μ, with this observation coming from Evseev, Paget and Wildon [10, Theorem 3.3] in the context of symmetric group characters (where the term t-decomposable is used rather than our t-tileable). In the precise form above this appears in [2, Theorem 3.1].

In an effort to keep this paper for the most part self-contained we now provide a proof using Macdonald's approach, which is the same as that of Farahat.

Proof of Theorem 3.2. The first step of the proof is clear: apply φ_t to the Jacobi–Trudi formula (3.1) to obtain

$$\varphi_t s_{\lambda/\mu} = \det_{1 \leqslant i,j \leqslant n} \big(\varphi_t h_{\lambda_i - \mu_j - i + j} \big), \tag{3.6}$$

where $n \geqslant l(\lambda)$ is a fixed integer. An entry (i,j) in this new determinant is nonzero only if $\lambda_i - i \equiv \mu_j - j \pmod{t}$. In order to group those entries within the same residue class, permute the rows and columns according to the permutations $\sigma_t(\lambda; n)$ and $\sigma_t(\mu; n)$. The resulting determinant has a block-diagonal structure. If $\kappa(\lambda) = (c_0, \dots, c_{t-1})$, $\kappa(\mu) = (d_0, \dots, d_{t-1})$ and $n = at + b$ $(0 \leqslant b \leqslant t-1)$, then the rth block along the main diagonal will have dimensions

$(c_r+a+[r\geqslant b])\times(d_r+a+[r\geqslant b])$. Here $[\cdot]$ denotes the *Iverson bracket* which is equal to one if the statement $\cdot$ is true and zero otherwise. These blocks will all be square if and only if $\kappa(\lambda)=\kappa(\mu)$, i.e., unless $t\text{-core}(\lambda)=t\text{-core}(\mu)$, and thus the determinant necessarily vanishes if this is not the case. It follows from our definition of the t-quotient that after applying the Verschiebung operator the indices of the complete homogeneous symmetric functions in the rth block along the diagonal are of the form $h_{\lambda_i^{(r)}-\mu_j^{(r)}-i+j}$ where $1\leqslant i\leqslant c_r+a+[r\geqslant b]$ and $1\leqslant j\leqslant d_r+a+[r\geqslant b]$. Thus we have shown that, if $t\text{-core}(\lambda)=t\text{-core}(\mu)$, then

$$\varphi_t s_{\lambda/\mu}=\operatorname{sgn}(\sigma_t(\lambda;n))\operatorname{sgn}(\sigma_t(\mu;n))\prod_{r=0}^{t-1}s_{\lambda^{(r)}/\mu^{(r)}}.$$

This product of skew Schur functions will further vanish unless $\mu^{(r)}\subseteq\lambda^{(r)}$ for each $0\leqslant r\leqslant t-1$. Putting this together with the previous vanishing we determine that $\varphi_t s_{\lambda/\mu}$ is zero unless λ/μ is t-tileable, in which case it is given by the above product. The sign is then equal to $\operatorname{sgn}_t(\lambda/\mu)$ by Lemma 2.7. □

3.2. **Generalised universal characters.** For a finite set of n variables the Schur polynomial $s_\lambda(x_1,\dots,x_n)$ is the character of the irreducible polynomial representation of GL_n indexed by λ. The classical groups O_{2n}, Sp_{2n} and SO_{2n+1} also carry irreducible representations indexed by partitions. The characters of these representations are rather Laurent polynomials symmetric under permutation and inversion of the n variables. Using the Jacobi–Trudi formulae for these characters, originally due to Weyl, they may still be expressed as determinants in the complete homogeneous symmetric functions of the form $h_r(x_1,1/x_1,\dots,x_n,1/x_n)$ [67, Theorems 7.8.E & 7.9.A]. Rather than working with these characters we will use the *universal characters*, as defined by Koike and Terada [26, 27]. These are symmetric function lifts of the ordinary characters given by 'forgetting' the variables in Weyl's Jacobi–Trudi formulae:

$$\mathrm{sp}_\lambda:=\frac{1}{2}\det_{1\leqslant i,j\leqslant k}\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+2}\big) \tag{3.7a}$$

$$\mathrm{o}_\lambda:=\det_{1\leqslant i,j\leqslant k}\big(h_{\lambda_i-i+j}-h_{\lambda_i-i-j}\big) \tag{3.7b}$$

$$\mathrm{so}_\lambda:=\det_{1\leqslant i,j\leqslant k}\big(h_{\lambda_i-i+j}+h_{\lambda_i-i-j+1}\big), \tag{3.7c}$$

where k is an integer such that $l(\lambda)\leqslant k$. We also have the dual forms

$$\mathrm{sp}_\lambda=\det_{1\leqslant i,j\leqslant \ell}\big(e_{\lambda'_i-i+j}-e_{\lambda'_i-i-j}\big) \tag{3.8a}$$

$$\mathrm{o}_\lambda=\frac{1}{2}\det_{1\leqslant i,j\leqslant \ell}\big(e_{\lambda'_i-i+j}+e_{\lambda'_i-i-j+2}\big) \tag{3.8b}$$

$$\mathrm{so}_\lambda=\det_{1\leqslant i,j\leqslant \ell}\big(e_{\lambda'_i-i+j}+e_{\lambda'_i-i-j+1}\big), \tag{3.8c}$$

where here ℓ is an integer such that $\lambda_1\leqslant\ell$. Comparing (3.7) and (3.8) it is clear that $\omega\mathrm{o}_\lambda=\mathrm{sp}_{\lambda'}$ and $\omega\mathrm{so}_\lambda=\mathrm{so}_{\lambda'}$.

Let Λ_n^{BC} denote the ring of Laurent polynomials in $x_1,\dots,x_n$ which are symmetric under permutation and inversion of the variables. Define for integers $n\geqslant 1$ the restriction maps $\pi_n:\Lambda\longrightarrow\Lambda_n^{\mathrm{BC}}$ by $\pi_n(e_r)=e_r(x_1,1/x_1,\dots,x_n,1/x_n)$. If $r>2n$ then $\pi_n(e_r)=0$, and moreover, $\pi_n(e_r-e_{2n-r})=0$ for each $0\leqslant r\leqslant n$. For a partition λ with $l(\lambda)\leqslant n$ the images of the universal characters under π_n are the actual characters of their respective groups indexed by λ. If $l(\lambda)>n$ then these specialisations either vanish or, up to a sign, produce an irreducible character of the same group associated to a different partition which is determined by the so-called "modification rules"; see [24] and [27, §2]. We also have the modified map π'_n which acts by $\pi'_n(e_r)=e_r(x_1,1/x_1,\dots,x_n,1/x_n,1)$ and satisfies $\pi_n(\mathrm{so}_\lambda)=\pi'_n(\mathrm{o}_\lambda)$ for λ with $l(\lambda)\leqslant n$.

In the introduction we already met the character so_λ in (1.6) and saw that it could be expanded as a signed sum over skew Schur functions where the inner shape is a self-conjugate (0-asymmetric) partition. In fact, all three of the characters (3.7) admit such expressions:

$$\mathrm{sp}_\lambda=\sum_{\mu\in\mathscr{P}_{-1}}(-1)^{|\mu|/2}s_{\lambda/\mu}, \tag{3.9a}$$

$$\mathrm{o}_\lambda = \sum_{\mu\in\mathscr{P}_1} (-1)^{|\mu|/2} s_{\lambda/\mu}, \tag{3.9b}$$

$$\mathrm{so}_\lambda = \sum_{\mu\in\mathscr{P}_0} (-1)^{(|\mu|-\mathrm{rk}(\mu))/2} s_{\lambda/\mu}. \tag{3.9c}$$

The Schur functions themselves may be simply expanded in terms of these universal characters:

$$s_\lambda = \sum_\mu \Bigg(\sum_{\substack{\nu \\ \nu \text{ even}}} c^\lambda_{\mu\nu}\Bigg)\mathrm{o}_\mu = \sum_\mu \Bigg(\sum_{\substack{\nu \\ \nu' \text{ even}}} c^\lambda_{\mu\nu}\Bigg)\mathrm{sp}_\mu = \sum_\mu \Bigg(\sum_\nu (-1)^{|\nu|} c^\lambda_{\mu\nu}\Bigg)\mathrm{so}_\lambda, \tag{3.10}$$

where here we write "ν even" meaning ν has only even parts. This last set of equalities are precisely the "Character Interrelation Theorem" of Koike and Terada; see [27, Theorem 2.3.1] and [26, Theorem 7.2].

While these three are the most well-known universal characters, we need two more. The first of these is the universal character associated with the negative part of the odd orthogonal group

$$\mathrm{so}^-_\lambda := \det_{1\leqslant i,j\leqslant k}\big(h_{\lambda_i-i+j} - h_{\lambda_i-i-j+1}\big) = \sum_{\mu\in\mathscr{P}_0} (-1)^{(|\mu|+\mathrm{rk}(\mu))/2} s_{\lambda/\mu}. \tag{3.11}$$

There is also an e-Jacobi–Trudi formula where the sum is replaced by a difference in (3.8c). Writing $-X := (-x_1, -x_2, x_3, \dots)$ we further have $\mathrm{so}^-_\lambda(X) = (-1)^{|\lambda|}\mathrm{so}^+_\lambda(-X)$, where, in order to avoid confusion, from now on we write so^+_λ in place of so_λ.

The next universal character we need is that of an irreducible rational representation of GL_n; see [25, 65]. (The universal characters of the polynomial representations are the Schur functions.) These representations are indexed by weakly decreasing sequences of integers with length exactly n, or, alternatively, pairs of partitions λ, μ such that $l(\lambda) + l(\mu) \leqslant n$. Given such a pair we let the i-th component of the associated GL_n weight $[\lambda,\mu]_n$ be given by $\lambda_i - \mu_{n-i+1}$. Recall that the Schur polynomial $s_\lambda(x_1,\dots,x_n)$ may be extended to weakly decreasing sequences of integers of length n by the relation

$$s_{(\lambda_1+1,\dots,\lambda_n+1)}(x_1,\dots,x_n) = (x_1\cdots x_n)s_{(\lambda_1,\dots,\lambda_n)}(x_1,\dots,x_n). \tag{3.12}$$

The character of an irreducible rational representation of GL_n is then simply $s_{[\lambda,\mu]_n}(x_1,\dots,x_n)$. Littlewood gave the following expansion in terms of skew Schur polynomials [37]:

$$s_{[\lambda,\mu]_n}(x_1,\dots,x_n) = \sum_\nu (-1)^{|\nu|} s_{\lambda/\nu}(x_1,\dots,x_n)s_{\mu/\nu'}(1/x_1,\dots,1/x_n). \tag{3.13}$$

Koike used this expression to define a universal character which depends on two independent alphabets $X = (x_1, x_2, x_3, \dots)$ and $Y = (y_1, y_2, y_3, \dots)$ as [25][6]

$$\mathrm{rs}_{\lambda,\mu}(X;Y) := \sum_\nu (-1)^{|\nu|} s_{\lambda/\nu}(X)s_{\mu/\nu'}(Y), \tag{3.14}$$

which is an element of $\Lambda_X \otimes \Lambda_Y$. Define the restriction map $\tilde\pi_n : \Lambda_X\otimes\Lambda_Y \longrightarrow \Lambda^\pm_n$, the space of symmetric Laurent polynomials in $x_1,\dots,x_n$, by

$$\tilde\pi_n(\mathrm{rs}_{\lambda,\mu}(X;Y)) = \mathrm{rs}_{\lambda,\mu}(x_1,\dots,x_n;1/x_1,\dots,1/x_n) = s_{[\lambda,\mu]}(x_1,\dots,x_n), \tag{3.15}$$

for $l(\lambda)+l(\mu)\leqslant n$. Again, if this final condition is violated then there are modification rules which allow for the specialisation to be expressed as the character of a different rational representation. This object also has Jacobi–Trudi-type expressions. In terms of the complete homogeneous symmetric functions the first of these is

$$\mathrm{rs}_{\lambda,\mu}(X;Y) = \det\begin{pmatrix} (h_{\lambda_i-i+j}(X))_{1\leqslant i,j\leqslant k} & (h_{\lambda_i-i-j+1}(X))_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant \ell}} \\ (h_{\mu_i-i-j+1}(Y))_{\substack{1\leqslant i\leqslant \ell\\ 1\leqslant j\leqslant k}} & (h_{\mu_i-i+j}(Y))_{1\leqslant i,j\leqslant \ell}\end{pmatrix}, \tag{3.16}$$

where $k\geqslant l(\lambda)$ and $\ell \geqslant l(\mu)$. Again we have the dual form

$$\mathrm{rs}_{\lambda,\mu}(X;Y) = \det\begin{pmatrix} (e_{\lambda'_i-i+j}(X))_{1\leqslant i,j\leqslant k} & (e_{\lambda'_i-i-j+1}(X))_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant \ell}} \\ (e_{\mu'_i-i-j+1}(Y))_{\substack{1\leqslant i\leqslant \ell\\ 1\leqslant j\leqslant k}} & (e_{\mu'_i-i+j}(Y))_{1\leqslant i,j\leqslant \ell}\end{pmatrix},$$

[6] Our $\mathrm{rs}_{\lambda,\mu}$ stands for "rational Schur function".

where now $k \geqslant \lambda_1$ and $\ell \geqslant \mu_1$. Exactly how these determinantal representations of $\mathrm{rs}_{\lambda,\mu}(X;Y)$ and the skew Schur expansion are related will be explained below. In what follows we will predominantly use this symmetric function for $X=Y$, in which case we suppress the alphabet and simply write $\mathrm{rs}_{\lambda,\mu} = \mathrm{rs}_{\lambda,\mu}(X;X)$. We have already used this in Theorem 1.4 of the introduction.

In analysing Goulden's combinatorial proof of the Jacobi–Trudi formula [16], Bressoud and Wei [7] discovered a uniform extension of (3.9) involving an integer $z \geqslant -1$ which reproduces the above for $z=-1,1,0$ respectively. This was generalised further by Hamel and King to an expression valid for all $z \in \mathbb{Z}$ and including an additional parameter q [19, 20]. Then the main result of Hamel and King is

$$\mathcal{X}_\lambda(z;q) := \det_{1\leqslant i,j\leqslant k}\big(h_{\lambda_i-i+j} + [j>-z]q h_{\lambda_i-i-j+1-z}\big) \tag{3.17a}$$

$$= \sum_{\mu\in\mathscr{P}_z} (-1)^{(|\mu|-\mathrm{rk}(\mu)(z+1))/2} q^{\mathrm{rk}(\mu)} s_{\lambda/\mu}, \tag{3.17b}$$

where k is an integer such that $k \geqslant l(\lambda)$ and we have used the Iverson bracket from the proof of Theorem 3.2. Their paper [19] provides a proof of the identity (3.17a) = (3.17b) using the Laplace expansion of the determinant, whereas in [20] a combinatorial proof is provided using the Lindström–Gessel–Viennot lemma [15]. The general symmetric function $\mathcal{X}(z;q)$ also reduces to the three classical cases, but in a slightly different manner to the determinant of Bressoud and Wei:

$$\mathrm{sp}_\lambda = \mathcal{X}_\lambda(-1;1), \quad \mathrm{o}_\lambda = \mathcal{X}_\lambda(1;-1), \quad \text{and} \quad \mathrm{so}^{\pm}_\lambda = \mathcal{X}_\lambda(0;\pm 1).$$

The expansion in terms of skew Schur functions immediately implies the following duality with respect to the involution ω:

$$\omega\mathcal{X}_\lambda(z;q) = \mathcal{X}_{\lambda'}(-z;(-1)^z q). \tag{3.18}$$

This extends $\omega \mathrm{o}_\lambda = \mathrm{sp}_{\lambda'}$.

The symmetric function $\mathcal{X}_\lambda(z;q)$ is the subject of the first main result of Hamel and King in [19, 20]. They also introduce a generalisation of the determinantal form of $\mathrm{rs}_{\lambda,\mu}(X;Y)$ (3.16) in a similar vein, involving two parameters u,v and a pair of (possibly negative) integers a,b. We express this as

$$\begin{aligned} &\mathrm{rs}_{\lambda,\mu}(X;Y;a,b;u,v) \\ &\quad := \det\begin{pmatrix} \big(h_{\lambda_i-i+j}(X)\big)_{1\leqslant i,j\leqslant k} & \big([j>-a]u h_{\lambda_i-i-j-a+1}(X)\big)_{\substack{1\leqslant i\leqslant k\\ 1\leqslant j\leqslant \ell}} \\ \big([j>-b]v h_{\mu_i-i-j-b+1}(Y)\big)_{\substack{1\leqslant i\leqslant \ell\\ 1\leqslant j\leqslant k}} & \big(h_{\mu_i-i+j}\big)_{1\leqslant i,j\leqslant \ell}. \end{pmatrix}, \end{aligned} \tag{3.19}$$

where as usual $k \geqslant l(\lambda)$ and $\ell \geqslant l(\mu)$ are integers. For $(a,b,u,v)=(0,0,1,1)$ we recover Koike's rational universal character. Observe that the structure of this determinant, including Iverson brackets, is clearly similar to that of $\mathcal{X}_\lambda(z;q)$. Since the determinant is quite complicated, let us give an example for $(\lambda,\mu,a,b,k,\ell) = ((3,2),(4,2,1,1),-1,2,2,4)$:

$$\det\begin{pmatrix} h_3(X) & h_2(X) & 0 & uh_2(X) & uh_1(X) & u \\ h_1(X) & h_2(X) & 0 & u & 0 & 0 \\ vh_1(Y) & v & h_4(Y) & h_5(Y) & h_6(Y) & h_7(Y) \\ 0 & 0 & h_1(Y) & h_2(Y) & h_3(Y) & h_4(Y) \\ 0 & 0 & 0 & 1 & h_1(Y) & h_2(Y) \\ 0 & 0 & 0 & 0 & 1 & h_1(Y) \end{pmatrix}. \tag{3.20}$$

Using both algebraic and lattice path techniques, Hamel and King show that this more general symmetric function expands nicely in terms of skew Schur functions [19, Theorem 2]

$$\mathrm{rs}_{\lambda,\mu}(X;Y;a,b;u,v) = \sum_{\nu} (-1)^{|\nu|}(uv)^{\mathrm{rk}(\nu)} s_{\lambda/(\nu+a^{\mathrm{rk}(\nu)})}(X) s_{\mu/(\nu'+b^{\mathrm{rk}(\nu)})}(Y),$$

where the sum is over all partitions $\nu = (a_1,\dots,a_k \mid b_1,\dots,b_k)$ of arbitrary Frobenius rank such that $a_r \geqslant \max\{0,-a\}$ and $b_r \geqslant \max\{0,-b\}$ for $1 \leqslant r \leqslant k = \mathrm{rk}(\nu)$. For example, in computing $\mathrm{rs}_{(3,2),(4,2,1,1)}(X;Y;-1,2;u,v)$ from (3.20) the term $\nu=(1)$ is excluded from the sum since $(1) = (0 \mid 0)$ in Frobenius notation. Intuitively, this ensures that the Frobenius rank of the partition $\nu + (a^{\mathrm{rk}(\nu)})$ is never less than the Frobenius rank of ν.

A variant of the Koike and Hamel–King determinants involving an additional positive integer c occurs naturally in our factorisation results for universal characters below. Here we write $[k,\ell] := (k+1,\dots,\ell)$, which we treat as empty for $k \geqslant \ell$. The modified Hamel–King determinant is defined by the identity

$$\begin{aligned}&u^c(-1)^{kc+\binom{c}{2}}\mathrm{rs}_{\lambda,\mu}(X;Y;a,b;c;u,v)\\ &=\det\begin{pmatrix} \big(h_{\lambda_i-i+j}(X)\big)_{\substack{i\in[0,k]\\ j\in[c,k]}} & \big([j>-a-c]uh_{\lambda_i-i-j-a+1}(X)\big)_{\substack{i\in[0,\max\{k,c\}]\\ j\in[-c,\ell]}} \\ \big([j>-b]vh_{\mu_i-i-j+1-b}(Y)\big)_{\substack{i\in[0,\ell]\\ j\in[c,k]}} & \big(h_{\mu_i-i+j}(Y)\big)_{\substack{i\in[0,\ell]\\ j\in[-c,\ell]}}\end{pmatrix}\end{aligned}$$

where $k \geqslant l(\lambda)$ and $\ell \geqslant l(\mu)$. For $c = 0$ this reduces to the Hamel–King determinant (3.19). While not entirely clear from the definition, this determinant does not depend on k or ℓ as long the length conditions hold. In the case that $c \geqslant k$ the two sub-matrices on the left do not appear due to having no valid column indices. Like Koike's character, this also has an expansion in terms of skew Schur functions. Recall from Theorem 2.3 that $\mathrm{rk}_c(\lambda)$ denotes the Frobenius rank of the partition obtained by removing the first c rows of λ.

Theorem 3.3. *For partitions λ, μ, and integers a, b, c such that $c \geqslant 0$ we have*

$$\mathrm{rs}_{\lambda,\mu}(X;Y;a,b;c;u,v) = \sum_{\nu}(-1)^{|\nu|}(uv)^{\mathrm{rk}_c(\nu)}s_{\lambda/(\nu+(a^{c+\mathrm{rk}_c(\nu)}))}(X)s_{\mu/(\nu'+(b^{\mathrm{rk}_c(\nu)}))}(Y),$$

where the sum is over all partitions ν for which $\mathrm{rk}_c(\nu) = \mathrm{rk}_c(\nu + (a^{c+\mathrm{rk}_c(\nu)}))$.

Proof. The technique is the same as in [25, p. 68] and [19, p. 553]. Without loss of generality assume that $k \geqslant c$. We apply the Laplace expansion to the determinantal form of $\mathrm{rs}_{\lambda,\mu}(X;Y;a,b;c;u,v)$ according to the given block structure, choosing the first k rows to be fixed. We index the sum by permutations $w \in \mathfrak{S}_{k+\ell}/(\mathfrak{S}_k \times \mathfrak{S}_\ell)$ acting on the set $\{c+1,\dots,k\} \cup \{c,\dots,1-\ell\}$. (In other words, the first k columns are labelled $c+1,\dots,k$ and the final ℓ columns $c,\dots,1-\ell$.) Define the sets $K_w := \{w(j) : 1 \leqslant j \leqslant k\}$ and $L_w := \{w(j) : 1-\ell \leqslant j \leqslant 0\}$. Then the Laplace expansion of $\mathrm{rs}_{\lambda,\mu}(X;Y;a,b;c;u,v)$ may be expressed as

$$(-1)^{kc+\binom{c}{2}}\sum_{w\in\mathfrak{S}_{k+\ell}/(\mathfrak{S}_k\times\mathfrak{S}_\ell)}\mathrm{sgn}(w)u^{r-c}v^s\det_{\substack{1\leqslant i\leqslant k\\ j\in K_w}}\big(\alpha_j h_{\lambda_i-i+p_j}(X)\big)\det_{\substack{1\leqslant i\leqslant \ell\\ j\in L_w}}\big(\beta_j h_{\mu_i-i-q_j+1}(Y)\big),$$

where we set

$$\alpha_j := \begin{cases}1 & \text{if } c+1\leqslant j\leqslant k,\\ 0 & \text{if } c-a+1\leqslant j\leqslant c,\\ 1 & \text{if } 1-\ell\leqslant j\leqslant c-a,\end{cases}\quad\text{and}\quad \beta_j := \begin{cases}0 & \text{if } c+1\leqslant j\leqslant c-b-1,\\ 1 & \text{if } c-b\leqslant j\leqslant k,\\ 1 & \text{if } 1-\ell\leqslant j\leqslant c,\end{cases}$$

which encode the Iverson brackets from the full determinant,

$$p_j := \begin{cases} j & \text{if } c+1\leqslant j\leqslant k,\\ j-a & \text{if } 1-\ell\leqslant j\leqslant c,\end{cases}\quad\text{and}\quad q_j := \begin{cases} j+b & \text{if } c+1\leqslant j\leqslant k,\\ j & \text{if } 1-\ell\leqslant j\leqslant c.\end{cases}\tag{3.21}$$

We also have the quantities $r = \{1 \leqslant j \leqslant k : 1-\ell \leqslant w(j) \leqslant c\}$ and $s = \{1-\ell \leqslant j \leqslant c : c+1 \leqslant w(j) \leqslant k\}$.

As a next step we reverse the order of the columns labelled $c,\dots,1$ and then move them to the left, which cancels the overall sign from the determinant defining our symmetric function. Treating the w as permutations of the set $\{1,\dots,k\} \cup \{0,\dots,1-\ell\}$ we then choose coset representatives such that $w(1) < \cdots < w(k)$ and $w(-1) > \cdots > w(1-\ell)$ ordered canonically as integers. For example, in two-line notation with $(k,\ell) = (3,2)$ one such coset representative is

$$\begin{pmatrix}1 & 2 & 3 & 0 & -1\\ -1 & 2 & 3 & 1 & 0\end{pmatrix}.$$

The coset representatives of $\mathfrak{S}_{k+\ell}/(\mathfrak{S}_k \times \mathfrak{S}_\ell)$ and partitions $\nu \subseteq (k^\ell)$ are in bijection; see [44, p. 3] or [19, p. 553]. The assignment $w(i) = i - \nu_i$ if $1 \leqslant i \leqslant k$ and $w(i) = \nu'_{1-i} + i$ for $1-\ell \leqslant i \leqslant 0$ gives the corresponding partition. In the above example we obtain $\nu = (2)$, and the sign of the permutation will be equal to $= (-1)^{|\nu|}$. Moreover, $r - c = s = \mathrm{rk}_c(\nu)$. To complete the proof we need only observe that the definitions of p_j and q_j from (3.21) imply that the terms in the sum which are nonzero come from partitions ν for which $\nu + (a^{c+\mathrm{rk}_c(\nu)}) \subseteq \lambda$ and $\nu' + (b^{\mathrm{rk}_c(\nu)}) \subseteq \mu$. We must also have that $\mathrm{rk}_c(\nu) = \mathrm{rk}_c(\nu + (a^{c+\mathrm{rk}_c(\nu)}))$. This completes the proof. □

3.3. **Restriction rules.** Later on we will require some general restriction rules due to Koike and Terada. The purpose of this subsection is to collect these results. The first of these rules gives the restriction of an irreducible GL_n character to any subgroup of the form $\mathrm{GL}_k \times \mathrm{GL}_{n-k}$ where $0 \leqslant k \leqslant n$. This uses the restriction homomorphism $\tilde{\pi}_n$ defined in (3.15).

Theorem 3.4 ([25, Proposition 2.6])**.** *Let λ, μ be partitions such that $l(\lambda) + l(\mu) \leqslant n$. Then for any integer k such that $0 \leqslant k \leqslant n$,*

$$\tilde{\pi}_n(\mathrm{rs}_{\lambda,\mu}(X;Y)) = \sum_{\nu,\xi,\rho,\tau} \bigg(\sum_{\eta} c^{\lambda}_{\nu,\rho,\eta} c^{\mu}_{\xi,\tau,\eta} \bigg) \tilde{\pi}_k(\mathrm{rs}_{\nu,\xi}(X;Y)) \tilde{\pi}_{n-k}(\mathrm{rs}_{\rho,\tau}(X;Y)). \tag{3.22}$$

The next result gives the restriction of SO_{2n+1} to a maximal parabolic subgroup $\mathrm{GL}_k \times \mathrm{SO}_{2(n-k)+1}$. We write $\pi_{n;Z}$ and $\tilde{\pi}_{n;X,Y}$ for the restriction maps acting on the labelled sets of variables.

Theorem 3.5 ([28, Theorem 2.1])**.** *For any partition λ and integers k, n such that $0 \leqslant k \leqslant n$, then*

$$\tilde{\pi}_{k;X,Y} \pi_{n-k;Z}(\mathrm{so}^+_\lambda(X,Y,Z)) = \sum_{\mu,\nu,\xi} \bigg(\sum_{\eta} c^{\lambda}_{\mu,\nu,\xi,\eta,\eta} \bigg) \tilde{\pi}_k(\mathrm{rs}_{\mu,\nu}(X;Y)) \pi_{n-k}(\mathrm{so}^+_\xi). \tag{3.23}$$

Care needs to be taken in considering the case $k = n$, in which case π_0 will extract the constant term of so^+_λ. This may be computed by (3.9c) and is

$$\pi_0(\mathrm{so}^+_\lambda) = \begin{cases} (-1)^{(|\lambda| - \mathrm{rk}(\lambda))/2} & \text{if } \lambda \in \mathscr{P}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we also have an expression for the restriction of SO_{2n+1} to GL_n, which is different to the $k = 0$ case above.

Theorem 3.6 ([28, Theorem A.1])**.** *For a partition λ of length at most n we have that*

$$\tilde{\pi}_n(\mathrm{so}^+_\lambda(X,Y)) = \sum_{\mu,\nu} \bigg(\sum_{\eta} c^{\lambda}_{\mu,\nu,\eta} \bigg) \tilde{\pi}_n(\mathrm{rs}_{\mu,\nu}(X;Y)). \tag{3.24}$$

The difference between the $k = n$ case of (3.23) and (3.24) is that for $n \geqslant 2l(\lambda)$ the latter will contain only positive terms, and there is no need for modification rules in the computation of the sum. In general, the restriction maps in Theorem 3.5 can be removed if $k \geqslant 2l(\lambda)$ and $n - k \geqslant l(\lambda)$, but of course this excludes the case $n = k$; see [28, Corollary 2.3].

4. Factorisations of universal characters

We now turn to the factorisation of the universal characters under the operator φ_t. As a first step we state the universal character lifts of the factorisation results of Lecouvey from [32], amounting to the computation of $\varphi_t \mathrm{o}_\lambda$, $\varphi_t \mathrm{sp}_\lambda$ and $\varphi_t \mathrm{so}_\lambda$ in our notation. These also include the results of Ayyer and Kumari [3] as special cases. As seen in the previous section, these universal characters have a uniform generalisation in Hamel and King's symmetric function $\mathcal{X}_\lambda(z;q)$. Our main result is the computation of $\varphi_t \mathcal{X}_\lambda(z;q)$ for all $z \in \mathbb{Z}$, amounting to a large generalisation of the results of Lecouvey and Ayyer and Kumari.

4.1. **Factorisations of classical characters.** In [32], Lecouvey sought generalisations of the LLT polynomials beyond type A. To achieve this goal he needed analogues of the action of the t-th Verschiebung operator on the Schur polynomials, Theorem 3.2, for the symplectic and orthogonal characters. These results, stated in [32, Section 3.2], give conditions on the vanishing of the characters under φ_t for all t. In addition, with the restriction that t must be odd in the symplectic and odd orthogonal cases, he expresses the result using branching coefficients involving a subgroup of Levi type. In subsequent work [33], he used these factorisations to give expressions for the plethysm $\mathrm{so}^+_\lambda \circ p_t$ and its cousins by passing from the characters to the universal characters. For $t = 2$ some preliminary work towards the computation of these twisted characters was done by Mizukawa [45].

Independently of the results of Lecouvey, Ayyer and Kumari also proved expressions for the action of the t-th Verschiebung operator on the (non-universal) symplectic and orthogonal characters, but rather phrased in terms of "twisting" by a root of unity [3]. There are, however, key differences between their results and those of Lecouvey. Their twisted character identities, when they are nonzero, factor as products of other characters. Moreover, they give nicer conditions for when the characters are nonzero. Namely, they show that $\mathrm{o}_\lambda, \mathrm{sp}_\lambda$ and so^+_λ vanish under φ_t if and only if the t-core of λ is 1-, -1- or 0-asymmetric respectively.[7] Lifts of the results of Lecouvey, Ayyer and Kumari to the universal characters were given in [2]. The proofs there, like Lecouvey's, are based on the Jacobi–Trudi formulae for the classical groups. Note that in [3], due to twisting by a primitive t-th root of unity, the characters are associated with a Lie group of rank nt. In [32], no such restriction on the rank n is assumed, only that the length of the partition indexing the character is at most n. Indeed, depending on the remainder of n modulo t the structure of the factorisation of the classical characters will change. We will discuss this more below in the case of SO_{2n+1}, and show that the construction of Lecouvey may be phrased in terms of the classical Littlewood decomposition. Note that in Schur case it is clear from Theorem 3.2 that cyclic permutations of the quotient do not change the result, and so no such distinction must be made in that case.

Recall that in the skew Schur function case, when it is nonzero, the sign of $\varphi_t s_{\lambda/\mu}$ may be expressed elegantly in terms of the t-ribbon tiling of the skew shape λ/μ. In all previous work the signs obtained by applying the operator φ_t to the ordinary and universal characters were not expressed in such a combinatorial manner, rather as the sign of a permutation multiplied by some further factors to account for matrix operations occurring in the proof. In the present work we are able to improve on this by giving explicit expressions for the signs based on tilings of skew shapes and statistics on the indexing partitions, as already exemplified in Theorem 1.4 of the introduction.

Let us now state the two missing cases, beginning with the even orthogonal universal character. In these results we again write $\tilde{\lambda}$ as shorthand for the t-core of λ. We also set $\mathrm{rs}_{\lambda,\mu} := \mathrm{rs}_{\lambda,\mu}(X;X;0,0;0;1,1)$.

Theorem 4.1. *For all $t \geqslant 2$ and a partition λ we have that $\varphi_t \mathrm{o}_\lambda$ vanishes unless t-core$(\lambda) \in \mathscr{P}_1$, in which case*

$$\varphi_t \mathrm{o}_\lambda = (-1)^{|\tilde{\lambda}|/2} \operatorname{sgn}_t(\lambda/\tilde{\lambda}) \mathrm{o}_{\lambda^{(0)}} \prod_{r=1}^{\lfloor (t-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r)}} \times \begin{cases} \mathrm{so}^-_{\lambda^{(t/2)}} & t \text{ even}, \\ 1 & t \text{ odd}. \end{cases}$$

If $l(\lambda) \leqslant n$ then restricting to Λ_n^{BC} recovers [3, Theorem 2.15]. Secondly, we have the symplectic case.

Theorem 4.2. *For all $t \geqslant 2$ and a partition λ we have that $\varphi_t \mathrm{sp}_\lambda$ vanishes unless $\tilde{\lambda} \in \mathscr{P}_{-1}$, in which case*

$$\varphi_t \mathrm{sp}_\lambda = (-1)^{(|\tilde{\lambda}|+\mathrm{rk}(\tilde{\lambda}))/2} \operatorname{sgn}_t(\lambda/\tilde{\lambda}) \mathrm{sp}_{\lambda^{(t-1)}} \prod_{r=0}^{\lfloor (t-3)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(t-r-2)}} \times \begin{cases} \mathrm{so}_{\lambda^{((t-2)/2)}} & t \text{ even}, \\ 1 & t \text{ odd}. \end{cases}$$

As for the previous theorem we may recover [3, Theorem 2.11]. The odd orthogonal case is given in Theorem 1.4 above and generalises [3, Theorem 2.17].

The aforementioned three theorems appeared in [2, Theorems 3.2–3.4] with the same signs as in [3]. The expressions for the signs we present here are not only of a more combinatorial flavour, but also easier to compute. Another upshot of these expressions is that they show that the algorithms for computing the action of φ_t on the classical characters in Lecouvey's work [32] can be phrased entirely in terms of the Littlewood decomposition of the underlying partition.

4.2. **A uniform (z,q)-analogue.** The universal characters and the Schur functions are all contained in the general symmetric function $\mathcal{X}_\lambda(z;q)$ of Hamel and King. Thus, a natural question is whether the operator φ_t acts as nicely on this symmetric function as it does for its special cases. Our main result is the affirmative answer to this question for all integers z and including the parameter q.

Recall from Corollary 2.5 that $\mu_{\mathbf{c}}$ denotes the minimal z-asymmetric partition with $\kappa_t(\mu_{\mathbf{c}}) = \mathbf{c}$. If $z < 0$ then the conditions in that corollary need to be conjugated. From here on out we write $\mathrm{rs}_{\lambda,\mu}(a;c;q) := \mathrm{rs}_{\lambda,\mu}(X;X;a,a;c;q,q)$ and extend this to negative c by $\mathrm{rs}_{\lambda,\mu}(a;-c;q) := \mathrm{rs}_{\mu,\lambda}(a;c;q)$.

[7] Note that 1-asymmetric partitions have several names including *threshold partitions* or *doubled distinct partitions*.

Theorem 4.3. *Let a, b, t, z be integers such that $t \geqslant 2$ and $z = at + b$ where $0 \leqslant b \leqslant t-1$. Then $\varphi_t \mathcal{X}_\lambda(z;q)$ vanishes unless $\kappa_t(t\text{-core}(\lambda)) := \mathbf{c} \in \mathcal{C}_{b;t}$ and $\lambda \supseteq \mu_{\mathbf{c}}$. If these conditions are satisfied, then*

$$\varphi_t \mathcal{X}_\lambda(z;q)$$
$$= \varepsilon(q) \prod_{r=0}^{\lfloor (b-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(r)},\lambda^{(b-r-1)}}(a+1; c_r; q) \prod_{s=b}^{\lfloor (t+b-2)/2 \rfloor} \mathrm{rs}_{\lambda^{(s)},\lambda^{(t+b-s-1)}}(a; c_s; q)$$
$$\times \begin{cases} 1 & \text{if } b \text{ even, } t \text{ even,} \\ \mathcal{X}_{\lambda^{((b-1)/2)}}(a+1;q) & \text{if } b \text{ odd, } t \text{ odd,} \\ \mathcal{X}_{\lambda^{((t+b-1)/2)}}(a;q) & \text{if } b \text{ even, } t \text{ odd,} \\ \mathcal{X}_{\lambda^{((b-1)/2)}}(a+1;q)\mathcal{X}_{\lambda^{((t+b-1)/2)}}(a;q) & \text{if } b \text{ odd, } t \text{ even,} \end{cases}$$

where the factor $\varepsilon(q)$ may be expressed as

$$\varepsilon(q) = (-1)^{(|\mu_{\mathbf{c}}| - (z+1)\mathrm{rk}(t\text{-core}(\lambda))/2} \operatorname{sgn}_t(\lambda/\mu_{\mathbf{c}}) q^{\mathrm{rk}(t\text{-core}(\lambda))}.$$

This result contains all of the factorisation theorems for ordinary and universal characters previously mentioned. Upon setting $q = 0$ all characters reduce to Schur functions (either one or a product of two) and so we recover the straight shape case of Theorem 3.2 which was stated as Theorem 1.3 in the introduction. However, Theorem 3.2 is a key ingredient in our proof below, so we are not able to claim a new proof of this result. Substituting $q = (-1)^z$ and then choosing $z = 0, 1$ or -1 gives the factorisations for the classical characters in Theorems 1.4, 4.1 and 4.2 respectively. If we instead keep the parameter q then we further obtain q-deformations of these factorisations.

Our proof is based on the skew Schur expansion of $\mathcal{X}_\lambda(z;q)$ (3.17b). This is in contrast to previous proofs of these characters factorisations which were all based on determinantal expressions. Our technique gives a better understanding of the structure of these factorisations. In particular, through Theorem 2.3 and its corollaries, it explains the combinatorial mechanism of these results. Of course, by using the determinantal forms of all the symmetric functions involved it is possible to give a purely determinantal proof, however again the sign will not be so easily expressed in this case.

4.3. **Proof of Theorem 4.3.** Since the proof has several components, we break it up into smaller sections. The initial step is obvious: apply the t-th Verschiebung operator to $\mathcal{X}_\lambda(z;q)$ using the skew Schur expansion (3.17b) and Theorem 3.2. This gives

$$\begin{aligned} \varphi_t \mathcal{X}_\lambda(z;q) &= \sum_{\mu \in \mathscr{P}_z} (-1)^{(|\mu| - \mathrm{rk}(\mu)(z+1))/2} q^{\mathrm{rk}(\mu)} \varphi_t s_{\lambda/\mu} \\ &= \sum_{\substack{\mu \in \mathscr{P}_z \\ \lambda/\mu \ t\text{-tileable}}} (-1)^{(|\mu| - \mathrm{rk}(\mu)(z+1))/2} q^{\mathrm{rk}(\mu)} \operatorname{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} s_{\lambda^{(r)}/\mu^{(r)}}. \end{aligned} \tag{4.1}$$

4.3.1. *Vanishing.* From here the vanishing part of the theorem is already evident. Firstly, λ/μ is t-tileable only if $t\text{-core}(\lambda) = t\text{-core}(\mu)$, so that $\kappa_t(t\text{-core}(\lambda))$ must lie in $\mathcal{C}_{b;t}$ since μ is z-asymmetric. If this is the case then Corollary 2.5 provides the minimal term in the sum. In the case $z \geqslant 0$, this term can only appear if, for $0 \leqslant r \leqslant b-1$ with $c_r > 0$ we have $\lambda^{(r)} \supseteq ((a+1)^{c_r})$ and for $b \leqslant s \leqslant t-1$ with $c_s > 0$ we have $\lambda^{(s)} \supseteq (a^{c_s})$. If $z < 0$ then we need only conjugate these conditions.

4.3.2. *Identification of the prefactor.* Now assume that we are in the case where $\varphi_t \mathcal{X}_\lambda(z)$ is nonzero. That is, $\kappa_t(t\text{-core}(\lambda)) \in \mathcal{C}_{b;t}$ and we have the minimal requirements on the t-quotient just given. Observe that

$$\operatorname{sgn}_t(\lambda/\mu) = \operatorname{sgn}_t(\lambda/\mu_{\mathbf{c}}) \operatorname{sgn}_t(\mu/\mu_{\mathbf{c}}),$$

so we may already pull out an overall sign of $\operatorname{sgn}_t(\lambda/\mu_{\mathbf{c}})$. Also, $\mathrm{rk}(\mu_{\mathbf{c}}) = \mathrm{rk}(t\text{-core}(\lambda))$ thanks to Lemma 2.6. The Littlewood decomposition implies that $|\mu_{\mathbf{c}}|$ is the minimal size of all partitions in the sum, so we in fact can remove an overall factor of

$$\varepsilon(q) := (-1)^{(|\mu_{\mathbf{c}}| - (z+1)\mathrm{rk}(t\text{-core}(\lambda)))/2} q^{\mathrm{rk}(t\text{-core}(\lambda))} \operatorname{sgn}_t(\lambda/\mu_{\mathbf{c}}),$$

as desired.

Collecting the above we now have that

$$\begin{aligned}
(4.2)\quad & \varphi_t \mathcal{X}_\lambda(z;q) \\
&= \varepsilon(q) \sum_{\substack{\mu\in\mathscr{P}_z \\ \lambda/\mu\ t\text{-tileable}}} (-1)^{\sum_{r=0}^{t-1}(t|\mu^{(r)}|-(z+1)\mathrm{rk}_{c_r}(\mu^{(r)}))/2}\, \mathrm{sgn}_t(\mu/\mu_{\mathbf{c}}) \prod_{r=0}^{t-1} q^{\mathrm{rk}_{c_r}(\mu^{(r)})} s_{\lambda^{(r)}/\mu^{(r)}}.
\end{aligned}$$

As a direct consequence of our Theorem 2.3 we can replace the sum over $\mu\in\mathscr{P}_z$ with a sum over t-tuples of partitions satisfying the conditions (2.5) such that

$$\mu = \phi_t^{-1}\big(\mathbf{c}, (\mu^{(0)},\dots,\mu^{(t-1)})\big).$$

In fact, the conditions (2.5) ensure that the product of skew Schur functions coincides with the product obtained by expanding the right-hand side of the theorem.

4.3.3. *Factorisation of the sign.* The only thing needed in order to show that the sum (4.1) decouples in the desired way is the factorisation of the interior sign. This will be achieved by an inductive argument by considering terms in the sum, say μ and ν, for which $|\mu|-|\nu|$ is as small as possible. Also, it is most convenient here to assume that $z\geqslant 0$. For $z\leqslant 0$ the same set of steps will yield the factorisation of the sign.

Consider the case where $t+b$ is odd and $b<t-1$ and fix all entries in the quotient of μ except for $\mu^{((t+b-1)/2)}$. Since $c_{(t+b-1)/2}=0$ the minimal choice of quotient entry is $\mu^{((t+b-1)/2)}=\varnothing$ and $\mathrm{rk}_{c_{(t+b-1)/2}}(\mu^{((t+b-1)/2)})=\mathrm{rk}(\mu^{((t+b-1)/2)})$. There are two ways to add cells to $\mu^{((t+b-1)/2))}$ whilst remaining in the set of z-asymmetric partitions: (i) we may add a row of $a+1$ cells, the left-most of which sits on the main diagonal of $\mu^{((t+z-1)/2)}$ or (ii) a pair of cells at either end of a principal hook of μ. In case (i), in terms of the t-Maya diagram, this corresponds to moving a bead directly to the left of the origin $a+1$ spaces to the right. As we know from Lemma 2.7, the sign $\mathrm{sgn}_t(\mu/\mu_{\mathbf{c}})$ will change by the number of beads passed over. The conditions (2.5a) ensure that there are no beads present in this region in any of the runners labelled $0\leqslant r\leqslant b-1$. Therefore the only beads counted when computing the sign lie above runner $(t+b-1)/2$ in the column directly to the left of the origin, and strictly between runners $b-1$ and $(t+b-1)/2$ in column $a+1$. However, conditions (2.5b) tell us that the number of such beads is always $(t-b-1)/2$, since the runners either side of $(t+b-1)/2$ form pairs up to a-shifted conjugation. This introduces a factor of $(-1)^{(t-b-1)/2}$, but since we have added $a+1$ cells to the t-quotient and the rank has further increased by one this sign change cancels with that coming from the exponent of -1, leading to no overall sign change. In case (ii) the rank is unchanged and the two ribbons must be conjugates of one another, so their heights sum to $t-1$. Putting this together, we see that the sign associated to $\mu^{((t+b-1)/2)}$ is equal to $(-1)^{(|\mu^{(t+b-1)}|-(a+1)\mathrm{rk}(\mu^{(t+b-1)}))/2}$.

Now assume that b is odd. Again since $c_{b-1}=0$ the minimal choice of $\mu^{(b-1)}$ is $\varnothing$ and $\mathrm{rk}_{c_{b-1}}(\mu^{(b-1)})=\mathrm{rk}(\mu^{(b-1)})$. The analysis is almost exactly the same as that of the previous paragraph. We again have two cases corresponding to either increasing the rank of $\mu^{(b-1)}$ or not. If we do not, then the sign will change since we add a pair of conjugate ribbons. If the rank does increase, then we are moving a bead from column -1 of runner $b-1$ to column $a+2$. In the range $b\leqslant s\leqslant t-1$ we will find precisely $t-b$ beads, since each runner will have a single bead in either column -1 or column $a+1$ by the conjugation conditions. Similar to the previous case we will find $(b-1)/2$ beads in the range $0\leqslant r\leqslant b-1$, thus contributing $(-1)^{(2t-b-1)/2}$ all together. But this is precisely the sign coming from the change in size and rank of the quotient, leading to no overall change in sign. Thus we have shown that the sign in this case changes by $(-1)^{(|\mu^{(b-1)}|-(a+2)\mathrm{rk}(\mu^{(b-1)}))/2}$.

For the next case take a pair of runners r and $t+b-r-1$ for $b\leqslant r\leqslant t-1$ such that $r\neq t+b-r-1$. The partitions $\mu^{(r)}$ and $\mu^{(t+b-r-1)}$ in the quotient are governed by a single partition, $\xi^{(r)}$, such that $\mu^{(r)}=\xi^{(r)}+(a^{c_r+\mathrm{rk}_{c_r}(\xi^{(r)})})$ and $\mu^{(t+z-r-1)}=(\xi^{(r)})'+(a^{\mathrm{rk}_{c_r}(\xi^{(r)})})$. Without loss of generality assume that $c_r\geqslant 0$. By the definition of the quotient partitions we have $\mathrm{rk}_{c_r}(\mu^{(r)})=\mathrm{rk}_{c_r}(\xi^{(r)})=\mathrm{rk}_{-c_r}(\xi^{(t+z-r-1)})=\mathrm{rk}_{c_{t+b-r-1}}(\mu^{(t+b-r-1)})$. The minimal partition in the sum, $\mu_{\mathbf{c}}$, has already absorbed some of the contribution from $\mu^{(r)}$, so we are left with the sign contribution $(-1)^{|\xi^{(r)}|-(z+1)\mathrm{rk}_{c_r}(\xi^{(r)})}$. As above there are two cases: (i) $\mathrm{rk}_{c_r}(\xi^{(r)})$ does not increase and (ii) $\mathrm{rk}_{c_r}(\xi^{(r)})$ increases. In case (i) then the analysis is exactly the same as before and the two ribbons added will be conjugates of one another so that the overall sign changes. For case (ii), it is convenient to use the t-Maya diagram. Indeed, increasing $\mathrm{rk}_{c_r}(\nu^{(r)})$ by one corresponds to the moving of two beads on runners r and $t+b-r-1$ from column -1 to column $a+1$. If we interpret the sign of these two ribbons in terms of bead-counting, then the only beads not double-counted are those strictly between the runners r and $t+b-r-1$. However, between the two runners in question all quotient elements are a-shifted conjugate pairs with the addition of an a-symmetric partition in the case $t+b$ is odd. This implies that the number of beads contributing to the sign is equal to the number of such runners, namely to $t+b-2r-2$. This procedure is exemplified in Figure 6. Since we have added a single cell to $\xi^{(r)}$ and increased its rank by 1 the overall sign changes in this case. In either case we see that the sign may be expressed as $(-1)^{|\xi^{(r)}|}$.

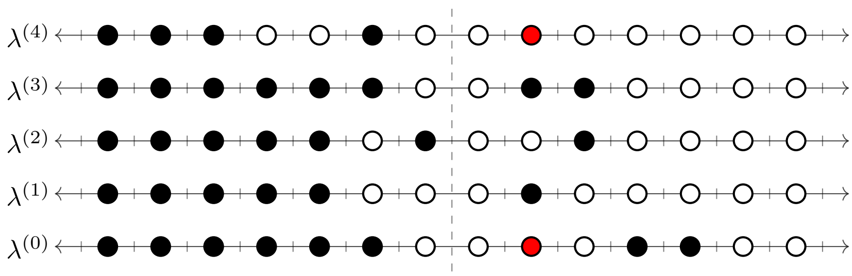


FIGURE 6. The 5-Maya diagram of the 5-asymmetric partition $\lambda = (20\ 15\ 13\ 12\ 9\ 8\ 6\ 5 \mid 15\ 10\ 8\ 7\ 4\ 3\ 1\ 0)$ with $\kappa_5(\lambda) = (2,-1,0,1,-2)$. The beads shaded red have been moved two spaces to the right, producing a sign of -1.

For our final cases we take the pair of runners r and $b-r-1$ where $0 \leqslant r \leqslant b-1$ and $r \neq b-r-1$. Without loss of generality again assume that $c_r \geqslant 0$. The associated pair of partitions is here governed by a single partition $\nu^{(r)}$ for which $\mu^{(r)} = \nu^{(r)} + ((a+1)^{c_r+\mathrm{rk}_{c_r}(\nu^{(r)})})$ and $\mu^{(z-r-1)} = (\nu^{(r)})' + ((a+1)^{\mathrm{rk}_{c_r}(\nu^{(r)})})$. However, the analysis of the previous paragraph applies in the same manner to this case. If we add a cell to $\nu^{(r)}$ such that the c_r-shifted rank does not change then this corresponds to a pair of conjugate ribbons and again giving an overall sign of -1. On the other hand, if $\mathrm{rk}_{c_r}(\nu^{(r)})$ increases then the sign also changes by -1, corresponding to a total sign change of $(-1)^{|\nu^{(r)}|}$ in either case.

Combining all of the above cases we have shown that if $z \geqslant 0$ the sign in the sum is equal to

$$\begin{aligned}(4.3)\qquad &\prod_{r=0}^{\lfloor (b-2)/2\rfloor} (-1)^{|\nu^{(r)}|} \prod_{r=b}^{\lfloor (b+t-2)/2\rfloor} (-1)^{|\xi^{(r)}|} \\ &\times \begin{cases} 1 & b \text{ even, } t \text{ even,} \\ (-1)^{(|\mu^{((b-1)/2)}|-(a+2)\mathrm{rk}(\mu^{((b-1)/2)})/2} & b \text{ odd, } t \text{ odd,} \\ (-1)^{(|\mu^{((t+b-1)/2)}|-(a+1)\mathrm{rk}(\mu^{((t+b-1)/2)}))/2} & b \text{ even, } t \text{ odd,} \\ (-1)^{(|\mu^{((b-1)/2)}|-(a+2)\mathrm{rk}(\mu^{((b-1)/2)})+|\mu^{((t+b-1)/2)}|-(a+1)\mathrm{rk}(\mu^{((t+b-1)/2)}))/2} & b \text{ odd, } t \text{ even.} \end{cases}\end{aligned}$$

It follows from the same set of steps that in the case $z \leqslant 0$ the same sign is obtained, and we spare the reader repeating the details.

4.3.4. *Final steps for factorisation.* To conclude the proof, note that the structure of the sign decomposition (4.3) is the same as that of the theorem. In particular, the sign factors completely over the quotient, and the sum now decouples into a product of sums. Recalling our convention regarding $\mathrm{rs}_{\lambda,\mu}(a;c;q)$ when $c<0$, the sums governed by the $\nu^{(r)}$ for $1 \leqslant r \leqslant \lfloor (b-2)/2\rfloor$ will each produce a copy of $\mathrm{rs}_{\lambda^{(r)},\lambda^{(b-r-1)}}(a+1;c_r;q)$. The sums governed by the $\xi^{(r)}$ for $b \leqslant r \leqslant \lfloor (t+b-2)/2\rfloor$ will give copies of $\mathrm{rs}_{\lambda^{(r)},\lambda^{(t+b-r-1)}}(a;c_r;q)$. If b is odd then we also pick up a copy of $\mathcal{X}_{\lambda^{((b-1)/2)}}(a+1;q)$, and if $b+t$ is odd then we pick up a copy of $\mathcal{X}_{\lambda^{((t+b-1)/2)}}(a;q)$, as desired.

5. Plethysm rules for universal characters

As we discussed in the introduction, the factorisation of the Schur function under φ_t is intimately related with the Schur expansion of the plethysm $s_\lambda \circ p_t$. This expression, known as the SXP rule, has several extensions, the most general of which we will reproduce here together with a short proof showing the equivalence with (the full) Theorem 3.2. Then our attention turns to generalisations of this rule to the universal characters due to Lecouvey.

5.1. Wildon's SXP rule.

One of the first applications of Littlewood's core and quotient construction is to the plethysm $s_\lambda \circ p_t$, his expression for which is now referred to as the *SXP rule* [38, p. 351]. The rule was reproved by Chen, Garsia and Remmel in [9], relying on the $\mu = \varnothing$ case of Theorem 3.2. It was later given an involutive proof by Remmel and Shimozono [56, §5]. Recently, Wildon proved an extension of the SXP rule which manifests as the Schur expansion for the expression $s_\tau(s_{\lambda/\mu} \circ p_t)$ and, moreover, his proof relies entirely on a sequence of bijections and involutions. Here, we wish to point out that Wildon's SXP rule is equivalent to the full Theorem 3.2.

Theorem 5.1 ([68, Theorem 1.1])**.** *For any integer $t \geqslant 2$ and partitions λ, μ, τ,*

$$s_\tau(s_{\lambda/\mu} \circ p_t) = \sum_{\substack{\nu \\ \nu/\tau\ t\text{-tileable}}} \operatorname{sgn}_t(\nu/\tau) c^{\lambda}_{\nu^{(0)}/\tau^{(0)},\dots,\nu^{(t-1)}/\tau^{(t-1)},\mu} s_\nu.$$

Proof. By the definition of the skew Schur functions we may express the coefficient of s_ν in the Schur expansion of the left-hand side as

$$\langle s_\tau(s_{\lambda/\mu} \circ p_t), s_\nu\rangle = \langle s_{\lambda/\mu}, \varphi_t s_{\nu/\tau}\rangle.$$

Applying Theorem 3.2 with $\lambda/\mu \mapsto \nu/\tau$ on the right-hand side of this equation then shows that the above vanishes unless ν/τ is t-tileable, in which case it is given by

$$\begin{aligned}\operatorname{sgn}_t(\nu/\tau)\Big\langle s_{\lambda/\mu}, \prod_{r=0}^{t-1} s_{\nu^{(r)}/\tau^{(r)}}\Big\rangle &= \operatorname{sgn}_t(\nu/\tau)\Big\langle s_\lambda, s_\mu \prod_{r=0}^{t-1} s_{\nu^{(r)}/\tau^{(r)}}\Big\rangle \\ &= \operatorname{sgn}_t(\nu/\tau) c^{\lambda}_{\nu^{(0)}/\tau^{(0)},\dots,\nu^{(t-1)}/\tau^{(t-1)},\mu}.\end{aligned}$$ ∎

5.2. SXP rules for universal characters.

Since, like the Schur functions, the universal characters admit nice factorisations under the map φ_t, it is natural to also seek SXP-type rules for these symmetric functions. This question has already been considered by Lecouvey, who, following his paper [32], gave analogues of the SXP rule for the universal symplectic and orthogonal characters [33]. In this section we wish to restate these rules more explicitly by using our combinatorial framework.

Define coefficients $a^{\bullet}_{\lambda,\nu}(t)$ where $\bullet$ is one of sp, o or so$^+$ by

$$\mathrm{o}_\lambda \circ p_t = \sum_\nu a^{\mathrm{o}}_{\lambda,\nu}(t)\mathrm{o}_\nu, \quad \mathrm{sp}_\lambda \circ p_t = \sum_\nu a^{\mathrm{sp}}_{\lambda,\nu}(t), \quad \text{and} \quad \mathrm{so}^+_\lambda \circ p_t = \sum_\nu a^{\mathrm{so}^+}_{\lambda,\nu}(t)\mathrm{so}^+_\nu.$$

To begin, we first point out that it is not difficult to give explicit, albeit cumbersome, expressions for these coefficients.

Lemma 5.2 ([32, Lemma 3.1.1])**.** *We have*

$$\begin{aligned}
a^{\mathrm{o}}_{\lambda,\nu}(t) &= \sum_{\mu\in\mathscr{P}_1} \sum_{\substack{\xi \\ t\text{-core}(\xi)=\varnothing}} \sum_{\substack{\eta \\ \eta\ \text{even}}} (-1)^{|\mu|/2} \operatorname{sgn}_t(\xi) c^{\lambda}_{\xi^{(0)},\dots,\xi^{(t-1)},\mu} c^{\xi}_{\nu,\eta},\\
a^{\mathrm{sp}}_{\lambda,\nu}(t) &= \sum_{\mu\in\mathscr{P}_{-1}} \sum_{\substack{\xi \\ t\text{-core}(\xi)=\varnothing}} \sum_{\substack{\eta \\ \eta'\ \text{even}}} (-1)^{|\mu|/2} \operatorname{sgn}_t(\xi) c^{\lambda}_{\xi^{(0)},\dots,\xi^{(t-1)},\mu} c^{\xi}_{\nu,\eta},\\
a^{\mathrm{so}^+}_{\lambda,\nu}(t) &= \sum_{\mu\in\mathscr{P}_0} \sum_{\substack{\xi \\ t\text{-core}(\xi)=\varnothing}} \sum_{\eta} (-1)^{|\nu|+(|\mu|-\operatorname{rk}(\mu))/2} \operatorname{sgn}_t(\xi) c^{\lambda}_{\xi^{(0)},\dots,\xi^{(t-1)},\mu} c^{\xi}_{\nu,\eta}.
\end{aligned}$$

Moreover, $a^{\mathrm{o}}_{\lambda,\nu}(t) = (-1)^{|\lambda|(t-1)} a^{\mathrm{sp}}_{\lambda',\nu'}(t)$.

Proof. We begin with the first identity. Expanding $\mathrm{o}_\lambda \circ p_t$ in terms of skew Schur functions and then applying the SXP rule of Theorem 5.1 with $\tau = \varnothing$ leads to

$$\mathrm{o}_\lambda \circ p_t = \sum_{\mu \in \mathscr{P}_1} \sum_{\substack{\xi \\ t\text{-core}(\xi)=\varnothing}} (-1)^{|\mu|/2} \operatorname{sgn}_t(\xi) c^{\lambda}_{\xi^{(0)},\dots,\xi^{(t-1)},\mu} s_\xi.$$

By the character interrelation formula (3.10) we have

$$\mathrm{o}_\lambda \circ p_t = \sum_{\nu} \mathrm{o}_\nu \Bigg(\sum_{\mu \in \mathscr{P}_1} \sum_{\substack{\xi \\ t\text{-core}(\xi)=\varnothing}} \sum_{\substack{\eta \\ \eta \text{ even}}} (-1)^{|\mu|/2} \operatorname{sgn}_t(\xi) c^{\lambda}_{\xi^{(0)},\dots,\xi^{(t-1)},\mu} c^{\xi}_{\nu,\eta} \Bigg).$$

The same steps yield the formulae for the other characters. For the duality between the coefficients one uses the involution ω combined with (3.4). Note that the universal characters are not homogeneous symmetric functions. However, the skew Schur expansions show that in the symplectic and even orthogonal cases they are sums of homogeneous symmetric functions whose degrees agree modulo two, and so the identity still holds in this case. ☐

In fact, Lecouvey shows that $a^{\mathrm{o}}_{\lambda,\nu}(t) = a^{\mathrm{so}^+}_{\lambda,\nu}(t)$, his argument being based on the fact that π'_n and the plethysm by p_t commute. Using this fact applied to the universal character o_λ shows the equality of the coefficients for $n \geqslant tl(\lambda)$. We have not found a simple explanation at the level of universal characters for why the expressions given above for the coefficients $a^{\mathrm{o}}_{\lambda,\nu}(t)$ and $a^{\mathrm{so}^+}_{\lambda,\nu}(t)$ coincide.

As we remarked in Subsection 4.1, Lecouvey has given algorithms for computing the action of φ_t on classical group characters. For the odd orthogonal group SO_{2n+1} this algorithm is crucial in stating his SXP-type rules. In view of Theorem 1.4, we may restate this algorithm entirely in terms of the classical Littlewood decomposition. What follows is a reinterpretation of the algorithm given in [33, §4].

Construction 5.3. Let $n, t \in \mathbb{N}$ be such that $n = at + b$ for $0 \leqslant b \leqslant t-1$. Further let λ be a partition of length at most n with $\kappa_t(\lambda) = (c_0, \dots, c_{t-1})$ and quotient $(\lambda^{(0)}, \dots, \lambda^{(t-1)})$. Reading indices modulo t we define for $0 \leqslant r \leqslant \lfloor \frac{t-2}{2} \rfloor$ sequences

$$\gamma^{(r)} := [\lambda^{(-r-b-1)}, \lambda^{(r-b)}]_{2a+d_r} + (c^{2a+d_r}_{-r-b-1}),$$

where we additionally set

$$d_r := \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant b-1, \\ 2 & \text{if } 0 \leqslant t-r-1 \leqslant b-1 \text{ and } 0 \leqslant r \leqslant b-1, \\ 0 & \text{otherwise.} \end{cases}$$

Also, if t is odd, $\gamma^{((t-1)/2)} := \lambda^{((t-1)/2-b)}$ where $l(\gamma^{((t-1)/2)}) \leqslant a + d_{(t-1)/2}$ and

$$d_{(t-1)/2} := \begin{cases} 1 & \text{if } b > (t-1)/2, \\ 0 & \text{if } b \leqslant (t-1)/2. \end{cases}$$

Given the above, write

$$\gamma_n(\lambda; t) := (\gamma^{(0)}, \dots, \gamma^{(\lfloor (t-1)/2 \rfloor)}).$$

The output of this construction is a dominant weight for

$$G_n(\lambda; t) := \mathrm{GL}_{2a+d_0} \times \cdots \times \mathrm{GL}_{2a+d_{\lfloor (t-2)/2 \rfloor}} \times \begin{cases} \mathrm{SO}_{2(a+d_{(t-1)/2})+1} & \text{if } t \text{ odd}, \\ 1 & \text{if } t \text{ even}, \end{cases}$$

a Levi subgroup of SO_{2n+1}. Let $\mathfrak{g}_n(\lambda; t)$ denote the corresponding Lie algebra. We write $V^{\mathfrak{so}_{2n+1}}(\lambda)$ for the irreducible finite-dimensional representation of SO_{2n+1} of highest weight λ, and similarly for $V^{\mathfrak{g}_n(\mu;t)}(\gamma(\mu; t))$. The branching coefficient $[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{g}_n(\mu;t)}(\gamma_n(\mu; t))]$ then gives the multiplicity of $V^{\mathfrak{g}_n(\mu;t)}(\gamma_n(\mu; t))$ when $V^{\mathfrak{so}_{2n+1}}(\lambda)$ is restricted to $G_n(\mu; t)$. Note that if $b = 0$, so that n is a multiple of t, then this construction will output the partitions in the quotient paired as in Theorem 1.4.

For an example, take $(n,t)=(8,5)$ so that $(a,b)=(1,3)$. Then for the partition $\lambda=(15,14,10,7,4,3,2,1)$ we have $\kappa_5(\lambda)=(0,-1,1,0,0)$ and

$$\big(\lambda^{(0)},\lambda^{(1)},\lambda^{(2)},\lambda^{(3)},\lambda^{(4)}\big)=\big(\varnothing,\varnothing,(2,2,1),(1),(3,1)\big).$$

Construction 5.3 will output

$$\gamma_8(\lambda;5)=\big((0,-1,-1),(0,0,-1),(3,1)\big),$$

and $G_8(\lambda;5)=\mathrm{GL}_3\times\mathrm{GL}_3\times\mathrm{SO}_5$.

Let $\mathscr{C}_{b;t}$ denote the set of sequences $(c_0,\dots,c_{t-1})\in\mathbb{Z}^t$ such that $c_{r-b}+c_{t-r-1-b}=0$, where indices are read modulo t. When viewed as encoding t-cores, this corresponds to the set of t-cores μ which, after shifting the indices of $\kappa_t(\mu)$ cyclicly b places to the right, are self-conjugate. We are now ready to state the SXP rule for the odd orthogonal characters.

Theorem 5.4 ([33, Theorem 4.1.1]). *Let $t\geqslant 2$ and n be integers such that $n=at+b$ where $0\leqslant b\leqslant t-1$. Then for any partition λ with $l(\lambda)\leqslant n$,*

$$\begin{aligned}(5.1)\quad &\pi_n(\mathrm{so}^+_\lambda)\circ p_t\\ &=\sum_{\substack{\mu\\ l(\mu)\leqslant n\\ \kappa_t(\mu)\in\mathscr{C}_{b;t}}}(-1)^{(|\tilde\mu|-\mathrm{rk}(\tilde\mu))/2}\,\mathrm{sgn}_t(\mu/\tilde\mu)[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_n(\mu;t)}(\gamma_n(\mu;t))]\pi_n(\mathrm{so}^+_\mu).\end{aligned}$$

There are also versions of this result for Sp_{2n} and O_{2n} in the case t is even, but they are not stated in [33]. For t odd there cannot be rules of this form since the coefficients describing the action of the Verschiebung operator on the characters are not branching coefficients. This is further clarified by the appearance of the "negative" odd orthogonal characters in Theorems 1.4 and 4.1. However, Theorem 5.4 is all that is needed to state the universal character lifts of these rules.

As remarked in [33, p. 769], it is possible to give explicit expressions for the branching coefficients occurring in (5.1) in terms of (multi-)Littlewood–Richardson coefficients. These are particularly simple for $n\geqslant tl(\lambda)$, since for these values of n the coefficients stabilise.

Lemma 5.5. *Assume that $n\geqslant tl(\lambda)$ and t-core$(\lambda)=\varnothing$. If t is even, then*

$$[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_n(\mu;t)}(\gamma_n(\mu;t))]=\sum_{\eta^1,\dots,\eta^{t/2}}c^{\lambda}_{\eta^1,\dots,\eta^{t/2},\mu^{(0)},\dots,\mu^{(t-1)}}.$$

If t is odd, then[8]

$$[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_n(\mu;t)}(\gamma_n(\mu;t))]=\sum_{\eta^1,\dots,\eta^{(t+1)/2}}c^{\lambda}_{\eta^1,\dots,\eta^{(t+1)/2},\eta^{(t+1)/2},\mu^{(0)},\dots,\mu^{(t-1)}}.$$

Else, if t-core$(\lambda)\neq\varnothing$ then

$$[V^{\mathfrak{so}_{2n+1}}(\lambda):V^{\mathfrak{g}_n(\mu;t)}(\gamma_n(\mu;t))]=0.$$

Proof. Assume that $n\geqslant tl(\lambda)$ and t-core$(\mu)=\varnothing$. The output of Construction 5.3 applied to μ yields a tuple of weights $(\gamma^{(0)},\dots,\gamma^{(\lfloor(t-1)/2\rfloor)})$ which are made up of pairs of partitions, with an additional single partition if t is odd. If t is even then we first use the restriction rule of Theorem 3.6, which is positive since $n\geqslant tl(\lambda)\geqslant 2l(\lambda)$. From here we then iterate the rule of Theorem 3.4 to branch onto the group $G_n(\mu;t)$. In the case t is odd, then we begin with the rule of Theorem 3.5, choosing $k=a+d_{(t-1)/2}$, and then iterate Theorem 3.4 to land in $G_n(\mu;t)$. Since we have assumed that $n\geqslant tl(\lambda)$, these rules will all contain only positive terms, expressed as sums of multi-Littlewood–Richardson coefficients as in the statement.

Now assume that t-core$(\mu)\neq\varnothing$ and that $n=tl(\lambda)$. We have that $\sum_{r=0}^{t-1}l(\mu^{(r)})\leqslant l(\mu)\leqslant l(\lambda)$, which may be seen from the t-Maya diagram. Since μ has nonempty t-core there exists some r for which $c_{t-r-1}\neq 0$ and $l(\mu^{(r)})+l(\mu^{(t-r-1)})\leqslant l(\lambda)$. This means that the length of at least one of the partitions which make up $\gamma^{(r)}$, which has been shifted by c_{t-r-1}, will be greater than the length of λ, and so the branching coefficients will vanish in this case. $\square$

[8]It is correct that $\eta^{(t+1)/2}$ occurs twice in the lower-index of the multi-Littlewood–Richardson coefficient.

Note that the above also shows that for any n such that $n \geqslant tl(\lambda)$ the branching coefficients are always the same, since increasing n by one merely permutes the $\mu^{(r)}$.

Let us denote the stablised version of the above coefficients from Lemma 5.5 by $b_{\lambda,\mu}(t)$. We may now state the SXP rules for the universal characters.

Theorem 5.6 ([33, Theorem 4.5.1])**.** *For λ a partition and $t \geqslant 2$ and integer we have*

$$\mathrm{so}^+_\lambda \circ p_t = \sum_{\substack{\mu \\ t\text{-core}(\mu)=\varnothing}} \mathrm{sgn}_t(\mu) b_{\lambda,\mu}(t) \mathrm{so}^+_\mu,$$

$$\mathrm{o}_\lambda \circ p_t = \sum_{\substack{\mu \\ t\text{-core}(\mu)=\varnothing}} \mathrm{sgn}_t(\mu) b_{\lambda,\mu}(t) \mathrm{o}_\mu,$$

and

$$\mathrm{sp}_\lambda \circ p_t = (-1)^{|\lambda|(t-1)} \sum_{\substack{\mu \\ t\text{-core}(\mu)=\varnothing}} \mathrm{sgn}_t(\mu') b_{\lambda',\mu'}(t) \mathrm{sp}_\mu.$$

where $b_{\lambda,\mu}(t)$ denotes the branching coefficients of Lemma 5.5.

Proof. The first equation is immediate from the large-n vanishing part of Lemma 5.5. As remarked after the proof of Lemma 5.2 the coefficients in the expansions of $\mathrm{so}^+_\lambda \circ p_t$ and $\mathrm{o}_\lambda \circ p_t$ coincide, which establishes the second equality. By the duality part of that same lemma, or by directly applying the ω involution,

$$a^{\mathrm{sp}}_{\lambda,\mu}(t) = (-1)^{|\lambda|(t-1)} a^{\mathrm{o}}_{\lambda',\mu'}(t) = (-1)^{|\lambda|(t-1)} \,\mathrm{sgn}_t(\mu') b_{\lambda',\mu'}(t). \qquad \square$$

As this section shows, SXP rules for symplectic and orthogonal characters are intimately connected with the representation theory of their associated groups. Thus, it is not clear if there exists a general SXP rule for the symmetric function $\mathcal{X}_\lambda(z;q)$ in the same manner. We have also not found a simple proof of the fact that the stabilised coefficients $\mathrm{sgn}_t(\mu) b_{\lambda,\mu}(t)$ agree with $a^{\mathrm{so}^+}_{\lambda,\mu}(t)$ as expressed in Lemma 5.2. Finally, it does not appear that adjoint relation between φ_t and the plethysm by p_t may be employed to give short proofs of the SXP rules based on the factorisations of Theorems 1.4, 4.1 and 4.2. This is because there is no orthonormality for the universal characters under the Hall inner product. In contrast, Lecouvey uses deformations of the Verschiebung operator with respect to the standard inner product on the character rings under which the Weyl characters are orthonormal.

6. Variations on factorisations

To conclude, we explain the connections between the results of this paper and very closely related results: symmetric functions twisted by roots of unity and characters of the symmetric group.

6.1. **Symmetric polynomials twisted by roots of unity.** A perspective we have not taken in this paper is that of “twisting” a symmetric polynomial by a primitive t-th root of unity. In fact, this is very closely connected to the original work of Littlewood and Richardson on this topic; see the papers [34, 40, 41] or Littlewood’s book [36, §7.3]. The interested reader should consult the recent paper of Ayyer and Kumari [4], which proves new results regarding twisting both ordinary and universal characters by roots of unity, as well as surveying some of the results we will now discuss.

A simple generating function argument shows that the action of the t-th Verschiebung operator on, for instance, the complete homogeneous symmetric functions, agrees with the result of replacing $X_n \mapsto (X_n, \xi X_n, \dots, \xi^{t-1} X_n)$ where $aX_n := (ax_1, \dots, ax_n)$ for any $a \in \mathbb{C}$ and evaluating. Littlewood and Richardson apply this twisting to the bialternant formula for the Schur functions and then through a sequence of matrix manipulations deduce the vanishing and factorisation. This is the same approach which is taken in the work of Ayyer and Kumari [3]. The advantage of this approach is it allows for slightly more general statements, such as the following theorem due to Littlewood and Richardson.

Theorem 6.1 ([42, Theorem XI])**.** *Let λ be a partition of length at most $nt+1$. Then for another variable y we have that*

$$s_\lambda(X_n, \xi X_n, \dots, \xi^{t-1}X_n, y) = 0$$

unless t-core$(\lambda) = (k)$ for some $0 \leqslant k \leqslant t-1$, in which case

$$s_\lambda(X_n, \xi X_n, \dots, \xi^{t-1}X_n, y) = \operatorname{sgn}_t(\lambda/(k)) y^k s_{\lambda^{(k-1)}}(X_n^t, y^t) \prod_{\substack{r=0 \\ r\neq k-1}}^{t-1} s_{\lambda^{(r)}}(X_n^t).$$

This has itself been generalised in several directions. For instance, Littlewood also characterises the vanishing and factorisation of $s_\lambda(1, \dots, \xi^m)$ where m is an arbitrary positive integer independent of t [36, §7.2] which has proved important in the context of cyclic sieving; see [55, Theorem 4.3], [57, Lemma 6.2] and [54, Theorem 4.4]. Recently Kumari extended this by replacing the variable y in Theorem 6.1 by a set of variables $y_1, \dots, y_r$, generalising Littlewood's result [30]. She also proves similar results for the characters of the symplectic and orthogonal groups for these same twists, however, the evaluations are not always products, and are quite complicated. None of these results extend elegantly through the Verschiebung operators. We have given a version of Theorem 6.1 involving a deformation of the Verschiebung operator [2, Proposition 6.2], but this is, in our opinion, not particularly natural. There is also a version of Theorem 3.2 for flagged skew Schur functions [29].

Outside of the realm of classical symmetric functions and classical group characters there has been little interest in the action of the operators φ_t. To our knowledge the only work in this direction is due to Mizukawa [45], who has given expressions for the action of the Verschiebung operators on the Schur Q-functions, as well as SXP-type rules. These involve variants of the Littlewood decomposition for partitions with distinct parts (also called *bar partitions*), the concepts of which were developed in the papers of Morris [47] and Olsson [51]. By considering the double of a strict partition which is 1-asymmetric, an idea Humphreys attributes to Macdonald [21], these results may be phrased in terms of the ordinary Littlewood decomposition. Using this, one may extend Mizukawa's results to skew Schur Q-functions by use of their definition as a Pfaffian, which plays the same role as the Jacobi–Trudi formula in the proof of Theorem 3.2.

6.2. **Characters of the symmetric group.** In this paper we have not discussed equivalent statements for the characters of the symmetric group, as in Littlewood and Richardson's original Theorem 1.1. Here we give the precise connection between these two perspectives.

The following may be found in, for instance, [44, §I.7]. Let R^n denote the space of class functions on $\mathfrak{S}_n$. The *characteristic map* $\mathrm{ch}^n : R^n \longrightarrow \Lambda^n$ is defined by

$$\mathrm{ch}^n(f) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} f(w) p_{\mathrm{cyc}(w)},$$

where Λ^n denotes the space of homogeneous symmetric functions of degree n and $\mathrm{cyc}(w)$ is a partition of n encoding the cycle type of w. Under this map $\mathrm{ch}^n(\chi^\lambda) = s_\lambda$. Now let $R := \bigoplus_{n \geqslant 0} R^n$. For $f \in R^n$ and $g \in R^m$ defining the induction product $f \cdot g := \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}}(f \otimes g)$ turns R into a graded algebra. We also have a scalar product on R which for $f = \sum_{n \geqslant 0} f_n$ and $g = \sum_{n \geqslant 0} g_n$ is given by

$$\langle f, g \rangle' := \sum_{n \geqslant 0} \langle f_n, g_n \rangle_{\mathfrak{S}_n}, \tag{6.1}$$

where $\langle f_n, g_n \rangle_{\mathfrak{S}_n}$ is the ordinary scalar product of $\mathfrak{S}_n$ characters. The map $\mathrm{ch} := \bigoplus_{n \geqslant 0} \mathrm{ch}^n$ is then an isometric isomorphism between R and Λ. We now define the actions of the t-th Verschiebung operator and its adjoint on R. On Λ this adjoint is the plethysm by a power sum p_t, but in general it is the *Frobenius operator* or *Adams operation* (the former is not to be confused with the Frobenius characteristic, another name given to ch). As in the case of the characteristic map we first define for $f \in R^n$ the operator φ_t^n by

$$\varphi_t^n(f)(\mu) = f(t\mu). \tag{6.2}$$

From this we see that if $f \in R^n$ then $\varphi_t(f) \in R^{n/t}$ if $t \mid n$ and is the zero function otherwise. Then $\varphi_t := \bigoplus_{n \geqslant 0} \varphi_t^n$. In particular if 1_n denotes the trivial representation of $\mathfrak{S}_n$ then $\varphi_t(1_n) = 1_{n/t}$ if t

divides n and is equal to zero otherwise. Since $\mathrm{ch}(1_n) = h_n$ it follows that $\mathrm{ch}\,\varphi_t = \varphi_t\,\mathrm{ch}$, where on the left we use (6.2) and on the right we use the Verschiebung operator on Λ. The same is true of ch^{-1}. The action of ψ_t^n may now be defined by

$$\psi_t^n(\chi^\lambda)(\mu) = \begin{cases} t^{l(\mu)}\chi^\lambda(\mu/t) & \text{if } t \mid \mu_i \text{ for all } i \geqslant 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

Then also set $\psi_t := \bigoplus_{n\geqslant 0} \psi_t^n$. Note similarity between (6.3) and the expression for $\varphi_t p_\lambda$ from Proposition 3.1. The fact that these operators are adjoint with respect to (6.1) then follows from the orthonormality of the irreducible characters. All in all, the point of the above constructions is that the characteristic map, when applied to the expression of Theorem 1.1, yields the expression for $\varphi_t s_\lambda$ of Theorem 1.3. By applying ch^{-1} to Theorem 3.2 we obtain the following theorem of Farahat.

Theorem 6.2 ([12]). *Let λ/μ be a skew shape with $|\lambda/\mu| = nt$. Then $\varphi_t(\chi^{\lambda/\mu}) = 0$ unless λ/μ is t-tileable, in which case*

$$\varphi_t(\chi^{\lambda/\mu}) = \mathrm{sgn}_t(\lambda/\mu) \prod_{r=0}^{t-1} \chi^{\lambda^{(r)}/\mu^{(r)}},$$

where the product on the right-hand side is the induction product.

The operators φ_t and ψ_t in the context of symmetric group characters first appeared in the relatively unknown paper of Kerber, Sänger and Wagner [23]. In particular, they state the actions (6.2) and (6.3) in Section 4 of that paper, together with the adjoint relation. These are then used to give a proof of Farahat's generalisation of Theorem 1.1, which describes the action of the Verschiebung operator on the skew character $\chi^{\lambda/\mu}$. This is notably different to Farahat's proof, which uses symmetric functions. They also prove the character-theoretic analogue of the SXP rule, our Theorem 1.2, which is equivalent to Littlewood's original rule for the plethysm $s_\lambda \circ p_t$. Another proof of Farahat's theorem is given in [10, §3]. For a more recent application of these ideas to characters of the symmetric group see the paper of Rhoades [58].

There has also been some recent interest in the character values $\chi^\lambda_{t\mu}$ from a slightly different perspective. Lübeck and Prasad [43] have shown that for λ a partition with empty 2-core the character value $\chi^\lambda_{2\mu}$ is equal, up to the sign $\mathrm{sgn}_2(\lambda)$, to the value of an irreducible character of the wreath product $\mathbb{Z}_2 \wr \mathfrak{S}_n$ (also known as the hyperoctahedral group) indexed by $(\lambda^{(0)}, \lambda^{(1)})$ evaluated at the conjugacy class $(\mu, \varnothing)$. (For the necessary background on characters of wreath products see [44, Chapter I, Appendix B].) Their proof is heavily algebraic, and along the way they prove and apply the $t = 2$ cases of Theorems 1.1 and 1.3. They also consider the case where 2-core$(\lambda) = (1)$, which itself hinges on the $t = 2$ case of Theorem 6.1 and its character-theoretic analogue, also contained in a theorem of Littlewood [38, p. 340]. This was generalised by Adin and Roichman [1], who further show that for t-core$(\lambda) = \varnothing$ the value $\mathrm{sgn}_t(\lambda)\chi^\lambda_{t\mu}$ may be expressed as the character of the wreath product $G \wr \mathfrak{S}_n$ indexed by $(\lambda^{(0)}, \dots, \lambda^{(t-1)})$ evaluated at the t-tuple $(\mu, \varnothing, \dots, \varnothing)$ where G is any finite abelian group of order t. Their proof is of a more combinatorial flavour, using Stembridge's Murnaghan–Nakayama rule for wreath products [66, §4] and ribbon combinatorics. Note that this does not cover the vanishing of the character values $\chi^\lambda(t\mu)$ in the case t-core(λ) is nonempty. Since Stembridge's rules work more generally for skew shapes, it would be interesting to investigate a skew extension of these results, putting Farahat's theorem into the picture. For further remarks on this side of the story we refer to the review of the paper of Lübeck and Prasad by Wildon [69], which includes a proof of Theorem 1.1 using the SXP rule.

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PROOF OF SOME LITTLEWOOD IDENTITIES CONJECTURED BY LEE, RAINS AND WARNAAR

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Abstract. We prove a novel pair of Littlewood identities for Schur functions, recently conjectured by Lee, Rains and Warnaar in the Macdonald case, in which the sum is over partitions with empty 2-core. As a byproduct we obtain a new Littlewood identity in the spirit of Littlewood's original formulae.

Contents

1. Introduction

The classical Littlewood identities are the following three summation formulae for Schur functions:

$$\sum_{\lambda} s_\lambda(x) = \prod_{i\geqslant 1}\frac{1}{1-x_i}\prod_{i<j}\frac{1}{1-x_ix_j}, \tag{1.1a}$$

$$\sum_{\substack{\lambda\\ \lambda \text{ even}}} s_\lambda(x) = \prod_{i\geqslant 1}\frac{1}{1-x_i^2}\prod_{i<j}\frac{1}{1-x_ix_j}, \tag{1.1b}$$

$$\sum_{\substack{\lambda\\ \lambda' \text{ even}}} s_\lambda(x) = \prod_{i<j}\frac{1}{1-x_ix_j}, \tag{1.1c}$$

where $x=(x_1,x_2,x_3,\dots)$ is a countable alphabet. Here and throughout the rest of the paper "λ even" means the partition λ has only even parts and λ' denotes the conjugate of λ. These identities were first written down together by Littlewood [16, p. 238], however (1.1a) was already known to Schur [27]. They have since afforded many far-reaching generalisations and have found applications in areas

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such as combinatorics, representation theory and elliptic hypergeometric series. In particular there are many generalisations of (1.1) at the Schur level [3, 7, 10, 11, 12, 13, 21, 22, 28]. Also see [25] for comprehensive references to the literature.

The purpose of this note is to prove the Schur function case of a pair of Littlewood identities for Macdonald polynomials recently conjectured by Lee, Rains and Warnaar [15, Conjecture 9.5]. To state these we need some notation. Denote the multiset of hook lengths of a partition λ by $\mathcal{H}_\lambda$. We refine this by writing $\mathcal{H}_\lambda^{\mathrm{e/o}}$ for the submultiset of even/odd hook lengths. The standard infinite q-shifted factorial is given by $(a;q)_\infty := \prod_{i\geqslant 0}(1-aq^i)$ and we define a statistic

$$\varsigma(\lambda) := \sum_{(i,j)\in\lambda} (-1)^{\lambda_i+\lambda_j'-i-j+1}(\lambda_i - i), \tag{1.2}$$

in terms of the Young diagram of λ; see Subsection 2.1 below. Finally, let $\hat{\Lambda}_{\mathbb{F}}$ denote the completion of the ring of symmetric functions over the field $\mathbb{F}$ with respect to the natural grading by degree.

Theorem 1.1. *As identities in $\hat{\Lambda}_{\mathbb{Q}(q)}$ at the alphabet $x=(x_1,x_2,x_3,\dots)$ we have that*

$$\sum_{\substack{\lambda\\ 2\text{-core}(\lambda)=0}} q^{\varsigma(\lambda)} \frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}}(1-q^h)}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}}(1-q^h)} s_\lambda(x) = \prod_{i\geqslant 1}\frac{(qx_i^2;q^2)_\infty}{(x_i^2;q^2)_\infty}\prod_{i<j}\frac{1}{1-x_ix_j}, \tag{1.3}$$

and

$$\sum_{\substack{\lambda\\ 2\text{-core}(\lambda)=0}} q^{\varsigma(\lambda')} \frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}}(1-q^h)}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}}(1-q^h)} s_\lambda(x) = \prod_{i\geqslant 1}\frac{(q^2x_i^2;q^2)_\infty}{(qx_i^2;q^2)_\infty}\prod_{i<j}\frac{1}{1-x_ix_j}. \tag{1.4}$$

The condition $2\text{-core}(\lambda)=0$ generalises both the even row and even column conditions of (1.1b) and (1.1c). Indeed, by Lemma 2.2 we have that $\varsigma(\lambda)=0$ if and only if λ is even. Thus when setting $q=0$ (1.3) and (1.4) collapse to (1.1b) and (1.1c) respectively. In this sense these identities are in the spirit of Kawanaka's identity [13, Theorem 1.1]

$$\sum_\lambda \prod_{h\in\mathcal{H}_\lambda}\left(\frac{1+q^h}{1-q^h}\right)s_\lambda(x) = \prod_{i\geqslant 1}\frac{(-qx_i;q)_\infty}{(x_i;q)_\infty}\prod_{i<j}\frac{1}{1-x_ix_j},$$

since this reduces to (1.1a) when $q=0$. Unlike Kawanaka's identity one can make sense of the $q\to 1$ limit of (1.3) and (1.4). In either case we obtain the following Littlewood-type identity.

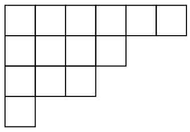
Corollary 1.2. *As an identity in $\hat{\Lambda}_{\mathbb{Q}}$ at the alphabet $x=(x_1,x_2,x_3,\dots)$,*

$$\sum_{\substack{\lambda\\ 2\text{-core}(\lambda)=0}} \frac{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{o}}} h}{\prod_{h\in\mathcal{H}_\lambda^{\mathrm{e}}} h} s_\lambda(x) = \prod_{i\geqslant 1}\frac{1}{(1-x_i^2)^{1/2}}\prod_{i<j}\frac{1}{1-x_ix_j}.$$

The outline of the paper is as follows. In the next section we give preliminaries regarding partitions, Schur functions and Koornwinder polynomials. In Section 3 we prove a pair of vanishing integrals for Schur functions again conjectured by Lee, Rains and Warnaar in the Macdonald case [15, Conjecture 9.2]. Then, in Section 4, we follow the techniques of [25] to prove the bounded analogues of Theorem 1.1 conjectured in [15, Conjecture 9.4]. The theorem then follows by taking an appropriate limit. We conclude with a derivation of Corollary 1.2.

2. Partitions and (BC_n)-symmetric functions

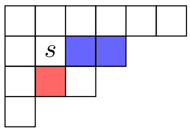
2.1. **Partitions.** A partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ is a weakly decreasing sequence of nonnegative integers such that finitely many λ_i are nonzero. The sum of the entries is denoted $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$ and if $|\lambda| = n$ we say λ is a partition of n. Nonzero entries are called parts, and the number of parts is called the length, denoted $l(\lambda)$. We denote by $\mathscr{P}$ the set of all partitions and by $\mathscr{P}_n$ the set of all partitions with length at most n. In particular $\mathscr{P}_0 = \{0\}$ where 0 denotes the unique partition of zero. If $\lambda \in \mathscr{P}_n$ we write $\lambda + \delta$ for the partition $(\lambda_1 + n - 1, \lambda_2 + n - 2, \dots, \lambda_n)$. The number $m_i(\lambda)$ of occurrences of an integer i as a part of λ is called the multiplicity. Sometimes we express a partition in terms of its multiplicities as $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} 3^{m_3(\lambda)} \cdots)$. We write $\mu \subset \lambda$ if the partition μ is contained in λ, i.e. if $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$. If $\lambda \subseteq (m^n)$ for some nonnegative integers m, n, then we write $(m^n) - \lambda$ for the complement of λ inside (m^n), that is, $(m^n) - \lambda := (m - \lambda_n, m - \lambda_{n-1}, \dots, m - \lambda_1)$. A partition is identified with its Young diagram, which is the left-justified array of squares with λ_i squares in row i with i increasing downward. For example



is the Young diagram of $(6, 4, 3, 1)$. The conjugate of a partition, written λ', is obtained by reflecting the Young diagram in the main diagonal, so that $(6, 4, 3, 1)' = (4, 3, 3, 2, 1, 1)$. The arm and leg lengths of a square $s = (i, j) \in \lambda$ are given by

$$a(s) := \lambda_i - j \quad \text{and} \quad l(s) := \lambda'_j - i,$$

which are the number of boxes strictly to the right and below s respectively. The hook length is the sum of these including s itself, so that $h(s) := a(s) + l(s) + 1$. Using the same example as above, in the Young diagram



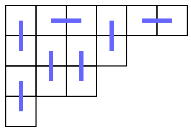
we have labelled the square $s = (2, 2)$ so that $a(s) = 2$, $l(s) = 1$ and $h(s) = 4$. As in the introduction we denote the multiset of hook lengths of λ by $\mathcal{H}_\lambda$. This is further refined as $\mathcal{H}^{\mathrm{e}}_\lambda$ and $\mathcal{H}^{\mathrm{o}}_\lambda$, the multisets of hook lengths which are even or odd, respectively. In terms of these we define the hook polynomials

$$\begin{aligned} H_\lambda(q) &:= \prod_{h \in \mathcal{H}_\lambda} (1 - q^h) \\ H^{\mathrm{e/o}}_\lambda(q) &:= \prod_{h \in \mathcal{H}^{\mathrm{e/o}}_\lambda} (1 - q^h), \end{aligned}$$

which are invariant under conjugation of λ. For $z \in \mathbb{C}$ we also need the content polynomials

$$\begin{aligned} C_\lambda(z; q) &:= \prod_{(i,j) \in \lambda} (1 - z q^{j-i}) \\ C^{\mathrm{e/o}}_\lambda(z; q) &:= \prod_{\substack{(i,j) \in \lambda \\ i + j \text{ even/odd}}} (1 - z q^{j-i}). \end{aligned}$$

In this paper we will frequently encounter partitions with empty 2-core, written 2-core$(\lambda) = 0$. One definition of such partitions is that their diagrams may be tiled by dominoes. Our running example of $(6,4,3,1)$ has empty 2-core since it admits the tiling



which is clearly not unique. We will now give some conditions which are equivalent to λ having empty 2-core which all easily follow by induction on $|\lambda|$. The reader interested in more general statements involving Littlewood's decomposition of a partition into its r-core and r-quotient for all $r \geqslant 2$ may consult, for example, [17] or [19, p. 12–15].

Lemma 2.1. *For $\lambda \in \mathscr{P}_{2n}$ the following are equivalent:*

(1) $2\text{-core}(\lambda) = 0$.
(2) $|\mathcal{H}^{\mathrm{o}}_\lambda| = |\mathcal{H}^{\mathrm{e}}_\lambda| = n$.
(3) *The set*

$$\{\lambda_1 + 2n - 1, \lambda_2 + 2n - 2, \dots, \lambda_{2n-1} + 1, \lambda_{2n}\}$$

contains n even and n odd integers.

2.2. **Auxiliary results.** Here we prove some properties of the statistic $\varsigma(\lambda)$ (1.2). Firstly, as we have already used in the introduction, we have the following characterisation of the vanishing of $\varsigma(\lambda)$.

Lemma 2.2. *Let $2\text{-core}(\lambda) = 0$. Then $\varsigma(\lambda) \geqslant 0$ with $\varsigma(\lambda) = 0$ if and only if λ is even.*

Proof. If λ is even then $\varsigma(\lambda) = 0$ since the number of even and odd hook lengths in each row is equal. Assume that λ is not even. Then λ has an even number of odd parts. Let λ_{i_1}, λ_{i_2} be the final two odd rows of λ. Since $2\text{-core}(\lambda)$ is empty these must be separated by an even number of even rows (possibly zero). Ignoring the rows above, the contribution to $\varsigma(\lambda)$ below and including row λ_{i_1} may be computed as

$$\lambda_{i_1} - \lambda_{i_2} + i_2 - i_1 + 2\sum_{j=i_1+1}^{i_2-1} (-1)^{i_1+j-1}(\lambda_j - j).$$

Since the numbers $\lambda_j - j$ are strictly decreasing this sum is positive. The next nonzero contribution to $\varsigma(\lambda)$ will come from the pair of odd rows above in the same fashion. Thus repeating the above shows that if λ has empty 2-core and contains at least two odd rows then $\varsigma(\lambda) > 0$. $\square$

Note that $\varsigma((2,1,1,1)) = 0$, so that $\varsigma(\lambda)$ may vanish for partitions with nonempty 2-core.

Lemma 2.3. *For $\lambda \in \mathscr{P}_{2n}$ there holds*

$$\varsigma(\lambda) = \sum_{(i,j)\in\lambda+\delta} (-1)^{\lambda_i - i - j + 1}(\lambda_i - i) - \sum_{1 \leqslant i < j \leqslant 2n} (-1)^{\lambda_i - \lambda_j + j - i}(\lambda_i - i). \tag{2.1}$$

Moreover, if $2\text{-core}(\lambda) = 0$, then

$$\varsigma(\lambda') = \frac{|\lambda|}{2} - n^2 - n + \sum_{1 \leqslant i < j \leqslant 2n} (-1)^{\lambda_i - \lambda_j + j - i}(\lambda_j - j). \tag{2.2}$$

Proof. We interpret the definition of $\varsigma(\lambda)$ as a sum over the Young diagram of λ where each square has weight $(-1)^{\lambda_i + \lambda'_j - i - j + 1}(\lambda_i - i)$. In the Young diagram of $\lambda+\delta$ place the integer $(-1)^{\lambda_i - i - j + 1}(\lambda_i - i)$ in box (i,j). Summing over i,j gives the first sum on the right of (2.1). To identify the second sum, we remove the columns with index $\lambda_j + 2n - j + 1$ for $2 \leqslant j \leqslant 2n$ whose entries are $(-1)^{\lambda_i - \lambda_j + j - i}(\lambda_i - i)$. The remaining diagram is that of λ with entries $(-1)^{\lambda_i + \lambda'_j - i - j + 1}(\lambda_i - i)$, which shows the first identity.

The proof of the second identity is similar. Note that by (1.2), $\varsigma(\lambda')$ may be written as

$$\varsigma(\lambda') = \sum_{(i,j)\in\lambda} (-1)^{\lambda_i + \lambda'_j - i - j + 1}(\lambda'_j - j).$$

We thus fill the diagram of $\lambda+\delta$ with integers $(-1)^{\lambda_i - i - j + 1}(2n-j)$, so that removing the same columns as before now gives

$$\varsigma(\lambda') = \sum_{(i,j)\in\lambda+\delta} (-1)^{\lambda_i - i - j + 1}(2n - j) - \sum_{1\leqslant i<j\leqslant 2n} (-1)^{\lambda_i - \lambda_j + j - i}(j - \lambda_j - 1).$$

A simple calculation shows that for 2-core$(\lambda) = 0$,

$$\sum_{(i,j)\in\lambda+\delta} (-1)^{\lambda_i - i - j + 1}(2n - j) + \sum_{1\leqslant i<j\leqslant 2n} (-1)^{\lambda_i - \lambda_j + j - i} = \frac{|\lambda|}{2} - n^2 - n,$$

completing the proof. □

2.3. **Schur functions.** For completeness we give a definition of the Schur functions in terms of the classical ratio of alternants. For $\lambda \in \mathscr{P}_n$ the Schur function is defined as

$$s_\lambda(x_1, \dots, x_n) := \frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j + n - j})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j})},$$

and $s_\lambda(x_1, \dots, x_n) := 0$ for $l(\lambda) > n$. The set of the $s_\lambda(x_1, \dots, x_n)$ indexed over $\mathscr{P}_n$ forms a $\mathbb{Z}$-basis for the ring of symmetric functions in n variables, denoted Λ_n. We also use the Schur functions in countably many variables $x = (x_1, x_2, x_3, \dots)$, such as in Theorem 1.1, which may be defined by the Jacobi–Trudi determinant [19, p. 41]. The set of such $s_\lambda(x)$ when indexed over all partitions λ form a $\mathbb{Z}$-basis for the ring of symmetric functions Λ. We also require the ring $\hat{\Lambda}$ which is the completion of Λ with respect to the natural grading by degree [23, p. 66].

Several of the results we need below are best stated in terms of Macdonald polynomials, which are a q,t-analogue of the Schur functions [19, §VI]. We simply note that the Macdonald polynomials $P_\lambda(x; q, t)$ are a basis for $\Lambda_{\mathbb{Q}(q,t)}$ and reduce to the Schur functions when $q = t$, i.e., $P_\lambda(x; q, q) = s_\lambda(x)$.

2.4. **Koornwinder polynomials and integrals.** The Koornwinder polynomials are a family of BC_n-symmetric functions depending on six parameters first introduced by Koornwinder [14] as a multivariate analogue of the Askey–Wilson polynomials [1]. Here we write $x = (x_1, \dots, x_n), x^{\pm} = (x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$ and for a single-variable function $g(x_i)$ we set

$$\begin{aligned} g(x_i^{\pm}) &:= g(x_i)g(x_i^{-1}) \\ g(x_i^{\pm}x_j^{\pm}) &:= g(x_i x_j)g(x_i^{-1}x_j)g(x_i x_j^{-1})g(x_i^{-1}x_j^{-1}). \end{aligned}$$

Below the function will be one of $g(x_i) = (x_i; q)_\infty$ or $g(x_i) = (1 - x_i)$. Also for the infinite q-shifted factorial we adopt the usual multiplicative notation

$$(a_1, \dots, a_n; q)_\infty := (a_1; q)_\infty \cdots (a_n; q)_\infty.$$

Let $W := \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ be the group of signed permutations on n letters. A Laurent polynomial $f(x) \in \mathbb{C}[x^{\pm}]$ is called BC_n-symmetric if it is invariant under the natural action of W on the n variables where the reflections act by $x_i \mapsto 1/x_i$. For $\lambda \in \mathscr{P}_n$ define the orbit-sum indexed by λ as

$$m_\lambda^{\mathrm{BC}}(x) := \sum_\alpha x^\alpha,$$

where the sum is over all elements α of the W-orbit of λ, the reflections act on sequences by $\alpha_i \mapsto -\alpha_i$, and $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The orbit-sums form a basis for the ring Λ_n^{BC} of BC_n-symmetric functions. For $q, t, t_0, t_1, t_2, t_3 \in \mathbb{C}$ with $|q|, |t|, |t_0|, |t_1|, |t_2|, |t_3| < 1$, define the Koornwinder density by

$$\Delta(x; q, t; t_0, t_1, t_2, t_3) := \prod_{i=1}^n \frac{(x_i^{\pm 2}; q)_\infty}{\prod_{r=0}^3 (t_r x_i^{\pm}; q)_\infty} \prod_{1 \leqslant i < j \leqslant n} \frac{(x_i^{\pm} x_j^{\pm}; q)_\infty}{(t x_i^{\pm} x_j^{\pm}; q)_\infty}.$$

This further allows one to define an inner product on Λ_n^{BC} by

$$\langle f, g \rangle_{q,t;t_0,t_1,t_2,t_3}^{(n)} := \int_{\mathbb{T}^n} f(x) g(x^{-1}) \Delta(x; q, t; t_0, t_1, t_2, t_3) \, \mathrm{d}T(x),$$

where $\mathbb{T}^n$ is the standard n-torus and the measure $T(x)$ is given by

$$\mathrm{d}T(x) := \frac{1}{2^n n! (2\pi \mathrm{i})^n} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}.$$

The Koornwinder polynomials are defined to be the unique BC_n-symmetric functions satisfying

$$K_\lambda = m_\lambda^{\mathrm{BC}} + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu^{\mathrm{BC}},$$

where $c_{\lambda\mu} \in \mathbb{C}(q, t, t_0, t_1, t_2, t_3)$, and for which

$$\langle K_\lambda, K_\mu \rangle_{q,t;t_0,t_1,t_2,t_3}^{(n)} = 0 \qquad \text{if } \lambda \neq \mu.$$

Note that $\mu \leqslant \lambda$ denotes the extension of the usual dominance order to all partitions $\lambda, \mu \in \mathscr{P}$: $\mu \leqslant \lambda$ if and only if $\mu_1 + \cdots + \mu_i \leqslant \lambda_1 + \cdots + \lambda_i$ for all $i \geqslant 1$. The Koornwinder polynomials satisfy many nice properties such as the quadratic norm evaluation and evaluation symmetry [4, 26]. The key identity we need is [25, Equation (2.6.9)] (see also [23, Corollary 7.2.1])

$$\begin{aligned}(2.3)\qquad &\lim_{m\to\infty} (x_1 \dots x_n)^m K_{(m^n)-\lambda}(x; q, t; t_0, t_1, t_2, t_3) \\ &\qquad = P_\lambda(x; q, t) \prod_{i=1}^n \frac{(t_0 x_i, t_1 x_i, t_2 x_i, t_3 x_i; q)_\infty}{(x_i^2; q)_\infty} \prod_{1 \leqslant i < j \leqslant n} \frac{(t x_i x_j; q)_\infty}{(x_i x_j; q)_\infty}.\end{aligned}$$

We will only use this for $\lambda = 0$, in which case $P_0(x; q, t) = 1$.

For a basis $\{f_\lambda\}$ of Λ_n^{BC} we write $[f_\lambda]g$ for the coefficient of f_λ in the expansion $g = \sum_\lambda c_\lambda f_\lambda$ where the c_λ lie in some coefficient ring. The virtual Koornwinder integral of a BC_n-symmetric function f is defined as

$$I_K^{(n)}(f; q, t; t_0, t_1, t_2, t_3) := [K_0(x; q, t; t_0, t_1, t_2, t_3)] f.$$

This is extended to allow for symmetric function arguments via the homomorphism $\Lambda_{2n} \longrightarrow \Lambda_n^{\mathrm{BC}}$ for which $f(x_1, \dots, x_{2n}) \mapsto f(x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$. Of course since $K_0 = 1$ the orthogonality of the Koornwinder polynomials allows us to express this as

$$I_K^{(n)}(f; q, t; t_0, t_1, t_2, t_3) = \frac{\langle f, 1 \rangle_{q,t;t_0,t_1,t_2,t_3}^{(n)}}{\langle 1, 1 \rangle_{q,t;t_0,t_1,t_2,t_3}^{(n)}}.$$

Note that the denominator has the explicit evaluation

$$\langle 1,1\rangle^{(n)}_{q,t;t_0,t_1,t_2,t_3} = \prod_{i=1}^{n} \frac{(t, t_0t_1t_2t_3t^{n+i-2};q)_\infty}{(q,t^i;q)_\infty \prod_{0\leqslant r<s\leqslant 3}(t_rt_st^{i-1};q)_\infty},$$

which is Gustafson's generalised Askey–Wilson integral [9]. The virtual Koornwinder integral can be evaluated for many choices of the argument f, see [15, 23, 24, 25]. In particular, the vanishing integrals of the next section may be expressed in terms of virtual Koornwinder integrals. We need one final identity involving virtual Koornwinder integrals. To state this conveniently, let

$$f^{(m)}_\lambda(q,t;t_0,t_1,t_2,t_3) := [P_\lambda(x;q,t)](x_1\cdots x_n)^m K_{(m^n)}(x;q,t;t_0,t_1,t_2,t_3).$$

Proposition 2.4 ([25, Proposition 4.9])**.** *For nonnegative integers n,m and $\lambda \subseteq (2m)^n$,*

$$f^{(m)}_\lambda(q,t;t_0,t_1,t_2,t_3) = (-1)^{|\lambda|} I^{(m)}_K\big(P_{\lambda'}(t,q);t,q;t_0,t_1,t_2,t_3\big).$$

3. Vanishing integrals

In this section we evaluate a pair of vanishing integrals for Schur functions conjectured by Lee, Rains and Warnaar in the Macdonald case [15, Conjecture 9.2].

For $a,b,q\in\mathbb{C}$ with $|a|,|b|,|q|<1$ we define

$$I^{(n)}_\lambda(a,b;q) := \frac{1}{Z_n(a,b;q)} \int_{\mathbb{T}^n} s_\lambda\big(x_1^{\pm},\dots,x_n^{\pm}\big) \prod_{i=1}^{n} \frac{(x_i^{\pm 2};q)_\infty}{(ax_i^{\pm 2}, bx_i^{\pm 2};q^2)_\infty} \times \prod_{1\leqslant i<j\leqslant n} \big(1-x_i^{\pm}x_j^{\pm}\big)\,\mathrm{d}T(x),$$

where λ is a partition with length at most $2n$ and the normalising factor is given by

$$\begin{aligned} Z_n(a,b;q) &:= \int_{\mathbb{T}^n} \prod_{i=1}^{n} \frac{(x_i^{\pm 2};q)_\infty}{(ax_i^{\pm 2}, bx_i^{\pm 2};q^2)_\infty} \prod_{1\leqslant i<j\leqslant n} \big(1-x_i^{\pm}x_j^{\pm}\big)\,\mathrm{d}T(x) \\ &= \prod_{i=1}^{n} \frac{(abq^{n+i-2};q)_\infty}{(q^i,-aq^{i-1},-bq^{i-1};q)_\infty (abq^{2i-2};q^2)^2_\infty}. \end{aligned}$$

Note that in terms of virtual Koornwinder integrals this is

$$I^{(n)}_\lambda(a,b;q) = I^{(n)}_K(s_\lambda;q,q,a^{1/2},-a^{1/2},b^{1/2},-b^{1/2}).$$

Lee, Rains and Warnaar prove the following properties of the above integral.

Proposition 3.1 ([15, Proposition 9.3])**.** *For $a,b,q\in\mathbb{C}$ with $|a|,|b|,|q|<1$ and λ a partition of length at most $2n$ the integral $I^{(n)}_\lambda(a,b;q)$ vanishes unless* 2-core$(\lambda)=0$*. Furthermore*

$$I^{(n)}_\lambda(q,q;q) = \prod_{i=1}^{n} \frac{(1-q^{2i-1})^{2n-2i+1}}{(1-q^{2i})^{2n-2i}} \times \operatorname*{Pf}_{1\leqslant i,j\leqslant 2n}\bigg(\frac{q^{(\lambda_i-\lambda_j+j-i-1)/2}}{1-q^{\lambda_i-\lambda_j+j-i}}\chi(\lambda_i-\lambda_j+j-i\ \textit{odd})\bigg), \tag{3.1a}$$

and

$$I^{(n)}_\lambda(1,q^2;q) = \frac{1}{2^{n-1}(1+q^n)} \prod_{i=1}^{n} \frac{(1-q^{2i-1})^{2n-2i+1}}{(1-q^{2i})^{2n-2i}} \times \operatorname*{Pf}_{1\leqslant i,j\leqslant 2n}\bigg(\frac{1+q^{\lambda_i-\lambda_j+j-i}}{1-q^{\lambda_i-\lambda_j+j-i}}\chi(\lambda_i-\lambda_j+j-i\ \textit{odd})\bigg). \tag{3.1b}$$

Lee, Rains and Warnaar also give a conjectural Macdonald polynomial analogue of this proposition [15, Conjecture 9.2]. There the generalisations of (3.1) are explicit products. Our next proposition gives the evaluation of the Pfaffians in the previous proposition, verifying the conjecture of Lee, Rains and Warnaar for $q = t$.

Proposition 3.2. *For λ with length at most $2n$ and 2-core$(\lambda) = 0$,*

$$I_\lambda^{(n)}(q, q; q) = q^{\varsigma(\lambda')} \frac{C_\lambda^{\mathrm{e}}(q^{2n}; q) H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{2n}; q) H_\lambda^{\mathrm{e}}(q)} \tag{3.2}$$

and

$$I_\lambda^{(n)}(1, q^2; q) = q^{\varsigma(\lambda)} \frac{1 + q^{n+2\varsigma(\lambda')-2\varsigma(\lambda)}}{1 + q^n} \frac{C_\lambda^{\mathrm{e}}(q^{2n}; q) H_\lambda^{\mathrm{o}}(q)}{C_\lambda^{\mathrm{o}}(q^{2n}; q) H_\lambda^{\mathrm{e}}(q)}. \tag{3.3}$$

Proof. Since the structure of the Pfaffians is similar, we focus on the second identity, and evaluate (3.1b).

Fix a partition $\lambda \in \mathscr{P}_{2n}$ with empty 2-core. Define the set $J \subseteq \{1, \dots, 2n\}$ as the collection of integers j for which column j has a nonzero entry in the first row, and set $I := \{1, \dots, 2n\} \setminus J$. Since 2-core$(\lambda) = 0$ it follows that $|I| = |J| = n$. The elements of I and J are labeled by i_k and j_k respectively, where $1 \leqslant k \leqslant n$ and ordered naturally. With this established we define the $n \times n$ matrix M with entries $M_{k,\ell}$ by

$$M_{k,\ell} := \frac{1 + q^{\lambda_{i_k} - \lambda_{j_\ell} + j_\ell - i_k}}{1 - q^{\lambda_{i_k} - \lambda_{j_\ell} + j_\ell - i_k}}.$$

The Pfaffian in (3.1b) may be expressed in terms of the determinant of M. Indeed, by pushing the rows with indices in J to the right we see that

$$\begin{aligned}
\operatorname*{Pf}_{1 \leqslant i,j \leqslant 2n} &\left(\frac{1 + q^{\lambda_i - \lambda_j + j - i}}{1 - q^{\lambda_i - \lambda_j + j - i}} \chi(\lambda_i - \lambda_j + j - i \text{ odd}) \right) \\
&= (-1)^{\binom{n}{2} + \sum_{j \in J} j} \operatorname{Pf} \begin{pmatrix} 0 & M \\ -M^t & 0 \end{pmatrix} \\
&= (-1)^{\sum_{j \in J} j} \det M.
\end{aligned}$$

The determinant may be evaluated simply by applying the following generalisation of Cauchy's double alternant which may be found in [5, Example 3.1; $a = 0$]:

$$\begin{aligned}
\det_{1 \leqslant i,j \leqslant n} \left(\frac{b x_i + c y_j}{x_i + y_j} \right) &= (b - c)^{n-1} \left(b \prod_{i=1}^n x_i + (-1)^{n-1} c \prod_{i=1}^n y_i \right) \\
&\quad \times \frac{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)(y_i - y_j)}{\prod_{i,j=1}^n (x_i + y_j)}.
\end{aligned}$$

We apply this with $(b, c, x_k, y_\ell) \mapsto (-1, 1, q^{\lambda_{i_k} - i_k}, -q^{\lambda_{j_\ell} - j_\ell})$ for $1 \leqslant k, \ell \leqslant n$. After some elementary manipulations the evaluation may now be expressed as

$$\begin{aligned}
&I_\lambda^{(n)}(1, q^2; q) \\
&\quad = \frac{\prod_{i \in I} q^{\lambda_i - i} + \prod_{j \in J} q^{\lambda_j - j}}{1 + q^n} \prod_{i=1}^n \frac{(1 - q^{2i-1})^{2n-2i+1}}{(1 - q^{2i})^{2n-2i}} \\
&\quad \times \prod_{\substack{1 \leqslant i < j \leqslant 2n \\ \lambda_i - \lambda_j + j - i \text{ even}}} \frac{1 - q^{\lambda_i - \lambda_j + j - i}}{q^{\lambda_j - j}} \prod_{\substack{1 \leqslant i < j \leqslant 2n \\ \lambda_i - \lambda_j + j - i \text{ odd}}} \frac{q^{\lambda_j - j}}{1 - q^{\lambda_i - \lambda_j + j - i}}.
\end{aligned}$$

The terms of the form $1 - q^x$ can be simplified thanks to the identity [19, p. 10–11]

$$\frac{C_\lambda(q^{2n}; q)}{H_\lambda(q)} = \prod_{s \in \lambda} \frac{1 - q^{n + c(s)}}{1 - q^{h(s)}} = \frac{\prod_{1 \leqslant i < j \leqslant n} 1 - q^{\lambda_i - \lambda_j + j - i}}{\prod_{i=1}^n (q; q)_i},$$

where $l(\lambda) \leqslant n$. Restricting all products to even/odd exponents implies that

$$\begin{aligned}
&\frac{C^{\mathrm{e}}_\lambda(q^{2n};q)H^{\mathrm{o}}_\lambda(q)}{C^{\mathrm{o}}_\lambda(q^{2n};q)H^{\mathrm{e}}_\lambda(q)} \\
&\quad = \prod_{\substack{1\leqslant i<j\leqslant 2n \\ \lambda_i-\lambda_j+j-i \text{ even}}} (1-q^{\lambda_i-\lambda_j+j-i}) \prod_{\substack{1\leqslant i<j\leqslant 2n \\ \lambda_i-\lambda_j+j-i \text{ odd}}} \frac{1}{1-q^{\lambda_i-\lambda_j+j-i}} \\
&\qquad \times \prod_{i=1}^{n} \frac{(1-q^{2i-1})^{2n-2i+1}}{(1-q^{2i})^{2n-2i}}.
\end{aligned}$$

It remains to show that the powers of q agree in the prefactor. Since

$$\prod_{i\in I} q^{\lambda_i-i} + \prod_{j\in J} q^{\lambda_j-j} = \prod_{\substack{i=1 \\ \lambda_i-i \text{ even}}}^{2n} q^{\lambda_i-i} + \prod_{\substack{i=1 \\ \lambda_i-i \text{ odd}}}^{2n} q^{\lambda_i-i},$$

this may be reduced to the pair of identities

$$\varsigma(\lambda) = \sum_{\substack{i=1 \\ \lambda_i-i \text{ even}}}^{2n} (\lambda_i-i) + \sum_{1\leqslant i<j\leqslant 2n} (-1)^{\lambda_i-\lambda_j+j-i}(\lambda_j-j),$$

and

$$n+2\varsigma(\lambda')-2\varsigma(\lambda) = \sum_{\substack{i=1 \\ \lambda_i-i \text{ odd}}}^{2n} (\lambda_i-i) - \sum_{\substack{i=1 \\ \lambda_i-i \text{ even}}}^{2n} (\lambda_i-i).$$

In the first of these write

$$\begin{aligned}
\sum_{\substack{i=1 \\ \lambda_i-i \text{ even}}}^{2n} (\lambda_i-i) &= \sum_{(i,j)\in\lambda+\delta} (-1)^{\lambda_i-i-j+1}(\lambda_i-i) + \sum_{i=1}^{2n}(\lambda_i-i) \\
&= \varsigma(\lambda) + \sum_{1\leqslant i<j\leqslant 2n} (-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i) + \sum_{i=1}^{2n}(\lambda_i-i),
\end{aligned}$$

where in the second equality we have applied (2.1) from Lemma 2.3. Since

$$\begin{aligned}
&\sum_{1\leqslant i<j\leqslant 2n} (-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i) + \sum_{1\leqslant i<j\leqslant 2n} (-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i) + \sum_{i=1}^{2n}(\lambda_i-i) \\
&\quad = \sum_{i,j=1}^{2n} (-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i) \\
&\quad = 0,
\end{aligned}$$

the first identity follows. For the second identity, a similar rewriting, now using (2.2) of Lemma 2.3, shows us that

$$\begin{aligned}
&\sum_{\substack{i=1 \\ \lambda_i-i \text{ odd}}}^{2n} (\lambda_i-i) - \sum_{\substack{i=1 \\ \lambda_i-i \text{ even}}}^{2n} (\lambda_i-i) \\
&\qquad = -2\sum_{(i,j)\in\lambda+\delta} (-1)^{\lambda_i-i-j+1}(\lambda_i-i) - \sum_{i=1}^{2n}(\lambda_i-i) \\
&\qquad = -2\varsigma(\lambda) - |\lambda| + 2n^2 + n - 2\sum_{1\leqslant i<j\leqslant 2n} (-1)^{\lambda_i-\lambda_j+j-i}(\lambda_i-i) \\
&\qquad = n + 2\varsigma(\lambda') - 2\varsigma(\lambda).
\end{aligned}$$

This finishes the evaluation of (3.1b). The evaluation of (3.1a) is almost identical except one directly applies (2.2) of Lemma 2.3 to compute the exponent of q in the prefactor. $\square$

4. Bounded Littlewood identities

Here we use the integral evaluations of the previous section to prove a bounded analogue of Theorem 1.1. This is followed by proofs of the theorem and of Corollary 1.2.

4.1. **A bounded analogue of Theorem 1.1.** Bounded Littlewood identities are generalisations of ordinary Littlewood identities in which the largest part of the indexing partition has an upper bound, say m, such that sending m to infinity recovers an ordinary (unbounded) Littlewood identity. The first example of such an identity was discovered by Macdonald [18, §1.5] where he used a bounded analogue of (1.1a) to prove the MacMahon and Bender–Knuth conjectures on plane partitions [2, 20]. Bounded analogues of the remaining two classical identities (1.1b) and (1.1c) were obtained by Désarménien, Proctor and Stembridge [7, 22, 28] and Okada [21] respectively. A host of other bounded identities for Hall–Littlewood and Macdonald polynomials may be found in [25] and references therein. For further discussion of the history of bounded Littlewood identities see [10]. We now state the bounded analogue of Theorem 1.1.

Theorem 4.1. *For nonnegative integers m and n,*

$$\sum_{\substack{\lambda \\ \text{2-core}(\lambda)=0}} q^{\varsigma(\lambda')} \frac{C^{\mathrm{e}}_{\lambda}(q^{-2m};q)H^{\mathrm{o}}_{\lambda}(q)}{C^{\mathrm{o}}_{\lambda}(q^{-2m};q)H^{\mathrm{e}}_{\lambda}(q)} s_{\lambda}(x) \\ = (x_1\cdots x_n)^m K_{(m^n)}\big(x;q,q;q^{1/2},-q^{1/2},q^{1/2},-q^{1/2}\big), \tag{4.1}$$

and

$$\sum_{\substack{\lambda \\ \text{2-core}(\lambda)=0}} \frac{q^{2\varsigma(\lambda')-\varsigma(\lambda)}+q^{m+\varsigma(\lambda)}}{1+q^m} \frac{C^{\mathrm{e}}_{\lambda}(q^{-2m};q)H^{\mathrm{o}}_{\lambda}(q)}{C^{\mathrm{o}}_{\lambda}(q^{-2m};q)H^{\mathrm{e}}_{\lambda}(q)} s_{\lambda}(x) \\ = (x_1\cdots x_n)^m K_{(m^n)}(x;q,q;1,-1,q,-q). \tag{4.2}$$

These identities are indeed bounded since $C^{\mathrm{e}}_{\lambda}(q^{-2m};q)$ vanishes if $\lambda_1 > 2m$. Since, by [15, Lemma 4.1], the Koornwinder polynomials on the right reduce to classical group characters for $q=0$, one recovers the previously mentioned Désarménien–Proctor–Stembridge and Okada identities respectively in this case. The Koornwinder polynomials for $q=t$ on the right-hand side may alternatively be expressed as a ratio of determinants of Askey–Wilson polynomials [1]; see, e.g., [6, Definition 4.1]. This, however, does not seem to shed light on a more explicit expression for the evaluation of these sums. In particular, the specialisations of $K_{(m^n)}$ above are not contained in [15, Lemma 4.1].

The following argument is sketched in [15, §9], but we give the details in the Schur case. Assuming the Macdonald polynomial version of the vanishing integrals [15, Conjecture 9.2], the same argument gives the conjectural Littlewood identities.

Proof of Theorem 4.1. The goal is to find an expression for the coefficient of $s_{\lambda}(x)$ in the Schur expansion of the right-hand side. By Proposition 2.4 this coefficient is

$$f^{(m)}_{\lambda}(x;q,q,t_0,t_1,t_2,t_3) = (-1)^{|\lambda|} I^{(m)}_K(s_{\lambda'}(x);q,q;t_0,t_1,t_2,t_3).$$

If we specialise $(t_0,t_1,t_2,t_3)=(q^{1/2},-q^{1/2},q^{1/2},-q^{1/2})$ then this reduces to

$$f^{(m)}_{\lambda}\big(x;q,q;q^{1/2},-q^{1/2},q^{1/2},-q^{1/2}\big) = (-1)^{|\lambda|} I^{(m)}_{\lambda'}(q,q;q).$$

The integral on the right is (3.2), as desired, and vanishes unless 2-core$(\lambda) = 0$. In this case the sign disappears since $|\lambda|$ is even and we obtain

$$(-1)^{|\lambda|} I_{\lambda'}^{(m)}(q,q;q) = q^{\varsigma(\lambda)} \frac{C_{\lambda'}^{\mathrm{e}}(q^{2m};q) H_{\lambda'}^{\mathrm{o}}(q)}{C_{\lambda'}^{\mathrm{o}}(q^{2m};q) H_{\lambda'}^{\mathrm{e}}(q)}.$$

By [15, Lemma 2.3] we may alternatively express this as

$$q^{\varsigma(\lambda)} \frac{C_{\lambda'}^{\mathrm{e}}(q^{2m};q) H_{\lambda'}^{\mathrm{o}}(q)}{C_{\lambda'}^{\mathrm{o}}(q^{2m};q) H_{\lambda'}^{\mathrm{e}}(q)} = q^{\varsigma(\lambda')} \frac{C_{\lambda}^{\mathrm{e}}(q^{-2m};q) H_{\lambda}^{\mathrm{o}}(q)}{C_{\lambda}^{\mathrm{o}}(q^{-2m};q) H_{\lambda}^{\mathrm{e}}(q)} \tag{4.3}$$

This establishes (4.1). For (4.2) the same procedure applies with the substitution $(t_0,t_1,t_2,t_3) = (1,-1,q,-q)$ and by using the integral (3.3). □

4.2. **Proof of Theorem 1.1.** With the bounded identities established we may take the $m \to \infty$ limit of both identities to obtain their unbounded counterparts. For the Koornwinder side we use (2.3) with $(\lambda,q,t) = (0,q,q)$ and $(t_0,t_1,t_2,t_3) = (q^{1/2},-q^{1/2},q^{1/2},-q^{1/2})$ or $(t_0,t_1,t_2,t_3) = (1,-1,q,-q)$. In the case of (4.1) this yields

$$\begin{aligned}
&\lim_{m\to\infty} (x_1\dots x_n)^m K_{(m^n)}\big(x;q,q;q^{1/2},-q^{1/2},q^{1/2},-q^{1/2}\big) \\
&\quad= \prod_{i=1}^n \frac{(q^{1/2}x_i,-q^{1/2}x_i,q^{1/2}x_i,-q^{1/2}x_i;q)_\infty}{(x_i^2;q)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_ix_j} \\
&\quad= \prod_{i=1}^n \frac{(qx_i^2;q^2)_\infty}{(x_i^2;q^2)_\infty} \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_ix_j},
\end{aligned}$$

where we have used

$$(a,-a;q)_\infty = (a^2;q^2)_\infty.$$

For the limit of the summand we use it in conjugate form (4.3) so that

$$\lim_{m\to\infty} q^{\varsigma(\lambda)} \frac{C_{\lambda'}^{\mathrm{e}}(q^{2m};q) H_{\lambda}^{\mathrm{o}}(q)}{C_{\lambda'}^{\mathrm{o}}(q^{2m};q) H_{\lambda}^{\mathrm{e}}(q)} = q^{\varsigma(\lambda)} \frac{H_{\lambda}^{\mathrm{o}}(q)}{H_{\lambda}^{\mathrm{e}}(q)}.$$

Thus we have proved (1.3). As before the same procedure yields (1.4).

4.3. **Proof of Corollary 1.2.** In order to obtain Corollary 1.2 we take $q \to 1$ in either (1.3) or (1.4). Let $(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$. Then we may take the limit of the product-side of (1.3) by using

$$\begin{aligned}
\lim_{q\to 1} \frac{(qx_i^2;q^2)_\infty}{(x_i^2;q^2)_\infty} &= \lim_{q\to 1} \sum_{n=0}^\infty \frac{(q;q^2)_n}{(q^2;q^2)_n} x_i^{2n} \\
&= \sum_{n=0}^\infty \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n} x_i^{2n} \\
&= \frac{1}{(1-x_i^2)^{1/2}},
\end{aligned}$$

where in the first line we have applied the q-binomial theorem [8, Equation (1.3.2)]:

$$\sum_{n=0}^\infty \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

The $q \to 1$ limit of the product-side of (1.4) gives the same result. The limit of either sum follows from the characterisation of partitions with empty 2-core in Lemma 2.1, namely that $|\mathcal{H}_\lambda^{\mathrm{e}}| = |\mathcal{H}_\lambda^{\mathrm{o}}|$.

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SKEW SYMPLECTIC AND ORTHOGONAL CHARACTERS THROUGH LATTICE PATHS

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Abstract. The skew Schur functions admit many determinantal expressions. Chief among them are the (dual) Jacobi–Trudi formula and the Lascoux–Pragacz formula, the latter being a skew analogue of the Giambelli identity. Comparatively, the skew characters of the symplectic and orthogonal groups, also known as the skew symplectic and orthogonal Schur functions, have received less attention in this direction. We establish analogues of the dual Jacobi–Trudi and Lascoux–Pragacz formulae for these characters. Our approach is entirely combinatorial, being based on lattice path descriptions of the tableaux models of Koike and Terada. Ordinary Jacobi–Trudi formulae are then derived in an algebraic manner from their duals.

1. Introduction

The classical groups, a term coined by Weyl, are the general linear groups over the real numbers, complex numbers and quaternions and certain subgroups thereof. We are concerned with the complex general linear, symplectic and orthogonal groups, which we write as $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{O}(n,\mathbb{C})$ respectively. Each of these groups carry families of irreducible representations indexed by partitions. In the case of $\mathrm{GL}(n,\mathbb{C})$, these representations are precisely the irreducible polynomial representations whose characters are the Schur polynomials, which are symmetric polynomials in n variables. For the symplectic and orthogonal groups, the irreducible characters in question are symmetric Laurent polynomials in the variables $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$ (also known as BC_n-symmetric polynomials). These characters sometimes go by the name of symplectic and orthogonal Schur polynomials, but we will refer to them simply as the symplectic and orthogonal characters.

The Schur polynomials have several different determinantal expressions. Among them are the Jacobi–Trudi formula and its dual, the Nägelsbach–Kostka identity, which express the Schur polynomial as (isobaric) determinants in the complete homogeneous or elementary symmetric functions respectively. There is also the Giambelli formula, which expresses the Schur polynomial as a determinant of Schur polynomials indexed by hook-shaped Young diagrams. These two types of determinantal expressions have analogues for skew Schur polynomials, the skew version of the Giambelli formula being due to Lascoux and Pragacz [17, 18]. Analogously, the symplectic and orthogonal characters were given Jacobi–Trudi-type expressions by Weyl [24, Theorems 7.8.E & 7.9.A], there expressed in terms of complete homogeneous or elementary symmetric functions with variables $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$. They also have Giambelli formulae, being first proved by Abramsky, Jahn and King [1], the structure of which is identical to the Schur case.

From a combinatorial point of view, the Schur polynomials may be defined as a weighted sum over semistandard Young tableaux or, equivalently, Gelfand–Tsetlin patterns. This extends easily to the skew case. By interpreting these tableaux as families of nonintersecting lattice paths, Gessel and Viennot [8] provided a beautiful proof of the skew Jacobi–Trudi formulae using what is now known as the Lindström–Gessel–Viennot lemma. Stembridge then applied this approach to the Giambelli and Lascoux–Pragacz formulae [22, §9].

In [5], Fulmek and Krattenthaler set out to provide similar lattice path proofs of the symplectic and orthogonal Jacobi–Trudi formulae and their duals as well as the Giambelli

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formulae in the straight case. For the symplectic identities they use the tableaux of King [13]. In the orthogonal case there are several different tableaux models, and of these Fulmek and Krattenthaler exploit the tableaux of Proctor, Sundaram as well as of King and Welsh [14, 21, 23]. The main tools in their proofs are the Lindström–Gessel–Viennot lemma and a modified reflection principle. They succeeded in proving the dual Jacobi–Trudi formulae for the symplectic and orthogonal characters in this way, and pass to the ordinary formulae by the dual paths of Gessel and Viennot. For the Giambelli identities, they prove the symplectic and odd orthogonal cases. However, they were unable to obtain a lattice path proof of the even orthogonal Giambelli formula. (This was due to the even orthogonal Sundaram-type tableaux they introduce not having an appropriate weight function; see the discussions in [3, §3.6] and [5, §8].)

Compared to the skew Schur polynomials, skew analogues of the symplectic and orthogonal characters have received little attention. Indeed, only very recently did Jing, Li and Wang obtain Jacobi–Trudi formulae for these characters [12, Propositions 3.2–3.3], and there only in terms of the complete symmetric functions. The goal of the present paper is to provide dual Jacobi–Trudi-type formulae for the skew symplectic and orthogonal characters. We accomplish this in a purely combinatorial way, using an approach that is somewhat in the vein of the one used by Fulmek and Krattenthaler with an extension to the skew setting. In addition, we also produce Lascoux–Pragacz-type skew analogues of the Giambelli formulae for these characters. What facilitates these more general formulae are the tableaux models for skew symplectic and orthogonal characters due to Koike and Terada [16]. These tableaux have already proved combinatorially useful in proving factorisation theorems for skew symplectic and orthogonal characters in work of Ayyer and the second named author [2]. Remarkably, our proofs are simpler than those of Fulmek and Krattenthaler, which further emphasises that the tableaux of Koike and Terada are the better tool in the combinatorial setting.

We begin in the next section by providing definitions and statements of our results. In Section 3 we introduce the tableaux of Koike and Terada as well as the corresponding families of nonintersecting lattice paths. In the following Section 4 we give proofs of the dual Jacobi–Trudi formulae. In order to relate these to the results of Jing, Li and Wang, we use the standard algebraic approach for proving the equivalence of the ordinary Jacobi–Trudi formula and its dual in Section 5. Following this, Section 6 contains the proofs of the Lascoux–Pragacz-type skew analogues of the Giambelli formulae.

2. Definitions and main results

A *partition* is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$. We call the λ_i its *parts*, $l(\mu) \coloneqq k$ its *length* and $|\lambda| \coloneqq \sum_{i=1}^{k} \lambda_i$ its *size*. The *Young diagram* of a partition λ is a collection of left-justified boxes (*cells*) with λ_i cells in the i-th row from the top. For the rest of the paper, we do not distinguish between a partition and its associated Young diagram. The *conjugate* $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ of λ is the partition where λ'_i is the number of boxes in the i-th column of the Young diagram of λ, counted from the left. For two partitions λ, μ, we say that μ is *contained* in λ, denoted by $\mu \subseteq \lambda$, if the Young diagram of μ can be obtained from the Young diagram of λ by removing boxes. In this case, we denote by λ/μ the *skew shaped Young diagram* (or *skew shape* for short) obtained by removing all boxes of the Young diagram of μ from the one of λ. The size of λ/μ is the number of boxes in λ/μ and denoted by $|\lambda/\mu|$. We will sometimes write (n^m) for the rectangular partition with m parts equal to n.

We are interested in irreducible characters of the classical Lie groups $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$ and $\mathrm{O}(n, \mathbb{C})$ indexed by partitions. The definitions of these objects may be found, for instance, in [6, §24] or [21, Appendix B], and we only cover the essentials needed to state our results. The characters we are interested in are all symmetric or BC_n-symmetric polynomials with variables $\mathbf{x} \coloneqq (x_1, \dots, x_n)$, the rings of which we denote by Λ_n and Λ_n^{BC} respectively. For $\mathrm{GL}(n, \mathbb{C})$, the characters of the irreducible polynomial representations are the *Schur* *polynomials* $s_\lambda(\mathbf{x})$ where λ runs over all partitions of length at most n. They are easily computed, for $l(\lambda) \leqslant n$, by

$$s_\lambda(\mathbf{x}) := \frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j})}, \tag{1}$$

setting $\lambda_j = 0$ for $j > l(\lambda)$. In fact, this convention is used throughout the whole paper. The formula is a special case of the Weyl character formula. If $l(\lambda) > n$ then $s_\lambda(\mathbf{x}) := 0$. From the definition it is clear that these are in fact symmetric polynomials of homogeneous degree $|\lambda|$. When λ is a single row or column of r boxes then the Schur polynomials reduce to the *complete homogeneous symmetric polynomials* h_r and the *elementary symmetric polynomials* e_r in $\mathbf{x}$ respectively.

For the symplectic group $\mathrm{Sp}(2n, \mathbb{C})$ we have irreducible characters given by the Weyl formula

$$\mathrm{sp}_\lambda(\mathbf{x}) := \frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j+1} - x_i^{-(\lambda_j+n-j+1)})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j+1} - x_i^{-(n-j+1)})},$$

where again $l(\lambda) \leqslant n$. The orthogonal case is little more delicate. For a statement P we write $[P]$ for the *Iverson bracket*: $[P] = 1$ if P is true and $[P] = 0$ otherwise. Then the even orthogonal group $\mathrm{O}(2n, \mathbb{C})$ has irreducible characters

$$\mathrm{o}_\lambda(\mathbf{x}) := 2^{[\lambda_n \neq 0]} \frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j} + x_i^{-(n-j)})}.$$

If $\lambda_n = 0$ then this is also the character of the irreducible representation of the special orthogonal group $\mathrm{SO}(2n, \mathbb{C})$ corresponding to λ. However, if $\lambda_n \neq 0$ then the above splits into a sum of two irreducible characters of $\mathrm{SO}(2n, \mathbb{C})$, one indexed by λ and the other by $(\lambda_1, \dots, \lambda_{n-1}, -\lambda_n)$, which is of course not a partition. In the odd orthogonal case $\mathrm{O}(2n+1, \mathbb{C})$ there is no such distinction, and therefore we will label the irreducible characters of $\mathrm{O}(2n+1, \mathbb{C})$ (or $\mathrm{SO}(2n+1, \mathbb{C})$) by

$$\mathrm{so}_\lambda(\mathbf{x}) := \frac{\det_{1\leqslant i,j\leqslant n}(x_i^{\lambda_j+n-j+1/2} - x_i^{-(\lambda_j+n-j+1/2)})}{\det_{1\leqslant i,j\leqslant n}(x_i^{n-j+1/2} - x_i^{-(n-j+1/2)})}.$$

We will refer to these two sets of orthogonal characters as even orthogonal characters and odd orthogonal characters respectively.

Each of the above four families of characters have skew variants. In Section 3 we will explicitly define each of these families in terms of skew tableaux, however for now let us explain where they come from. We say two partitions *interlace*, written $\mu \preccurlyeq \lambda$, if $\mu \subseteq \lambda$ and

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \cdots.$$

One of the fundamental properties of the Schur polynomials is the *branching rule* [20, p. 72]

$$s_\lambda(x_1, \dots, x_n) = \sum_{\mu \preccurlyeq \lambda} x_n^{|\lambda/\mu|} s_\mu(x_1, \dots, x_{n-1}).$$

Since λ and μ interlace, the skew shape λ/μ has no two boxes in the same column. Iterating the branching rule n times naturally leads to the notion of semistandard Young tableaux. Alternatively, one could iterate only k times for $1 \leqslant k \leqslant n-1$. The coefficient of $s_\mu(x_1, \dots, x_{n-k})$ in this expansion is the skew Schur polynomial $s_{\lambda/\mu}(x_{n-k+1}, \dots, x_n)$. In other words, we have the more general branching rule

$$s_\lambda(x_1, \dots, x_n) = \sum_{\mu \subseteq \lambda} s_\mu(x_1, \dots, x_{n-k}) s_{\lambda/\mu}(x_{n-k+1}, \dots, x_n).$$

In representation-theoretic terms, the branching rule describes the restriction of the irreducible representation indexed by λ to the subgroup $\mathrm{GL}(n-1, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$, or more generally to $\mathrm{GL}(n-k, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{C})$.

Koike and Terada [16] carried out this same procedure for the symplectic and orthogonal groups. They use the branching rules of Zhelobenko [25] to define skew analogues of the symplectic and orthogonal characters. In their most general form these characters depend on a skew shape λ/μ and integers n, m such that $l(\mu) \leqslant m$ and $l(\lambda) \leqslant n+m$. Following our notation from above, we denote these by $\mathrm{sp}^m_{\lambda/\mu}, \mathrm{so}^m_{\lambda/\mu}$ and $\mathrm{o}^m_{\lambda/\mu}$. These objects are, like their non-skew variants, symmetric Laurent polynomials in $x_1, x_1^{-1} \ldots, x_n, x_n^{-1}$. While we give a combinatorial definition of these characters below, we should mention that Jing, Li and Wang have given alternative definitions in terms of vertex operators [12].

The main goal of this paper is to prove, combinatorially, two types of determinantal formulae for $\mathrm{sp}^m_{\lambda/\mu}$, $\mathrm{so}^m_{\lambda/\mu}$ and $\mathrm{o}^m_{\lambda/\mu}$: (i) dual Jacobi–Trudi-type formulae and (ii) Giambelli–Lascoux–Pragacz-type formulae. The Jacobi–Trudi-type formulae are then derived from their duals in an algebraic manner.

For $\mu \subseteq \lambda$, $l(\lambda) \leqslant n$ and $\lambda_1 \leqslant N$, and the Jacobi–Trudi identity and its dual are

$$s_{\lambda/\mu}(\mathbf{x}) = \det_{1\leqslant i,j\leqslant n}(h_{\lambda_i-\mu_j-i+j}(\mathbf{x})) = \det_{1\leqslant i,j\leqslant N}(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{x})), \tag{2}$$

where we remind the reader that $\mathbf{x} = (x_1, \ldots, x_n)$ for a non-negative integer n that is fixed throughout the paper. Here we have stated this as an identity for the Schur polynomials, but it also holds in the ring of symmetric functions on a countable set of variables, in which case the $s_{\lambda/\mu}$ are the *skew Schur functions*.

We set $\mathbf{x}^{\pm} := (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$ and are now ready to state our theorems.

Theorem 2.1. *Let m, n, N be non-negative integers and λ, μ partitions such that $\mu \subseteq \lambda$, $l(\mu) \leqslant m$, $l(\lambda) \leqslant n+m$ and $\lambda_1 \leqslant N$. Then*

$$\mathrm{sp}^m_{\lambda/\mu}(\mathbf{x}) = \det_{1\leqslant i,j\leqslant N}\left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{x}^{\pm}) - e_{\lambda'_i+\mu'_j-i-j-2m}(\mathbf{x}^{\pm})\right), \tag{3a}$$

$$\mathrm{so}^m_{\lambda/\mu}(\mathbf{x}) = \det_{1\leqslant i,j\leqslant N}\left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{x}^{\pm}) + e_{\lambda'_i+\mu'_j-i-j-2m+1}(\mathbf{x}^{\pm})\right), \tag{3b}$$

$$\mathrm{o}^m_{\lambda/\mu}(\mathbf{x}) = \frac{1}{2^{[m=l(\mu)]}}\det_{1\leqslant i,j\leqslant N}\left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{x}^{\pm}) + e_{\lambda'_i+\mu'_j-i-j-2m+2}(\mathbf{x}^{\pm})\right). \tag{3c}$$

A main contribution of our work is to provide, as we believe, enlightening combinatorial explanations of these formulae. By using an algebraic approach, we can dualise these three identities and obtain the following ordinary Jacobi–Trudi-type formulae for these characters.

Theorem 2.2. *Let m, n, N be non-negative integers and λ, μ partitions such that $\mu \subseteq \lambda$, $l(\mu) \leqslant m$, $l(\lambda) \leqslant n+m$ and $l(\lambda) \leqslant N$. Then*

$$\mathrm{sp}^m_{\lambda/\mu}(\mathbf{x}) = \det_{1\leqslant i,j\leqslant N}\left(h_{\lambda_i-\mu_j-i+j}(\mathbf{x}^{\pm}) + [j>m+1]h_{\lambda_i-i-j+2m+2}(\mathbf{x}^{\pm})\right), \tag{4a}$$

$$\mathrm{so}^m_{\lambda/\mu}(\mathbf{x}) = \det_{1\leqslant i,j\leqslant N}\left(h_{\lambda_i-\mu_j-i+j}(\mathbf{x}^{\pm}) + [j>m]h_{\lambda_i-i-j+2m+1}(\mathbf{x}^{\pm})\right), \tag{4b}$$

$$\mathrm{o}^m_{\lambda/\mu}(\mathbf{x}) = \det_{1\leqslant i,j\leqslant N}\left(h_{\lambda_i-\mu_j-i+j}(\mathbf{x}^{\pm}) - [j>m]h_{\lambda_i-i-j+2m}(\mathbf{x}^{\pm})\right). \tag{4c}$$

For μ empty and $m=0$, these formulae are due to Weyl [24, Theorems 7.8.E & 7.9.A]. The identities (4a) and (4c) as stated were recently obtained by Jing, Li and Wang [12, Propositions 3.2–3.3].

Our combinatorial approach also admits the derivation of Giambelli-type formulae, and, in order to formulate them, we need the *Frobenius notation* of a partition. For a partition λ, let (p,p) be the diagonal cell with maximal p which is still contained in the Young diagram of λ (p is the size of the *Durfee square*). For $1 \leqslant i \leqslant p$, let α_i be the number of cells right of (i,i) in the same row and β_i be the number of cells below (i,i) in the same column. We then write $\lambda = (\alpha_1, \ldots, \alpha_p|\beta_1, \ldots, \beta_p)$. Using Frobenius notation for λ and μ, that is, $\lambda = (\alpha_1, \ldots, \alpha_p|\beta_1, \ldots, \beta_p)$ and $\mu = (\gamma_1, \ldots, \gamma_q|\delta_1, \ldots, \delta_q)$, the *Lascoux–Pragacz formula* reads as

$$s_{\lambda/\mu}(\mathbf{x}) = (-1)^q \det\begin{pmatrix} \left(s_{(\alpha_i|\beta_j)}(\mathbf{x})\right)_{1\leqslant i,j\leqslant p} & \left(h_{\alpha_i-\gamma_j}(\mathbf{x})\right)_{1\leqslant i\leqslant p,\ 1\leqslant j\leqslant q} \\ \left(e_{\beta_j-\delta_i}(\mathbf{x})\right)_{1\leqslant i\leqslant q,\ 1\leqslant j\leqslant p} & 0 \end{pmatrix}. \tag{5}$$

This was shown by Lascoux and Pragacz in [17, 18]. The μ empty case is much older, being due to Giambelli [9], and is therefore known as the *Giambelli identity*. We provide the following analogues.

Theorem 2.3. *Let $\lambda = (\alpha_1, \dots, \alpha_p | \beta_1, \dots, \beta_p)$ and $\mu = (\gamma_1, \dots, \gamma_q | \delta_1, \dots, \delta_q)$ be two partitions such that $\mu \subseteq \lambda$, $l(\mu) \leqslant m$ and $l(\lambda) \leqslant n + m$. Then*

$$\mathrm{sp}^m_{\lambda/\mu}(\mathbf{x}) = (-1)^q \det \begin{pmatrix} \big(\mathrm{sp}^m_{(\alpha_i|\beta_j)}(\mathbf{x})\big)_{1\leqslant i,j\leqslant p} & \big(\mathrm{sp}^m_{(\alpha_i)/(\gamma_j)}(\mathbf{x})\big)_{\substack{1\leqslant i\leqslant p,\\ 1\leqslant j\leqslant q}} \\ \big(\mathrm{sp}^m_{(1^{\beta_j+1})/(1^{\delta_i+1})}(\mathbf{x})\big)_{\substack{1\leqslant i\leqslant q,\\ 1\leqslant j\leqslant p}} & 0 \end{pmatrix}, \tag{6a}$$

$$\mathrm{so}^m_{\lambda/\mu}(\mathbf{x}) = (-1)^q \det \begin{pmatrix} \big(\mathrm{so}^m_{(\alpha_i|\beta_j)}(\mathbf{x})\big)_{1\leqslant i,j\leqslant p} & \big(\mathrm{so}^m_{(\alpha_i)/(\gamma_j)}(\mathbf{x})\big)_{\substack{1\leqslant i\leqslant p,\\ 1\leqslant j\leqslant q}} \\ \big(\mathrm{so}^m_{(1^{\beta_j+1})/(1^{\delta_i+1})}(\mathbf{x})\big)_{\substack{1\leqslant i\leqslant q,\\ 1\leqslant j\leqslant p}} & 0 \end{pmatrix}, \tag{6b}$$

$$\mathrm{o}^m_{\lambda/\mu}(\mathbf{x}) = (-1)^q \det \begin{pmatrix} \big(\mathrm{o}^m_{(\alpha_i|\beta_j)}(\mathbf{x})\big)_{1\leqslant i,j\leqslant p} & \big(\mathrm{o}^m_{(\alpha_i)/(\gamma_j)}(\mathbf{x})\big)_{\substack{1\leqslant i\leqslant p,\\ 1\leqslant j\leqslant q}} \\ \big(\mathrm{o}^m_{(1^{\beta_j+1})/(1^{\delta_i+1})}(\mathbf{x})\big)_{\substack{1\leqslant i\leqslant q,\\ 1\leqslant j\leqslant p}} & 0 \end{pmatrix}. \tag{6c}$$

As mentioned in the introduction, for μ empty these formulae reduce to those of Abramsky, Jahn and King [1]. Like their formulae, ours have the exact same structure as in the Schur case since $s_{(1^{\beta_j+1})/(1^{\delta_i+1})}(\mathbf{x}) = e_{\beta_j-\delta_i}(\mathbf{x})$ and $s_{(\alpha_i)/(\gamma_j)}(\mathbf{x}) = h_{\alpha_i-\gamma_j}(\mathbf{x})$.

Weyl's Jacobi–Trudi formulae for the symplectic and orthogonal characters are used by Koike and Terada to define the *universal characters* for these groups [15]. This is achieved by "forgetting" the variables $\mathbf{x}^{\pm}$ in either (3) and (4) and then treating the determinants as polynomials in the e_r or h_r respectively, and thus as elements of the ring of symmetric functions at an arbitrary alphabet. By specialising the arbitrary alphabet to $\mathbf{x}^{\pm}$ the actual characters are recovered. Many identities between the general linear characters (Schur functions) and the orthogonal and symplectic characters are derived by Koike and Terada using only the universal characters. It would be interesting to investigate whether using (3) and (4) to define skew analogues of the universal characters leads to similar "universal" proofs of identities such as, for example, branching rules.

Note that Hamel [10] has given determinantal formulae for skew analogues of the symplectic and odd orthogonal characters using outside decompositions as introduced in [11]. However, the skew tableaux she defines are different to those of Koike and Terada, and so are the associated skew characters.

3. Tableaux and lattice paths

In this section, we introduce the underlying combinatorial models for $s_{\lambda/\mu}, \mathrm{sp}^m_{\lambda/\mu}, \mathrm{so}^m_{\lambda/\mu}$ and $\mathrm{o}^m_{\lambda/\mu}$ in terms of tableaux. There are several possibilities for the models underlying $\mathrm{so}^m_{\lambda/\mu}$ and $\mathrm{o}^m_{\lambda/\mu}$, and we choose those defined by Koike and Terada in [16]. In the non-skew case, all of the various combinatorial models and their equivalences may be found in [21] (also see [4]). We then also introduce the corresponding families of non-intersecting lattice paths.

3.1. Semistandard Young tableaux.

Let $\mu \subseteq \lambda$ be two partitions. A *semistandard Young tableau* of shape λ/μ is a filling of the cells of the Young diagram λ/μ with positive integers such that the entries increase weakly along rows and strictly down columns; see Figure 1 for an example. We denote by $\mathrm{SSYT}^n_{\lambda/\mu}$ the set of semistandard Young tableaux of shape λ/μ and maximal entry n. The weight $\mathbf{x}^T$ of a semistandard Young tableau T is defined as the monomial

$$\mathbf{x}^T = x_1^{\#\text{ of 1's in } T} \cdots x_n^{\#\text{ of } n\text{'s in } T}.$$

The *skew Schur polynomial* $s_{\lambda/\mu}(\mathbf{x})$ corresponding to the shape λ/μ is defined as the multivariate generating function of semistandard Young tableaux of shape λ/μ, i.e.,

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}^n_{\lambda/\mu}} \mathbf{x}^T.$$

Our path models depend on arbitrary integers $N \geqslant \lambda_1$ and $n \geqslant l(\lambda)$. A *family of Schur paths* associated with the shape λ/μ is a family of N non-intersecting lattice paths with starting points $S_i = (\mu'_i - i + 1, 2l(\mu) - \mu'_i + i - 1)$, end points $E_j = (\lambda'_j - j + 1, n - \lambda'_j + j - 1 + 2l(\mu))$ for $1 \leqslant i, j \leqslant N$, where also here we set $\lambda'_j = 0$ and $\mu'_i = 0$ for $j > l(\lambda')$ and $i > l(\mu')$, and with step set $\{(1,0),(0,1)\}$. The weight of a family of paths is the product of the weights of its steps, where the i-th step of a path has weight x_i if it is horizontal and 1 otherwise. Phrased differently, the weight of a horizontal step starting at (a,b) has weight $x_{a+b-2l(\mu)+1}$ (this is called the *e-labelling*).

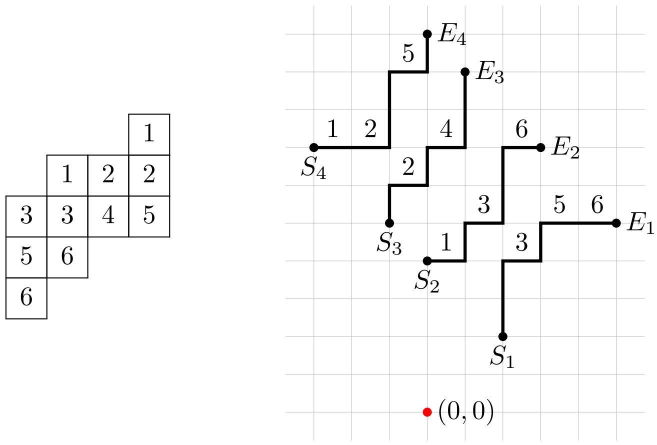


FIGURE 1. A semistandard Young tableau of shape $(4,4,4,2,1)/(3,1)$ (left) and the corresponding family of Schur paths (right) for $N = 4$ and $n = 6$.

The bijection between semistandard Young tableaux T of shape λ/μ and families of Schur paths associated with the shape λ/μ is as follows. The path starting at S_i corresponds to the i-th column of T and the j-th step in the path is a horizontal step if and only if j appears as a filling in the i-th column. See Figure 1 for an example.

3.2. **Skew (n,m)-symplectic semistandard Young tableaux.** Let $\mu \subseteq \lambda$ be two partitions and n, m integers with $l(\mu) \leqslant m$ and $l(\lambda) \leqslant n+m$. A *skew (n,m)-symplectic semistandard Young tableau* (or *skew (n,m)-symplectic tableau*) of skew shape λ/μ is a semistandard Young tableau of skew shape λ/μ which is filled by

$$\overline{1} < 1 < \overline{2} < 2 < \cdots < \overline{n} < n,$$

and satisfies the

- *m-symplectic condition*: the entries in row $m+i$ are at least $\overline{i}$.

We denote by $\mathrm{SPT}^{(n,m)}_{\lambda/\mu}$ the set of skew (n,m)-symplectic tableaux of shape λ/μ. For a skew (n,m)-symplectic tableau T, the weight $\mathbf{x}^T$ is defined as

$$\mathbf{x}^T = \prod_{i=1}^{n} x_i^{(\#\text{ of } i\text{'s in } T) - (\#\text{ of } \overline{i}\text{'s in } T)}.$$

The *skew m-symplectic character* $\mathrm{sp}^m_{\lambda/\mu}(\mathbf{x})$ is defined as the multivariate generating function of skew (n,m)-symplectic tableaux of shape λ/μ:

$$\mathrm{sp}^m_{\lambda/\mu}(\mathbf{x}) = \sum_{T\in \mathrm{SPT}^{(n,m)}_{\lambda/\mu}} \mathbf{x}^T.$$

For our path model, we again fix an integer $N \geqslant \lambda_1$. A *family of (n,m)-symplectic paths* associated with the shape λ/μ a family of N non-intersecting lattice paths with starting points $S_i = (\mu'_i - i + 1, 2m - \mu'_i + i - 1)$ end points $E_j = (\lambda'_j - j + 1, 2n + 2m - \lambda'_j + j - 1)$ for $1 \leqslant i, j \leqslant N$. The step set of these paths is $\{(1,0),(0,1)\}$, and additionally each path must stay weakly above the line $y = x - 1$. The weight of a vertical step is 1 and a horizontal step has weight x_i if it is the $(2i)$-th step of a path and weight x_i^{-1} if it is the $(2i-1)$-st step of a path. Equivalently, the weight of a horizontal step starting at (a,b) is $x^{-1}_{(a+b)/2-m+1}$ if $a+b$ is even and $x_{(a+b+1)/2-m}$ if $a+b$ is odd.

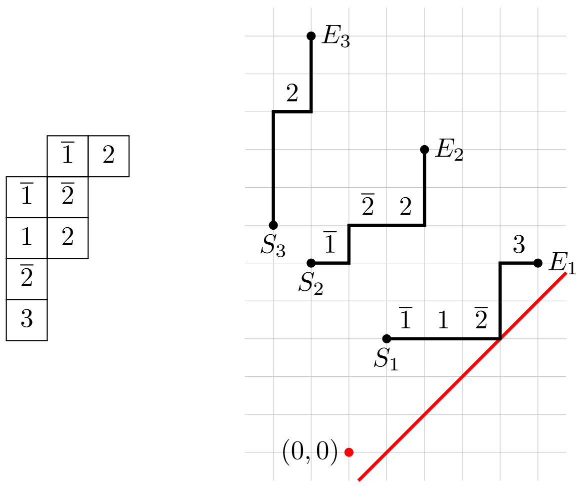


FIGURE 2. A skew $(3,2)$-symplectic tableau of shape $(3,2,2,1,1)/(1)$ (left) and its associated family of $(3,2)$-symplectic paths (right).

There is a bijection between skew (n,m)-symplectic tableaux and families of (n,m)-symplectic paths that is very similar to the one in the Schur case. Namely, given a skew (n,m)-symplectic tableau T, the path starting at S_i of the corresponding family of symplectic paths is obtained by associating the j-th step of the path with the j-th element in the sequence $\overline{1}, 1, \overline{2}, 2, \ldots$ and letting a step being horizontal if and only if the corresponding element in the sequence appears in the i-th column. An example is given in Figure 2.

3.3. **Skew (n,m)-odd orthogonal semistandard Young tableaux.** Let λ, μ be partitions such that $\mu \subseteq \lambda$, $l(\mu) \leqslant m$ and $l(\lambda) \leqslant n+m$. A *skew (n,m)-odd orthogonal semistandard Young tableau* (or *skew (n,m)-odd orthogonal tableau*) associated with the shape λ/μ is a skew semistandard Young tableau of shape λ/μ which is filled by

$$\widehat{1} < \overline{1} < 1 < \cdots < \widehat{n} < \overline{n} < n$$

and satisfies

- the *modified m-symplectic condition*: the entries in row $m+i$ are at least $\widehat{i}$, and
- the *m-odd orthogonal condition*: the symbol $\widehat{i}$ can only appear in the first column of the $(m+i)$-th row.

Note that, unlike in [16], we use $\widehat{i}$ instead of $\natural_i$. Denote by $\mathrm{SOT}^{(n,m)}_{\lambda/\mu}$ the set of skew (n,m)-odd orthogonal tableaux of shape λ/μ. We define the weight $\mathbf{x}^T$ of a tableau $T \in \mathrm{SOT}^m_{\lambda/\mu}$

as

$$\mathbf{x}^T = \prod_{i=1}^{n} x_i^{(\# \text{ of } i\text{'s in } T) - (\# \text{ of } \bar{i}\text{'s in } T)}.$$

The *skew m-odd orthogonal character* $\mathrm{so}^m_{\lambda/\mu}(\mathbf{x})$ is the multivariate generating function of skew (n, m)-odd orthogonal tableaux of shape λ/μ:

$$\mathrm{so}^m_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in \mathrm{SOT}^{(n,m)}_{\lambda/\mu}} \mathbf{x}^T.$$

A *family of (n, m)-odd orthogonal paths* associated with the shape λ/μ is a family of non-intersecting (n, m)-symplectic paths for which we extend the step set by a diagonal step $(1, 1)$ which can only occur when the starting point is of the form (i, i). The weight of a family of (n, m)-odd orthogonal paths is the product of the weights of all steps, where horizontal and vertical steps have the same weight as in the symplectic case and diagonal steps have weight 1.

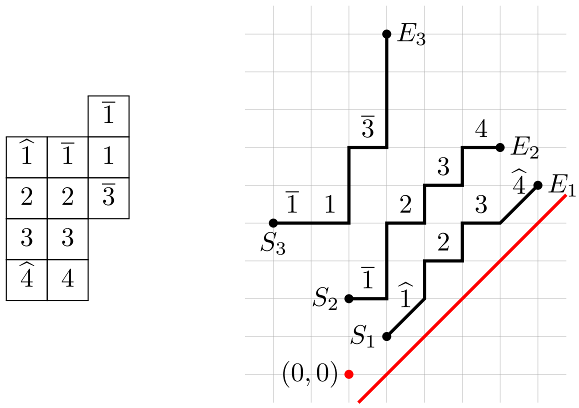


Figure 3. A skew $(4, 1)$-odd orthogonal tableau of shape $(3, 3, 3, 2, 2)/(2)$ (left) and its associated family of $(4, 1)$-odd orthogonal paths (right).

We obtain a weight preserving bijection from skew (n, m)-odd orthogonal tableaux of shape λ/μ to families of (n, m)-odd orthogonal paths associated with the shape λ/μ as follows. We follow the same procedure as in the symplectic setting by interpreting first each $\widehat{i}$ entry as $\bar{i}$. Note that each horizontal step corresponding to an $\widehat{i}$ entry ends at the line $y = x - 1$ and therefore has to be followed by a vertical step. Now replace each of these horizontal steps coming from an $\widehat{i}$ entry and its following vertical step by a diagonal step. See Figure 3 for an example.

3.4. **Skew (n, m)-even orthogonal semistandard Young tableaux.** As before, let $\mu \subseteq \lambda$ be partitions such that $l(\mu) \leqslant m$ and $l(\lambda) \leqslant n + m$. A *skew (n, m)-even orthogonal semistandard Young tableau* (or *skew (n, m)-even orthogonal tableau*) of shape λ/μ is a semistandard Young tableau of shape λ/μ which is filled by the symbols

$$\check{1} < \widehat{1} < \bar{1} < 1 < \cdots < \check{n} < \widehat{n} < \bar{n} < n$$

and satisfies

- the *modified m-symplectic condition*: the entries in row $m + i$ are at least $\widehat{i}$,
- the *modified m-odd orthogonal condition*: the symbol $\widehat{i}$ can only appear in the first column of the $(m + i)$-th row and it appears if and only if the entry above is $\check{i}$, and
- the *m-even orthogonal condition*: if $\bar{i}$ appears in the first column of the $(m + i)$-th row and i also appears in the same row, then there is an $\bar{i}$ immediately above this i entry.

Note that, unlike in [16], we use the symbols $\check{\imath}$ and $\widehat{\imath}$ instead of $\sharp_i$ $\flat_i$; this corresponds to interchanging the roles of $\check{\imath}$ and $\widehat{\imath}$ if compared to [2].

We denote the set of skew (n, m)-even orthogonal tableaux of shape λ/μ by $\mathrm{OT}^{(n,m)}_{\lambda/\mu}$. The weight $\mathbf{x}^T$ of a skew (n, m)-even orthogonal tableau T is defined as

$$\mathbf{x}^T = \prod_{i=1}^{n} x_i^{(\#\text{ of } i \text{ entries in } T)-(\#\text{ of } \bar{\imath} \text{ entries in } T)}.$$

The *skew m-even orthogonal character* $\mathrm{o}^m_{\lambda/\mu}(\mathbf{x})$ is the multivariate generating function of skew (n, m)-even orthogonal tableaux of shape λ/μ, so

$$\mathrm{o}^m_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in \mathrm{OT}^{(n,m)}_{\lambda/\mu}} \mathbf{x}^T.$$

We say that a family of lattice paths is *strongly non-intersecting* if there is no intersection between any pair of paths when considering them as subsets of $\mathbb{R}^2$. (For our application of the Lindström–Gessel–Viennot Lemma 4.2, we will also need the notion of *weakly non-intersecting* in Section 4.2, where we only forbid intersections at lattice points that are endpoints of steps in both paths.)

A *family of (n, m)-even orthogonal paths* associated with the shape λ/μ is a family of strongly non-intersecting symplectic paths for which we extend the step set by a horizontal step $(2, 0)$, called an *o-horizontal step*, which can only occur starting at a point of the form $(i-2, i)$, and where the family of paths does not have any trapped position as defined below. We draw o-horizontal steps as arcs to avoid confusion.

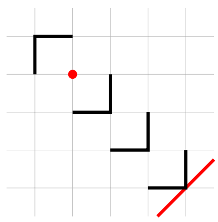


FIGURE 4. The local configuration around a trapped position which is marked as a red dot.

For positive integers i, d, we call the position $(m + i - d, m + i + d - 1)$ *trapped* if the following is satisfied:

- the lattice point $(m + i - d, m + i + d - 1)$ is not contained in any path,
- for each $d' \in \{0, \ldots, d - 1\}$, there is a lattice path which passes through $(m + i - d', m + i + d' - 1)$ by a horizontal step followed by a vertical step, and
- there is a path passing through the point $(m + i - d - 1, m + i + d)$ by a vertical step followed by a horizontal step.

See Figure 4 for an example with $d = 3$.

We now describe the weight preserving bijection from (n, m)-even orthogonal tableaux of shape λ/μ to families of (n, m)-even orthogonal paths associated with the same shape that are strongly non-intersecting. First we interpret each $\check{\imath}$ as $\bar{\imath}$ and each $\widehat{\imath}$ as i and apply the map from (n, m)-symplectic tableaux to families of (n, m)-symplectic paths. Then we replace the pairs of horizontal steps associated with $\check{\imath}, \widehat{\imath}$ coming from the modified m-odd orthogonal condition by an o-horizontal step. For an example see Figure 5. In order to see that this map is a bijection it suffices to check that a tableau contradicts the m-even orthogonal condition if and only if the corresponding family of paths has a trapped position, which is done next.

Assume that T is a tableau contradicting the m-even orthogonal-condition, i.e., there exists an integer i such that the $(m + i)$-th row starts with an entry $\bar{\imath}$ and also contains

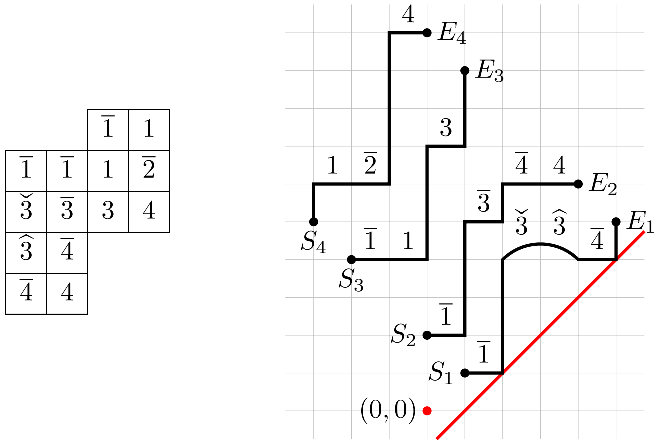


Figure 5. A skew $(4,1)$-even orthogonal tableau of shape $(4,4,4,2,2)/(2)$ (left) and its associated family of $(4,1)$-even orthogonal paths (right).

an entry i whose entry above is not $\bar{i}$. It is immediate that the top neighbour of the first i in this row can not be $\bar{i}$ since this would contradict the condition that rows are weakly increasing. Denote by d the number of $\bar{i}$ entries in the $(m+i)$-th row. When looking at the corresponding family of paths, the $(d'+1)$-st entry $\bar{i}$ corresponds to a horizontal step ending at the point $(m+i-d', m+i+d'-1)$. Since the bottommost of these paths touches the line $y = x-1$ at $(m+i, m+i-1)$ and all paths are strongly non-intersecting, each of these horizontal steps are followed by a vertical step. The first entry i in the $(m+i)$-th row corresponds to a horizontal step starting at $(m+i-d-1, m+i+d)$. Since the top neighbour of this i is not $\bar{i}$, the step before has to be a vertical step. This implies that the position $(m+i-d, m+i+d-1)$ is trapped.

4. Combinatorial proofs of the dual Jacobi–Trudi formulae

The proofs of the dual Jacobi–Trudi formulae for all of the skew characters under consideration — skew symplectic, skew odd orthogonal and skew even orthogonal — follow a similar scheme: we interpret the respective tableaux columnwise as non-intersecting lattice paths as seen in Section 3. Thus, each column corresponds to a lattice path whose generating function can be written in terms of elementary symmetric functions in the alphabet $\mathbf{x}^{\pm} = (x_1^{-1}, x_1, \dots, x_n^{-1}, x_n)$. Applying the Lindström–Gessel–Viennot Lemma 4.2 then yields a determinantal formula.

The proofs increase in complexity and a brief summary is as follows.

- For the *skew symplectic case*, we have to compute the generating function of lattice paths consisting of unit horizontal and vertical steps in the positive direction which do not cross the line $y = x-1$. This is achieved by using a modified reflection principle (Lemma 4.1) by Fulmek and Krattenthaler [5], which provides a difference of two elementary symmetric functions as the generating function, see Lemma 4.3.
- In the case of *skew odd orthogonal characters*, we need to compute the generating function of lattice paths which may additionally have diagonal steps along the line $y = x$. For each fixed number of such diagonal steps in the lattice path, we obtain a difference of elementary symmetric functions in Lemma 4.4. Adding these differences together allow telescopic cancelling, which finally yields a sum of two elementary symmetric functions in Corollary 4.5.
- In the case of *skew even orthogonal characters*, we allow horizontal double steps ending on the line $y = x$ instead of diagonal steps. By similar means (Lemma 4.6), we also obtain a sum of two elementary symmetric functions as the generating function

for these lattice paths in Corollary 4.7. However, applying the Lindström–Gessel–Viennot Lemma 4.2 in this case also results in families of lattice paths that do not correspond to a skew (n, m)-even orthogonal tableau. We provide a sign-reversing involution between those and families of lattice paths with trapped positions at the end of the section.

4.1. **Modified reflection principle.** One of the main tools in proving the dual Jacobi–Trudi formulae is a *modified reflection principle* which we present next.

A lattice point $(x, y) \in \mathbb{Z}^2$ is said to be *even* if $x + y$ is even, otherwise it is said to be *odd*. Suppose we have a lattice path starting in $P = (a, b)$ that consists of unit horizontal and vertical steps in the positive direction. In addition, we assume that P is an even point. We assign weights to the steps as follows. Vertical steps have weight 1, whereas the weights of horizontal steps are given by a modified e-labelling: the step from (i, j) to $(i + 1, j)$ has weight $x^{-1}_{(i+j-a-b)/2+1}$ if (i, j) is even and $x_{(i+j-a-b+1)/2}$ if (i, j) is odd. Now the weight of the path is the product of the weights of its steps.

Furthermore, consider a line $y = x + d$ such that d is even and P lies above that diagonal line, that is, $b > a + d$. The modified reflection principle will enable us to derive combinatorially a formula for the generating function of such lattice paths that start at P and that have no intersection with the line $y = x + d$. This is done by computing the generating function of those paths that have an intersection with $y = x + d$ and then subtracting it from the generating function of all lattice paths.

Lemma 4.1. *Let $a, b, c, d, f \in \mathbb{Z}$ such that $a + b$ and d are even, $b > a + d$ and $f > c + d$. There is a weight-preserving bijection between lattice paths with unit horizontal and vertical steps that start at the point $P = (a, b)$ and end at (c, f), and that have an intersection with the line $y = x + d$, and lattice paths with unit horizontal and vertical steps that start at the reflected point $P' = (b - d, a + d)$ of P along $y = x + d$ and also end at (c, f).*

Proof. The proof is illustrated in Figure 6. Suppose our path touches the line for the first

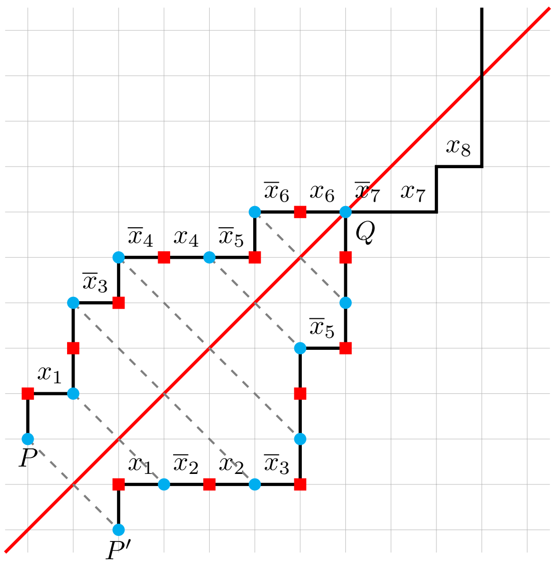


FIGURE 6. The modified reflection principle where $\overline{x}_i := x_i^{-1}$.

time at the point Q, when traversing it starting at P. The modified reflection of the path from P to Q along the line $y = x + d$ works as follows: P is reflected in the usual way, so it is mapped to $P' = (b - d, a + d)$. The same applies to all other even points and to odd points at which the path does not turn; these are odd points that lie in between either two vertical or two horizontal steps. The remaining points are odd points in which the path turns. We reflect these points in such a way that the directions of the turns are maintained. Concretely, if an odd point (x, y) comes with a left turn (a horizontal step followed by a vertical step), it is mapped to $(y - d + 1, x + d - 1)$; if it comes with a right turn (a vertical step followed by a horizontal step), it is mapped to $(y - d - 1, x + d + 1)$.

Note that the modification of the usual reflection ensures that the mapping is weight-preserving. □

4.2. **The Lindström–Gessel–Viennot Lemma.** Another important tool is the well-known Lindström–Gessel–Viennot Lemma [7, 8, 19]. We state it in a form that is convenient for us. We need the following two notions.

- We say that a family of lattice paths is *weakly non-intersecting* if no pair of paths has an intersection at a lattice point that is the endpoint of steps in both paths.
- We say that a family of lattice paths is *strongly non-intersecting* if no pair of paths intersects when considering them as subsets of $\mathbb{R}^2$.

One may interpret the first notion of non-intersecting when considering the paths as graphs, and the second notion when considering their natural embeddings in $\mathbb{R}^2$. The background for these notions is that the Gessel–Viennot sign-reversing involution can only be applied if there is a pair of paths that intersect at a lattice point that is the endpoint of steps in both paths, which is precisely the case when the family is *not* weakly non-intersecting. Also note that the two notions of non-intersecting are equivalent for families of Schur paths, of m-symplectic paths, and of m-odd orthogonal paths, however, they differ for families of m-even orthogonal paths due to the steps of length 2. Thus we may simply say non-intersecting in the first two cases.

Lemma 4.2 (The Lindström–Gessel–Viennot Lemma)**.** *We consider lattice paths on the lattice $\mathbb{Z}^2$ which is equipped with edge weights and where the allowed steps are given by a finite subset of $\mathbb{Z}^2$ such that the step set does not allow self-intersections of paths. Let $S_1, \ldots, S_n$ and $E_1, \ldots, E_n$ be lattice points and let $\mathcal{P}(S_i \to E_j)$ denote the generating function of lattice paths from S_i and E_j with respect to the step set and the edge weights. Then*

$$\det_{1 \leqslant i,j \leqslant n} (\mathcal{P}(S_i \to E_j))$$

is the signed generating function of families of n lattice paths such that the following is satisfied.

- *The $S_1, S_2, \ldots, S_n$ are the starting points and the $E_1, E_2, \ldots, E_n$ are the ending points.*
- *The paths are weakly non-intersecting.*
- *The sign is $\operatorname{sgn} \sigma$ where σ is the permutation of $\{1, 2, \ldots, n\}$ such that S_i is connected to $E_{\sigma(i)}$ for $i = 1, 2, \ldots, n$.*

4.3. **Skew symplectic characters.** For the skew symplectic character $\mathrm{sp}^m_{\lambda/\mu}$, we have to compute the generating function of families of non-intersecting lattice paths from $(\mu'_j - j + 1, 2m - \mu'_j + j - 1)$ to $(\lambda'_i - i + 1, 2n + 2m - \lambda'_i + i - 1)$ for $1 \leqslant i, j \leqslant N$ with $\lambda_1 \leqslant N$, $l(\mu) \leqslant m$ and $l(\lambda) \leqslant n + m$ that consist of unit horizontal and vertical steps in the positive direction and that do not cross the line $y = x - 1$, see Section 3.2. In order to apply Lemma 4.2, we first compute the generating function of paths between pairs of starting and ending points.

Lemma 4.3. *Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a + b$ is even and the points (a, b) and $(c, 2n+a+b-c)$ lie strictly above the line $y = x-1$. Then, with the path set-up of Lemma 4.1, the generating function of paths from (a, b) to $(c, 2n + a + b - c)$ is given by*

$$e_{c-a}(\mathbf{x}^{\pm}) - e_{c-b-2}(\mathbf{x}^{\pm}),$$

where $\mathbf{x}^{\pm} = (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$.

Proof. We use the modified reflection principle from Lemma 4.1 with $(a, b, c, d, f) \mapsto (a, b, c, -2, 2n+a+b-c)$ for this proof. The generating function of all lattice paths from (a, b) to $(c, 2n+a+b-c)$ with the given step set regardless of whether they cross the line $y = x - 1$ is $e_{c-a}(\mathbf{x}^{\pm})$.

Next, we derive the generating function of all lattice paths from (a,b) to $(c,2n+a+b-c)$ that do cross that line. Those paths touch at least once the line $y=x-2$. We take the initial part of a path from (a,b) to that first intersection point and reflect it in a weight-preserving way according to the modified reflection principle. This yields paths from $(b+2,a-2)$ to $(c,2n+a+b-c)$. The generating function of these paths is $e_{c-b-2}(\mathbf{x}^{\pm})$, which we need to subtract from $e_{c-a}(\mathbf{x}^{\pm})$ to obtain the result. $\square$

In view of the above, the generating function of lattice paths from $(\mu'_j-j+1,2m-\mu'_j+j-1)$ to $(\lambda'_i-i+1,2n+2m-\lambda'_i+i-1)$ which do not cross the line $y=x-1$ is given by

$$e_{\lambda'_i-\mu'_j-i+j}(\mathbf{x}^{\pm})-e_{\lambda'_i+\mu'_j-i-j-2m}(\mathbf{x}^{\pm}).$$

Note the requirement that $(\lambda'_i-i+1,2n+2m-\lambda'_i+i-1)$ lies above the line $y=x-1$ is equivalent to $l(\lambda)\leqslant n+m$. Applying the Lindström–Gessel–Viennot Lemma 4.2 finally yields

$$\mathrm{sp}^m_{\lambda/\mu}(\mathbf{x})=\det_{1\leqslant i,j\leqslant N}\left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{x}^{\pm})-e_{\lambda'_i+\mu'_j-i-j-2m}(\mathbf{x}^{\pm})\right),$$

and this concludes the combinatorial proof of Theorem 2.1 (3a).

4.4. **Skew odd orthogonal characters.** For the skew odd orthogonal character $\mathrm{so}^m_{\lambda/\mu}$, we have to compute as before the generating function of families of non-intersecting lattice paths from $(\mu'_j-j+1,2m-\mu'_j+j-1)$ to $(\lambda'_i-i+1,2n+2m-\lambda'_i+i-1)$ for $1\leqslant i,j\leqslant N$ with $\lambda_1\leqslant N$, $l(\mu)\leqslant m$ and $l(\lambda)\leqslant n+m$ that do not cross the line $y=x-1$; see Section 3.3. In this case, however, we have a bigger step set. We allow unit horizontal and vertical steps in the positive direction and, in addition, diagonal steps $(1,1)$ on the line $y=x$ which have weight 1. In order to compute this generating function, we first derive the generating function of individual paths with a given number k of diagonal steps and then we sum over all non-negative integers k.

Lemma 4.4. *Let k be a positive integer and let $\{(1,0),(0,1),(1,1)\}$ be the step set such that diagonal steps are only allowed along the line $y=x$. Also let $a,b,c\in\mathbb{Z}$ and $n\in\mathbb{N}$ such that $a+b$ is even and the points (a,b) and $(c,2n+a+b-c)$ lie strictly above the line $y=x-1$. Then the generating function of lattice paths from (a,b) to $(c,2n+a+b-c)$ that do not cross the line $y=x-1$ and that have exactly k diagonal steps is*

$$e_{c-b-k}(\mathbf{x}^{\pm})-e_{c-b-k-2}(\mathbf{x}^{\pm}),$$

where $\mathbf{x}^{\pm}=(x_1,x_1^{-1},\dots,x_n,x_n^{-1})$.

Proof. First, we map a path with k diagonal steps to a path which only consists of unit horizontal and vertical steps as follows. Replace every diagonal step by two vertical steps. In the process, we keep the terminal part of the path and shift the initial part accordingly. This is illustrated in Figure 7. We obtain a lattice path from $(a+k,b-k)$ to $(c,2n+a+b-c)$ that does not cross the line $y=x-2k-1$ but intersects the line $y=x-2k$.

We observe that this mapping is a weight-preserving bijection. For the inverse mapping, we consider a lattice path from $(a+k,b-k)$ to $(c,2n+a+b-c)$ not crossing the line $y=x-2k-1$ but intersects the line $y=x-2k$. Since the path goes to $(c,2n+a+b-c)$, there must be a point where the path touches the line $y=x-2k$ and continues with at least two vertical steps. We take the rightmost of such points, replace the first two vertical steps by a diagonal step and shift the initial part of the path by $(-1,1)$. We thus obtain a path from $(a+k-1,b-k+1)$ to $(c,2n+a+b-c)$ not crossing the line $y=x-2k+1$ but intersecting the line $y=x-2k+2$. As before, there is a point where the path intersects with the line $y=x-2k+2$ and continues with at least two vertical step. We iteratively repeat the step of replacing two vertical steps by a diagonal step at the rightmost occurrence and shifting the initial part of the path by $(-1,1)$ until we have got a path that starts at (a,b) and has k diagonal steps.

With this weight-preserving bijection in mind, the statement in the theorem follows by applying twice the modified reflection principle from Lemma 4.1 as follows.

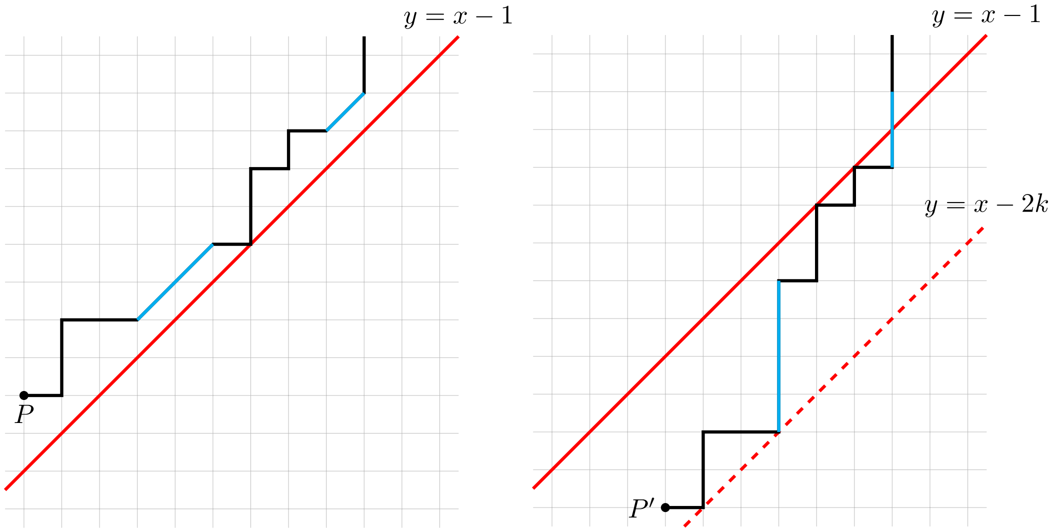


FIGURE 7. Situation in Lemma 4.4 when replacing diagonal steps by vertical steps of length 2.

To compute the generating function of paths from $(a+k, b-k)$ to $(c, 2n+a+b-c)$ that intersect the line $y = x - 2k$, we reflect $(a+k, b-k)$ along $y = x - 2k$, which yields $(b+k, a-k)$. Hence, the generating function is $e_{c-b-k}(\mathbf{x}^{\pm})$.

From the latter generating function, we have to subtract the generating function of paths that cross the line $y = x - 2k - 1$; note that such paths always have to intersect the line $y = x - 2k$ because of the end point $(c, 2n+a+b-c)$. Paths that cross the line $y = x - 2k - 1$ definitely intersect the line $y = x - 2k - 2$. Reflecting $(a+k, b-k)$ along $y = x - 2k - 2$ gives $(b+k+2, a-k-2)$, which implies that the generating function of paths crossing the line $y = x - 2k - 1$ is $e_{c-b-k-2}(\mathbf{x}^{\pm})$. This concludes the proof of the lemma. □

The generating function of paths from (a, b) to $(c, 2n+a+b-c)$ without diagonal steps is $e_{c-a}(\mathbf{x}^{\pm}) - e_{c-b-2}(\mathbf{x}^{\pm})$, which follows from Lemma 4.3. Next we sum the generating functions for paths with exactly k diagonal steps for all non-negative k. This sum turns out to be a telescoping sum which reduces to a sum of two terms after cancellation:

$$\left(e_{c-a}(\mathbf{x}^{\pm}) - e_{c-b-2}(\mathbf{x}^{\pm})\right) + \sum_{k \geqslant 1} \left(e_{c-b-k}(\mathbf{x}^{\pm}) - e_{c-b-k-2}(\mathbf{x}^{\pm})\right) = e_{c-a}(\mathbf{x}^{\pm}) + e_{c-b-1}(\mathbf{x}^{\pm}).$$

This leaves us with the following.

Corollary 4.5. *Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a+b$ is even and the points (a, b) and $(c, 2n+a+b-c)$ lie strictly above the line $y = x-1$. Then the generating function of lattice paths from (a, b) to $(c, 2n+a+b-c)$ with step set $\{(1,0), (0,1), (1,1)\}$ that do not cross the line $y = x - 1$ and where diagonal steps are only allowed on the line $y = x$ is*

$$e_{c-a}(\mathbf{x}^{\pm}) + e_{c-b-1}(\mathbf{x}^{\pm}),$$

where $\mathbf{x}^{\pm} = (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$.

By setting $a = \mu'_j - j + 1$, $b = 2m - \mu'_j + j - 1$ and $c = \lambda'_i - i + 1$ in Corollary 4.5 and applying the Lindström–Gessel–Viennot Lemma (Lemma 4.2), we finally obtain

$$\mathrm{so}^m_{\lambda/\mu}(\mathbf{x}) = \det_{1 \leqslant i,j \leqslant N} \left(e_{\lambda'_i - \mu'_j - i + j}(\mathbf{x}^{\pm}) + e_{\lambda'_i + \mu'_j - i - j - 2m + 1}(\mathbf{x}^{\pm})\right),$$

and this concludes the combinatorial proof of Theorem 2.1 (3b).

4.5. **Skew even orthogonal characters.** Recall that in the case of skew even orthogonal characters, we have to compute the generating function of strongly non-intersecting lattice paths from $(\mu'_j - j + 1, 2m - \mu'_j + j - 1)$ to $(\lambda'_i - i + 1, 2n + 2m - \lambda'_i + i - 1)$ for $1 \leqslant i, j \leqslant N$ with $\lambda_1 \leqslant N$, $l(\mu) \leqslant m$ and $l(\lambda) \leqslant n + m$ that do not cross the line $y = x - 1$ and where a certain pattern that corresponds to the m-even orthogonal condition is not allowed; see Section 3.4. The step set is $\{(1,0), (0,1), (2,0)\}$, where the horizontal steps $(2,0)$, called o-horizontal steps, are only allowed to start at points on the line $y = x + 2$ (and thus end on the line $y = x$). The o-horizontal steps are equipped with the weight 1 and the unit steps have weights according to the standard e-labelling.

We compute the generating function of such paths in Corollary 4.7 in a similar manner to the previous section. However, we also need to deal with the avoidance of the patterns that take care of the m-even orthogonal condition, that is, with trapped positions. It turns out that this combines nicely with the application of the Lindström–Gessel–Viennot Lemma 4.2. Namely, when applying the lemma to the generating functions of paths from Corollary 4.7, we may have pairs of paths that are weakly non-intersecting but not strongly non-intersecting. With our step set such intersections may only occur when o-horizontal steps intersect with two unit vertical steps. Therefore, we present a sign-reversing involution that shows that families of lattice paths with such intersections and with trapped positions indeed cancel after applying the Lindström–Gessel–Viennot Lemma 4.2.

Lemma 4.6. *Let k be a positive integer and let $\{(1,0), (0,1), (2,0)\}$ be the step set such that* o*-horizontal steps are only allowed to end on the line $y = x$. Also let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a + b$ is even and (a, b) and $(c, 2n + a + b - c)$ lie strictly above the line $y = x - 1$. Then the generating function of lattice paths from (a, b) to $(c, 2n + a + b - c)$ that do not cross the line $y = x - 1$ and that have exactly k* o*-horizontal steps is*

$$\begin{cases} e_{c-b-2k+2}(\mathbf{x}^{\pm}) - e_{c-b-2k-2}(\mathbf{x}^{\pm}) & \text{if } b - a \geqslant 2, \\ e_{c-a-2k}(\mathbf{x}^{\pm}) - e_{c-b-2k-2}(\mathbf{x}^{\pm}) & \text{otherwise}, \end{cases}$$

where $\mathbf{x}^{\pm} = (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$.

Proof. We proceed in a way similar to the proof of Lemma 4.4. First, we replace every o-horizontal step by a double vertical step whilst keeping the terminal part of the path but shifting the initial part accordingly. See Figure 8 for an illustration. This results in a path from $(a + 2k, b - 2k)$ to $(c, 2n + a + b - c)$ that intersects the line $y = x - 4k + 2$ but that does not cross the line $y = x - 4k - 1$.

This mapping is again a weight-preserving bijection. We continue by using the modified reflection principle in Lemma 4.1.

In order to compute the generating function of paths from $(a+2k, b-2k)$ to $(c, 2n+a+b-c)$ that intersect the line $y = x - 4k + 2$, we have to distinguish two cases. If $(a + 2k, b - 2k)$ already lies below $y = x - 4k + 2$, that is, if $b - a = 0$, then the generating function is simply $e_{c-a-2k}(\mathbf{x}^{\pm})$. Otherwise, we reflect $(a + 2k, b - 2k)$ along $y = x - 4k + 2$ and obtain $(b + 2k - 2, a - 2k + 2)$. Thus, the generating function of these paths is $e_{c-b-2k+2}(\mathbf{x}^{\pm})$.

Lattice paths from $(a+2k, b-2k)$ to $(c, 2n+a+b-c)$ which cross the line $y = x-4k-1$ touch also the line $y = x - 4k - 2$. Reflecting $(a + 2k, b - 2k)$ along $y = x - 4k - 2$ gives $(b + 2k + 2, a - 2k - 2)$, and thus the generating function of these paths is $e_{c-b-2k-2}(\mathbf{x}^{\pm})$. This completes the proof. □

Using Lemma 4.3, we know that the generating function of lattice paths from (a, b) to $(c, 2n + a + b - c)$ that do not go below the line $y = x - 1$ and have no o-horizontal steps is $e_{c-a}(\mathbf{x}^{\pm}) - e_{c-b-2}(\mathbf{x}^{\pm})$. By summing over all possible numbers of o-horizontal steps, we finally obtain in the case $b - a \geqslant 2$:

$$\left(e_{c-a}(\mathbf{x}^{\pm}) - e_{c-b-2}(\mathbf{x}^{\pm})\right) + \sum_{k \geqslant 1} \left(e_{c-b-2k+2}(\mathbf{x}^{\pm}) - e_{c-b-2k-2}(\mathbf{x}^{\pm})\right) = e_{c-a}(\mathbf{x}^{\pm}) + e_{c-b}(\mathbf{x}^{\pm}).$$

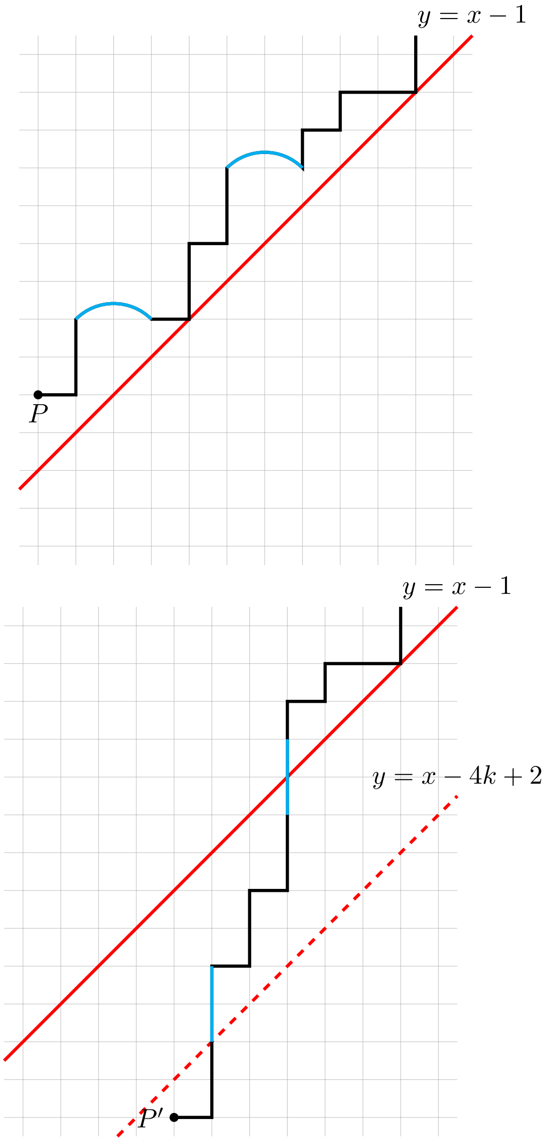


FIGURE 8. Situation in Lemma 4.6 when replacing o-horizontal steps by vertical steps of length 2.

On the other hand, if $b - a = 0$, we obtain

$$\left(e_{c-a}(\mathbf{x}^{\pm}) - e_{c-a-2}(\mathbf{x}^{\pm})\right) + \sum_{k\geqslant 1}\left(e_{c-a-2k}(\mathbf{x}^{\pm}) - e_{c-a-2k-2}(\mathbf{x}^{\pm})\right) = e_{c-a}(\mathbf{x}^{\pm}).$$

Corollary 4.7. *Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a + b$ is even and the points (a, b) and $(c, 2n+a+b-c)$ lie strictly above the line $y = x-1$. Then the generating function of lattice paths from (a, b) to $(c, 2n+a+b-c)$ with step set $\{(1,0),(0,1),(2,0)\}$ that do not cross the line $y = x - 1$ and where* o*-horizontal steps are only allowed to end on the line $y = x$ is*

$$\begin{cases} e_{c-a}(\mathbf{x}^{\pm}) & \text{if } b - a = 0, \\ e_{c-a}(\mathbf{x}^{\pm}) + e_{c-b}(\mathbf{x}^{\pm}) & \text{if } b - a \geqslant 2, \end{cases}$$

where $\mathbf{x} = (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$.

We now apply the Lindström–Gessel–Viennot Lemma 4.2 to all lattice paths from $(\mu'_j - j + 1, 2m - \mu'_j + j - 1)$ to $(\lambda'_i - i + 1, 2n + 2m - \lambda'_i + i - 1)$ for $1 \leqslant i, j \leqslant N$ with $\lambda_1 \leqslant N$, $l(\mu) \leqslant m$, $l(\lambda) \leqslant n + m$ and step set $\{(1,0),(0,1),(2,0)\}$ such that the paths weakly stay above the line $y = x - 1$ and o-horizontal steps are only allowed to end on the line $y = x$. By Corollary 4.7 this gives

$$\frac{1}{2^{[m=l(\mu)]}} \det_{1\leqslant i,j\leqslant N} \left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{x}^{\pm}) + e_{\lambda'_i+\mu'_j-i-j-2m+2}(\mathbf{x}^{\pm}) \right). \tag{7}$$

Note that the prefactor $\frac{1}{2}$ takes care of the case $m = l(\mu)$ because then the first column of the underlying matrix in the previous determinant is $(2e_{\lambda'_1-\mu'_1}(\mathbf{x}^{\pm}), 2e_{\lambda'_2-\mu'_1-1}(\mathbf{x}^{\pm}), \ldots, 2e_{\lambda'_N-\mu'_1-N+1}(\mathbf{x}^{\pm}))^{\top}$.

To complete the proof, we want to show that (7) indeed represents a formula for $o^m_{\lambda/\mu}(\mathbf{x})$. This is done by a sign-reversing involution. First, we recall that the Lindström–Gessel–Viennot Lemma 4.2 (7) yields a signed enumeration of weakly non-intersecting lattice paths where o-horizontal steps might intersect with vertical steps. Figure 9 shows an example of lattice paths with such intersections. On the other hand, there might still be trapped positions. The sign-reversing involution will transform an intersection into a trapped position, or vice versa.

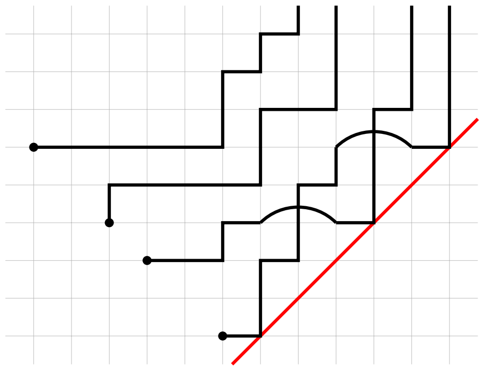


FIGURE 9. Weakly non-intersecting lattice paths that are enumerated by the Lindström–Gessel–Viennot Lemma.

The intersections between o-horizontal steps and a pair of unit vertical steps only occur along the line $y = x+1$. Consider such an intersection and assume it is at the point $(d, d+1)$. That means we have an o-horizontal step $(d-1, d+1) \to (d+1, d+1)$ in one path, and two unit vertical steps $(d, d) \to (d, d+1) \to (d, d+2)$ in the other path. We perform the following local changes to the family of lattice paths along the line $y + x = 2d + 1$ as indicated in Figure 10: First we resolve the crossing by replacing it by the following two turns $(d, d) \to (d+1, d) \to (d+1, d+1)$ and $(d-1, d+1) \to (d, d+1) \to (d, d+2)$, which will then appear in different paths. Note that this changes the sign of the permutation, however, it does not change the weight. Then we look for the unoccupied point (x_0, y_0) on the line $y + x = 2d + 1$ that is above the line $y = x - 1$ and has minimal $y_0 - x_0$. This point is unique and between that point and the line $y = x - 1$ there is a series of left turns along the line $y + x = 2d + 1$. We take the left turn $(x_0, y_0 - 1) \to (x_0 + 1, y_0 - 1) \to (x_0 + 1, y_0)$ and replace it by $(x_0, y_0 - 1) \to (x_0, y_0) \to (x_0 + 1, y_0)$.

Note that this local transformation only changes the sign of the weight. In addition, we see that the resulting constellations such as on the right of Figure 10 are exactly the trapped positions that cannot occur in the lattice path interpretation of skew (n, m)-even orthogonal tableaux due to the m-even orthogonal condition. This observation leads us to the announced sign-reversing involution.

Consider the families of lattice paths whose signed enumeration is given by (7). If it contains a crossing of an o-horizontal step with two vertical steps or if it contains a trapped

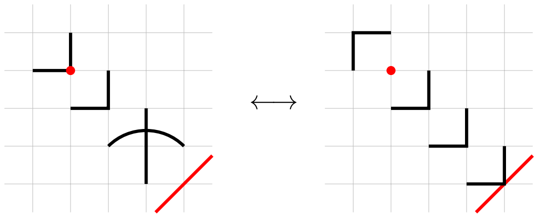


FIGURE 10. Local changes between lattice paths with an intersection of an o-horizontal step with two vertical steps (left) and with a trapped position (right).

position then choose a canonical occurrence as follows. Both the crossings and the trapped position are located on lines $y + x = 2d + 1$ for integers d. We consider the crossing or the trapped position for which the d is minimal. Then we perform the local changes as exemplified in Figure 10. This mapping changes the sign of the weight and is readily invertible. We are left with families of lattice paths that have neither crossings nor trapped positions, which are exactly those that correspond to skew (n, m)-even orthogonal tableaux. This finally proves that

$$\mathrm{o}^m_{\lambda/\mu}(\mathbf{x}) = \frac{1}{2^{[m=l(\mu)]}} \det_{1\leqslant i,j\leqslant N} \left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{x}^{\pm}) + e_{\lambda'_i+\mu'_j-i-j-2m+2}(\mathbf{x}^{\pm})\right),$$

and concludes the combinatorial proof of Theorem 2.1 (3c).

5. Proofs of the Jacobi–Trudi formulae

The purpose of this section is to derive ordinary Jacobi–Trudi-type formulae for the characters $\mathrm{sp}^m_{\lambda/\mu}$, $\mathrm{so}^m_{\lambda/\mu}$ and $\mathrm{o}^m_{\lambda/\mu}$. We achieve this by an algebraic approach using complementary cofactors, which is one of the standard ways to prove the equivalence of the Jacobi–Trudi formula for Schur functions and its dual. These computations all work in the ring of symmetric functions on a countable alphabet $\mathbf{X} = (x_1, x_2, x_3, \dots)$, and so in this section we work in such generality. The formulae for the actual characters are obtained by substituting $\mathbf{x}^{\pm} = (x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$ for $\mathbf{X}$.

Given an $N \times N$ matrix A and a pair of sequences σ, τ of the same length, we let A^{σ}_{τ} denote the matrix obtained by extracting rows and columns with indices from σ and τ respectively in the given order. The basic lemma we need is the following [6, Lemma A.42].

Lemma 5.1. *Let A and B be $N \times N$ mutually inverse matrices and let (σ, σ') and (τ, τ') be pairs of complementary subsets of $\{1, \dots, N\}$ such that $|\sigma| = |\tau|$. Then*

$$\det(A^{\sigma}_{\tau}) = \varepsilon \det(A) \det\left(B^{\tau'}_{\sigma'}\right),$$

where ε is the product of the signs of the permutations formed by the words $\sigma\sigma'$ and $\tau\tau'$.

Recall that the matrices

$$\left(e_{i-j}(\mathbf{X})\right)_{1\leqslant i,j\leqslant r} \quad \text{and} \quad \left((-1)^{i-j} h_{i-j}(\mathbf{X})\right)_{1\leqslant i,j\leqslant r},$$

are lower-unitriangular, i.e., lower-triangular with diagonal entries all 1, and where as before $\mathbf{X} = (x_1, x_2, x_3, \dots)$. Moreover, thanks to the relationship [20, p. 21]

$$\sum_{k=0}^{r} (-1)^k e_{r-k}(\mathbf{X}) h_k(\mathbf{X}) = \delta_{r,0} \tag{8}$$

where $\delta_{a,b}$ is the usual Kronecker delta, they are mutually inverse. We actually require the following slightly more general pair of matrices.

Lemma 5.2. *For $m, N \in \mathbb{N}$, $k \in \mathbb{Z}$ and a parameter t the matrices*

$$\mathcal{E}(N, m, k; t) := \left(e_{i-j}(\mathbf{X}) + [j < m + \lceil k/2 \rceil]\, t e_{i+j-2m-k}(\mathbf{X})\right)_{1\leqslant i,j\leqslant N}$$

and

$$\mathcal{H}(N,m,k;t) := \Big((-1)^{i-j}\big(h_{i-j}(\mathbf{X}) - [i > m + \lfloor k/2\rfloor]\,(-1)^k t h_{2m-i-j+k}(\mathbf{X})\big)\Big)_{1\leqslant i,j\leqslant N}$$

are lower-unitriangular and mutually inverse.

Proof. The (i,j)-th entry in the product of the two matrices is

$$\sum_{\ell=0}^{N}(-1)^{\ell-j}e_{i-\ell}(\mathbf{X})h_{\ell-j}(\mathbf{X}) + \sum_{\ell=0}^{m+\lceil k/2\rceil-1}(-1)^{\ell-j}te_{i+\ell-2m-k}(\mathbf{X})h_{\ell-j}(\mathbf{X}) \\ - \sum_{\ell=m+\lfloor k/2\rfloor+1}^{N}(-1)^{\ell-j+k}te_{i-\ell}(\mathbf{X})h_{2m-\ell-j+k}(\mathbf{X}),$$

where we assume that $i \geqslant j$ since the lower-triangularity is clear from the definition. The first sum in this expression simplifies to

$$\sum_{\ell=0}^{N}(-1)^{\ell-j}e_{i-\ell}(\mathbf{X})h_{\ell-j}(\mathbf{X}) = \sum_{\ell=0}^{i-j}(-1)^{\ell}e_{i-j-\ell}(\mathbf{X})h_{\ell}(\mathbf{X}) = \delta_{i-j,0},$$

by (8). For the remaining sums, we obtain by substituting $\ell \mapsto m - \ell + \lceil k/2\rceil - 1$ in the first sum and $\ell \mapsto m + \ell + \lfloor k/2\rfloor + 1$ in the second sum

$$\begin{aligned}
&\sum_{\ell=0}^{m+\lceil k/2\rceil-1}(-1)^{\ell-j}te_{i+\ell-2m-k}(\mathbf{X})h_{\ell-j}(\mathbf{X}) - \sum_{\ell=m+\lfloor k/2\rfloor+1}^{N}(-1)^{\ell-j+k}te_{i-\ell}(\mathbf{X})h_{2m-\ell-j+k}(\mathbf{X})\\
&\quad= \sum_{\ell=0}^{m+\lceil k/2\rceil-1}(-1)^{\ell-j+\lceil k/2\rceil+m+1}te_{i-\ell-m-\lfloor k/2\rfloor-1}(\mathbf{X})h_{m+\lceil k/2\rceil-\ell-j-1}(\mathbf{X})\\
&\qquad- \sum_{\ell=0}^{N-m-\lfloor k/2\rfloor-1}(-1)^{\ell-j+\lceil k/2\rceil+m+1}te_{i-\ell-m-\lfloor k/2\rfloor-1}(\mathbf{X})h_{m+\lceil k/2\rceil-\ell-j-1}(\mathbf{X})\\
&\quad= 0,
\end{aligned}$$

where the last equality follows since the summands are equal and vanish unless the index satisfies $\ell \leqslant \min\{m + \lceil k/2\rceil - 1, N - m - \lfloor k/2\rfloor - 1\}$. □

Recall from [20, p. 3] that for a partition $\lambda \subseteq (N^M)$ the sets

$$\{\lambda_i + M - i + 1 : 1 \leqslant i \leqslant M\} \quad\text{and}\quad \{M + j - \lambda'_j : 1 \leqslant j \leqslant N\} \tag{9}$$

form a disjoint union of $\{1,\dots,N+M\}$. This implies that for partitions $\mu \subseteq \lambda \subseteq (N^M)$ the sequences

$$\begin{aligned}
(\sigma,\sigma') &:= (\lambda'_1 + N, \dots, \lambda'_N + 1, N - \lambda_1 + 1, \dots, N + M - \lambda_M)\\
(\tau,\tau') &:= (\mu'_1 + N, \dots, \mu'_N + 1, N - \mu_1 + 1, \dots, N + M - \mu_M).
\end{aligned}$$

form permutations of $\{1,\dots,N+M\}$. Applying Lemma 5.1 with these choices of (σ,σ') and (τ,τ'), $A = \mathcal{E}(N+M,m,k;t)$ and $B = \mathcal{H}(N+M,m,k;t)$ we obtain that the determinants

$$\det_{1\leqslant i,j\leqslant N}\left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{X}) + [N+\mu'_j-j+1 < m+\lceil k/2\rceil]te_{\lambda'_i+\mu'_j-i-j+2(N-m+1)-k}(\mathbf{X})\right) \tag{10a}$$

and

$$\det_{1\leqslant i,j\leqslant M}\left(h_{\lambda_i-\mu_j-i+j}(\mathbf{X}) - [N-\mu_j+j > m+\lfloor k/2\rfloor](-1)^k th_{\lambda_i+\mu_j-i-j+2(m-N)+k}(\mathbf{X})\right), \tag{10b}$$

are equal. Note that in this case the product of the signs of the permutations is $\varepsilon = (-1)^{|\lambda|+|\mu|}$, which is a consequence of the proof that (9) form a disjoint union of $\{1,\dots,N+M\}$. If one sets $t=0$ in (10) then

$$\det_{1\leqslant i,j\leqslant N}\left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{X})\right) = \det_{1\leqslant i,j\leqslant M}\left(h_{\lambda_i-\mu_j-i+j}(\mathbf{X})\right),$$

showing that the Jacobi–Trudi formula and its dual are equal. The dual forms of our Jacobi–Trudi formulae for the skew characters $\mathrm{sp}^m_{\lambda/\mu}, \mathrm{so}^m_{\lambda/\mu}$ and $\mathrm{o}^m_{\lambda/\mu}$ are similarly contained in (10). We can now prove Theorem 2.2.

Proof of Theorem 2.2. Fix partitions $\mu \subseteq \lambda$ such that $l(\mu) \leqslant m$, $l(\lambda) \leqslant n+m$ and $\lambda \subseteq (N^M)$. For the symplectic case we set $(m,k,t) \mapsto (N+m, 2, -1)$ in (10), which gives

$$\det_{1\leqslant i,j\leqslant N}\left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{X}) - [\mu'_j - j < m]e_{\lambda'_i+\mu'_j-i-j-2m}(\mathbf{X})\right)$$
$$= \det_{1\leqslant i,j\leqslant M}\left(h_{\lambda_i-\mu_j-i+j}(\mathbf{X}) + [j-\mu_j > m+1]h_{\lambda_i+\mu_j-i-j+2m+2}(\mathbf{X})\right).$$

Since $\mu'_j \leqslant m$ for $1 \leqslant j \leqslant N$ we always have two terms in each entry of the determinant on the left. After replacing $\mathbf{X}$ by $(x_1, x_1^{-1}, \dots, x_n, x_n^{-1})$ it is equal to $\mathrm{sp}^m_{\lambda/\mu}$ by (3a). Further, $m + \mu_j - j + 1 < 0$ is only true for $j > m+1$, so the determinant on the right-hand side becomes (4a) with $N \mapsto M$ and the same variable substitution.

Instead taking $(m,k,t) \mapsto (N+m, 1, 1)$ in (10) produces

$$\det_{1\leqslant i,j\leqslant N}\left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{X}) + [\mu'_j - j < m]e_{\lambda'_i+\mu'_j-i-j-2m+1}(\mathbf{X})\right)$$
$$= \det_{1\leqslant i,j\leqslant M}\left(h_{\lambda_i-\mu_j-i+j}(\mathbf{X}) + [j-\mu_j > m]h_{\lambda_i+\mu_j-i-j+2m+1}(\mathbf{X})\right).$$

By the same arguments as above the left-hand side is equal to $\mathrm{so}^m_{\lambda/\mu}$ and the right-hand side to (4b) with $N \mapsto M$, after appropriate substitution of variables.

Finally, choosing $(m,k,t) \mapsto (N+m, 0, 1)$ in (10) yields

$$\det_{1\leqslant i,j\leqslant N}\left(e_{\lambda'_i-\mu'_j-i+j}(\mathbf{X}) + [\mu'_j - j < m-1]e_{\lambda'_i+\mu'_j-i-j-2m+2}(\mathbf{X})\right)$$
$$= \det_{1\leqslant i,j\leqslant M}\left(h_{\lambda_i-\mu_j-i+j}(\mathbf{X}) - [j-\mu_j > m]h_{\lambda_i+\mu_j-i-j+2m}(\mathbf{X})\right).$$

The right-hand clearly gives (4c) with $N \mapsto M$. If $m = l(\mu) = \mu'_1$ then the $(1,j)$-th entry of the determinant on the left-hand side is equal to $e_{\lambda'_i-\mu'_j-i+j}(\mathbf{X})$ which is half of the $(1,j)$-th entry of (3c). Hence by factoring out $\frac{1}{2}$ from the first column, the left-hand side of the above equation is equal to (3c). $\square$

In [5], Fulmek and Krattenthaler derive the Jacobi–Trudi formulae for sp_λ and o_λ using Gessel and Viennot's technique of dual paths, which is, in essence, a combinatorial realisation of the approach of this section.

6. Combinatorial proofs of the Giambelli formulae

Fulmek and Krattenthaler [5] provided combinatorial proofs of the Giambelli identities for ordinary symplectic and odd orthogonal characters based on Stembridge's proof of the Giambelli identity for ordinary Schur functions [22]. We adapt their ideas in order to prove combinatorially the respective skew Giambelli identities analogous to formula (5). By means of a sign-reversing involution similar to the one in Section 4.5, we are also able to provide a combinatorial proof of the Giambelli identity for skew even orthogonal characters.

In Section 3, we obtained the lattice path models (families of (n,m)-symplectic/odd orthogonal/even orthogonal paths) that we used to prove the dual Jacobi–Trudi formulae by reading off the corresponding tableaux columnwise. Ordinary Giambelli-type identities express characters of arbitrary shapes as determinants whose entries are characters of hook shapes. That observation suggests to read off tableaux hookwise.

Consider the Young diagram of shape λ/μ for two partitions $\mu \subseteq \lambda$. The *hook* at position (i,j) consists of the cells

- $(i,j) \cup \{(i,k) : j+1 \leqslant k \leqslant \lambda_i\} \cup \{(k,j) : i+1 \leqslant k \leqslant \lambda'_j\}$ if $(i,j) \in \lambda/\mu$ and
- $\{(i,k) : \mu_i + 1 \leqslant k \leqslant \lambda_i\} \cup \{(k,j) : \mu'_j + 1 \leqslant k \leqslant \lambda'_j\}$ if $(i,j) \notin \lambda/\mu$.

In other words, the hook at a position (i,j) is the union of the cell (i,j) (the *corner* of the hook) together with the cells to the right of it in the i-th row (the *arm* of the hook) and with the cells below of it in the j-th column (the *leg* of the hook) provided that (i,j) is part of the Young diagram of shape λ/μ. If (i,j) is not an element of the skew shape λ/μ, that is, $(i,j) \notin \lambda/\mu$, then the hook is not connected and we call it *broken*. Figure 11 illustrates the two different types of hooks.

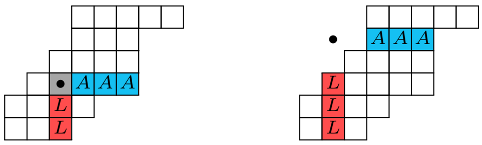


FIGURE 11. A Young diagram of shape $(8,6,6,6,4,3)/(3,3,2,1)$ with the hook at position $(4,3)$ (left) and with the broken hook at position $(2,2)$ (right) marked. The arms of the hooks are shaded blue with inscribed A's and the legs shaded red with inscribed L's.

Next, we introduce the set-up common to all three considered models for the lattice paths that we obtain by reading off tableaux hookwise; for the specific details of each model see the following subsections. Let $\lambda = (\alpha_1,\dots,\alpha_p|\beta_1,\dots,\beta_p)$ and $\mu = (\gamma_1,\dots,\gamma_q|\delta_1,\dots,\delta_q)$ be two partitions $\mu \subseteq \lambda$ with $l(\mu) \leqslant m$ and $l(\lambda) \leqslant n+m$. Define $A_i \coloneqq (-\alpha_i, 2n+2m-1)$ and $B_i \coloneqq (\beta_i+1, 2n+2m-\beta_i-1)$ for $1 \leqslant i \leqslant p$ and $C_i \coloneqq (-\gamma_i, 2m)$ and $D_i \coloneqq (\delta_i+1, 2m-\delta_i-1)$ for $1 \leqslant i \leqslant q$.

We will associate lattice paths corresponding to the *principal hooks* of the skew tableaux: the i-th principal hook is the hook at the diagonal position (i,i). For $q+1 \leqslant i \leqslant p$, the paths from A_i to B_i correspond to the i-th principal hook read from right to left and from top to bottom. For $1 \leqslant i \leqslant q$, the i-th principal hook is broken. The paths from A_i to C_i correspond to the arm of the i-th principal hook read from right to left, whereas the paths from D_i to B_i correspond to the leg of the i-th principal hook read from top to bottom.

The lattice paths fulfil the following properties:

- In the region $\{(x,y) \in \mathbb{Z}^2 : x \leqslant 0\}$, lattice paths stay weakly above the line $x = 2m$ and horizontal unit steps $(i, 2m+j) \to (i+1, 2m+j)$ get the weight $x_{(j+2)/2}^{-1}$ if j is even and $x_{(j+1)/2}$ if j is odd.
- In the region $\{(x,y) \in \mathbb{Z}^2 : x \geqslant 0\}$, lattice paths stay weakly above the lines $y = -x+2m$ and $y = x-1$ and horizontal unit steps $(i, 2m+j) \to (i+1, 2m+j)$ are assigned the weight $x_{(i+j+2)/2}^{-1}$ if $i+j$ is even and $x_{(i+j+1)/2}$ if $i+j$ is odd.

All other steps have weight 1, and the weight of a family of paths is the product of the weights of all its steps. Note that this set-up constitutes a combined e- and h-labelling: We have an e-labelling in the region $x > 0$ for the legs of the hooks and an h-labelling in the region $x \leqslant 0$ for the arms and the corners of the hooks. The exact lattice path model for each of the skew characters is specified in the respective section.

A crucial observation will be be the following: For each of the different lattice path models, the families of $p+q$ lattice paths from $\{A_1,\dots,A_p,D_1,\dots,D_q\}$ to $\{B_1,\dots,B_p, C_1,\dots,C_q\}$ are strongly non-intersecting if and only if we have paths from A_i to C_i as well as from D_i to B_i for $1 \leqslant i \leqslant q$ and from A_i to B_i for $q+1 \leqslant i \leqslant p$. By applying the Lindström–Gessel–Viennot Lemma 4.2 and — in the case of skew even orthogonal characters — by a sign-reversing involution, we will show that

$$(-1)^q \det \begin{pmatrix} (\mathcal{P}(A_i \to B_j))_{1\leqslant i,j\leqslant p} & (\mathcal{P}(A_i \to C_j))_{\substack{1\leqslant i\leqslant p,\\ 1\leqslant j\leqslant q}} \\ (\mathcal{P}(D_i \to B_j))_{\substack{1\leqslant i\leqslant q,\\ 1\leqslant j\leqslant p}} & (\mathcal{P}(D_i \to C_j))_{1\leqslant i,j\leqslant q} \end{pmatrix} \tag{11}$$

yields the respective Giambelli-type formulae, where $(-1)^q$ is the sign of the permutation

$$\begin{pmatrix} 1 & \cdots & q & q+1 & \cdots & p & p+1 & \cdots & p+q \\ p+1 & \cdots & p+q & q+1 & \cdots & p & 1 & \cdots & q \end{pmatrix}.$$

The sign can be readily computed since the permutation is equal to

$$(1 \quad p+1)(2 \quad p+2)\cdots(q \quad p+q)$$

in cycle notation, which is a product of exactly q transpositions.

In the following three subsections, we prove Theorem 2.3.

6.1. **Proof of the skew symplectic Giambelli identity (6a).** A skew (n, m)-symplectic tableau is encoded by non-intersecting lattice paths from A_i to C_i and from D_i to B_i for $1 \leqslant i \leqslant q$ as well as from A_i to B_i for $q+1 \leqslant i \leqslant p$ with step set $\{(1,0),(0,-1)\}$ in the region $x \leqslant 0$ and step set $\{(1,0),(0,1)\}$ in the region $x \geqslant 0$ that stay all weakly above the line $y = x - 1$; see Figure 12 for an example. Given a skew shape λ/μ, the generating function

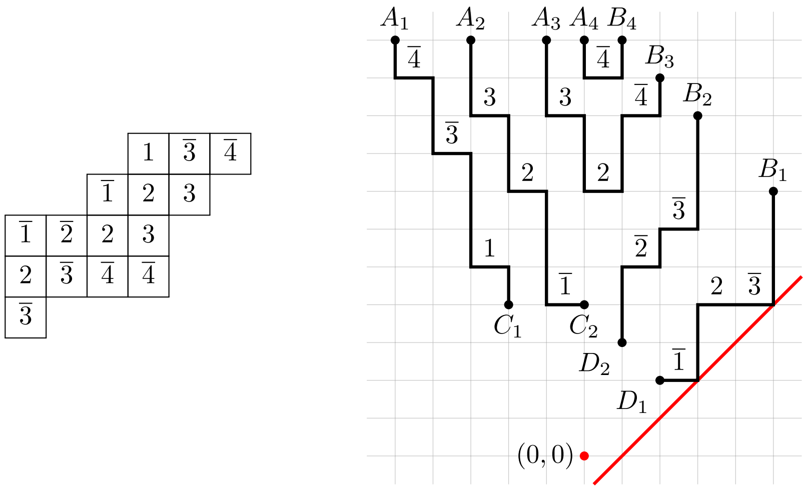


FIGURE 12. A skew $(4,2)$-symplectic tableau of shape $(5,3,1,0|4,2,1,0)/(2,0|1,0)$ (left) and its associated family of non-intersecting lattice paths (right).

$\mathrm{sp}^m_{\lambda/\mu}(\mathbf{x})$ is thus the sum over all families of $p+q$ strongly non-intersecting lattice paths with starting points $\{A_1, \ldots, A_p, D_1, \ldots, D_q\}$ and endpoints $\{B_1, \ldots, B_p, C_1, \ldots, C_q\}$ such that A_i is connected to C_i and D_i to B_i for $1 \leqslant i \leqslant q$ as well as having A_i connected to B_i for $q+1 \leqslant i \leqslant p$. Applying the Lindström–Gessel–Viennot Lemma 4.2 implies that $\mathrm{sp}^m_{\lambda/\mu}(\mathbf{x})$ is given by (11). There are clearly no lattice paths from any D_i to C_j, hence $\mathcal{P}(D_i \to C_j) = 0$. Moreover, the paths from A_i to C_j correspond to complete homogeneous symmetric functions, that is, $\mathcal{P}(A_i \to C_j) = h_{\alpha_i - \gamma_j}(\mathbf{x}^{\pm}) = \mathrm{sp}^m_{(\alpha_i)/(\gamma_j)}(\mathbf{x})$. Regarding the paths from D_i to B_j, we have to keep in mind that no lattice paths are allowed to touch or cross the line $y = x - 2$. Hence, Lemma 4.3 implies that $\mathcal{P}(D_i \to B_j) = e_{\beta_j - \delta_i}(\mathbf{x}^{\pm}) - e_{\beta_j + \delta_i - 2m}(\mathbf{x}^{\pm})$. Finally, the paths from A_i to B_j correspond to skew m-symplectic characters indexed by hooks, so $\mathcal{P}(A_i \to B_j) = \mathrm{sp}^m_{(\alpha_i|\beta_j)}$.

6.2. **Proof of the skew odd orthogonal Giambelli identity (6b).** In the case of skew m-odd orthogonal characters $\mathrm{so}^m_{\lambda/\mu}$, we consider similar lattice paths as in the previous section with the only addition that we also allow diagonal steps $(1,1)$ along the line $y = x$. These steps correspond to the entries $\widehat{i}$ in the $(m+i)$-th row of the first column of the skew (n, m)-odd orthogonal tableau as before and are equipped with the weight 1. See Figure 13 for an example. As seen before, it suffices to compute the determinantal expression (11)

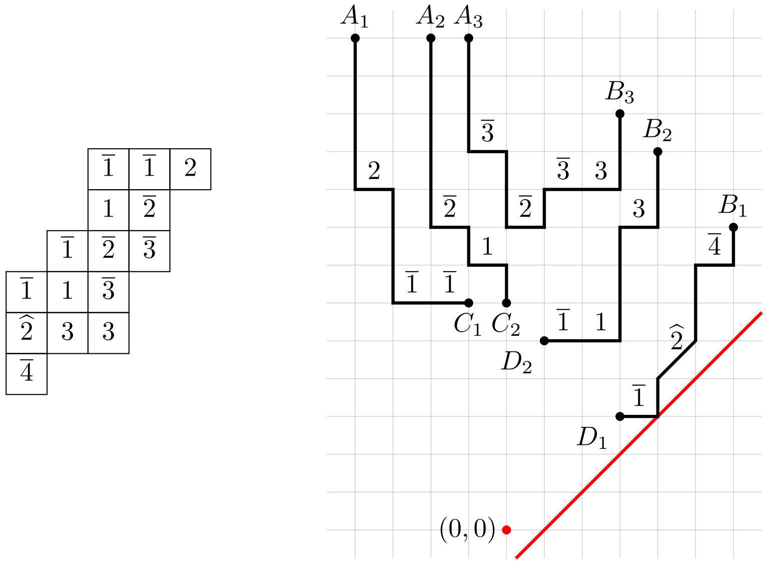


FIGURE 13. A skew $(4, 3)$-odd orthogonal tableau of shape $(4, 2, 1|5, 3, 2)/(1, 0|2, 0)$ (left) and its associated family of non-intersecting lattice paths (right).

in our setting. The expressions for $\mathcal{P}(D_i \to C_j)$ and $\mathcal{P}(A_i \to C_j)$ are the same as in the symplectic case, so $\mathcal{P}(D_i \to C_j) = 0$ and $\mathcal{P}(A_i \to C_j) = h_{\alpha_i-\gamma_j}(\mathbf{x}^{\pm}) = \mathrm{so}^m_{(\alpha_i)/(\gamma_j)}(\mathbf{x})$. The entry $\mathcal{P}(D_i \to B_j) = e_{\beta_j-\delta_i}(\mathbf{x}^{\pm}) + e_{\beta_j+\delta_i-2m+1}(\mathbf{x}^{\pm}) = \mathrm{so}^m_{(1^{\beta_j+1})/(1^{\delta_i+1})}(\mathbf{x})$ is a simple consequence of Corollary 4.5, and $\mathcal{P}(A_i \to B_j)$ corresponds to the skew m-odd orthogonal character $\mathrm{so}^m_{(\alpha_i|\beta_j)}(\mathbf{x})$.

6.3. **Proof of the skew even orthogonal Giambelli identity (6c).** Now, we consider lattice paths in our general set-up with step set $\{(1, 0), (0, -1)\}$ in the region $x \leqslant 0$ and step set $\{(1, 0), (0, 1), (2, 0)\}$ in the region $x \geqslant 0$ such that all lattice paths stay weakly above the line $y = x - 1$ and o-horizontal steps are only allowed if they end on the line $y = x$. As before, we interpret the entries of a skew (n, m)-even orthogonal tableau of shape λ/μ hookwise as lattice paths. Yet again, consecutive entries $\check{i}$ and $\hat{i}$ in the first column are interpreted as an o-horizontal step with weight 1 drawn as an arch. As a result, we obtain a family of $p + q$ strongly non-intersecting lattice paths, where $\lambda = (\alpha_1, \dots, \alpha_p|\beta_1, \dots, \beta_p)$ and $\mu = (\gamma_1, \dots, \gamma_q|\delta_1, \dots, \delta_q)$. See Figure 14 for an example. However, not every family of non-intersecting lattice paths in our set-up corresponds to a skew (n, m)-even orthogonal tableau. In fact, the m-even orthogonal condition implies that the configurations illustrated in Figure 15 cannot occur. To be more precise, assume we have a skew (n, m)-even orthogonal tableau with entry $\bar{i}$ in the first column of the $(m + i)$-th row and i also appears in the j-th column of that row. Then the entry in the same column one row above is supposed to be $\bar{i}$. If that entry at position $(m + i - 1, j)$ is not $\bar{i}$, then we obtain one of the three cases in Figure 15; namely (a) if $j < m + i$, (b) if $j = m + i$ and (c) if $j > m + i$. In particular, the vacancies indicated by the red points imply that the families of lattice paths are not obtained by skew (n, m)-even orthogonal tableaux. Note that case (a) is equivalent to the case of the trapped position in Section 3.4. The reason for the other cases is that the trapped position moves from a region with e-labelling to a region with h-labelling.

The corners of the nested left turns in Figure 15 are each lying on the line $y + x = 2m + 2i - 1$; we say that $2m + 2i - 1$ is the distance between the origin and the trapped positions.

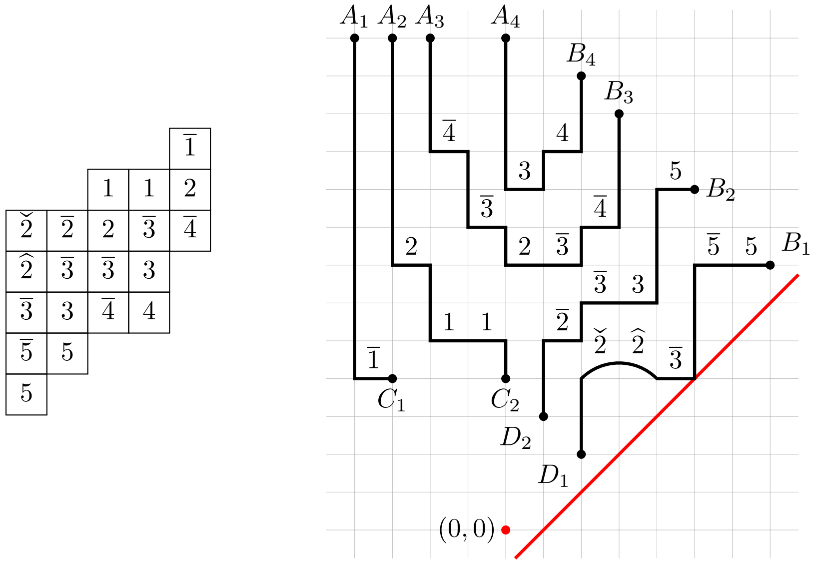


FIGURE 14. A skew $(5,2)$-even orthogonal tableau of shape $(4,3,2,0|6,4,2,1)/(3,0|1,0)$ (left) and its associated family of non-intersecting lattice paths (right).

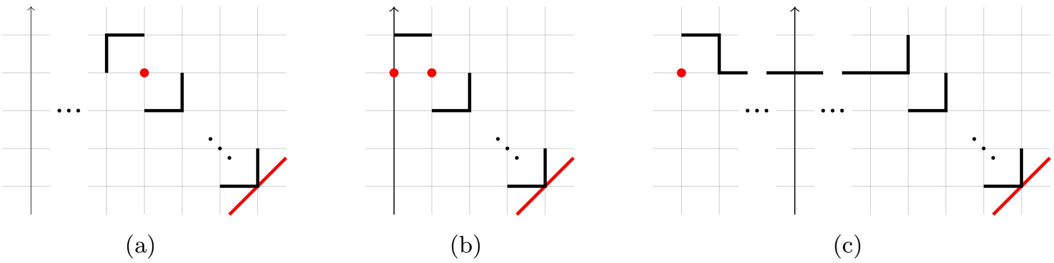


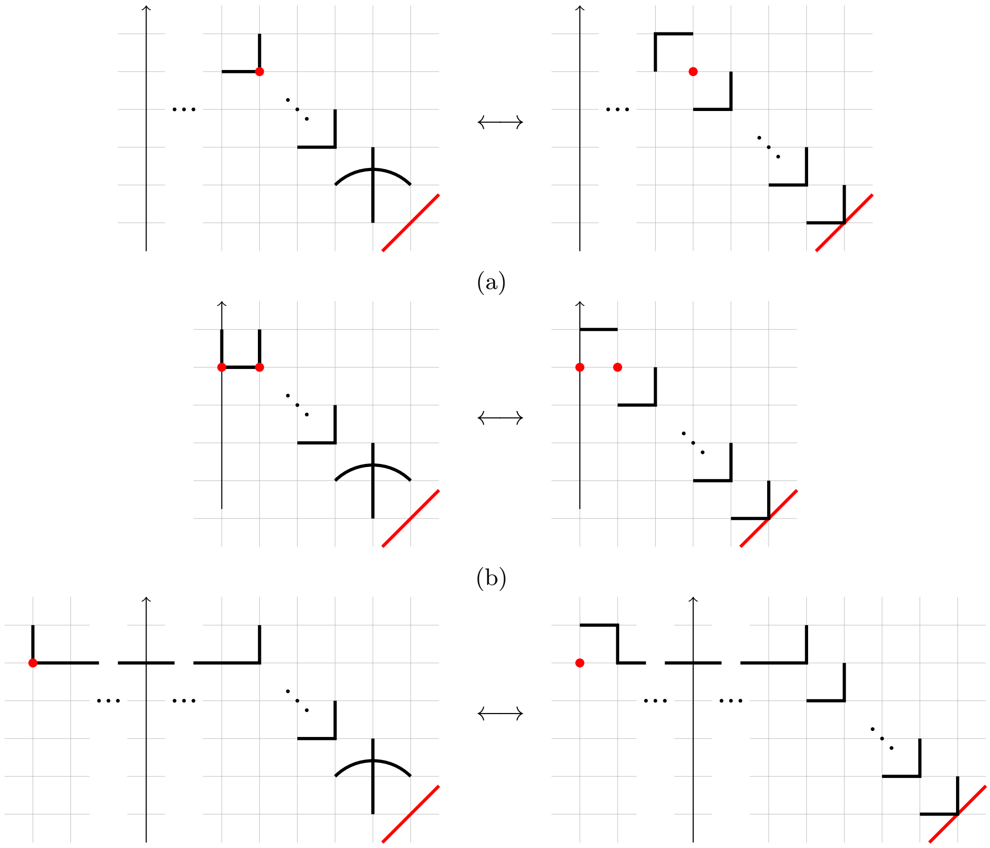
FIGURE 15. Local configurations of trapped positions at $(m+i-1,j)$ if (a) $j < m+i$, (b) $j = m+1$ or (c) $j > m+1$. The large points indicate vacancies.

As in the cases before, we want to apply the Lindström–Gessel–Viennot Lemma 4.2. The evaluation of the determinant (11) in the current set-up yields

$$\frac{(-1)^q}{2^{[m=l(\mu)\wedge m\neq 0]}} \det \begin{pmatrix} (\mathrm{o}^m_{(\alpha_i|\beta_j)}(\mathbf{x}))_{1\leqslant i,j\leqslant p} & (h_{\alpha_i-\gamma_j}(\mathbf{x}^{\pm}))_{\substack{1\leqslant i\leqslant p,\\ 1\leqslant j\leqslant q}} \\ (e_{\beta_j-\delta_i}(\mathbf{x}^{\pm})+e_{\beta_j+\delta_i-2m+2}(\mathbf{x}^{\pm}))_{\substack{1\leqslant i\leqslant q,\\ 1\leqslant j\leqslant p}} & 0 \end{pmatrix},$$

which, by rewriting the entries in terms of the even orthogonal characters, is equal to (6c). However, a priori this is not the generating function of skew (n,m)-even orthogonal tableaux since it also enumerates the trapped positions in Figure 15 as well as families of weakly but not strongly non-intersecting lattice paths with intersections of o-horizontal steps with vertical steps along the line $y = x+1$. We provide a sign-reversing involution under which these families of lattice paths cancel out, ultimately showing that the above is indeed equal to $\mathrm{o}^m_{\lambda/\mu}(\mathbf{x})$.

Consider the families of lattice paths enumerated by (6c) as described above. Assume a given family of lattice paths has intersections involving o-horizontal steps or trapped positions that we have specified above. If an intersection lies on the line $y+x=d$, then we say that d is the distance between the origin and that intersection. We take the unique trapped position or intersection with the smallest distance to the origin and perform the corresponding local changes shown in Figure 16; the rest of the paths are left unchanged. Thus, the weight of the family of lattice paths is exactly changed by a factor of -1.



(a) (b) (c)

FIGURE 16. Local changes on the families of lattice paths enumerated by (6c).

This sign-reversing involution cancels all families of lattice paths that contain one of the trapped positions or an intersection of an o-horizontal step with two vertical steps. We are left with exactly those families of strongly non-intersecting lattice paths that correspond to skew (n, m)-even orthogonal tableaux. This completes the proof of Theorem 2.3.

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