Nonintersecting lattice paths in Combinatorics, Commutative Algebra and Statistical Mechanics

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Für Anita, Christian, Miriam und Theresia.

Contents

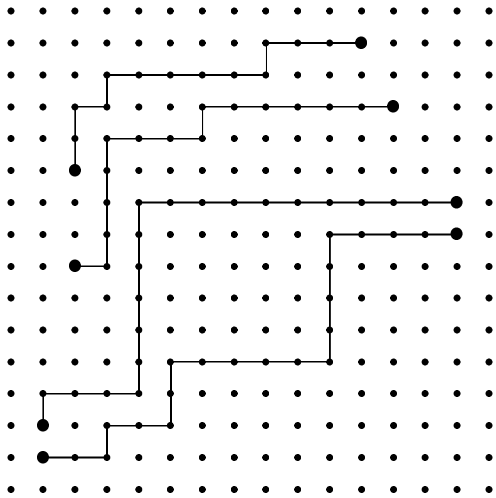


Figure 1. Four non-intersecting lattice paths with steps $(0, 1)$ and $(1, 0)$

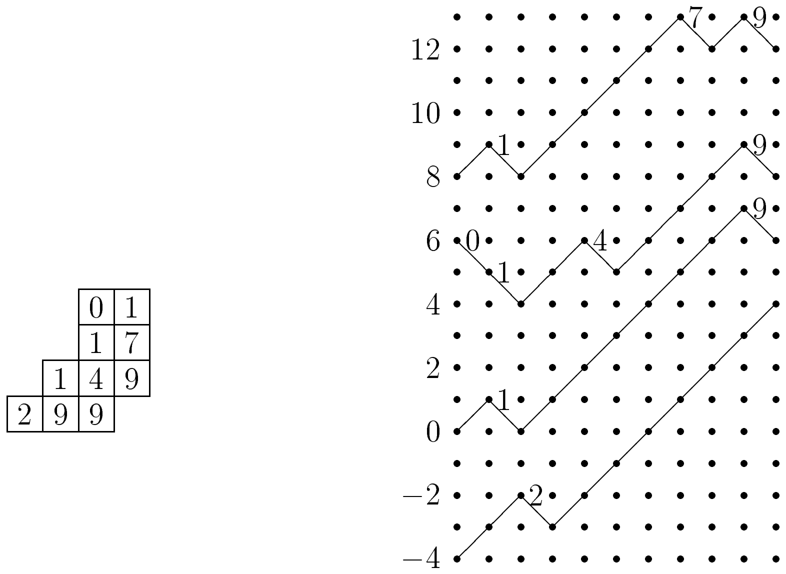
Introduction

A lattice path is a polygonal line in the discrete Cartesian plane $\mathbb{Z}^2$. However, in this thesis we will only be concerned with lattice paths with steps $(1, 1)$ and $(1, -1)$ or with steps $(1, 0)$ and $(0, 1)$. A family of lattice paths is called non-intersecting, if no two paths have a lattice point in common. In Figure 1 an example for such a family is shown.

Families of non-intersecting lattice paths are objects of great importance in combinatorics: They can be used to count plane partitions and different sorts of tableaux, and thus may be used to proof certain determinantal formulas for Schur functions and symplectic and orthogonal characters, see [4, 5, 6]. In statistical mechanics non-intersecting lattice paths are known as 'vicious walkers', and are used to describe wetting and melting processes, see [3]. In commutative algebra families of non-intersecting lattice paths can be used to describe the Hilbert series of determinantal and Pfaffian rings, see [8]. In this thesis we present new results in these three areas.

The first chapter is titled "A 'nice' bijection for a content formula for skew semistandard Young tableaux". In this chapter we give a bijective proof of a formula relating the generating functions for Young tableaux and the generating function for reverse semistandard Young tableaux to each other.

As already mentioned before, there is a close connection between semistandard Young tableaux and families of non-intersecting lattice paths. In the following we will explain these matters briefly. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ be two weakly decreasing sequences of non-negative integers such that $s < r$ and $\lambda_i < \mu_i$ for $i \in \{1, 2, \ldots, s\}$. For convenience we set $\mu_i = 0$ for $i > s$. The shape λ/μ is an array of r rows of boxes, such that the i^{th} row contains $\lambda_i - \mu_i$ boxes and the first box of the i^{th} row is placed in column $\mu_i + 1$, for $i \in \{1, 2, \ldots, r\}$.



a. a semistandard Young tableau of shape $(4, 4, 4, 3)/(2, 2, 1)$

b. the corresponding family of non-intersecting lattice paths

Figure 2.

A semistandard Young tableau of shape λ/μ is a filling of these boxes with non-negative integers such that the entries are weakly increasing along rows and strictly increasing along columns. An example for a semistandard Young tableau can be found in Figure 2.a. Similarly, a reverse semistandard Young tableau of shape λ/μ is a filling of the boxes with non-negative integers such that the entries are weakly decreasing along rows and strictly decreasing along columns.

Let P be a semistandard Young tableaux of shape λ/μ, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$. For any shape λ let λ' denote the transposed shape, i.e., λ'_i is the length of the i^{th} column of λ. Define $a_i = 2i - 2\mu'_i$ and $e_i = 2i - 2\lambda'_i + m + 1$ for $i \in \{1, 2, \ldots, \lambda_1\}$, where m is the maximum of all entries in P. Now we translate P into a family of λ_1 lattice paths starting at $(0, a_i)$ and terminating at $(m+1, e_i)$, $i \in \{1, 2, \ldots, \lambda_1\}$, as follows: The j^{th} path does a $(1, -1)$-step for the i^{th} time after $P_{\mu'_j + i, j}$ steps, all the other steps are $(1, 1)$. Because each row of P is weakly increasing, this family of paths is non-intersecting. An example of this bijection is shown in Figure 2.

Similarly, reverse semistandard Young tableaux can also be interpreted as families of non-intersecting lattice paths. Thus, the formula we prove in Chapter 1 can be translated into a formula relating the generating functions of certain families of non-intersecting lattice paths.

In fact, the bijection relating semistandard Young tableaux and families of non-intersecting lattice paths described above was used in [12] to find an asymptotic approximation of the number of certain families of non-intersecting lattice paths. In the second chapter, "Asymptotic analysis of vicious walkers with arbitrary endpoints", we generalize these results. However, we have to use a different method.

'Vicious walkers' are families of p non-intersecting lattice paths with steps $(1,1)$ and $(1,-1)$ that have given starting points $(0,2a_i)$, $i \in \{1,2,\ldots,p\}$ and end somewhere on the line $x = m$. In this setting, m is called the length of the walkers.

Thus the question arises how to count families of non-intersecting lattice paths. This task is accomplished by the famous Lindström-Gessel-Viennot Theorem, see Theorem 2.1 on page 22 or [6, Corollary 2]. This theorem reduces the enumeration of families of non-intersecting lattice paths to the evaluation of a determinant, whose entry in row i and column j is the number of lattice paths from the i^{th} starting to the j^{th} end point.

Originally, this result was discovered by Lindström in 1973, in the context of matroid theory, see [13]. It is a curious coincidence that his result was independently rediscovered in the 1980s in three different communities at about the same time: in statistical physics by Fisher [3, Section 5.3], in order to apply it to the analysis of vicious walkers as a model of wetting and melting, in combinatorial chemistry by John and Sachs [9] and Gronau, Just, Schade, Scheffler and Wojciechowski [7] in order to compute Pauling's bond order in benzenoid hydrocarbon molecules, and in enumerative combinatorics by Gessel and Viennot [5, 6] in order to count tableaux and plane partitions. Finally, it should be mentioned that the same idea appeared even earlier in work by Karlin and McGregor [10, 11] in a probabilistic framework.

Using the Lindström-Gessel-Viennot Theorem, some knowledge of ordinary and odd orthogonal characters, the Poisson summation Theorem and some limit cases of Selberg's integral formula, we find an asymptotic approximation of the number of vicious walkers as described above, as their length m tends to infinity. Also, we obtain such an asymptotic approximation if the walkers are not allowed to go below the x-axis.

The last two chapters of this thesis are concerned with the Hilbert series of ladder determinantal rings. These rings are very important objects in commutative algebra and Schubert calculus, see [2, 8, 1]. Thus, it is a natural question to ask for a 'nice' formula for their Hilbert series. It is known that the Hilbert series of a ladder determinantal ring equals $\sum_{\ell \geq 0} h_\ell z^\ell/(1-z)^d$, where, d is the Krull dimension of the ring and h_ℓ denotes the number of families of non-intersecting lattice paths with steps $(1,0)$ and $(0,1)$, and ℓ north-east turns. Here, a north-east turn is a point of the lattice path which is the end point of a $(0,1)$-step and the starting point of a $(1,0)$-step.

It is the aim of Chapter 3 to present a formula for the generating function mentioned above. Naturally, we would like to have a determinantal expression similar to that in the Lindström-Gessel-Viennot Theorem. In order to obtain such a formula, it helps to know how the latter theorem is proved. The key ingredient is the following involution on the set of families of paths which contain two paths that intersect: Let $(P_1, P_2, \ldots, P_n)$ be such a family of paths and suppose that P_i and P_j are two paths that intersect in a lattice point x. Let $\tilde{P}_i$ be the path which is identical to P_i up to x but then follows P_j, and, similarly, let $\tilde{P}_j$ be the path which is identical to P_j up to x but then follows P_i. For $k \notin \{i,j\}$, let $\tilde{P}_k = P_k$. Clearly, mapping

$(P_1, P_2, \ldots, P_n)$ to $(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)$ is an involution.

However, if we want to count families of non-intersecting lattice paths with a given total number of north-east turns, we cannot apply this involution, because $(P_1, P_2, \ldots, P_n)$ and $(\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_n)$ may have a different total number of north-east turns. The solution is to consider so called two-rowed arrays, as defined in Section 3 of Chapter 3, that are more general than paths.

In fact, the main idea of the proof of the Lindström-Gessel-Viennot Theorem remains valid: The involution as defined in Section 4.4 on page 50 is still based on the idea of switching tails, but, as mentioned before, acts on two-rowed arrays.

The last chapter, "The h-vector of a ladder determinantal ring cogenerated by 2×2 minors is log-concave", is also concerned with families of non-intersecting lattice paths with steps $(1,0)$ and $(0,1)$ that have a given number of north-east turns. In fact, in the case of ladder determinantal rings, the 'h-vector' is exactly the generating function $\sum_{\ell \geq 0} h_\ell z^\ell$ described above. It was conjectured that this h-vector is log-concave, i.e., it satisfies $h_{i-1}h_{i+1} \leq h_i^2$ for $i \in \{1, 2, \ldots, p\}$. Corollary 4.6 on page 73 provides an affirmative answer in the simplest case, where there is only a single path. It remains a challenging problem to prove the conjecture for arbitrarily large families of non-intersecting lattice paths.

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Chapter 1
A 'nice' bijection for a content formula for skew semistandard Young tableaux

Abstract

Based on Schützenberger's evacuation and a modification of jeu de taquin, we give a bijective proof of an identity connecting the generating function of reverse semistandard Young tableaux with bounded entries with the generating function of all semistandard Young tableaux. This solves Exercise 7.102 b of Richard Stanley's book 'Enumerative Combinatorics 2'.

1 Introduction

The purpose of this article is to present a solution for Exercise 7.102 b of Richard Stanley's book 'Enumerative Combinatorics 2' [5]. There, Stanley asked for a 'nice' bijective proof of the identity

$$\sum_{\substack{R \text{ reverse SSYT} \\ \text{of shape } \lambda/\mu \\ \text{with } R_{ij} \leq a+\mu_i - i}} q^{n(R)} = \left(\sum_{\substack{P \text{ SSYT} \\ \text{of shape } \lambda/\mu}} q^{n(P)} \right) \cdot \prod_{\rho \in \lambda/\mu} (1 - q^{a+c(\rho)}), \tag{1}$$

where a is an arbitrary integer such that $a + c(\rho) > 0$ for all cells $\rho \in \lambda/\mu$.[1] Here, and in the sequel, we use notation defined below:

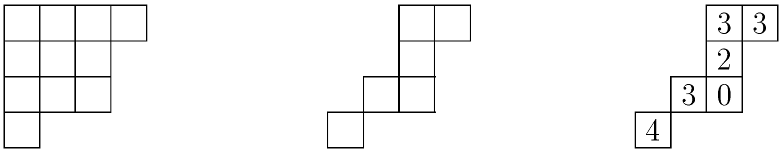
Definition 1.1. A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, for some r.

The *Ferrers diagram* of a partition λ is an array of cells with r left-justified rows and λ_i cells in row i. Figure 1.a shows the Ferrers diagram corresponding to $(4, 3, 3, 1)$. We label the cell in the i^{th} row and j^{th} column of the Ferrers diagram of λ by the pair (i, j). Also, we write $\rho \in \lambda$, if ρ is a cell of λ.

A partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ is *contained* in a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, if $s \leq r$ and $\mu_i \leq \lambda_i$ for $i \in \{1, 2, \ldots, s\}$.

The *skew diagram* λ/μ of partitions λ and μ, where μ is contained in λ, consists of the cells of the Ferrers diagram of λ which are not cells of the Ferrers diagram

[1] In fact, this is the corrected version of the identity originally given in [5], to be found at `http://www-math.mit.edu/~rstan/ec`. Stanley took it from [1], Theorem 3.1, where the formula is stated incorrectly, too.



a. Ferrers diagram b. skew Ferrers diagram c. reverse SSYT

Figure 1.

of μ. Figure 1.b shows the skew diagram corresponding to $(4,3,3,1)/(2,2,1)$. The *content* $c(\rho)$ of a cell $\rho = (i,j)$ of λ/μ is $j - i$.

Given partitions λ and μ, a *tabloid of shape* λ/μ is a filling T of the cells of the skew diagram λ/μ with non-negative integers. T_ρ denotes the entry of T in cell ρ. The *norm* $n(T)$ of a tabloid T is simply the sum of all entries of T. The *content weight* $w_c(T)$ of a tabloid T is $\sum_{\rho\in\lambda/\mu} T_\rho \cdot (a + c(\rho))$, where a is a given integer such that $a + c(\rho) > 0$ for all cells $\rho \in \lambda/\mu$.

A *semistandard Young tableau of shape* λ/μ, short *SSYT*, is a tabloid P such that the entries are weakly increasing along rows and strictly increasing along columns.

A *reverse semistandard Young tableau of shape* λ/μ is a tabloid R such that the entries are weakly decreasing along rows and strictly decreasing along columns. In Figure 1.c a reverse SSYT of shape $(4,3,3,1)/(2,2,1)$ is shown.

2 A Bijective proof of Identity 1

In fact, we will give a bijective proof of the following rewriting of Identity 1:

$$\sum_{\substack{P \text{ SSYT} \\ \text{of shape } \lambda/\mu}} q^{n(P)} = \Bigg(\sum_{\substack{R \text{ reverse SSYT} \\ \text{of shape } \lambda/\mu \\ \text{with } R_{ij} \le a + \mu_i - i}} q^{n(R)} \Bigg) \cdot \prod_{\rho\in\lambda/\mu} \frac{1}{1 - q^{a+c(\rho)}}$$

$$= \sum_{\substack{(R,T) \\ R \text{ reverse SSYT} \\ \text{of shape } \lambda/\mu \\ \text{with } R_{ij} \le a + \mu_i - i, \\ T \text{ tabloid} \\ \text{of shape } \lambda/\mu}} q^{n(R)} q^{w_c(T)}.$$

So all we have to do is to set up a bijection that maps SSYT'x P onto pairs (R,T), where R is a reverse SSYT with $R_{ij} \le a + \mu_i - i$ and T is an arbitrary tabloid, such that $n(P) = n(R) + w_c(T)$.

The bijection consists of two parts. The first step is a modification of a mapping known as 'evacuation', which consists of a special sequence of so called 'jeu de taquin slides'. An in depth description of these procedures can be found, for example, in Bruce Sagan's Book 'The symmetric group' [4], Sections 3.9 and 3.11. We use evacuation to bijectively transform the given SSYT P in a reverse SSYT Q which has the same shape and the same norm as the original one.

The second step of our bijection also consists of a sequence of – modified – jeu de taquin slides and bijectively maps a reverse SSYT Q onto a pair (R, T) as described above. This procedure is very similar to bijections discovered by Christian Krattenthaler, proving Stanley's hook-content formula. [2, 3]

$$\begin{array}{cccc} & & 0 & 1 \\ & & 1 & 7 \\ & 1 & 4 & 9 \\ 2 & 9 & 9 & \end{array} \quad \Leftrightarrow \quad \begin{array}{cccc} & & 9 & 9 \\ & & 7 & 4 \\ & 9 & 1 & 1 \\ 2 & 1 & 0 & \end{array} \quad \Leftrightarrow \quad \left(\begin{array}{cccc} & & 4 & 3 \\ & & 2 & 2 \\ & 4 & 1 & 0 \\ 2 & 1 & 0 & \end{array}, \begin{array}{cccc} & & 0 & 0 \\ & & 0 & 0 \\ & 1 & 0 & 2 \\ 0 & 0 & 1 & \end{array} \right)$$

$$n(.) = 43 \qquad\qquad n(.) = 43 \qquad\qquad n(.) = 19, w_c(.) = 24$$

Figure 2.

A complete example for the bijection can be found in the appendix. There we chose $a = 6$ and map the SSYT P of shape $(4, 4, 4, 3)/(2, 2, 1)$ on the left of Figure 2 to the reverse SSYT Q in the middle of Figure 2, which in turn is mapped to the pair on the right of Figure 2, consisting of a reverse SSYT R, where the entry of the cell $\rho = (i, j)$ is less or equal to $a + \mu_i - i$, and a tabloid T so that $n(Q) = n(R) + w_c(T)$.

In the algorithm described below we will produce a filling of a skew diagram step by step, starting with the 'empty tableau' of the given shape.

Theorem 2.1. *The following two maps define a correspondence between SSYT'x and reverse SSYT'x of the same shape λ/μ and the same norm:*

$\Rrightarrow$ Given a SSYT P of shape λ/μ, produce a reverse SSYT Q of the same shape and the same norm as follows:

Let Q be the empty tableau of shape λ/μ.

WHILE there is a cell of P which contains an entry

Let e be the minimum of all entries of P. Among all cells τ with $P_\tau = e$, let $\rho = (i, j)$ be the cell which is situated most right.

WHILE ρ has a bottom or right neighbour in P that contains an entry

Denote the entry to the right of ρ by x and the entry below ρ by y. We allow also that there is only an entry to the right or below ρ and the other cell is missing or empty.

If $x < y$, or there is no entry below ρ, then replace

$$\begin{array}{cc} e & x \\ y & \end{array} \quad \text{by} \quad \begin{array}{cc} x & e \\ y & \end{array},$$

and let ρ be the cell $(i, j + 1)$.

Otherwise, if $x \geq y$, or there is no empty to the right, replace

$$\begin{array}{|c|c|}\hline e & x \\ \hline y & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \text{by} \quad \begin{array}{|c|c|}\hline y & x \\ \hline e & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},$$

and let ρ be the cell $(i+1, j)$.

END WHILE.

Put Q_ρ equal to e and delete the entry of the cell ρ from P. Note that cells of P which contain an entry still form a SSYT. In the proof below, ρ will be called the cell where the jeu de taquin slide stops.

END WHILE.

$\Leftarrow$ Given a reverse SSYT Q of shape λ/μ, produce a SSYT P of the same shape and the same norm as follows:

Let P be the empty tableau of shape λ/μ.

WHILE there is a cell of Q which contains an entry

Let e be the maximum of all entries of Q. Among all cells τ with $Q_\tau = e$, let $\rho = (i, j)$ be the cell which is situated most left.

Set $P_\rho = e$ and delete the entry of the cell ρ from Q.

WHILE ρ has a top or left neighbour in P that contains an entry

Denote the entry to the left of ρ by x and the entry above ρ by y. We allow also that there is only an entry to the left or above ρ and the other cell is missing or empty.

If $x > y$, or there is no entry above ρ, then replace

$$\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & y \\ \hline x & e \\ \hline\end{array} \quad \text{by} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & y \\ \hline e & x \\ \hline\end{array},$$

and let ρ be the cell $(i, j-1)$.

Otherwise, if $x \leq y$, or there is no entry to the left, replace

$$\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & y \\ \hline x & e \\ \hline\end{array} \quad \text{by} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & e \\ \hline x & y \\ \hline\end{array},$$

and let ρ be the cell $(i-1, j)$.

END WHILE.

The cells of P which contain an entry now form a SSYT. In the proof below, ρ will be called the cell where the jeu de taquin slide stops.

END WHILE.

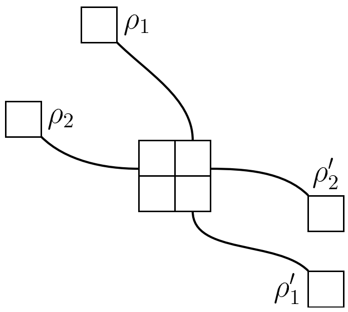


Figure 3.

Proof. Note that what happens during the execution of the inner loop of $\textcircled{$\Rightarrow$}$ ($\textcircled{$\Leftarrow$}$) is a *jeu de taquin forward (backward) slide* performed on Q into the cell ρ, see Section 3.9 of [4].

First we have to show that $\textcircled{$\Rightarrow$}$ is well defined. I.e., we have to check that after each jeu de taquin forward slide, after the entry e in the cell ρ is deleted from P, the cells of P which contain an entry form a SSYT as stated in the algorithm. This follows, because after either type of replacement in the inner loop the only possible violations of increase along rows and strict increase along columns in P can only involve e and the entries to its right and below. When the jeu de taquin forward slide is finished, ρ is a bottom-right corner of P, hence after deleting the entry in ρ no violations of increase or strict increase can occur.

Next we show that $\textcircled{$\Rightarrow$}$ indeed produces a reverse SSYT. In fact, we even show that the tabloid defined by the cells of Q which have been filled already, is a reverse SSYT at every stage of the algorithm.

Clearly, every cell of Q is filled with an entry exactly once. Furthermore, at the time the cell ρ is filled, the cells in Q to the right and to the bottom of ρ – if they exist – are filled already, otherwise ρ would not be a bottom-right corner of P. Because the sequence of entries chosen is monotonically increasing, rows and columns of Q are decreasing.

So it remains to show that the columns of Q are in fact strictly decreasing. Suppose that ρ_1 and ρ_2 are cells both containing the same minimal entry e, and ρ_1 is right of ρ_2.

When the jeu de taquin forward slide in $\textcircled{$\Rightarrow$}$ is performed into the cell ρ_1, the entry e describes a path from ρ_1 to the cell where the slide stops, which we will denote by ρ'_1. Similarly, we have a path from ρ_2 to a cell ρ'_2.

Now suppose ρ'_1 is in the same column as, but below ρ'_2, as depicted in Figure 3. Clearly, in this case the two paths would have to cross and we had the following situation:

First, (the star is a placeholder for an entry we do not know)

$$\begin{array}{|c|c|}\hline * & c \\ \hline z & y \\ \hline\end{array} \quad \text{would be replaced by} \quad \begin{array}{|c|c|}\hline * & y \\ \hline z & c \\ \hline\end{array}.$$

In this situation, z would have to be smaller then y.

Then, when the jeu de taquin forward slide into the cell ρ_2 is performed, the following situation would arise at the same four cells:

$$\begin{array}{|c|c|}\hline c & y \\ \hline z & * \\ \hline\end{array} \quad \text{would have to be replaced by} \quad \begin{array}{|c|c|}\hline y & c \\ \hline z & * \\ \hline\end{array}.$$

But this cannot happen, because then y would have to be strictly smaller than z.

It can be shown in a very similar manner that $\Leftarrow$ indeed produces a SSYT. We leave the details to the reader.

Finally, we want to prove that $\Leftarrow$ is inverse to $\Rightarrow$. Suppose that in $\Rightarrow$, a jeu de taquin forward slide into the cell ρ containing the entry e is performed on P. Suppose that the slide stopped in ρ', $Q_{\rho'}$ is set to e and the entry in ρ' is deleted from P. Among the entries of Q, e is maximal, because smallest entries are chosen first in $\Rightarrow$. Furthermore, among those cells of Q containing the entry e, the cell ρ' is most left. This follows, because the tabloid defined by the cells of Q which have been filled already, is a reverse SSYT, and the paths defined by the jeu de taquin slides cannot cross, as we have shown above.

It is straightforward to check that in this situation the jeu de taquin backward slide into ρ' performed on P in $\Leftarrow$ stops in the original cell ρ. By induction we find that $\Leftarrow$ is inverse to $\Rightarrow$. $\square$

The second step of the bijection is just as easy:

Theorem 2.2. *The following two maps define a correspondence between reverse SSYT'x Q to pairs (R, T), where R is a reverse SSYT with $R_{ij} \le a + \mu_i - i$ and T is an arbitrary tabloid, so that $n(Q) = n(R) + w_c(T)$, Q, R and T being of shape λ/μ:*

- $\Rightarrow$ Given a reverse SSYT Q of shape λ/μ, produce a pair (R, T) as described above as follows:

  Set $R = Q$ and set all entries of T equal to 0.

  WHILE there is a cell $\tau = (i, j)$ such that $R_\tau > a + \mu_i - i$

    Let e be maximal so that there is a cell τ with $R_\tau - \big(a + c(\tau)\big) = e$. Among all cells τ with $R_\tau - \big(a + c(\tau)\big) = e$, let $\rho = (i, j)$ be the cell which is situated most bottom. Set $R_\rho = e$.

    WHILE $e < R_{(i,j+1)}$ or $e \le R_{(i+1,j)}$

Denote the entry to the right of ρ by x and the entry below ρ by y. We allow also that there is only a cell to the right or below ρ and the other cell is missing.

If $x-1>y$, or there is no cell below ρ, then replace

$$\begin{array}{|c|c|}\hline e & x \\ \hline y \\ \cline{1-1}\end{array} \quad \text{by} \quad \begin{array}{|c|c|}\hline x-1 & e \\ \hline y \\ \cline{1-1}\end{array},$$

and let ρ be the cell $(i,j+1)$.

Otherwise, if $y+1 \geq x$, or there is no cell to the right, replace

$$\begin{array}{|c|c|}\hline e & x \\ \hline y \\ \cline{1-1}\end{array} \quad \text{by} \quad \begin{array}{|c|c|}\hline y+1 & x \\ \hline e \\ \cline{1-1}\end{array},$$

and let ρ be the cell $(i+1,j)$.

END WHILE.

Increase T_ρ by one.

END WHILE.

$\Leftarrow$ Given a pair (R,T) as described above, produce a reverse SSYT Q of shape λ/μ as follows:

Set $Q=R$.

WHILE there is a cell $\tau=(i,j)$ such that $T_\tau \neq 0$

Let e be minimal so that there is a cell τ with $Q_\tau = e$ and $T_\tau \neq 0$. Among these cells τ let $\rho=(i,j)$ be the cell which is situated most right. Decrease T_ρ by one.

WHILE $e+a+c(\rho) > Q_{(i,j-1)}$ or $e+a+c(\rho) \geq Q_{(i-1,j)}$

Denote the entry to the left of ρ by x and the entry above ρ by y. We allow also that there is only a cell to the left or above ρ and the other cell is missing.

If $y > x+1$, or there is no cell above ρ, then replace

$$\begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & y \\ \hline x & e \\ \hline\end{array} \quad \text{by} \quad \begin{array}{|c|c|}\cline{2-2} \multicolumn{1}{c|}{} & y \\ \hline e & x+1 \\ \hline\end{array},$$

and let ρ be the cell $(i,j-1)$.

Otherwise, if $x \geq y-1$, or there is no cell to the left, replace

$$\begin{array}{|c|c|}\hline \multicolumn{1}{c|}{} & y \\ \hline x & e \\ \hline \end{array} \quad \text{by} \quad \begin{array}{|c|c|}\hline \multicolumn{1}{c|}{} & e \\ \hline x & y-1 \\ \hline \end{array},$$

and let ρ be the cell $(i-1, j)$.

END WHILE.

Increase Q_ρ by $a + c(\rho)$.

END WHILE.

Remark. Because of the obvious similarity to jeu de taquin slides, we will call what happens in the inner loop of $\textcircled{$\Rightarrow$}$ ($\textcircled{$\Leftarrow$}$) a *modified jeu de taquin (backward) slide* into ρ performed on R (Q).

Lemma 2.3. *The two maps 2.2.*$\textcircled{$\Rightarrow$}$ *and 2.2.*$\textcircled{$\Leftarrow$}$ *are well defined. I.e. the tabloid R produced by* $\textcircled{$\Rightarrow$}$ *is indeed a reverse SSYT with $R_{ij} \leq a + \mu_i - i$ and the tabloid Q produced by* $\textcircled{$\Leftarrow$}$ *is indeed a reverse SSYT. Also, the equation $n(Q) = n(R) + w_c(T)$ holds.*

Furthermore, the following statement is true: Suppose that $\textcircled{$\Rightarrow$}$ *performs a modified jeu de taquin slide on R into a cell ρ_1 with $R_{\rho_1} = e$. After this, suppose that another modified jeu de taquin slide on R into a cell ρ_2 with the same entry e is performed. Let ρ_1' and ρ_2' be the cells where the slides stop. Then ρ_1' is left of ρ_2' or $\rho_1' = \rho_2'$. A corresponding statement holds for Algorithm 2.2.*$\textcircled{$\Leftarrow$}$.

Proof. First of all, we have to prove that Algorithm 2.2.$\textcircled{$\Rightarrow$}$ terminates. We required that $a + c(\tau) > 0$ for all cells τ, which implies that every time when we replace the entry in cell ρ by e (see the beginning of the outer loop of the algorithm) we decrease $\max_{\tau=(i,j)}(R_\tau - a - \mu_i + i)$. It is easy to see that this maximum is never increased in the subsequent steps of the algorithm.

It is easy to check that after every type of replacement within the modified jeu de taquin slides, the validity of the equation $n(Q) = n(R) + w_c(T)$ is preserved.

So it remains to show that after every modified jeu de taquin slide of $\textcircled{$\Rightarrow$}$, the resulting filling R of λ/μ is in fact a reverse SSYT: We have that $Q_\tau - \big(a + c(\tau)\big) = e$ is maximal at the very left of λ/μ, because rows are decreasing in Q. Therefore, when $Q_\tau > a + \mu_i - i$, as required for the execution of the outer loop of $\textcircled{$\Rightarrow$}$, we have

$$e = Q_\tau - \big(a + c(\tau)\big) > a + \mu_i - i - (a + \mu_i + 1 - i) = -1,$$

so e is non-negative. Furthermore, after either type of replacement during the modified jeu de taquin slide, the only possible violations of decrease along rows or strict decrease along columns can involve only the entry e and the entries to the right and below. By induction, R must be a reverse SSYT.

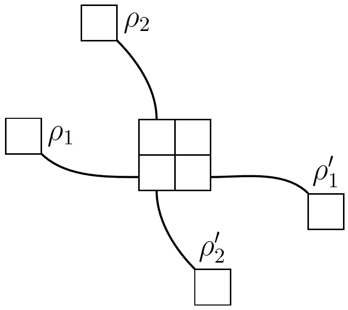


Figure 4.

The second statement of the lemma is shown with an argument similar to that used in the proof of Theorem 2.1.

When the jeu de taquin forward slide in $\textcircled{\scriptsize$\Rightarrow$}$ is performed into the cell ρ_1, the entry e describes a path from ρ_1 to the cell ρ'_1, where the slide stops. Similarly, we have a path from ρ_2 to ρ'_2. We conclude that, if ρ'_1 were strictly to the right of ρ'_2, that these paths would have to cross. (See Figure 4). Hence we had the following situation:

First, (the star is a placeholder for an entry we do not know)

$$\begin{array}{|c|c|}\hline * & z \\ \hline e & x \\ \hline\end{array} \quad \text{would be replaced by} \quad \begin{array}{|c|c|}\hline * & z \\ \hline x-1 & e \\ \hline\end{array}.$$

In this situation, x would have to be strictly smaller then z.

Then, when the modified jeu de taquin slide into ρ_2 is performed, the following situation would arise at the same four cells:

$$\begin{array}{|c|c|}\hline e & z \\ \hline x-1 & * \\ \hline\end{array} \quad \text{would have to be replaced by} \quad \begin{array}{|c|c|}\hline x & z \\ \hline e & * \\ \hline\end{array}.$$

But this cannot happen, because then x would have to be at least as big as z is.

The corresponding statement for Algorithm 2.2.$\textcircled{\scriptsize$\Leftarrow$}$ is shown similarly. $\square$

Proof of Theorem 2.2. It remains to show, that $\textcircled{\scriptsize$\Rightarrow$}$ and $\textcircled{\scriptsize$\Leftarrow$}$ are inverse to each other. This is pretty obvious considering the lemma:

Suppose that the pair (R, T) is an intermediate result obtained after a modified jeu de taquin slide into the cell ρ. After this, $T_{\rho'}$ is increased, where ρ' is the cell where the slide stopped. Then the entry in ρ' must be among the smallest entries of R, so that $T_{\rho'} \neq 0$, because the sequence of e's in the cells chosen for the modified jeu de taquin slides is monotonically decreasing. If there is more than one cell ρ which contains a minimal entry of R and satisfies $T_\rho \neq 0$, the lemma asserts that the right-most cell was the last cell chosen for the modified jeu de taquin slide $\textcircled{\scriptsize$\Rightarrow$}$.

Hence it is certain that the right-most cell containing a minimal entry as selected before the modified jeu de taquin slide of $\Leftarrow$ is ρ'. It is easy to check, that the replacements done in $\Leftarrow$ are exactly inverse to those in $\Rightarrow$. For example, suppose the following replacement is performed in $\Rightarrow$:

$$\begin{array}{cc} & \boxed{z} \\ \boxed{e} & \boxed{x} \\ \boxed{y} & \end{array} \quad \text{is replaced by} \quad \begin{array}{cc} & \boxed{z} \\ \boxed{x-1} & \boxed{e} \\ \boxed{y} & \end{array}.$$

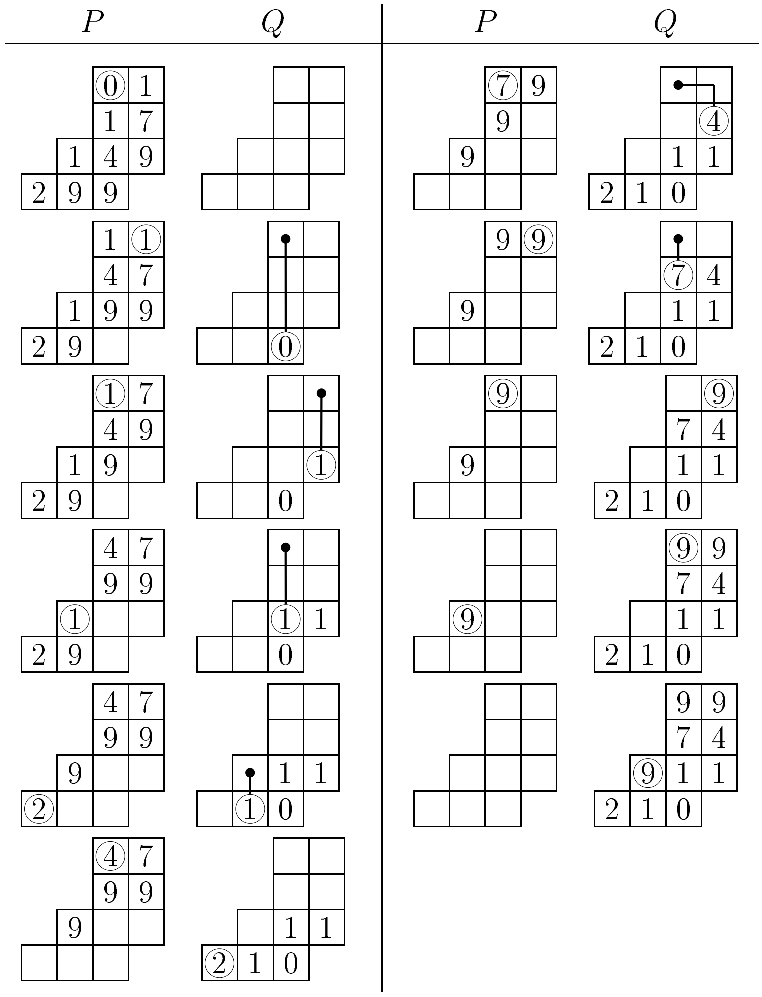
Then we had $x-1 > y$ and, because of strictly decreasing columns, $z > x$. Therefore, in $\Leftarrow$, this is reversed and we end up with the original situation.

Similarly, we can show that $\Rightarrow$ is inverse to $\Leftarrow$, too. $\square$

Appendix A: Step by step example

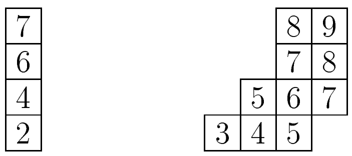
This appendix contains a complete example for the algorithms described above for a SSYT of shape $(4,4,4,3)/(2,2,1)$ and $a = 6$.

First the SSYT P on the left of Figure 2 is transformed into the reverse SSYT Q in the middle of Figure 2 using Algorithm 2.1.$\Rightarrow$. The example has to be read in the following way: Each pair (P, Q) in the table depicts an intermediate result of the algorithm. The cell of P containing the encircled entry is the cell into which the next jeu de taquin slide is performed. The jeu de taquin path is indicated by the line in Q.



The inverse transformation $\Leftarrow$ of the reverse SSYT Q into the SSYT P can be traced in the same table, we only have to start at the right bottom, where the tableau P is empty, and work our way upwards to the top left of the table. Note that the jeu de taquin paths are the same.

In the second step of the bijection, this reverse SSYT Q is mapped onto a pair (R,T), where R is a reverse SSYT with $R_{ij} \leq a + \mu_i - i$, T is a tabloid and $n(Q) = n(R) + w_c(T)$.

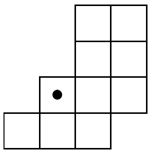
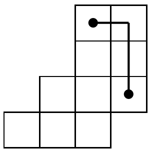
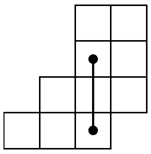
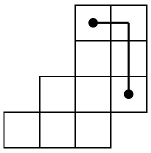


a. $a + \mu_i - i$ b. The tabloid with entries $a + c(\rho)$

Figure 5.

First, the algorithm initialises R to Q and sets all entries of T to zero. Using modified jeu de taquin slides, R is then transformed into a reverse SSYT where the entries are bounded as required. First the algorithm checks whether there are still cells in R which are too large. For reference, we give the relevant bounds in Figure 5.a. Then, for selecting the cell into which the modified jeu de taquin slide is performed, we need to calculate $R_\rho - (a + c(\rho))$. Again, for reference we display these values for each cell in Figure 5.b.

Each row of the table below depicts an intermediate result of Algorithm 2.2.$\Rightarrow$. The cells containing the encircled entry are the cells into which the modified jeu de taquin slide will be performed, the cells containing the boxed entry indicate, where the last modified jeu de taquin slide stopped. In the third column the jeu de taquin path for the selected cell is indicated.

R	T	jeu de taquin path
$\begin{array}{cccc} & & 9 & 9 \\ & & 7 & 4 \\ & 9 & 1 & 1 \\ 2 & 1 & 0 & \end{array}$	$\begin{array}{cccc} & & 0 & 0 \\ & & 0 & 0 \\ & 0 & 0 & 0 \\ 0 & 0 & 0 & \end{array}$	
$\begin{array}{cccc} & & 9 & 9 \\ & & 7 & 4 \\ & \boxed{4} & 1 & 1 \\ 2 & 1 & 0 & \end{array}$	$\begin{array}{cccc} & & 0 & 0 \\ & & 0 & 0 \\ & 1 & 0 & 0 \\ 0 & 0 & 0 & \end{array}$	
$\begin{array}{cccc} & & 8 & 5 \\ & & 7 & 2 \\ & 4 & 1 & \boxed{1} \\ 2 & 1 & 0 & \end{array}$	$\begin{array}{cccc} & & 0 & 0 \\ & & 0 & 0 \\ & 1 & 0 & 1 \\ 0 & 0 & 0 & \end{array}$	
$\begin{array}{cccc} & & 8 & 5 \\ & & 2 & 2 \\ & 4 & 1 & 1 \\ 2 & 1 & \boxed{0} & \end{array}$	$\begin{array}{cccc} & & 0 & 0 \\ & & 0 & 0 \\ & 1 & 0 & 1 \\ 0 & 0 & 1 & \end{array}$	
$\begin{array}{cccc} & & 4 & 3 \\ & & 2 & 2 \\ & 4 & 1 & \boxed{0} \\ 2 & 1 & 0 & \end{array}$	$\begin{array}{cccc} & & 0 & 0 \\ & & 0 & 0 \\ & 1 & 0 & 2 \\ 0 & 0 & 1 & \end{array}$	

Again, the inverse transformation $\Leftarrow$ can be traced in the same table, starting at the bottom, moving upwards. Now the cells containing the boxed entry are the cells into which the next modified jeu de taquin slide will be performed, the cells containing the encircled entry indicate where the last slide stopped. Of course, the jeu de taquin paths are the same as for $\Rightarrow$.

Appendix B: A complete matchup for SSYT'x of shape $(3,2)/(1)$ with norm 5, where $a = 2$

In the table below you find a complete matchup for SSYT'x of shape $(3,2)/(1)$ with norm 5 where $a = 2$. The first column contains all SSYT'x of shape $(3,2)/(1)$ and norm 5. In the second column, the corresponding reverse SSYT'x obtained by evacuation are displayed. Finally, in columns three and four, the results of Algorithm 2.2.⇔ can be found.

This table was produced with a Common-LISP-implementation of the algorithms above, which can be found on the author's homepage.[2]

P		Q		R	T
$\begin{array}{ccc} & 0 & 0 \\ 0 & 5 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 5 & 0 \\ 0 & 0 & \end{array}$	$\Leftrightarrow$	$\Bigl(\begin{array}{ccc} & 2 & 0 \\ 0 & 0 & \end{array}$,	$\begin{array}{ccc} & 1 & 0 \\ 0 & 0 & \end{array}\Bigr)$
$\begin{array}{ccc} & 0 & 0 \\ 1 & 4 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 4 & 0 \\ 1 & 0 & \end{array}$	$\Leftrightarrow$	$\Bigl(\begin{array}{ccc} & 1 & 0 \\ 0 & 0 & \end{array}$,	$\begin{array}{ccc} & 1 & 0 \\ 1 & 0 & \end{array}\Bigr)$
$\begin{array}{ccc} & 0 & 0 \\ 2 & 3 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 3 & 0 \\ 2 & 0 & \end{array}$	$\Leftrightarrow$	$\Bigl(\begin{array}{ccc} & 1 & 0 \\ 0 & 0 & \end{array}$,	$\begin{array}{ccc} & 0 & 0 \\ 2 & 1 & \end{array}\Bigr)$
$\begin{array}{ccc} & 0 & 1 \\ 0 & 4 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 1 & 0 \\ 4 & 0 & \end{array}$	$\Leftrightarrow$	$\Bigl(\begin{array}{ccc} & 1 & 0 \\ 0 & 0 & \end{array}$,	$\begin{array}{ccc} & 0 & 0 \\ 4 & 0 & \end{array}\Bigr)$
$\begin{array}{ccc} & 0 & 1 \\ 1 & 3 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 3 & 0 \\ 1 & 1 & \end{array}$	$\Leftrightarrow$	$\Bigl(\begin{array}{ccc} & 2 & 0 \\ 0 & 0 & \end{array}$,	$\begin{array}{ccc} & 0 & 0 \\ 1 & 1 & \end{array}\Bigr)$
$\begin{array}{ccc} & 0 & 1 \\ 2 & 2 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 2 & 0 \\ 2 & 1 & \end{array}$	$\Leftrightarrow$	$\Bigl(\begin{array}{ccc} & 2 & 0 \\ 0 & 0 & \end{array}$,	$\begin{array}{ccc} & 0 & 0 \\ 1 & 1 & \end{array}\Bigr)$
$\begin{array}{ccc} & 0 & 2 \\ 0 & 3 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 2 & 0 \\ 3 & 0 & \end{array}$	$\Leftrightarrow$	$\Bigl(\begin{array}{ccc} & 2 & 0 \\ 0 & 0 & \end{array}$,	$\begin{array}{ccc} & 0 & 0 \\ 3 & 0 & \end{array}\Bigr)$
$\begin{array}{ccc} & 0 & 2 \\ 1 & 2 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 2 & 2 \\ 1 & 0 & \end{array}$	$\Leftrightarrow$	$\Bigl(\begin{array}{ccc} & 2 & 2 \\ 0 & 0 & \end{array}$,	$\begin{array}{ccc} & 0 & 0 \\ 1 & 0 & \end{array}\Bigr)$
$\begin{array}{ccc} & 1 & 1 \\ 0 & 3 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 1 & 1 \\ 3 & 0 & \end{array}$	$\Leftrightarrow$	$\Bigl(\begin{array}{ccc} & 1 & 1 \\ 0 & 0 & \end{array}$,	$\begin{array}{ccc} & 0 & 0 \\ 3 & 0 & \end{array}\Bigr)$
$\begin{array}{ccc} & 1 & 1 \\ 1 & 2 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 2 & 1 \\ 1 & 1 & \end{array}$	$\Leftrightarrow$	$\Bigl(\begin{array}{ccc} & 2 & 1 \\ 0 & 0 & \end{array}$,	$\begin{array}{ccc} & 0 & 0 \\ 0 & 1 & \end{array}\Bigr)$

[2] `http://www.mat.univie.ac.at/~rubey/biject.lisp`

P		Q		R	T
$\begin{array}{ccc} & 0 & 3 \\ 0 & 2 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 3 & 2 \\ 0 & 0 & \end{array}$	$\Leftrightarrow$	$\left(\begin{array}{ccc} & 1 & 0 \\ 0 & 0 & \end{array} \right.,$	$\left. \begin{array}{ccc} & 0 & 1 \\ 0 & 0 & \end{array} \right)$
$\begin{array}{ccc} & 0 & 3 \\ 1 & 1 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 3 & 1 \\ 1 & 0 & \end{array}$	$\Leftrightarrow$	$\left(\begin{array}{ccc} & 1 & 1 \\ 0 & 0 & \end{array} \right.,$	$\left. \begin{array}{ccc} & 0 & 0 \\ 1 & 1 & \end{array} \right)$
$\begin{array}{ccc} & 1 & 2 \\ 0 & 2 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 2 & 1 \\ 2 & 0 & \end{array}$	$\Leftrightarrow$	$\left(\begin{array}{ccc} & 2 & 1 \\ 0 & 0 & \end{array} \right.,$	$\left. \begin{array}{ccc} & 0 & 0 \\ 2 & 0 & \end{array} \right)$
$\begin{array}{ccc} & 0 & 4 \\ 0 & 1 & \end{array}$	$\Leftrightarrow$	$\begin{array}{ccc} & 4 & 1 \\ 0 & 0 & \end{array}$	$\Leftrightarrow$	$\left(\begin{array}{ccc} & 1 & 1 \\ 0 & 0 & \end{array} \right.,$	$\left. \begin{array}{ccc} & 1 & 0 \\ 0 & 0 & \end{array} \right)$

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[1] Sara C. Billey, William Jockusch, and Richard P. Stanley, *Some combinatorial properties of Schubert polynomials*, Journal of Algebraic Combinatorics **2** (1993), no. 4, 345–374.

[2] Christian Krattenthaler, *An involution principle-free bijective proof of Stanley's hook-content formula*, Discrete Mathematics and Theoretical Computer Science (1998), no. 3, 11–32.

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[4] Bruce E. Sagan, *The symmetric group*, Wadsworth & Brooks/Cole, Pacific Grove, California, 1987.

[5] Richard P. Stanley, *Enumerative combinatorics*, vol. 2, Cambridge University Press, 1999.

Chapter 2
Asymptotic analysis of vicious walkers with arbitrary endpoints

Abstract

We derive asymptotic results for the number of configurations of vicious and ∞-friendly walkers with given starting points and varying end points, both in absence and presence of a wall. Thus we extend previous results by Krattenthaler, Guttmann and Viennot [*J. Phys. A: Math. Gen.* **33** (2000), 8835-8866]. In our proofs we follow closely arguments given in the latter article.

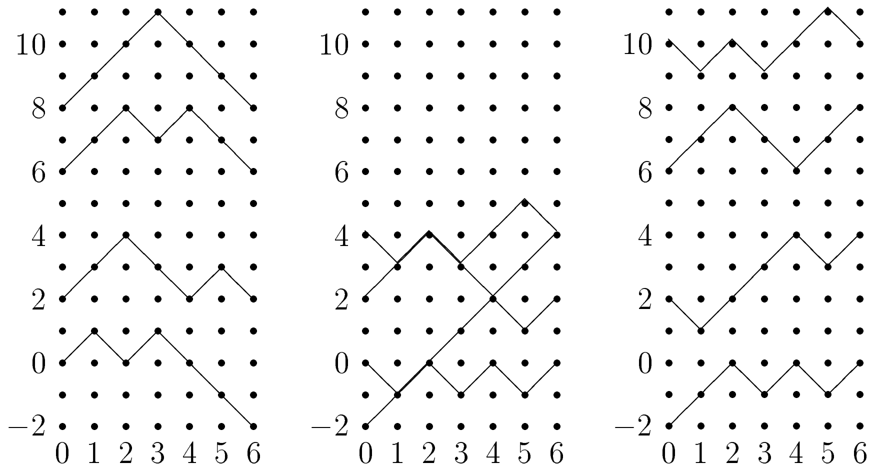
1 Introduction

A configuration of *vicious walkers* is a set of p non-intersecting lattice paths in $\mathbb{Z}^2$ with steps $(1,1)$ and $(1,-1)$ that start at $(0,2a_i)$ and terminate at (m,e_i), where e_i must have the same parity as m, for $i \in \{1,2,\ldots,p\}$. A family of lattice paths is called non-intersecting, if no two paths have a lattice point in common. The term *vicious* comes from the conception, that two walkers that arrive at the same lattice site annihilate each other. An example of such a configuration is depicted in Figure 1.a.

We will also consider vicious walkers that are additionally constrained by an impenetrable wall, that is, non-intersecting lattice paths that must not run below the x-axis. In this paper we derive exact asymptotics for the number of configurations of vicious walkers in both models, where the starting points are fixed but the end points may vary, see Theorems 3.1 and 4.1.

As corollaries, we obtain asymptotics for the number of configurations of so called ∞-friendly walkers. In this model, any number of paths may share an arbitrary number of lattice sites, but paths never change sides. In Figure 1.b an example for a configuration in this model can be found. For more information on the various models the reader is referred to the introduction of [5]. For the proofs, we closely follow derivations which can be found in the latter article.

In Section 3 we will be concerned with the case where the lattice paths are unconstrained. As ingredients for our calculations we will need the following: The Lindström-Gessel-Viennot theorem on non-intersecting lattice paths, some knowledge of Schur functions (the irreducible characters of special linear groups), the Poisson summation theorem and Mehta's integral.



a. a configuration of vicious walkers with $a_1 = 0$, $a_2 = 1$, $a_3 = 3$, $a_4 = 4$ and $m = 6$

b. a configuration of ∞-friendly walkers with $a_1 = -2$, $a_2 = 0$, $a_3 = 2$ and $a_4 = 4$

c. a family of non-intersecting lattice paths corresponding to the configuration in b.

Figure 1.

The other case, where the lattice paths must not go below the x-axis, is treated in Section 4. There we additionally need the reflection principle by D. André. Instead of Schur functions, we need the odd orthogonal characters and instead of Mehta's integral we need an 'orthogonal' analogue of Selberg's integral.

In the following section we will state these theorems and give appropriate references.

2 Ingredients

In this section we want to outline the method we use to obtain our results. Also, we list the various theorems we apply.

First we consider the problem of enumerating non-intersecting lattice paths with *fixed* starting and end points. The Lindström–Gessel–Viennot determinant reduces this problem to the problem of counting the number of *single* lattice paths with a given starting and end point:

Theorem 2.1 (Lindström-Gessel-Viennot). *Let $A_1, A_2, \ldots, A_p$ and $E_1, E_2, \ldots, E_p$ be lattice points, with the property that if $1 \leq i < j \leq p$ and $1 \leq k < l \leq p$, then any path from A_i to E_l must intersect any path from A_j to E_k. Then the number of families $(P_1, P_2, \ldots, P_p)$ of non-intersecting lattice paths, where P_i runs from A_i to E_i, $i \in \{1, 2, \ldots, p\}$, is given by*

$$\det_{1 \leq i,j \leq p} \big(|P(A_j \to E_i)| \big),$$

where $P(A \to E)$ denotes the set of all lattice paths from A to E.

Proof. See [6, Lemma 1] or [4, Corollary 2]. $\square$

In the case of vicious walkers without a wall, the number of lattice paths with steps $(1,1)$ and $(1,-1)$ from $(0,2a)$ to (m,e) is equal to $\binom{m}{\frac{1}{2}(m-e)+a}$. In the case of the presence of a wall, the reflection principle by D. André, see [1] or [2, page 22], shows that the number of lattice paths with steps $(1,1)$ and $(1,-1)$ from $(0,2a)$ to (m,e) which do not go below the x-axis equals $\binom{m}{\frac{1}{2}(m-e)+a} - \binom{m}{\frac{1}{2}(m+e+2)+a}$.

Thus, returning to our problem, we obtain a sum over all possible end points of the lattice paths in question of the respective Lindström-Gessel-Viennot determinant. In a second step, we extract some factors from each of the determinants, such that the leading term of the resulting determinant – viewed as polynomial in m – is roughly the numerator of a Weyl character.

Lemma 2.2. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ be a partition, i.e. a weakly decreasing sequence of non-negative integers. Then the Schur function $s_\lambda(x_1, x_2, \ldots, x_p)$ is defined by*

$$s_\lambda(x_1, x_2, \ldots, x_p) = \frac{\det_{1\le i,j\le p}(x_j^{\lambda_i+p-i})}{\det_{1\le i,j\le p}(x_j^{p-i})}.$$

We have

$$s_\lambda(1,1,\ldots,1) = \prod_{1\le i<j\le p} \frac{\lambda_i - i - \lambda_j + j}{j-i} \text{ and}$$

$$\det_{1\le i,j\le p}(x_j^{i-1}) = \prod_{1\le i<j\le p}(x_j - x_i).$$

The odd orthogonal character $so_\lambda(x_1^{\pm1}, x_2^{\pm1}, \ldots, x_p^{\pm1}, 1)$ is defined by

$$so_\lambda(x_1^{\pm1}, x_2^{\pm1}, \ldots, x_p^{\pm1}, 1) = \frac{\det_{1\le i,j\le p}(x_j^{\lambda_i+p-i+1/2} - x_j^{-(\lambda_i+p-i+1/2)})}{\det_{1\le i,j\le p}(x_j^{p-i+1/2} - x_j^{-(p-i+1/2)})}$$

and we have

$$so_\lambda(1,1,\ldots,1) = \prod_{1\le i<j\le p} \frac{\lambda_i - i - \lambda_j + j}{j-i} \prod_{1\le i\le j\le p} \frac{2p+1+\lambda_i - i + \lambda_j - j}{2p+1-i-j} \text{ and}$$

$$\det_{1\le i,j\le p}(x_j^{i-1/2} - x_j^{-(i-1/2)}) = \prod_{j=1}^{p} x_j^{-p+1/2}(x_j - 1) \prod_{1\le i<j\le p}(x_i - x_j)(1 - x_i x_j).$$

Proof. See, for example, [3, (24.29) and (A.30 (ii))]. $\square$

After that, it remains to approximate a sum of the form

$$\sum_{e_1<e_2<\cdots<e_p} f(e_1, e_2, \ldots, e_p) e^{-\sum_{j=1}^{p} e_j^2/m},$$

where f is a polynomial in $e_1, e_2, \ldots, e_p$. Applying the following lemma, which relies on the Poisson summation theorem, we transform the sum into an integral:

Lemma 2.3. *Let N be a non-negative integer and let $b : \mathbb{N} \to \mathbb{N}$ be an arbitrary function. Furthermore, let $p : \mathbb{N} \to \mathbb{R}$ be a function of at most polynomial growth. Then, as m tends to infinity,*

$$\sum_{k=b(m)}^{\infty} p(k)e^{-k^2/m} = \int_{b(m)}^{\infty} p(y)e^{-y^2/m}dy + O(1),$$

where the constant in the error term $O(1)$ is independent of b.

Proof. See [5, Lemma A1]. Note, however, that the second equality stated there is incorrect in the case of arbitrary b. However, we do not need this equality. $\square$

Finally, the integral can be computed by the following:

Lemma 2.4. *Let k be a complex number with positive real part. Then*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \le i < j \le p} |x_j - x_i|^{2k} \exp\left[-\frac{1}{2}\sum_{j=1}^{p} x_j^2\right] dx_1 dx_2 \dots dx_p$$
$$= (2\pi)^{p/2} \prod_{l=1}^{p} \frac{\Gamma(lk+1)}{\Gamma(k+1)}. \quad (1)$$

Let k_1, k_2 and k_3 be a complex number with positive real part. Then

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{p} |x_j|^{k_1+k_3} \prod_{1 \le i < j \le p} \left|x_j^2 - x_i^2\right|^{k_2} \exp\left[-\frac{1}{2}\sum_{j=1}^{p} x_j^2\right] dx_1 dx_2 \dots dx_p$$
$$= (2^{1-k_1-k_3}\pi)^{p/2} \prod_{l=1}^{p} \frac{\Gamma(\frac{1}{2}lk_2+1)\Gamma(k_1+k_3+(l-1)k_2+1)}{\Gamma(\frac{1}{2}k_2+1)\Gamma(\frac{1}{2}(k_1+k_3+(l-1)k_2)+1)}. \quad (2)$$

Proof. Proofs can be found in [7, (4.1) and Conjecture 6.1] $\square$

3 Vicious walkers without a wall

In this section we derive results for vicious walkers with given starting points and varying end points, where there is no wall restriction.

Theorem 3.1. *The number of configurations of p vicious walkers with starting points $(0, 2a_i)$, $i \in \{1, 2, \dots, p\}$, where each walker does m steps, is asymptotically*

$$2^{mp+p^2/4} m^{-p^2/4+p/4} \pi^{-p/4} \frac{\prod_{1\le i<j\le p}(a_j - a_i)}{\prod_{l=1}^{p/2}(2l-1)!} \times \left(1 + O(m^{-1/2}(\log m)^3)\right) \qquad \text{if } p \text{ is even}$$

$$2^{mp+p^2/4-1/4} m^{-p^2/4+p/4} \pi^{-p/4+1/4} \frac{\prod_{1\le i<j\le p}(a_j - a_i)}{\prod_{l=1}^{(p-1)/2}(2l)!} \times \left(1 + O(m^{-1/2}(\log m)^3)\right) \qquad \text{if } p \text{ is odd}$$

as m tends to infinity.

If we set in the theorem above $a_i = 2i - 2$ for $i \in 1, 2, \ldots, p$, we regain [5, Theorem 3], albeit with a worse error term:

Corollary 3.2. *The number of stars with p branches of length m, i.e., configurations of p vicious walkers with starting points $(0, 2i-2)$, $i \in \{1, 2, \ldots, p\}$, where each walker does m steps, is asymptotically*

$$\begin{aligned} &2^{mp+p^2/4} m^{-p^2/4+p/4} \pi^{-p/4} \prod_{l=1}^{p/2} (2l-2)! \\ &\qquad \times \left(1 + O(m^{-1/2} (\log m)^3)\right) && \text{if } p \text{ is even} \\ &2^{mp+p^2/4-1/4} m^{-p^2/4+p/4} \pi^{-p/4+1/4} \prod_{l=1}^{(p-1)/2} (2l-1)! \\ &\qquad \times \left(1 + O(m^{-1/2} (\log m)^3)\right) && \text{if } p \text{ is odd} \end{aligned}$$

as m tends to infinity.

Asymptotics for the model concerning ∞-friendly walkers can also be deduced from Theorem 3.1: Given a family of non-crossing paths with starting points $(0, 2a_i)$, shifting the i^{th} path by $2i-2$ units up, $i \in \{1, 2, \ldots, p\}$, we obtain a family of non-intersecting paths with starting points $(0, a_i + 2i - 2)$. An instance of this correspondence is depicted in Figure 1.b and Figure 1.c. It is obvious, that this correspondence is a bijection. Thus, if we replace a_i in Theorem 3.1 by $a_i + 2i - 2$ for $i \in 1, 2, \ldots, p$, we obtain asymptotics for the number of configurations of ∞-friendly walkers with starting points $(0, 2a_i)$.

In this vein, if we set $a_i = 4i - 4$ for $i \in 1, 2, \ldots, p$ in Theorem 3.1, we regain [5, Theorem 4]:

Corollary 3.3. *The number of ∞-friendly stars in the TK model with p branches of length m, i.e., configurations of p vicious walkers with starting points $(0, 2i-2)$, $i \in \{1, 2, \ldots, p\}$, where each walker does m steps, and any number of walkers may share an arbitrary number of steps, is asymptotically*

$$\begin{aligned} &2^{mp+3p^2/4-p/2} m^{-p^2/4+p/4} \pi^{-p/4} \prod_{l=1}^{p/2} (2l-2)! \\ &\qquad \times \left(1 + O(m^{-1/2} (\log m)^3)\right) && \text{if } p \text{ is even} \\ &2^{mp+3p^2/4-p/2-1/4} m^{-p^2/4+p/4} \pi^{-p/4+1/4} \prod_{l=1}^{(p-1)/2} (2l-1)! \\ &\qquad \times \left(1 + O(m^{-1/2} (\log m)^3)\right) && \text{if } p \text{ is odd} \end{aligned}$$

as m tends to infinity.

Proof of Theorem 3.1. Given lattice points $(0, 2a)$ and (m, e), where e has the same parity as m, there are $\binom{m}{\frac{1}{2}(m-e)+a}$ lattice paths with steps $(1, 1)$ and $(1, -1)$ from $(0, 2a)$ to (m, e). Applying Theorem 2.1, we obtain the following expression for the number of families of non-intersecting lattice paths with starting points $(0, 2a_i)$, $i \in \{1, 2, \ldots, p\}$, where each path has m steps:

$$\sum_{\substack{-m+2a_1 \le e_1 < e_2 < \cdots < e_p \le m+2a_p \\ e_i \equiv m \bmod 2 \text{ for } i \in \{1,2,\ldots,p\}}} \det_{1 \le i,j \le p} \left(\binom{m}{\frac{m-e_j}{2} + a_i} \right).$$

We now distinguish between two cases, depending on whether m is even or odd. The computations in both cases are, however, rather similar. Therefore, we carry them out in detail only for the first case: If m is even we may replace each summation index e_i by $2e_i$ for $i \in \{1, 2, \ldots, p\}$, and obtain

$$\begin{aligned} &\sum_{-\frac{m}{2}+a_1 \le e_1 < e_2 < \cdots < e_p \le \frac{m}{2}+a_p} \det_{1 \le i,j \le p} \left(\binom{m}{\frac{m}{2} + a_i - e_j} \right) \\ &= \sum_{-\frac{m}{2}+a_1 \le e_1 < e_2 < \cdots < e_p \le \frac{m}{2}+a_p} \left(\prod_{j=1}^{p} \frac{m!}{(\frac{m}{2} + a_p - e_j)!(\frac{m}{2} + a_p + e_j)!} \right) \\ &\qquad \cdot \det_{1 \le i,j \le p} \left(\left(\frac{m}{2} + a_p - e_j \right)^{\underline{a_p - a_i}} \left(\frac{m}{2} + a_p + e_j \right)^{\underline{a_p + a_i}} \right). \qquad (3) \end{aligned}$$

Here, $x^{\underline{m}}$ denotes the falling factorial power $x(x+1)\ldots(x-m+1)$. We will find asymptotic approximations for the product and the determinant in the last line of (3) separately.

We start by considering the determinant. We are only interested in its leading term considered as a polynomial in m. However, it turns out to be necessary to regard the determinant as a polynomial in m and the $e_1, e_2, \ldots, e_p$ in a first stage. Truncation of each entry of the original matrix to its leading terms yields the determinant

$$
\begin{aligned}
&\det_{1\le i,j\le p}\left(\left(\frac{m}{2}-e_j\right)^{a_p-a_i}\left(\frac{m}{2}+e_j\right)^{a_p+a_i}\right)\\
&\quad=\left(\prod_{j=1}^{p}\left(\frac{m}{2}-e_j\right)\left(\frac{m}{2}+e_j\right)\right)^{a_p}\det_{1\le i,j\le p}\left(\left(\frac{m+2e_j}{m-2e_j}\right)^{a_i}\right)\\
&\quad=\left(\prod_{j=1}^{p}\left(\frac{m}{2}-e_j\right)\left(\frac{m}{2}+e_j\right)\right)^{a_p}\det_{1\le i,j\le p}\left(\left(\frac{m+2e_j}{m-2e_j}\right)^{i-1}\right)\\
&\qquad\qquad\times s_{(a_p-p+1,a_{p-1}-p+2,\dots,a_1)}\left(\frac{m+2e_1}{m-2e_1},\frac{m+2e_2}{m-2e_2},\dots,\frac{m+2e_p}{m-2e_p}\right)\\
&\quad=\left(\prod_{j=1}^{p}\left(\frac{m}{2}-e_j\right)\left(\frac{m}{2}+e_j\right)\right)^{a_p}\left(\prod_{1\le i<j\le p}\frac{m+2e_j}{m-2e_j}-\frac{m+2e_i}{m-2e_i}\right)\\
&\qquad\qquad\times s_{(a_p-p+1,a_{p-1}-p+2,\dots,a_1)}\left(\frac{m+2e_1}{m-2e_1},\frac{m+2e_2}{m-2e_2},\dots,\frac{m+2e_p}{m-2e_p}\right)\\
&\quad=\left(\prod_{j=1}^{p}\left(\frac{m}{2}-e_j\right)\left(\frac{m}{2}+e_j\right)\right)^{a_p}\left(\prod_{1\le i<j\le p}\frac{4m(e_j-e_i)}{(m-2e_j)(m-2e_i)}\right)\\
&\qquad\qquad\times s_{(a_p-p+1,a_{p-1}-p+2,\dots,a_1)}\left(\frac{m+2e_1}{m-2e_1},\frac{m+2e_2}{m-2e_2},\dots,\frac{m+2e_p}{m-2e_p}\right)\\
&\quad=2^{-2pa_p+2\binom{p}{2}}m^{\binom{p}{2}}\left(\prod_{j=1}^{p}(m-2e_j)^{a_p-p+1}(m+2e_j)^{a_p}\right)\left(\prod_{1\le i<j\le p}(e_j-e_i)\right)\\
&\qquad\qquad\times s_{(a_p-p+1,a_{p-1}-p+2,\dots,a_1)}\left(\frac{m+2e_1}{m-2e_1},\frac{m+2e_2}{m-2e_2},\dots,\frac{m+2e_p}{m-2e_p}\right).
\end{aligned}
$$

We now observe that both the original determinant in (3) and the result of the calculation above have degree $2pa_p$, when considered as polynomials in m and $e_1, e_2, \dots, e_p$. Thus, the coefficients of the leading terms must be equal, since we omitted only terms of lower degree. Furthermore, both polynomials are divisible by $\prod_{1\le i<j\le p}(e_j-e_i)$, which implies that, considered as polynomials in m only, their leading terms are equal, too.

On ignoring again terms whose contribution to the overall asymptotics are negligible and applying Lemma 2.2, we obtain that the determinant in (3) is equal to

$$2^{-2pa_p+2\binom{p}{2}}m^{2pa_p-\binom{p}{2}}\frac{\prod_{1\le i<j\le p}(a_j-a_i)(e_j-e_i)}{\prod_{l=1}^{p-1}l!}\left(1+O(1/m)\right). \tag{4}$$

Next, we want to find an asymptotic approximation for the product in Equation (3). If e_j is positive, we have

$$\begin{aligned}
&\frac{m!}{(\frac{m}{2}+a_p-e_j)!(\frac{m}{2}+a_p+e_j)!}\\
&=\frac{m!}{\left((\frac{m}{2}+a_p)!\right)^2}\prod_{l=1}^{e_j}\frac{1+\frac{2}{m}(a_p+l-e_j)}{1+\frac{2}{m}(a_p+l)}\\
&=\frac{m!}{\left((\frac{m}{2}+a_p)!\right)^2}\exp\left[-\frac{2e_j^2}{m}+O(e_j^3/m^2)\right]\\
&=2^{m+2a_p+\frac{1}{2}}m^{-2a_p-\frac{1}{2}}\pi^{-\frac{1}{2}}\left(1+O(1/m)\right)\exp\left[-\frac{2e_j^2}{m}\right]\left(1+O(e_j^3/m^2)\right).
\end{aligned}$$

If e_j is negative, there is a similar calculation that leads to the same result. Now we see that the dominant terms of the sum in (3) are those with $-\sqrt{m}\log m < e_1 < e_2 < \cdots < e_p < \sqrt{m}\log m$: For these terms, we obtain

$$\begin{aligned}
&\prod_{j=1}^{p}\frac{m!}{(\frac{m}{2}+a_p-e_j)!(\frac{m}{2}+a_p+e_j)!}\\
&\qquad=2^{p(m+2a_p+\frac{1}{2})}m^{-p(2a_p+\frac{1}{2})}\pi^{-\frac{p}{2}}\exp\left[-\frac{2}{m}\sum_{j=1}^{p}e_j^2\right]\left(1+O(m^{-1/2}(\log m)^3)\right). \qquad (5)
\end{aligned}$$

However, if, for example, $e_p \geq \sqrt{m}\log m$, then $\exp\left[-\frac{2e_p^2}{m}\right] \leq m^{-2\log m}$. Because of $1+O(e_j^3/m^2)=O(m)$ for $j\in\{1,2,\ldots,p\}$ and $\exp\left[-\frac{2e_j^2}{m}\right]\leq 1$ for $j\in\{1,2,\ldots,p-1\}$ we obtain

$$\prod_{j=1}^{p}\frac{m!}{(\frac{m}{2}+a_p-e_j)!(\frac{m}{2}+a_p+e_j)!}=O\left(2^{p(m+2a_p+\frac{1}{2})}m^{-p(2a_p-\frac{1}{2})-2\log m}\right).$$

Since the asymptotic approximation of the determinant as given in (4) is of polynomial order in m, these summands are asymptotically negligible.

Substituting Equations (4) and (5) into (3) we obtain

$$\begin{aligned}
&\sum \left(\prod_{j=1}^{p} \frac{m!}{(\frac{m}{2}+a_p-e_j)!(\frac{m}{2}+a_p+e_j)!} \right. \\
&\qquad\qquad \left. \cdot \det_{1\le i,j\le p} \left(\left(\frac{m}{2}+a_i-e_j+1\right)_{a_p-a_i} \left(\frac{m}{2}-a_i+e_j+1\right)_{a_p+a_i} \right) \right) \\
&= \sum 2^{p(m+2a_p+\frac{1}{2})} m^{-p(2a_p+\frac{1}{2})} \pi^{-\frac{p}{2}} \exp\left[-\frac{2}{m} \sum_{j=1}^{p} e_j^2 \right] \\
&\qquad\qquad \cdot 2^{-2pa_p+2\binom{p}{2}} m^{2pa_p-\binom{p}{2}} \frac{\prod_{1\le i<j\le p}(a_j-a_i)(e_j-e_i)}{\prod_{l=1}^{p-1} l!} \\
&\qquad\qquad \times \left(1+O(m^{-1/2}(\log m)^3)\right) \\
&= 2^{p(m+p-\frac{1}{2})} m^{-\frac{p^2}{2}} \pi^{-\frac{p}{2}} \frac{\prod_{1\le i<j\le p}(a_j-a_i)}{\prod_{l=1}^{p-1} l!} \\
&\qquad\qquad \times \sum \left(\prod_{1\le i<j\le p} (e_j-e_i) \right) \exp\left[-\frac{2}{m} \sum_{j=1}^{p} e_j^2 \right] \\
&\qquad\qquad \times \left(1+O\big(m^{-1/2}(\log m)^3\big)\right),
\end{aligned} \tag{6}$$

where the sum is over $-\sqrt{m}\log m \le e_1 < e_2 < \cdots < e_p \le \sqrt{m}\log m$. In fact, we may extend the range of summation and sum over $-\sqrt{m}\log m \le e_1 \le e_2 \le \cdots \le e_p \le \sqrt{m}\log m$, since the expression inside the sum is zero if any two e_i should be the same. It remains to find an asymptotic approximation of this sum. To achieve this, we apply Lemma 2.3 successively for $j = p, p-1, \ldots, 1$ to each of the individual sums. We start with the sum over e_p:

$$\begin{aligned}
&\sum_{e_p=e_{p-1}}^{\sqrt{m}\log m} \left(\prod_{1\le i<p} (e_p-e_i) \right) \exp\left[-\frac{2e_p^2}{m} \right] \\
&\qquad = \sum_{e_p=e_{p-1}}^{\infty} \left(\prod_{1\le i<p} (e_p-e_i) \right) \exp\left[-\frac{2e_p^2}{m} \right] \times (1+O(1/m)) \\
&\qquad = \left(\int_{e_{p-1}}^{\infty} \left(\prod_{1\le i<p} (e_p-e_i) \right) \exp\left[-\frac{2e_p^2}{m} \right] de_p + O(1) \right) \\
&\qquad\quad \times (1+O(1/m)).
\end{aligned}$$

This integral is of at most polynomial growth in the variables $e_1, e_2, \ldots, e_{p-1}$, so we can apply Lemma 2.3 again and iterate. The result is that we obtain

$$\int \left(\prod_{1\le i<j\le p} (e_j-e_i) \right) \exp\left[-\frac{2}{m} \sum_{j=1}^{p} e_j^2 \right] de_1 de_2 \ldots de_p \times (1+O(1/m))$$

where the integral is over $-\infty < e_1 \le e_2 \le \cdots \le e_p < \infty$. Now we substitute $\frac{\sqrt{m}}{2}x_i$ for e_i, and introduce – seemingly superfluous – absolute values:

$$\left(\frac{\sqrt{m}}{2}\right)^{\binom{p+1}{2}} \int \left(\prod_{1\le i<j\le p} |x_j - x_i|\right) \exp\left[-\frac{1}{2}\sum_{j=1}^{p} x_j^2\right] dx_1 dx_2 \dots dx_p \\ \times \left(1 + O\left(1/m\right)\right).$$

With the absolute values, the integrand is invariant under permutations of the x_i. Thus, we can rewrite the last line as

$$\left(\frac{\sqrt{m}}{2}\right)^{\binom{p+1}{2}} \frac{1}{p!} \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\prod_{1\le i<j\le p} |x_j - x_i|\right) \exp\left[-\frac{1}{2}\sum_{j=1}^{p} x_j^2\right] dx_1 dx_2 \dots dx_p \\ \times \left(1 + O\left(1/m\right)\right).$$

Using (1) of Lemma 2.4 with $k = \frac{1}{2}$ gives

$$\left(\frac{\sqrt{m}}{2}\right)^{\binom{p+1}{2}} \frac{1}{p!}(2\pi)^{p/2} \prod_{l=1}^{p} \frac{\Gamma(l/2+1)}{\Gamma(3/2)} \times \left(1 + O\left(1/m\right)\right).$$

Combining this expression with the last line of Equation (6) we obtain the claimed expression for even m. The computation for odd m is similar. We leave the details to the reader. □

4 Vicious walkers with a wall

In this section we derive results for vicious walkers with given starting points and varying end points, where the walkers must not go below the x-axis.

Theorem 4.1. *The number of configurations of p vicious walkers with starting points $(0, 2a_i)$, $i \in \{1, 2, \dots, p\}$, where each walker does m steps and must not go below the x-axis, is asymptotically*

$$2^{mp+p^2-p/2} m^{-p^2/2} \pi^{-p/2} \prod_{l=1}^{p-1} \frac{l!}{(2l+1)!} \prod_{1\le i<j\le p} (a_j - a_i) \prod_{1\le i\le j\le p} (a_j + a_i + 1) \\ \times \left(1 + O(m^{-1/2}(\log m)^3)\right)$$

as m tends to infinity.

If we set in the theorem above $a_i = 2i - 2$ for $i \in 1, 2, \dots, p$, we regain [5, Theorem 8], albeit with a worse error term:

Corollary 4.2. *The number of stars with p branches of length m which do not go below the x-axis, i.e., configurations of p vicious walkers which do not go below the x-axis with starting points $(0, 2i-2)$, $i \in \{1, 2, \ldots, p\}$, where each walker does m steps, is asymptotically*

$$2^{mp+p^2-p/2} m^{-p^2/2} \pi^{-p/2} \left(\prod_{l=1}^{p-1} l! \right) \left(1 + O(m^{-1/2} (\log m)^3)\right)$$

as m tends to infinity.

Similarly, if we set $a_i = 4i - 4$ for $i \in 1, 2, \ldots, p$ in Theorem 4.1, we regain [5, Theorem 9]:

Corollary 4.3. *The number of ∞-friendly stars in the TK model with p branches of length m which do not go below the x- axis, i.e., configurations of p vicious walkers which do not go below the x-axis with starting points $(0, 2i-2)$, $i \in \{1, 2, \ldots, p\}$, where each walker does m steps, and any number of walkers may share an arbitrary number of steps, is asymptotically*

$$2^{mp+p^2-3p/2} m^{-p^2/2} \pi^{-p/2} \left(\prod_{l=1}^{p} \frac{(l-1)!(2l-2)!(4l-2)!}{(l+p-1)!^2} \right) \left(1 + O(m^{-1/2} (\log m)^3)\right)$$

as m tends to infinity.

More generally, if we replace a_i in Theorem 4.1 by $a_i + 2i - 2$ for $i \in 1, 2, \ldots, p$, we obtain asymptotics for the number of configurations of ∞-friendly walkers with starting points $(0, 2a_i)$.

Proof of Theorem 4.1. Let $(0, 2a)$ and (m, e) be lattice points such that e has the same parity as m. Applying the reflection principle of André, we find that there are $\binom{m}{\frac{1}{2}(m-e)+a} - \binom{m}{\frac{1}{2}(m+e+2)+a}$ lattice paths with steps $(1, 1)$ and $(1, -1)$ from $(0, 2a)$ to (m, e), which do not go below the x-axis.

By Theorem 2.1, we obtain the following expression for the number of families of non-intersecting lattice paths with starting points $(0, 2a_i)$, $i \in \{1, 2, \ldots, p\}$, which do not go below the x axis and where each path has m steps:

$$\sum_{\substack{0 \le e_1 < e_2 < \cdots < e_p \le m+2a_p \\ e_i \equiv m \bmod 2 \text{ for } i \in \{1,2,\ldots,p\}}} \det_{1 \le i,j \le p} \left(\binom{m}{\frac{1}{2}(m-e_j)+a_i} - \binom{m}{\frac{1}{2}(m+e_j+2)+a_i} \right).$$

Again we have to distinguish between two cases, depending on whether m is even or odd. Because the computations in both cases are rather similar, we carry them out in detail only for the case where m is odd: We replace each summation index e_i

by $2e_i - 1$ for $i \in \{1, 2, \dots, p\}$, and obtain

$$\sum_{0 \le e_1 < e_2 < \dots < e_p \le \frac{m+1}{2} + a_p} \det_{1 \le i,j \le p} \left(\binom{m}{\frac{m+1}{2} + a_i - e_j} - \binom{m}{\frac{m+1}{2} + a_i + e_j} \right)$$

$$= \sum_{0 \le e_1 < e_2 < \dots < e_p \le \frac{m+1}{2} + a_p} \left(\prod_{j=1}^{p} \frac{m!}{(\frac{m+1}{2} + a_p - e_j)!(\frac{m+1}{2} + a_p + e_j)!} \right)$$

$$\cdot \det_{1 \le i,j \le p} \left(\left(\frac{m+1}{2} + a_p - e_j \right)^{\underline{a_p - a_i}} \left(\frac{m-1}{2} + a_p + e_j + 1 \right)^{\underline{a_p + a_i + 1}} \right.$$

$$\left. - \left(\frac{m+1}{2} + a_p + e_j \right)^{\underline{a_p - a_i}} \left(\frac{m-1}{2} + a_p - e_j + 1 \right)^{\underline{a_p + a_i + 1}} \right), \quad (7)$$

where, $x^{\underline{m}}$ denotes the falling factorial power $x(x+1)\dots(x-m+1)$. We will find asymptotic approximations for the product and the determinant in the last line of (7) separately.

We start by considering the determinant. We are only interested in its leading term considered as a polynomial in m. However, it turns out to be necessary to regard the determinant as a polynomial in m and the $e_1, e_2, \dots, e_p$ in a first stage. Truncation of each entry of the original matrix to its leading terms yields the determinant

$$\det_{1 \le i,j \le p} \left(\left(\frac{m}{2} - e_j \right)^{a_p - a_i} \left(\frac{m}{2} + e_j \right)^{a_p + a_i + 1} - \left(\frac{m}{2} + e_j \right)^{a_p - a_i} \left(\frac{m}{2} - e_j \right)^{a_p + a_i + 1} \right)$$

$$= \left(\prod_{j=1}^{p} \left(\frac{m}{2} - e_j \right) \left(\frac{m}{2} + e_j \right) \right)^{a_p + \frac{1}{2}}$$

$$\times \det_{1 \le i,j \le p} \left(\left(\frac{m + 2e_j}{m - 2e_j} \right)^{a_i + \frac{1}{2}} - \left(\frac{m + 2e_j}{m - 2e_j} \right)^{-a_i - \frac{1}{2}} \right)$$

$$= \left(\prod_{j=1}^{p} \left(\frac{m}{2} - e_j \right) \left(\frac{m}{2} + e_j \right) \right)^{a_p + \frac{1}{2}}$$

$$\times \det_{1 \le i,j \le p} \left(\left(\frac{m + 2e_j}{m - 2e_j} \right)^{i - \frac{1}{2}} - \left(\frac{m + 2e_j}{m - 2e_j} \right)^{-i + \frac{1}{2}} \right)$$

$$\times so_{(a_p - p + 1, a_{p-1} - p + 2, \dots, a_1)} \left(\left(\frac{m + 2e_1}{m - 2e_1} \right)^{\pm 1}, \left(\frac{m + 2e_2}{m - 2e_2} \right)^{\pm 1}, \dots, \left(\frac{m + 2e_p}{m - 2e_p} \right)^{\pm 1}, 1 \right).$$

Applying Lemma 2.2 we obtain

$$\begin{aligned}
&\left(\prod_{j=1}^{p}\left(\frac{m}{2}-e_j\right)\left(\frac{m}{2}+e_j\right)\right)^{a_p+\frac{1}{2}}\\
&\qquad\qquad\times \det_{1\le i,j\le p}\left(\left(\frac{m+2e_j}{m-2e_j}\right)^{i-\frac{1}{2}}-\left(\frac{m+2e_j}{m-2e_j}\right)^{-i+\frac{1}{2}}\right)\\
&=\left(\prod_{j=1}^{p}\left(\frac{m}{2}-e_j\right)^{a_p+\frac{1}{2}}\left(\frac{m}{2}+e_j\right)^{a_p+\frac{1}{2}}\left(\frac{m+2e_j}{m-2e_j}\right)^{-p+\frac{1}{2}}\left(\left(\frac{m+2e_j}{m-2e_j}\right)-1\right)\right)\\
&\qquad\qquad\times\left(\prod_{1\le i<j\le p}\left(\frac{m+2e_j}{m-2e_j}-\frac{m+2e_i}{m-2e_i}\right)\left(1-\frac{m+2e_i}{m-2e_i}\frac{m+2e_j}{m-2e_j}\right)\right)\\
&=2^{-2pa_p+2p^2-p}m^{p^2-p}\left(\prod_{j=1}^{p}(m-2e_j)(m+2e_j)\right)^{a_p-p+1}\\
&\qquad\qquad\times\left(\prod_{j=1}^{p}e_j\right)\left(\prod_{1\le i<j\le p}(e_j^2-e_i^2)\right)
\end{aligned}$$

We now observe that both the original determinant in (7) and the result of the calculation above have degree $p^(2a_p+1)$, when considered as polynomials in m and $e_1,e_2,\ldots,e_p$. Thus, the coefficients of the leading terms must be equal. Furthermore, both polynomials are divisible by $\left(\prod_{j=1}^{p}e_j\right)\left(\prod_{1\le i<j\le p}(e_j^2-e_i^2)\right)$, which implies that, considered as polynomials in m only, their leading terms are equal, too.

On ignoring again terms whose contribution to the overall asymptotics are negligible we obtain that the determinant in (7) is equal to

$$2^{-2pa_p+2p^2-p}m^{2pa_p-p^2+p}\frac{\prod_{1\le i<j\le p}(a_j-a_i)\prod_{1\le i\le j\le p}(a_i+a_j+1)}{\prod_{l=1}^{p-1}(2l+1)!}\\
\left(\prod_{j=1}^{p}e_j\right)\left(\prod_{1\le i<j\le p}(e_j^2-e_i^2)\right)(1+O(1/m)). \quad (8)$$

Next, we want to find an asymptotic approximation for the product in Equation (7). If e_j is positive, we have

$$\begin{aligned}
&\frac{m!}{(\frac{m+1}{2}+a_p-e_j)!(\frac{m+1}{2}+a_p+e_j)!}\\
&= \frac{m!}{\left((\frac{m+1}{2}+a_p)!\right)^2}\prod_{l=1}^{e_j}\frac{1+\frac{2}{m}(\frac{1}{2}+a_p-e_j+l)}{1+\frac{2}{m}(\frac{1}{2}+a_p+l)}\\
&= \frac{m!}{\left((\frac{m+1}{2}+a_p)!\right)^2}\exp\left[-\frac{2e_j^2}{m}+O(e_j^3/m^2)\right]\\
&= 2^{m+2a_p+\frac{3}{2}}m^{-2a_p-\frac{3}{2}}\pi^{-\frac{1}{2}}\left(1+O(1/m)\right)\exp\left[-\frac{2e_j^2}{m}\right]\left(1+O(e_j^3/m^2)\right).
\end{aligned}$$

If e_j is negative, there is a similar calculation that leads to the same result. Now we see that the dominant terms of the sum in (7) are those with $0<e_1<e_2<\cdots<e_p<\sqrt{m}\log m$: For these terms, we obtain

$$\begin{aligned}
&\prod_{j=1}^{p}\frac{m!}{(\frac{m+1}{2}+a_p-e_j)!(\frac{m+1}{2}+a_p+e_j)!}\\
&\qquad= 2^{p(m+2a_p+\frac{3}{2})}m^{-p(2a_p+\frac{3}{2})}\pi^{-\frac{p}{2}}\exp\left[-\frac{2}{m}\sum_{j=1}^{p}e_j^2\right]\left(1+O(m^{-1/2}(\log m)^3)\right). \quad (9)
\end{aligned}$$

However, if $e_p\geq\sqrt{m}\log m$, then $\exp\left[-\frac{2e_p^2}{m}\right]\leq m^{-2\log m}$. Because of $1+O(e_j^3/m^2)=O(m)$ for $j\in\{1,2,\dots,p\}$ and $\exp\left[-\frac{2e_j^2}{m}\right]\leq 1$ for $j\in\{1,2,\dots,p-1\}$ we obtain

$$\begin{aligned}
&\prod_{j=1}^{p}\frac{m!}{(\frac{m+1}{2}+a_p-e_j)!(\frac{m+1}{2}+a_p+e_j)!}\\
&\qquad\qquad= O\left(2^{p(m+a_p-a_1+\frac{1}{2})}m^{-p(a_p-a_1-\frac{1}{2})-2\log m}\right).
\end{aligned}$$

Since the asymptotic approximation of the determinant as given in (8) is of polynomial order in m, these summands are asymptotically negligible.

Substituting Equations (8) and (9) into (7) we obtain

$$
\begin{aligned}
&\sum \left(\prod_{j=1}^{p} \frac{m!}{(\frac{m+1}{2} + a_p - e_j)!(\frac{m+1}{2} + a_p + e_j)!} \right) \\
&\quad \cdot \det_{1 \le i,j \le p} \left(\left(\frac{m+1}{2} + a_i - e_j + 1 \right)_{a_p - a_i} \left(\frac{m-1}{2} - a_i + e_j + 1 \right)_{a_p + a_i + 1} \right. \\
&\qquad\qquad \left. - \left(\frac{m+1}{2} + a_i + e_j + 1 \right)_{a_p - a_i} \left(\frac{m-1}{2} - a_i - e_j + 1 \right)_{a_p + a_i + 1} \right) \\
&= \sum 2^{p(m+2a_p+\frac{3}{2})} m^{-p(2a_p+\frac{3}{2})} \pi^{-\frac{p}{2}} \exp\left[-\frac{2}{m} \sum_{j=1}^{p} e_j^2 \right] \\
&\quad \cdot 2^{-2pa_p+2p^2-p} m^{2pa_p-p^2+p} \frac{\prod_{1 \le i<j \le p}(a_j - a_i) \prod_{1 \le i \le j \le p}(a_i + a_j + 1)}{\prod_{l=1}^{p-1}(2l+1)!} \\
&\quad \cdot \left(\prod_{j=1}^{p} e_j \right) \left(\prod_{1 \le i<j \le p} (e_j^2 - e_i^2) \right) \left(1 + O(m^{-1/2} (\log m)^3) \right) \\
&= 2^{pm+2p^2+\frac{p}{2}} m^{-p^2-\frac{p}{2}} \pi^{-\frac{p}{2}} \frac{\prod_{1 \le i<j \le p}(a_j - a_i) \prod_{1 \le i \le j \le p}(a_i + a_j + 1)}{\prod_{l=1}^{p-1}(2l+1)!} \\
&\quad \times \sum \left(\prod_{j=1}^{p} e_j \right) \left(\prod_{1 \le i<j \le p} (e_j^2 - e_i^2) \right) \exp\left[-\frac{2}{m} \sum_{j=1}^{p} e_j^2 \right] \\
&\quad \times \left(1 + O\big(m^{-1/2} (\log m)^3\big) \right),
\end{aligned}
\tag{10}
$$

where the sum is over $0 \le e_1 < e_2 < \cdots < e_p \le \sqrt{m} \log m$. In fact, we may extend the range of summation and sum over $0 \le e_1 \le e_2 \le \cdots \le e_p \le \sqrt{m} \log$, since the expression inside the sum is zero if any two e_i should be the same. It remains to find an asymptotic approximation of this sum. To achieve this, we apply Lemma 2.3 successively for $j = p, p-1, \ldots, 1$ to each of the individual sums. We start with the sum over e_p:

$$
\begin{aligned}
&\sum_{e_p = e_{p-1}}^{\sqrt{m} \log m} e_p \left(\prod_{1 \le i<p} (e_p^2 - e_i^2) \right) \exp\left[-\frac{2e_p^2}{m} \right] \\
&= \sum_{e_p = e_{p-1}}^{\infty} e_p \left(\prod_{1 \le i<p} (e_p^2 - e_i^2) \right) \exp\left[-\frac{2e_p^2}{m} \right] \times (1 + O(1/m)) \\
&= \left(\int_{e_{p-1}}^{\infty} e_p \left(\prod_{1 \le i<p} (e_p^2 - e_i^2) \right) \exp\left[-\frac{2e_p^2}{m} \right] de_p + O(1) \right) \times (1 + O(1/m)).
\end{aligned}
$$

This integral is of at most polynomial growth in the variables $e_1, e_2, \ldots, e_{p-1}$, so we can apply Lemma 2.3 again and iterate. The result is that we obtain

$$\int \left(\prod_{j=1}^{p} e_j\right)\left(\prod_{1\le i<j\le p}(e_j^2-e_i^2)\right)\exp\left[-\frac{2}{m}\sum_{j=1}^{p}e_j^2\right]de_1de_2\dots de_p \\ \times\left(1+O\left(1/m\right)\right)$$

where the integral is over $0<e_1\le e_2\le\cdots\le e_p<\infty$. Now we substitute $\frac{\sqrt{m}}{2}x_i$ for e_i, and introduce – seemingly superfluous – absolute values:

$$\left(\frac{\sqrt{m}}{2}\right)^{p^2+p}\int \left(\prod_{j=1}^{p}|x_j|\right)\left(\prod_{1\le i<j\le p}|x_j^2-x_i^2|\right)\exp\left[-\frac{1}{2}\sum_{j=1}^{p}x_j^2\right]dx_1dx_2\dots dx_p \\ \times\left(1+O\left(1/m\right)\right).$$

With the absolute values, the integrand is invariant under permutations of the x_i. Thus, we can rewrite the last line as

$$\left(\frac{\sqrt{m}}{2}\right)^{p^2+p}\frac{1}{2^pp!} \\ \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{j=1}^{p}|x_j|\right)\left(\prod_{1\le i<j\le p}|x_j^2-x_i^2|\right)\exp\left[-\frac{1}{2}\sum_{j=1}^{p}x_j^2\right]dx_1dx_2\dots dx_p \\ \times\left(1+O\left(1/m\right)\right).$$

Using (2) of Lemma 2.4 with $k_1+k_3=1$ and $k_2=1$ gives

$$\left(\frac{\sqrt{m}}{2}\right)^{p^2+p}\frac{1}{2^pp!}\pi^{p/2}\prod_{l=1}^{p}\frac{\Gamma(l+1)}{\Gamma(3/2)}\times\left(1+O\left(1/m\right)\right).$$

Combining this expression with the last line of Equation (10) we obtain the claimed expression for odd m. The computation for even m is similar, therefore we leave the details to the reader. ☐

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Chapter 3
A determinantal formula for the Hilbert series of one-sided ladder determinantal rings*

Dedicated to Shreeram Abhyankar

Abstract

We give a formula that expresses the Hilbert series of one-sided ladder determinantal rings, up to a trivial factor, in form of a determinant. This allows the convenient computation of these Hilbert series. The formula follows from a determinantal formula for a generating function for families of nonintersecting lattice paths that stay inside a one-sided ladder-shaped region, in which the paths are counted with respect to turns.

1 Introduction

Work of Abhyankar and Kulkarni [1, 2, 20, 21], Bruns, Conca, Herzog, and Trung [4, 5, 6, 11] showed that the computation of the Hilbert series of ladder determinantal rings (see Section 2 for precise definitions and background) boils down to counting families of n nonintersecting lattice paths with a given total number of turns in a certain ladder-shaped region. Thus, this raises the question of establishing an explicit formula for the number of these families of nonintersecting lattice paths.

In the case that there is no ladder restriction, Abhyankar [1, (20.14.4)] has found a determinantal formula for the Hilbert series (actually not just one, but a great number of them). As was made explicit in [6, 7, 21, 22], he thereby solved the aforementioned counting problem in the case of *no* ladder restriction. For direct proofs of the corresponding counting formula see [14, 22]. In the case of *one-sided* ladders, Kulkarni [20] established an explicit solution to the counting problem for $n = 1$ (i.e., if there is just one path; this corresponds to considering one-sided ladder determinantal rings defined by 2×2 minors). For arbitrary n, a determinantal formula for the number of families of n nonintersecting lattice paths in a one-sided ladder, where the starting and end points of the paths are successive, was given by the first author and Prohaska [17] (this corresponds to one-sided ladder determinantal rings defined by $(n + 1) \times (n + 1)$ minors), thereby proving a conjecture by Conca and Herzog [6, last paragraph]. Finally, Ghorpade [9] has recently proposed a solution to the counting problem with more general starting and end points of the paths, even in the

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case of *two-sided* ladders (this corresponds to *two-sided* ladder determinantal rings *cogenerated* by a given minor). This solution is based on an explicit formula for the counting problem for one path (i.e., $n = 1$), which is then summed over a large set of indices with complicated dependencies. Thus, this solution cannot be regarded as equally satisfying as the determinantal formula of Abhyankar and the determinantal formula of the first author and Prohaska, which are, however, only formulas in the case of a trivial ladder and in the case of a one-sided ladder, respectively.

The purpose of this paper is to provide a determinantal formula for the case of *one-sided* ladders where the starting and end points are more general than in [17] (see Corollary 3.2; this corresponds to *one-sided* ladder determinantal rings *cogenerated* by a given minor). This formula must be considered as superior to the aforementioned one by Ghorpade [9] in this case (i.e., the case of one- instead of two-sided ladders). It specializes directly to Abhyankar's formula [1, (20.14.4), $L = 2$, $k = 2$, with $F^{(22)}(m, p, a, V)$ defined on p. 50] in the case of no ladder restriction. On the other hand, if starting and end points are successive, then it does not specialize to the formula in [17]. (As already mentioned in Section 7 of [17], it seems that the formula in [17] cannot be extended in any direction.)

The entries in the determinant in our formula (5), respectively (6), are given by certain generating functions for two-rowed arrays, which are easy to compute as we show in Section 5. (The concept of two-rowed arrays was introduced in [12, 18] and developed to full power in [13, 14]. Also the proof of the main theorem in [17] depended heavily on two-rowed arrays.)

In the next section we recall the basic setup. In particular, we define ladder determinantal rings and state, in Theorem 2.1, the connection between the Hilbert series of such rings and the enumeration of nonintersecting lattice paths with respect to turns. Our main result, the determinantal formula for the Hilbert series of one-sided ladder determinantal rings cogenerated by a given fixed minor, is stated in Corollary 3.2 in Section 3. It follows from a determinantal formula for counting nonintersecting lattice paths in a one-sided ladder with respect to turns, where the starting and end points are allowed to be even more general than is needed for our main result. This counting formula is stated in Theorem 3.1, and it is proved in Section 4. In Section 5 we show how to compute the generating functions for two-rowed arrays that appear in the determinant of our formula.

2 Ladder determinantal rings and the enumeration of nonintersecting lattice paths with respect to turns

Let $X = (X_{i,j})_{0 \le i \le b,\ 0 \le j \le a}$ be a $(b + 1) \times (a + 1)$ matrix of indeterminates. Let $= (Y_{i,j})_{0 \le i \le b,\ 0 \le j \le a}$ be another $(b + 1) \times (a + 1)$ matrix with the property that $Y_{i,j} = X_{i,j}$ or 0, and if $Y_{i,j} = X_{i,j}$ and $Y_{i'j'} = X_{i'j'}$, where $i \le i'$ and $j \le j'$, then $Y_{s,t} = X_{s,t}$ for all s, t with $i \le s \le i'$ and $j \le t \le j'$. An example for such a matrix

Y, with $b = 15$ and $a = 13$ is displayed in Figure 1. (Note that there could be 0's in the bottom-right corner of the matrix also.) Such a "submatrix" Y of X is called a *ladder*. This terminology is motivated by the identification of such a matrix Y with the set of all points $(j, b-i)$ in the plane for which $Y_{i,j} = X_{i,j}$. For example, the set of all such points for the special matrix in Figure 1 is shown in Figure 2. (It should be apparent from comparison of Figures 1 and 2 that the reason for taking $(j, b-i)$ instead of (i, j) is to take care of the difference in "orientation" of row and column indexing of a matrix versus coordinates in the plane.) In general, this set of points looks like a (two-sided) ladder-shaped region. If, on the other hand, we have either $Y_{0,0} = X_{0,0}$ or $Y_{b,a} = X_{b,a}$ then we call Y a *one-sided* ladder. In the first case we call Y a *lower ladder*, in the second an *upper ladder*. Thus, the matrix in Figure 1 is an upper ladder.

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{0,8} & X_{0,9} & X_{0,10} & X_{0,11} & X_{0,12} & X_{0,13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{1,8} & X_{1,9} & X_{1,10} & X_{1,11} & X_{1,12} & X_{1,13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{2,8} & X_{2,9} & X_{2,10} & X_{2,11} & X_{2,12} & X_{2,13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{3,7} & X_{3,8} & X_{3,9} & X_{3,10} & X_{3,11} & X_{3,12} & X_{3,13} \\
0 & 0 & 0 & 0 & 0 & 0 & X_{4,6} & X_{4,7} & X_{4,8} & X_{4,9} & X_{4,10} & X_{4,11} & X_{4,12} & X_{4,13} \\
0 & 0 & 0 & 0 & 0 & X_{5,5} & X_{5,6} & X_{5,7} & X_{5,8} & X_{5,9} & X_{5,10} & X_{5,11} & X_{5,12} & X_{5,13} \\
0 & 0 & 0 & 0 & X_{6,4} & X_{6,5} & X_{6,6} & X_{6,7} & X_{6,8} & X_{6,9} & X_{6,10} & X_{6,11} & X_{6,12} & X_{6,13} \\
0 & 0 & 0 & 0 & X_{7,4} & X_{7,5} & X_{7,6} & X_{7,7} & X_{7,8} & X_{7,9} & X_{7,10} & X_{7,11} & X_{7,12} & X_{7,13} \\
0 & 0 & 0 & 0 & X_{8,4} & X_{8,5} & X_{8,6} & X_{8,7} & X_{8,8} & X_{8,9} & X_{8,10} & X_{8,11} & X_{8,12} & X_{8,13} \\
X_{9,0} & X_{9,1} & X_{9,2} & X_{9,3} & X_{9,4} & X_{9,5} & X_{9,6} & X_{9,7} & X_{9,8} & X_{9,9} & X_{9,10} & X_{9,11} & X_{9,12} & X_{9,13} \\
X_{10,0} & X_{10,1} & X_{10,2} & X_{10,3} & X_{10,4} & X_{10,5} & X_{10,6} & X_{10,7} & X_{10,8} & X_{10,9} & X_{10,10} & X_{10,11} & X_{10,12} & X_{10,13} \\
X_{11,0} & X_{11,1} & X_{11,2} & X_{11,3} & X_{11,4} & X_{11,5} & X_{11,6} & X_{11,7} & X_{11,8} & X_{11,9} & X_{11,10} & X_{11,11} & X_{11,12} & X_{11,13} \\
X_{12,0} & X_{12,1} & X_{12,2} & X_{12,3} & X_{12,4} & X_{12,5} & X_{12,6} & X_{12,7} & X_{12,8} & X_{12,9} & X_{12,10} & X_{12,11} & X_{12,12} & X_{12,13} \\
X_{13,0} & X_{13,1} & X_{13,2} & X_{13,3} & X_{13,4} & X_{13,5} & X_{13,6} & X_{13,7} & X_{13,8} & X_{13,9} & X_{13,10} & X_{13,11} & X_{13,12} & X_{13,13} \\
X_{14,0} & X_{14,1} & X_{14,2} & X_{14,3} & X_{14,4} & X_{14,5} & X_{14,6} & X_{14,7} & X_{14,8} & X_{14,9} & X_{14,10} & X_{14,11} & X_{14,12} & X_{14,13} \\
X_{15,0} & X_{15,1} & X_{15,2} & X_{15,3} & X_{15,4} & X_{15,5} & X_{15,6} & X_{15,7} & X_{15,8} & X_{15,9} & X_{15,10} & X_{15,11} & X_{15,12} & X_{15,13}
\end{pmatrix}$$

Figure 1.

Now fix a "bivector" $M = [u_1, u_2, \dots, u_n \mid v_1, v_2, \dots, v_n]$ of positive integers with $u_1 < u_2 < \cdots < u_n$ and $v_1 < v_2 < \cdots < v_n$. Let $K[Y]$ denote the ring of all polynomials over some field K in the $Y_{i,j}$'s, $0 \le i \le b$, $0 \le j \le a$, and let $I_M(Y)$ be the ideal in $K[Y]$ that is generated by those $t \times t$ minors of Y that contain only nonzero entries, whose rows form a subset of the last $u_t - 1$ rows, or whose columns form a subset of the last $v_t - 1$ columns, $t = 1, 2, \dots, n+1$. Here, by convention, u_{n+1} is set equal to $b+2$, and v_{n+1} is set equal to $a+2$. (Thus, for $t = n+1$ the rows and columns of minors are unrestricted.) The ideal $I_M(Y)$ is called a *ladder determinantal ideal cogenerated by the minor M*. (That one speaks of 'the minor M' has its explanation in the identification of the bivector M with a particular minor

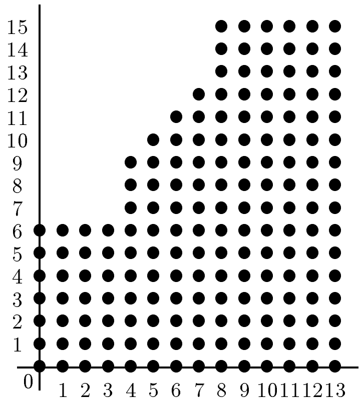

Figure 2.

of Y, cf. [11, Sec. 2]. It should be pointed out that our conventions here deviate slightly from the ones in [11]. In particular, we defined the ideal $I_M(Y)$ by restricting rows and columns of minors to a certain number of *last* rows or columns, while in [11] it is *first* rows, respectively columns. Clearly, a rotation of the matrix by 180° transforms one convention into the other.) The associated *ladder determinantal ring cogenerated by M* is $R_M(Y) := K[Y]/I_M(Y)$. (We remark that the definition of ladder is more general in [1, 2, 5, 11]. However, there is in effect no loss of generality since the ladders of [1, 2, 5, 11] can always be reduced to our definition by discarding superfluous 0's.)

When Abhyankar introduced ladder determinantal rings in the early 1980s, he was motivated by the study of singularities of Schubert varieties. Indeed, as was shown recently by Gonciulea and Lakshmibai in [10] (see also [3, Ch. 12]), the associated varieties (called ladder determinantal varieties) can be identified with opposite cells of certain Schubert varieties of type A. This connection allowed them to identify the irreducible components of such Schubert varieties in many cases, thus making substantial progress on a long-standing problem in algebraic geometry.

Results of Abhyankar [1, 2] or Herzog and Trung [11] allow to express the Hilbert series of the ladder determinantal ring $R_M(Y)$ in combinatorial terms. Before we can state the corresponding result, we need to introduce a few more terms.

When we say *lattice path* we always mean a lattice path in the plane consisting of unit horizontal and vertical steps in the positive direction, see Figure 3 for an example. We shall frequently abbreviate the fact that a lattice path P goes from A to E by $P : A \to E$.

Also, given lattice points A and E, we denote the set of all lattice paths from A to E by $\mathcal{P}(A \to E)$. A family $(P_1, P_2, \dots, P_n)$ of lattice paths P_i, $i = 1, 2, \dots, n$, is said to be *nonintersecting* if no two lattice paths of this family have a point in common. Given n-tuples of lattice points $\mathbf{A} = (A^{(1)}, A^{(2)}, \dots, A^{(n)})$ and $\mathbf{E} = (E^{(1)}, E^{(2)}, \dots, E^{(n)})$, we denote the set of all families $(P_1, P_2, \dots, P_n)$ of nonintersecting lattice paths, where P_i runs from $A^{(i)}$ to $E^{(i)}$, $i = 1, 2, \dots, n$, by $\mathcal{P}^+(\mathbf{A} \to \mathbf{E})$.

A point in a lattice path P which is the end point of a vertical step and at the

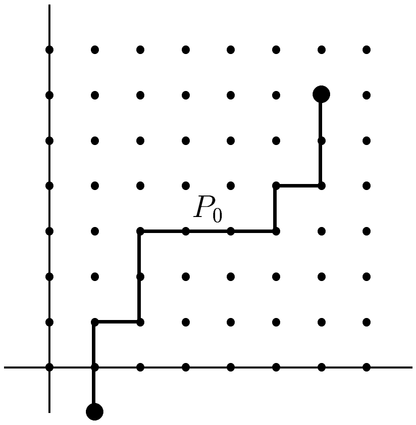

Figure 3.

same time the starting point of a horizontal step will be called a *north-east turn* (*NE-turn* for short) of the lattice path P. The NE-turns of the lattice path in Figure 3 are $(1,1)$, $(2,3)$, and $(5,4)$. We write $\mathrm{NE}(P)$ for the number of NE-turns of P. Also, given a family $\mathbf{P}=(P_1,P_2,\dots,P_n)$ of paths P_i, we write $\mathrm{NE}(\mathbf{P})$ for the number $\sum_{i=1}^n \mathrm{NE}(P_i)$ of all NE-turns in the family.

Our lattice paths will be restricted to ladder-shaped regions L corresponding to the nonzero entries of a given matrix Y in the way that was explained earlier (cf. Figures 1 and 2). We extend our lattice path notation in the following way. By $\mathcal{P}_L(A\to E)$ we mean the set of all lattice paths P from A to E *all of whose NE-turns lie in the ladder region L.* (It should be noted that, in the case of a two-sided ladder, it is possible that a path is *not* totally inside L while its NE-turns *are*. However, in the case of an upper ladder L, which is the case of interest in our paper, a path is inside L *if and only if* all of its NE-turns are.) Similarly, by $\mathcal{P}_L^+(\mathbf{A}\to\mathbf{E})$ we mean the set of all families $(P_1,P_2,\dots,P_n)$ of nonintersecting lattice paths, where P_i runs from $A^{(i)}$ to $E^{(i)}$ *and where all the NE-turns of P_i lie in the ladder region L.*

Finally, given any weight function w defined on a set $\mathcal{M}$, by the generating function $\mathrm{GF}(\mathcal{M};w)$ we mean $\sum_{x\in\mathcal{M}} w(x)$.

Theorem 2.1. *Let $Y=(Y_{i,j})_{0\le i\le b,\ 0\le j\le a}$ be a one-sided ladder, and let L be the associated ladder region, i.e., $Y_{i,j}=X_{i,j}$ if and only if $(j,b-i)\in L$. Let $M=[u_1,u_2,\dots,u_n \mid v_1,v_2,\dots,v_n]$ be a bivector of positive integers with $u_1<u_2<\dots<u_n$ and $v_1<v_2<\dots<v_n$. Furthermore, let $A^{(i)}=(0,u_{n+1-i}-1)$ and $E^{(i)}=(a-v_{n+1-i}+1,b)$, $i=1,2,\dots,n$. Then, under the assumption that all of the points $A^{(i)}$ and $E^{(i)}$, $i=1,2,\dots,n$, lie inside the ladder region L, the Hilbert series of the ladder determinantal ring $R_M(Y)=K[Y]/I_M(Y)$ equals*

$$\sum_{\ell=0}^{\infty}\dim_K R_M(Y)_\ell\, z^\ell=\frac{\mathrm{GF}(\mathcal{P}_L^+(\mathbf{A}\to\mathbf{E});z^{\mathrm{NE}(.)})}{(1-z)^{(a+b+3)n-\sum_{i=1}^n(u_i+v_i)}},\tag{1}$$

where $R_M(Y)_\ell$ denotes the homogeneous component of degree ℓ in $R_M(Y)$, and where, according to our definitions, $\mathrm{GF}(\mathcal{P}_L^+(\mathbf{A}\to\mathbf{E});z^{\mathrm{NE}(.)})$ is the generating func-

tion $\sum_{\mathbf{P}} z^{\mathrm{NE}(\mathbf{P})}$ *for all families* $\mathbf{P} = (P_1, P_2, \dots, P_n)$ *of nonintersecting lattice paths,* P_i *running from* $A^{(i)}$ *to* $E^{(i)}$*, such that all of its NE-turns stay inside the ladder region* L*.*

Remark. (a) The condition that all of the points $A^{(i)}$ and $E^{(i)}$ lie inside the ladder region L restricts the choice of ladders. In particular, for an upper ladder it means that $Y_{b-u_n+1,0} = X_{b-u_n+1,0}$ and $Y_{0,a-v_n+1} = X_{0,a-v_n+1}$, which will be relevant for us. Still, one could prove an analogous result even if this condition is dropped. In that case, however, the points $A^{(i)}$ and $E^{(i)}$ have to be modified in order to lie inside L and, thus, make the right-hand side of formula (1) meaningful.

(b) For an extension of Theorem 2.1 for the case of two-sided ladders see [24, Theorem 3.1].

Sketch of Proof. In [17, proof of Theorem 2], we gave two proofs of this assertion in the special case of a one-sided ladder and $u_i = v_i = i$, $i = 1, 2, \dots, n$ (cf. Example (1) on p. 10 of [11]). The first proof followed basically considerations by Kulkarni [20, 21] (see also [8]), and was based on an explicit basis for $R_M(Y)$ given by Abhyankar [1, Theorem (20.10)(5)]. The second proof was based on combinatorial descriptions of the dimensions $R_M(Y)_\ell$ of the homogeneous components of $R_M(Y)$ due to Herzog and Trung [11, Cor. 4.3 + Lemma 4.4]. Both proofs carry over verbatim to our more general situation because both Abhyankar's as well as Herzog and Trung's results are in fact theorems for the general ladder determinantal rings that we consider here. (However, the reader must be warned that the explicit form of Abhyankar's basis was misquoted in [17]. The correct assertion is that, given a multiset S as described in [17], the associated basis element is the product of a certain monomial in the X_{ij}'s and a certain minor of the matrix Y, see [1, definition of $w_v(t)$ in Theorem (20.10)] or [8, Theorem (6.7)(iii)] Also, the definition of the multisets S contained an error: Item 2 at the bottom of p. 1019 in [17] must be replaced by: The length of any sequence (i_1, j_1), (i_2, j_2), $\dots$, (i_k, j_k) of elements of S is at most n. The subsequent argument was however based on this corrected definition.) □

3 The determinantal formula

In view of Theorem 2.1, the computation of Hilbert series of ladder determinantal rings requires to solve the problem of counting families of nonintersecting lattice paths in a ladder-shaped region with respect to turns. We provide such a solution for one-sided ladders in Theorem 3.1. In order to formulate the result, we need to introduce the notion of *two-rowed arrays.*

From now on we restrict our attention to one-sided ladders. Without loss of generality it suffices to consider upper ladders. We encode upper ladder-shaped regions (such as the one in Figure 2) concisely by means of weakly increasing functions as follows: given an upper ladder region L, let f be the weakly increasing function from $[0, a]$ to $[1, b+1]$ with the property that it describes L by means of

$$L = \{(x, y) : x \in [0, a] \text{ and } 0 \le y < f(x)\}. \tag{2}$$

Here, by $[c,d]$ we mean the set of all integers $\geq c$ and $\leq d$. In essence, the function f describes the upper border of the region L. For example, the function f corresponding to the ladder region in Figure 2 (where $a=13$ and $b=15$) is given by $f(0)=7$, $f(1)=7$, $f(2)=7$, $f(3)=7$, $f(4)=10$, $f(5)=11$, $f(6)=12$, $f(7)=13$, $f(8)=16$, $f(9)=16$, $f(10)=16$, $f(11)=16$, $f(12)=16$, $f(13)=16$.

By a *two-rowed array* we mean two rows of integers

$$\begin{matrix} a_{-l+1} & a_{-l+2} & \dots & a_{-1} & a_0 & a_1 & \dots & a_k \\ & & & & & b_1 & \dots & b_k, \end{matrix} \tag{3}$$

where entries along both rows are *strictly increasing*. We call l the *type* of the two-rowed array. We allow l to be also negative. In this case the representation (3) has to be taken symbolically, in the sense that the first row of the two-rowed array is (by $-l$) shorter than the second row, i.e., looks like

$$\begin{matrix} & & & & a_{-l+1} & \dots & a_k \\ b_1 & b_2 & \dots & b_{-l} & b_{-l+1} & \dots & b_k. \end{matrix} \tag{4}$$

We define the *size* $|T|$ of a two-rowed array T to be the number of its entries. (Thus, the size of the two-rowed array in (3) is $l+2k$, as is the size of the one in (4).) We extend this definition and notation to families $\mathbf{T}=(T_1,T_2,\dots,T_n)$ of two-arrays by letting $|\mathbf{T}|$ denote the total number $|T_1|+|T_2|+\dots+|T_n|$ of entries in $\mathbf{T}$.

Now we define the basic set of objects which is crucial in our formulas. Given a function f as above, and pairs $A=(\alpha_1,\alpha_2)$ and $E=(\varepsilon_1,\varepsilon_2)$, we denote by $TA(l;A,E;f,d)$ the set of all two-rowed arrays of type l such that

- the entries in the first row are bounded below by α_1 and bounded above by ε_1,
- the entries in the second row are bounded below by α_2 and bounded above by ε_2,
- if the two-rowed array is represented as in (3) (respectively (4)), we have

  $$b_s < f(a_{s+d}), \tag{5}$$

  for all s such that both b_s and a_{s+d} exist in the two-rowed array.

If we want to make the lower and upper bounds transparent, then we will write such two-rowed arrays in the form

$$\begin{matrix} \alpha_1 \leq & a_{-l+1} & a_{-l+2} & \dots & a_{-1} & a_0 & a_1 & \dots & a_k & \leq \varepsilon_1 \\ \alpha_2 \leq & & & & & & b_1 & \dots & b_k & \leq \varepsilon_2. \end{matrix} \tag{6}$$

Our key theorem is the following.

Theorem 3.1. *Let n, a, b be positive integers and let L be an upper ladder-shaped region determined by the weakly increasing function $f : [0, a] \to [1, b+1]$ by means of* (2). *For convenience, extend f to all negative integers by setting $f(x) := f(0)$ for $x < 0$. Furthermore, let $A^{(i)} = (A_1^{(i)}, A_2^{(i)})$ and $E^{(i)} = (E_1^{(i)}, E_2^{(i)})$ for $i = 1, 2, \dots, n$ be lattice points in the region L satisfying*

$$f(x) = f\big(A_1^{(1)}\big) \quad \textit{for all } x \le A_1^{(1)}, \tag{7}$$

and

$$A_1^{(1)} \le A_1^{(2)} \le \dots \le A_1^{(n)}, \quad A_2^{(1)} > A_2^{(2)} > \dots > A_2^{(n)}, \tag{8}$$

and

$$E_1^{(1)} < E_1^{(2)} < \dots < E_1^{(n)}, \quad E_2^{(1)} \ge E_2^{(2)} \ge \dots \ge E_2^{(n)}. \tag{9}$$

Then the generating function $\sum z^{\mathrm{NE}(\mathbf{P})}$, where the sum is over all families $\mathbf{P} = (P_1, P_2, \dots, P_n)$ of nonintersecting lattice paths $P_i : A^{(i)} \to E^{(i)}$, $i = 1, 2, \dots, n$ lying in the region L, can be expressed as

$$\begin{aligned} &\mathrm{GF}(\mathcal{P}_L^+(\mathbf{A} \to \mathbf{E}); z^{\mathrm{NE}(.)}) \\ &\qquad\qquad = \det_{1 \le s,t \le n} \big(\mathrm{GF}(\mathit{TA}(t-s; \tilde{A}^{(t)}, \tilde{E}^{(s)}; f, s-1); z^{|.|/2}) \big), \end{aligned} \tag{10}$$

where $\tilde{A}^{(i)} = A^{(i)} + (-i+1, i)$ and $\tilde{E}^{(i)} = E^{(i)} + (-i, i-1)$, $i = 1, 2, \dots, n$. Here, by our definitions, $\mathrm{GF}(\mathit{TA}(t-s; \tilde{A}^{(t)}, \tilde{E}^{(s)}; f, s-1); z^{|.|/2})$ is the generating function $\sum_T z^{|T|/2}$, where the sum is over all two-rowed arrays of the form (6) *with $l = t - s$, $d = s - 1$, $\alpha_1 = A_1^{(i)} - i + 1$, $\alpha_2 = A_2^{(i)} + i$, $\varepsilon_1 = E_1^{(i)} - i$, and $\varepsilon_2 = E_2^{(i)} + i - 1$, which satisfy* (5).

Remark. (a) The condition (7) is equivalent to saying that to the left of $A^{(1)}$, which by (8) is the left-most starting point of the lattice paths, the boundary of the ladder region is horizontal. Clearly, this can be assumed without loss of generality because this part of the ladder (i.e., the ladder to the left of $A^{(1)}$) does not impose any restriction on the lattice paths, and, hence, on the left-hand side of (10).

(b) The formula (10) clearly reduces the problem of enumerating families of nonintersecting lattice paths in the ladder region L with respect to NE-turns to the problem of enumerating certain two-rowed arrays. We are going to address this problem in Section 5.

Thus, if we combine Theorems 2.1 and 3.1, we obtain the promised determinantal formula for the Hilbert series of one-sided ladder determinantal rings.

Corollary 3.2. *Let $Y = (Y_{i,j})_{0 \le i \le b,\ 0 \le j \le a}$ be an upper ladder, and let L be the associated ladder region, i.e., $Y_{i,j} = X_{i,j}$ if and only if $(j, b-i) \in L$, and let $f : [0, a] \to [1, b+1]$ be the function that describes this ladder region by means of* (2), *i.e., $Y_{i,j} = X_{i,j}$ if and only if $b - i < f(j)$. For convenience, extend f to all negative* *integers by setting $f(x) := f(0)$ for $x < 0$. Let $M = [u_1, u_2, \dots, u_n \mid v_1, v_2, \dots, v_n]$ be a bivector of positive integers with $u_1 < u_2 < \dots < u_n$ and $v_1 < v_2 < \dots < v_n$ such that $Y_{b-u_n+1,0} = X_{b-u_n+1,0}$ and $Y_{0,a-v_n+1} = X_{0,a-v_n+1}$ (cf. Remark 2.(a) after Theorem 2.1). Furthermore, we let $\tilde{A}^{(i)} = (-i+1, u_{n+1-i}+i-1)$ and $\tilde{E}^{(i)} = (a - v_{n+1-i} - i + 1, b + i - 1)$, $i = 1, 2, \dots, n$. Then the Hilbert series of the ladder determinantal ring $R_M(Y) = K[Y]/I_M(Y)$ equals*

$$\sum_{\ell=0}^{\infty} \dim_K R_M(Y)_\ell \, z^\ell \\ = \frac{\det_{1\le s,t\le n}\left(\mathrm{GF}(\mathit{TA}(t-s; \tilde{A}^{(t)}, \tilde{E}^{(s)}; f, s-1); z^{|\cdot|/2})\right)}{(1-z)^{(a+b+3)n - \sum_{i=1}^{n}(u_i+v_i)}}. \quad (11)$$

Remark. (a) Theorem 3.1 specializes to Theorem 1 in [14] in the case of a trivial ladder (i.e., if the function f is equal to $b+1$ for all x). For, in that case, by (37) the generating functions $\mathrm{GF}(\mathit{TA}(t-s; \tilde{A}_t, \tilde{E}_s; f, s-1); z^{|\cdot|/2})$ can be expressed in terms of binomial sums. To see that the resulting formula is indeed equivalent, one extracts the coefficient of z^K.

(b) For the same reason, Corollary 3.2 specializes to Abhyankar's formula [1, (20.14.4), $L = 2$, $k = 2$, with $F^{(22)}(m, p, a, V)$ defined on p. 50] in the case of a trivial ladder. Although Abhyankar's formula gives an expression for the Hilbert function (instead of for the Hilbert series), it is easy to see that it is equivalent to ours in this special case.

(c) The formula for the Hilbert series in [17, Theorem 2] addresses the special case $u_i = v_i = i$, $i = 1, 2, \dots, n$. However, Corollary 3.2 does not generalize this formula, as it does not directly specialize to Theorem 2 in [17]. Whereas in the latter formula the entries of the determinant are generating functions for paths, there is no such interpretation for the entries of the determinant in (11).

(d) Unfortunately, we do not know how to generalize Theorem 3.1, and, thus, Corollary 3.2, to the case of two-sided ladders. It seems that a completely new idea is needed to find such a generalization. In particular, the combinatorial formula [24, Theorem 3.1] for the Hilbert series in the case of two-sided ladders, on which such a generalization would have to be based, is already considerably more complicated than its special case for the case of one-sided ladders, stated in Theorem 2.1.

(e) More modest, but equally desirable, would it be to find an extension of Corollary 3.2 in the one-sided case to ladders L and bivectors M which do not satisfy the conditions of the statement, i.e., for which either $Y_{b-u_n+1,0} = 0$, or $Y_{0,a-v_n+1} = 0$, or both. This would require to find an extension of Theorem 3.1 to situations where the inequality chains (8) and (9) may be relaxed so that some starting and end points are allowed to lie on the boundary of the ladder region L (cf. Remark 2.(a) after Theorem 2.1). It seems again that a completely new idea is needed to find such an extension.

(f) In Section 5 of [17] it is shown that the proof of the main counting theorem yields in fact a weighted generalization thereof. An analogous weighted generalization of Theorem 3.1 can be obtained as well, which is again directly implied by the proof of Theorem 3.1 in Section 4. However, we omit the statement of this generalization for the sake of brevity.

Example 3.3. Let $a = 13$, $b = 15$, $n = 4$, let $Y = (Y_{i,j})$ be the matrix of Figure 1 and $M = [1, 2, 4, 6 \mid 1, 2, 3, 6]$. Our formula (11) gives for the Hilbert series of $R_M(Y) = K[Y]/I_M(Y)$, using (45) for determining the generating function $\sum_{T \in TA(l;A,E;f,d)} z^{|\cdot|/2}$ for two-rowed arrays T in the corresponding ladder region L of Figure 2,

$$\begin{aligned}
&(1+71z+2556z^2+61832z^3+1115762z^4+15750005z^5+178390279z^6+1647137174z^7\\
&\quad+12534233703z^8+79245271879z^9+418852424787z^{10}+1859941402206z^{11}+6965987806143z^{12}\\
&\quad+22071622313567z^{13}+59298706514083z^{14}+135299444287353z^{15}+262400571075662z^{16}\\
&\quad+432640455645309z^{17}+606103694379729z^{18}+720535170430557z^{19}+725289798304502z^{20}\\
&\quad+616230022969392z^{21}+439998448014899z^{22}+262469031030333z^{23}+129776697745621z^{24}\\
&\quad+52622863698472z^{25}+17241967478923z^{26}+4468021840695z^{27}+885721405230z^{28}\\
&\quad+126901720400z^{29}+11760999250z^{30}+532021875z^{31})/(1-z)^{99}.
\end{aligned}$$

4 Proof of Theorem 3.1

The basic idea of the proof is simple. It largely follows the proof of Theorem 4 in [14]. As a first step, we expand the determinant on the right-hand side of (10) according to the definition of a determinant, see Subsection 4.1. Thus, we obtain a sum of terms, each of which is indexed by a family of two-rowed arrays, see (12). Some of the terms have positive sign, some of them negative sign. In the second step, we identify the terms which cancel each other, see Subsection 4.2. Finally, in the third step, we identify the remaining terms with the families of nonintersecting lattice paths in the statement of the theorem, see Subsection 4.3.

However, the details are sometimes intricate. To show that the terms described in Subsection 4.2 do indeed cancel, we define an involution on families of two-rowed arrays in Subsection 4.4. (This involution is copied from [14, Proof of Theorem 4].) In order that our claims follow, this involution must have several properties, which are listed in Subsection 4.5. While most of these are either obvious or are already established in [14] and [23], we are only able to provide a rather technical justification of the one pertaining to the ladder condition. This is done in Subsection 4.6.

4.1 Expansion of the determinant

Let $\mathfrak{S}_n$ denote the symmetric group of order n. We start by expanding the determinant on the right-hand side of (10), to obtain

$$\det_{1\le s,t\le n}\left(\mathrm{GF}(\,TA(t-s;\tilde A^{(t)},\tilde E^{(s)};f,s-1);z^{|\cdot|/2})\right)$$
$$=\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}\sigma\prod_{i=1}^{n}\mathrm{GF}(\,TA(\sigma(i)-i;\tilde A^{(\sigma(i))},\tilde E^{(i)};f,i-1);z^{|\cdot|/2})$$
$$=\sum_{(\mathbf{T},\sigma)}\operatorname{sgn}\sigma\, z^{|\mathbf{T}|},\tag{12}$$

where the sum is over all pairs $(\mathbf{T},\sigma)$ of permutations σ in $\mathfrak{S}_n$, and families $\mathbf{T}=(T_1,T_2,\dots,T_n)$ of two-rowed arrays, T_i being of type $\sigma(i)-i$ (i.e., the second row containing k_i entries and the first row containing $k_i+\sigma(i)-i$ entries, for some k_i), and the bounds for the entries of T_i being as follows,

$$\begin{matrix}\tilde A_1^{(\sigma(i))}\le & a^{(i)}_{-\sigma(i)+i+1}\cdots & a^{(i)}_1 & \dots & a^{(i)}_{k_i} & \le \tilde E_1^{(i)}\\ \tilde A_2^{(\sigma(i))}\le & & b^{(i)}_1 & \dots & b^{(i)}_{k_i} & \le \tilde E_2^{(i)},\end{matrix}\tag{13}$$

with the property that

$$b_s^{(i)}<f(a^{(i)}_{s+i-1}),\qquad s=1,2,\dots,k_i-i+1,\tag{14}$$

$i=1,\dots,n$.

4.2 Which terms in (12) cancel?

Now we claim that the total contribution to the sum (12) of the families $(T_1,T_2,\dots,T_n)$ of two-rowed arrays as above which have the property that there exist T_i and T_{i+1}, T_i represented by

$$\begin{matrix}\tilde A_1^{(\sigma(i))}\le & a_{-\sigma(i)+i+1} & \cdots & a_1 & \dots & a_k & \le \tilde E_1^{(i)}\\ \tilde A_2^{(\sigma(i))}\le & & & b_1 & \dots & b_k & \le \tilde E_2^{(i)},\end{matrix}\tag{15a}$$

and T_{i+1} represented by

$$\begin{matrix}\tilde A_1^{(\sigma(i+1))}\le & c_{-\sigma(i+1)+i+2} & \cdots & c_1 & \dots & c_l & \le \tilde E_1^{(i+1)}\\ \tilde A_1^{(\sigma(i+1))}\le & & & d_1 & \dots & d_l & \le \tilde E_1^{(i+1)},\end{matrix}\tag{15b}$$

and indices I and J such that

$$c_J<a_I\tag{15c}$$
$$b_{I-1}<d_J\tag{15d}$$

and

$$1\le I\le k+1,\quad 0\le J\le l,\tag{15e}$$

equals 0. The inequalities (15c) and (15d) should be understood to hold only if all variables are defined, including the conventional definitions $a_{k+1} := \tilde{E}_1^{(i)} + 1$, $b_0 := \tilde{A}_2^{(\sigma(i))} - 1$, and $c_{-\sigma(i+1)+i+1} := \tilde{A}_1^{(\sigma(i))} - 1$. (These artificial settings apply for $I = k+1$, $I = 1$, and $J = -\sigma(i+1)+i+1$, respectively. It should be noted that the indexing conventions that we have chosen here deviate slightly from [14, Sec. 3, proof of Theorem 4], but are completely equivalent.)

We call the point (a_I, d_J) a *crossing point* of T_i and T_{i+1}, and, more generally, a *crossing point* of the family $\mathbf{T}$.

4.3 The remaining terms correspond to nonintersecting lattice paths

Suppose that we would have shown that the contribution to (12) of these families of two-rowed arrays equals zero. It implies that only those families $\mathbf{T} = (T_1, T_2, \dots, T_n)$ of two-rowed arrays, T_i being of the form (13) and satisfying (14), contribute to (12) where T_i and T_{i+1} have no crossing point for all i.

So, let $\mathbf{T}$ be such a family of two-rowed arrays without any crossing point. By using the arguments from [23][1] (with $A_1^{(i)}$, $A_2^{(i)}$, $E_1^{(i)}$, $E_2^{(i)}$ in [23] replaced by our $\tilde{A}_1^{(i)}$, $\tilde{A}_2^{(i)} - 1$, $\tilde{E}_1^{(i)} + 1$, $\tilde{E}_2^{(i)}$, respectively, $i = 1, 2, \dots, n$), it then follows that the permutation σ associated to $\mathbf{T}$ must be the identity permutation. Thus, the two-rowed array T_i has the form (recall (13))

$$\begin{array}{llcll} \tilde{A}_1^{(i)} \leq & a_1^{(i)} & \dots & a_{k_i}^{(i)} & \leq \tilde{E}_1^{(i)} \\ \tilde{A}_2^{(i)} \leq & b_1^{(i)} & \dots & b_{k_i}^{(i)} & \leq \tilde{E}_2^{(i)}, \end{array} \tag{16}$$

and satisfies (14). Moreover, we assumed that there is no crossing point, meaning that there are no consecutive two-rowed arrays T_i and T_{i+1} and indices I and J such that (15) holds.

By interpreting the two-rowed array (16) as a lattice path $\tilde{P}_i$ from $\tilde{A}^{(i)} - (0,1)$ to $\tilde{E}^{(i)} + (1,0)$ whose NE-turns are exactly $(a_1^{(i)}, b_1^{(i)}), \dots, (a_{k_i}^{(i)}, b_{k_i}^{(i)})$, $i = 1, 2, \dots, n$, the family $\mathbf{T}$ of two-rowed arrays is translated into a family $\widetilde{\mathbf{P}} = (\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_n)$ of paths. Clearly, under this translation we have $|\mathbf{T}|/2 = \mathrm{NE}(\widetilde{\mathbf{P}})$, and, hence,

$$z^{|\mathbf{T}|/2} = z^{\mathrm{NE}(\widetilde{\mathbf{P}})}. \tag{17}$$

The fact that (15) does not hold simply means that the paths $\tilde{P}_i$ and $\tilde{P}_{i+1}$ do not cross each other (that is, they may touch each other, but they never change sides), $i = 1, 2, \dots, n-1$. We refer the reader to the explanations in Section 2 (between Theorems 3 and 4) in [14]. Here, we content ourselves with an illustration. Suppose two paths Q_1 and Q_2 cross each other (see Figure 4). Furthermore suppose that the NE-turns of Q_1 are $(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)$, and the NE-turns of Q_2 are

[1]The proof in the original paper [14, last paragraph of the proof of Theorem 4] contained an error at this point. The inequality $A_1^{(\sigma(i+1))} - 1 \leq A^{(\sigma(i))}$ on page 12 of [14] is not true in general.

(c_1, d_1), (c_2, d_2), $\dots$, (c_l, d_l). Then it is obvious from Figure 4 that there exist I and J such that (15c)–(15e) hold.

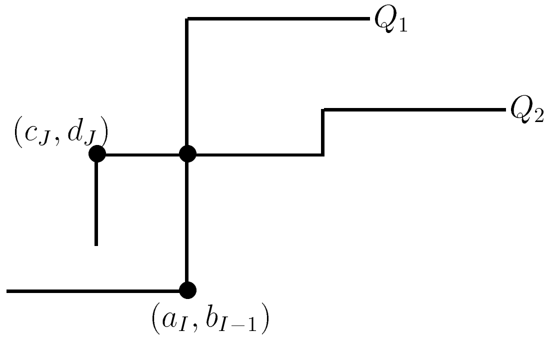

Figure 4.

To finally match with the claim of Theorem 3.1, we shift $\tilde{P}_i$ by $(i-1, -i+1)$, $i = 1, 2, \dots, n$. Thus we obtain a family $(P_1, P_2, \dots, P_n)$ of lattice paths, P_i running from $A^{(i)}$ to $E^{(i)}$. Clearly, under this shift, the condition that $\tilde{P}_i$ and $\tilde{P}_{i+1}$ do not cross each other translates into the condition that P_i and P_{i+1} do not touch each other, $i = 1, 2, \dots, n-1$. If we combine this fact with the observation that the first path, $P_1 = \tilde{P}_1$, stays inside the ladder region L because of (14) with $i = 1$, then we conclude that all the P_i's must also stay inside L because P_1 forms a barrier.

Thus, in view of (17), we have proved that the right-hand side of (10) is equal to the generating function $\sum_{\mathbf{P}} z^{\mathrm{NE}(\mathbf{P})}$, where the sum is over all families $\mathbf{P} = (P_1, P_2, \dots, P_n)$ of nonintersecting lattice paths, P_i running from $A^{(i)}$ to $E^{(i)}$ and staying inside the ladder region L. But this is exactly the left-hand side of (10). Thus Theorem 3.1 would be proved.

4.4 The involution

To show that the contribution to the sum (12) of the families $\mathbf{T} = (T_1, T_2, \dots, T_n)$ of two-rowed arrays, T_i being of the form (13) and satisfying (14) for $i = 1, 2, \dots, n$, which contain consecutive arrays T_i and T_{i+1} that have a crossing point (cf. (15)), indeed equals 0, we construct an involution, φ say, on this set of families that maps a family $(T_1, T_2, \dots, T_n)$ with associated permutation σ to a family $\overline{\mathbf{T}} = (\overline{T}_1, \overline{T}_2, \dots, \overline{T}_n)$ with associated permutation $\overline{\sigma}$, such that

$$\operatorname{sgn} \sigma = -\operatorname{sgn} \overline{\sigma}, \tag{18}$$

and such that

$$|\mathbf{T}| = |\overline{\mathbf{T}}|. \tag{19}$$

Clearly, this implies that the contribution to (12) of families that are mapped to each other cancels.

The definition of the involution φ can be copied from [14, Sec. 3, proof of Theorem 4]. For convenience, we repeat it here. Let $(\mathbf{T}, \sigma)$ be a pair under consideration for the sum (12). Besides, we assume that $\mathbf{T}$ has a crossing point. Consider all crossing points of two-rowed arrays with consecutive indices (see (15)). Among these points choose those with maximal x-coordinate, and among all those choose the crossing point with maximal y-coordinate. Denote this crossing point by S. Let i be minimal such that S is a crossing point of T_i and T_{i+1}. Let T_i and T_{i+1} be given by (15a) and (15b), respectively. By (15), S being a crossing point of T_i and T_{i+1} means that there exist I and J such that T_i looks like

$$\begin{matrix} \tilde{A}_1^{(\sigma(i))} \le & \dots & a_{I-1} & a_I & \dots & a_{k_i} & \le \tilde{E}_1^{(i)} \\ \tilde{A}_2^{(\sigma(i))} \le & \dots & b_{I-1} & b_I & \dots & b_{k_i} & \le \tilde{E}_2^{(i)}, \end{matrix} \tag{20}$$

T_{i+1} looks like

$$\begin{matrix} \tilde{A}_1^{(\sigma(i+1))} \le & \dots\dots\dots & & c_J & c_{J+1} & \dots & c_{k_{i+1}} & \le \tilde{E}_1^{(i)} \\ \tilde{A}_2^{(\sigma(i+1))} \le & \dots & d_{J-1} & d_J & & \dots\dots\dots & d_{k_{i+1}} & \le \tilde{E}_2^{(i)}, \end{matrix} \tag{21}$$

$S = (a_I, d_J)$,

$$c_J < a_I \tag{22a}$$

$$b_{I-1} < d_J \tag{22b}$$

and

$$1 \le I \le k_i + 1, \quad 0 \le J \le k_{i+1}. \tag{22c}$$

Because of the construction of S, the indices I *and* J are maximal with respect to (22).

We map $(\mathbf{T}, \sigma)$ to the pair $(\overline{\mathbf{T}}, \sigma \circ (i, i+1))$ ($(i, i+1)$ denotes the transposition exchanging i and $i+1$), where $\overline{\mathbf{T}} = (\overline{T}_1, \overline{T}_2, \dots, \overline{T}_n)$, with $\overline{T}_j = T_j$ for all $j \ne i, i+1$, with $\overline{T}_i$ being given by

$$\begin{matrix} \dots & c_J & a_I & \dots & a_{k_i} \\ \dots & d_{J-1} & b_I & \dots & b_{k_i}, \end{matrix} \tag{23a}$$

and with $\overline{T}_{i+1}$ being given by

$$\begin{matrix} \dots\dots\dots & & a_{I-1} & c_{J+1} & \dots & c_{k_{i+1}} \\ \dots & b_{I-1} & d_J & & \dots\dots\dots & d_{k_{i+1}}. \end{matrix} \tag{23b}$$

4.5 The properties of the involution

What we have to prove is that this operation is well-defined, i.e., that all the rows in (23a) and (23b) are strictly increasing, that $\overline{T}_i$ is of type $(\sigma \circ (i, i+1))(i) - i = \sigma(i+1) - i$, that $\overline{T}_{i+1}$ is of type $(\sigma \circ (i, i+1))(i+1) - i - 1 = \sigma(i) - i - 1$, that the bounds for the entries of $\overline{T}_i$ are given by

$$\begin{matrix} \tilde{A}_1^{(\sigma(i+1))} \le & \dots & c_J & a_I & \dots & a_{k_i} & \le \tilde{E}_1^{(i)} \\ \tilde{A}_2^{(\sigma(i+1))} \le & \dots & d_{J-1} & b_I & \dots & b_{k_i} & \le \tilde{E}_2^{(i)}, \end{matrix}$$

that those for $\overline{T}_{i+1}$ are given by

$$\begin{array}{cccccccc} \tilde{A}_1^{(\sigma(i))} \leq & \dots\dots\dots & a_{I-1} & c_{J+1} & \dots & c_{k_{i+1}} & \leq \tilde{E}_1^{(i+1)} \\ \tilde{A}_2^{(\sigma(i))} \leq & \dots \; b_{I-1} & d_J & \dots\dots\dots & & d_{k_{i+1}} & \leq \tilde{E}_2^{(i+1)}, \end{array}$$

and that (14) is satisfied for $\overline{T}_i$ and $\overline{T}_{i+1}$. Furthermore we have to prove that φ is indeed an involution (for which it suffices to show that (22) also holds for $\overline{T}_i$ and $\overline{T}_{i+1}$), and finally we must prove (18) (with $\overline{\sigma} = \sigma \circ (i, i+1)$) and (19).

The claim that (18) and (19) hold is trivial. All other claims, except for the claim about (14), can be proved by copying the according arguments from the proof of Theorem 4 in [14] (see the paragraphs after [14, Eq. (27)]).

4.6 The involution respects the ladder condition

It remains to show that (14) is satisfied for $\overline{T}_i$ and $\overline{T}_{i+1}$. Unfortunately, it is necessary to supplement and refine the according arguments in the proof of the main theorem in [17] (see the proof of (4.27) and (4.28) in [17, pp. 1035–37]) substantially in order to cope with the situation that we encounter here. Besides, we use the opportunity to correct an inaccuracy in [17].

We have to prove that for $1 \leq r \leq i-1$ we have

$$d_{J-i+r} < f(a_{I-1+r}), \tag{24}$$

provided both a_{I-1+r} and d_{J-i+r} exist (if either a_{I-1+r} or d_{J-i+r} does not exist there is nothing to show), and

$$b_{I-i+r} < f(c_{J+r}), \tag{25}$$

provided both b_{I-i+r} and c_{J+r} exist (if either b_{I-i+r} or c_{J+r} does not exist there is nothing to show).

Proof of (24). In the following, let r be fixed. We distinguish between two cases. If $E_1^{(1)} \leq a_I$, then we have the following chain of inequalities:

$$\begin{aligned} d_{J-i+r} &\leq d_{J-1} + 1 - i + r \leq b_I - i + r \leq b_{I-1+r} - i + 1 \\ &\leq \tilde{E}_2^{(i)} - i + 1 = E_2^{(i)} \leq E_2^{(1)} < f(E_1^{(1)}) \leq f(a_I) \leq f(a_{I-1+r}), \end{aligned} \tag{26}$$

as required. (The second inequality in (26) follows from the fact that the rows in (23a) are strictly increasing.)

Otherwise, if $E_1^{(1)} > a_I$, let us assume for the purpose of contradiction that (24) does not hold. Then, because of the first two inequalities in (26) we have $d_{J-i+r} \leq b_I$, and hence

$$f(a_I) \leq f(a_{I-1+r}) \leq d_{J-i+r} \leq b_I. \tag{27}$$

In more colloquial terms, the point (a_I, b_I) lies outside the ladder region L defined by (2).

For the following, we make the conventional definitions $a^{(j)}_{-\sigma(j)+j} := \tilde{A}_1^{(\sigma(j))} - 1$, $a^{(j)}_{k_j+1} := \tilde{E}_1^{(j)} + 1$ (which is in accordance with the conventional definition for a_{k+1} in (15)), and $b_0^{(j)} := \tilde{A}_2^{(\sigma(j))} - 1$ (which is in accordance with the conventional definition for b_0 in (15)).

For any $j < i$ we claim that, if for the two-rowed array T_{j+1} (given by (13) with i replaced by $j+1$) we find a pair $(a^{(j+1)}_{s_{j+1}}, b^{(j+1)}_{s_{j+1}})$ of entries (i.e., $a^{(j+1)}_{s_{j+1}}$ and $b^{(j+1)}_{s_{j+1}}$ exist in T_{j+1} or are defined by means of one of the above conventional definitions) such that[2]

$$a_I \geq a^{(j+1)}_{s_{j+1}} \quad \text{and} \quad b_I \leq b^{(j+1)}_{s_{j+1}}, \tag{28}$$

then we can find an $h \leq j$ such that the two-rowed array T_h contains a pair $(a^{(h)}_{s_h}, b^{(h)}_{s_h})$ satisfying the same condition, that is

$$a_I \geq a^{(h)}_{s_h} \quad \text{and} \quad b_I \leq b^{(h)}_{s_h}. \tag{29}$$

In other words, we claim that if in T_{j+1} we find a pair of entries which, when considered as a lattice point, is located (weakly) northwest of (a_I, b_I), then we will also find such a pair in T_h for some $h \leq j$.

Let us for the moment assume that we have already established the claim. Clearly, for $j = i-1$ the condition (28) is satisfied with $s_{j+1} = I$, in which case we have $a^{(j+1)}_{s_{j+1}} = a^{(i)}_I = a_I$ and $b^{(j+1)}_{s_{j+1}} = b^{(i)}_I = b_I$. Then, by iterating the assertion of our claim, we will find that (29) is satisfied for $h = 1$ and some s_1. Using this and (27) we obtain

$$f(a^{(1)}_{s_1}) \leq f(a_I) \leq b_I \leq b^{(1)}_{s_1}.$$

However, this inequality contradicts the fact that T_1 obeys the ladder condition (14) with $i = 1$ and $s = s_1$. Hence, inequality (24) must be actually true.

For the proof of the claim, we distinguish between four cases:

(i) $\sigma(j) \geq j$ and $a^{(j)}_1 \leq a_I$;
(ii) $\sigma(j) < j$ and $a^{(j)}_{-\sigma(j)+j+1} \leq a_I$;
(iii) $\sigma(j) \geq j$ and $a^{(j)}_1 > a_I$;
(iv) $\sigma(j) < j$ and $a^{(j)}_{-\sigma(j)+j+1} > a_I$.

4.6.1 Case $\sigma(j) \geq j$ and $a^{(j)}_1 \leq a_I$.

Because we are assuming $E^{(1)}_1 > a_I$, we have $a_I \leq E^{(1)}_1 - 1 = \tilde{E}^{(1)}_1 \leq \tilde{E}^{(j)}_1$. Therefore it is impossible that $a^{(j)}_1 = \tilde{E}^{(j)}_1 + 1$ (by one of our conventional assignments), and hence $a^{(j)}_1$ does indeed exist, i.e., $k_j \geq 1$ (cf. (13) with i replaced by j).

Let s_j be maximal such that $a^{(j)}_{s_j} \leq a_I$. By the above we have $1 \leq s_j \leq k_j$. Therefore $b^{(j)}_{s_j}$ exists. If $b^{(j)}_{s_j} < b_I$, then we have $a^{(j+1)}_{s_{j+1}} \leq a_I < a^{(j)}_{s_j+1}$ and $b^{(j)}_{s_j} < b_I \leq$

[2]It is at the corresponding place where the inaccuracy in [17] occurs. On p. 1036 the inequality chain $a_I \geq x_s \geq \cdots \geq u_t$ has to be replaced by $a_I \geq x_s, \ldots, a_i \geq u_t$, and the inequality chain $b_I \leq y_s \leq \cdots \leq v_t$ has to be replaced by $b_I \leq y_s, \ldots, b_I \leq v_t$.

$b^{(j+1)}_{s_{j+1}}$. But that means that $(a^{(j)}_{s_j+1}, b^{(j+1)}_{s_{j+1}})$ is a crossing point of T_j and T_{j+1} (cf. (15c)–(15e)) with larger x-coordinate than (a_I, d_J), contradicting the maximality of the crossing point (a_I, d_J). Hence, we actually have $b^{(j)}_{s_j} \geq b_I$, and thus (29) holds with $h = j$ and with s_j as above.

4.6.2 Case $\sigma(j) < j$ and $a^{(j)}_{-\sigma(j)+j+1} \leq a_I$.

The arguments from the above case apply verbatim if one replaces $a^{(j)}_1$ by $a^{(j)}_{-\sigma(j)+j+1}$ everywhere.

4.6.3 Case $\sigma(j) \geq j$ and $a^{(j)}_1 > a_I$.

We show that this case actually cannot occur. Because of (7), we have $f(A^{(1)}_1) \leq f(a_I)$, and therefore

$$b^{(j)}_0 = \tilde{A}^{(\sigma(j))}_2 - 1 \leq \tilde{A}^{(1)}_2 - 1 = A^{(1)}_2 < f(A^{(1)}_1) \leq f(a_I) \leq b_I \leq b^{(j+1)}_{s_{j+1}},$$

the two last inequalities being due to (27) and (28). On the other hand, we have $a^{(j+1)}_{s_{j+1}} \leq a_I < a^{(j)}_1$. This means that $(a^{(j)}_1, b^{(j+1)}_{s_{j+1}})$ is a crossing point of T_j and T_{j+1} with larger x-coordinate than (a_I, d_J), which contradicts again the maximality of (a_I, d_J).

4.6.4 Case $\sigma(j) < j$ and $a^{(j)}_{-\sigma(j)+j+1} > a_I$.

If $b^{(j)}_{-\sigma(j)+j} < b_I$, then we have $a^{(j+1)}_{s_{j+1}} \leq a_I < a^{(j)}_{-\sigma(j)+j+1}$ and $b^{(j)}_{-\sigma(j)+j} < b_I \leq b^{(j+1)}_{s_{j+1}}$. This means that $(a^{(j)}_{-\sigma(j)+j+1}, b^{(j+1)}_{s_{j+1}})$ is a crossing point of T_j and T_{j+1} with larger x-coordinate than (a_I, d_J), a contradiction. Therefore we actually have $b^{(j)}_{-\sigma(j)+j} \geq b_I$.

If $a^{(j)}_{-\sigma(j)+j} = \tilde{A}^{(\sigma(j))}_1 - 1 \leq a_I$ then (29) is satisfied with $h = j$ and $s_j = -\sigma(j) + j$. If, on the other hand, $a^{(j)}_{-\sigma(j)+j} > a_I$, then of course (29) cannot be satisfied for $h = j$ and any legal s_j. However, we can show that it is satisfied for some smaller h.

Let us pause for a moment and summarize the conditions that we are encountering in the current case:

$$\sigma(j) < j,\ a^{(j)}_{-\sigma(j)+j} > a_I \text{ and } b^{(j)}_{-\sigma(j)+j} \geq b_I. \tag{30}$$

Clearly, there is a maximal s with $s \leq \sigma(j) \leq \sigma(s)$. We are going to show that we can either find an $h \leq j$ and a legal s_h such that (29) is satisfied, or we find an index $\ell < j$ such that (30) is satisfied with j replaced by ℓ (in which case we repeat the subsequent arguments), or we can construct a sequence of pairs $(a^{(\ell)}_{r_\ell}, b^{(\ell)}_{r_\ell})$, $r_\ell \in \{1, 2, \ldots, k_\ell\}$ for $\ell \in \{s+1, s+2, \ldots, j-1\}$ that satisfy

$$a^{(\ell+1)}_{r_{\ell+1}} \geq a^{(\ell)}_{r_\ell} > a_I \text{ and } b^{(\ell)}_{r_\ell} \geq b^{(\ell+1)}_{r_{\ell+1}} \geq b_I, \tag{31}$$

where, in order that (31) makes sense for $\ell = j-1$, we set $r_j = -\sigma(j) + j$.

However, if we have found such pairs for $\ell \in \{s+1, s+2, \dots, j-1\}$, then we have

$$\begin{aligned} a_I < a_{r_{s+1}}^{(s+1)} < a_{-\sigma(j)+j}^{(j)} + 1 = \tilde{A}_1^{(\sigma(j))} \\ \le \tilde{A}_1^{(\sigma(j))} + \sigma(j) - s \le \tilde{A}_1^{(\sigma(s))} + \sigma(s) - s \le a_1^{(s)} \end{aligned}$$

and

$$b_0^{(s)} = \tilde{A}_2^{(\sigma(s))} - 1 \le \tilde{A}_2^{(\sigma(j))} - 1 < b_{-\sigma(j)+j}^{(j)} = b_{r_j}^{(j)} \le b_{r_{s+1}}^{(s+1)}.$$

This means that $(a_1^{(s)}, b_{r_{s+1}}^{(s+1)})$ is a crossing point of T_s and T_{s+1} with larger x-coordinate than (a_I, d_J), contradicting again the maximality of (a_I, d_J). Therefore we will actually find an $h \le j$ such that (29) is satisfied.

We prove our claim in (31) by a reverse induction on ℓ. (The last two inequalities in (30) guarantee that the induction can be started.) Suppose that we have already found indices $r_{j-1}, r_{j-2}, \dots, r_{\ell+1}$ satisfying (31). Then we distinguish between the two cases $\sigma(\ell) \ge \ell$ and $\sigma(\ell) < \ell$.

First let us consider the case $\sigma(\ell) \ge \ell$. If $a_1^{(\ell)} > a_{r_{\ell+1}}^{(\ell+1)}$, then we have $a_I < a_{r_{\ell+1}}^{(\ell+1)} < a_1^{(\ell)}$ and, if in addition $\ell \ge \sigma(j)$, we have

$$b_0^{(\ell)} = \tilde{A}_2^{(\sigma(\ell))} - 1 \le \tilde{A}_2^{(\sigma(j))} - 1 < b_{-\sigma(j)+j}^{(j)} = b_{r_j}^{(j)} \le b_{r_{\ell+1}}^{(\ell+1)},$$

where the first inequality is due to $\sigma(\ell) \ge \ell \ge \sigma(j)$. This means that $(a_1^{(\ell)}, b_{r_{\ell+1}}^{(\ell+1)})$ is a crossing point of T_ℓ and $T_{\ell+1}$ with larger x-coordinate than (a_I, d_J), again a contradiction.

If $\ell < \sigma(j)$, we can also prove that $b_0^{(\ell)} < b_{r_{\ell+1}}^{(\ell+1)}$, giving the same contradiction. However, this time we must argue differently. Since $\ell > s$, all of $\sigma(\ell+1), \sigma(\ell+2), \dots, \sigma(\sigma(j))$ must be less than $\sigma(j)$. For that reason, because of $\sigma(\ell) \ge \ell$ and the pigeon hole principle, there must be a $t \in \{\ell+1, \ell+2, \dots, \sigma(j)\}$ with $\sigma(t) < \sigma(\ell)$. Then, by (31), we obtain

$$b_0^{(\ell)} = \tilde{A}_2^{(\sigma(\ell))} - 1 \le \tilde{A}_2^{(\sigma(t))} - 1 < b_{r_t}^{(t)} \le b_{r_{\ell+1}}^{(\ell+1)}.$$

Hence, we actually have $a_1^{(\ell)} \le a_{r_{\ell+1}}^{(\ell+1)}$.

We also have $\tilde{E}_1^{(s)} \ge \tilde{A}_1^{(\sigma(s))} + \sigma(s) - s - 1$, because otherwise there would not be any two-rowed array T_s (see (13) with $i = s$), i.e., the family $\mathbf{T}$ of two-rowed arrays that we are considering would not exist, which is absurd. This implies the inequality chain

$$\begin{aligned} \tilde{E}_1^{(\ell)} \ge \tilde{E}_1^{(s)} \ge \tilde{A}_1^{(\sigma(s))} + \sigma(s) - s - 1 \\ \ge \tilde{A}_1^{(\sigma(j))} + \sigma(j) - s - 1 \ge \tilde{A}_1^{(\sigma(j))} - 1 = a_{-\sigma(j)+j}^{(j)} \ge a_{r_{\ell+1}}^{(\ell+1)}. \end{aligned}$$

Therefore it is impossible that $a_1^{(\ell)} = \tilde{E}_1^{(\ell)} + 1$ (by one of our conventional assignments), and hence $a_1^{(\ell)}$ does indeed exist, i.e., $k_\ell \ge 1$.

Now let r_ℓ be maximal, such that $a^{(\ell)}_{r_\ell} \le a^{(\ell+1)}_{r_{\ell+1}}$. By the above we have $1 \le r_\ell \le k_\ell$. If $b^{(\ell)}_{r_\ell} < b^{(\ell+1)}_{r_{\ell+1}}$, then we have $a_I < a^{(\ell+1)}_{r_{\ell+1}} < a^{(\ell)}_{r_\ell+1}$ and $b^{(\ell)}_{r_\ell} < b^{(\ell+1)}_{r_{\ell+1}}$. This means that $(a^{(\ell)}_{r_\ell+1}, b^{(\ell+1)}_{r_{\ell+1}})$ is a crossing point of T_ℓ and $T_{\ell+1}$ with larger x-coordinate than (a_I, d_J), which is once more a contradiction.

Hence, we actually have $b^{(\ell)}_{r_\ell} \ge b^{(\ell+1)}_{r_{\ell+1}}$. Therefore, if $a^{(\ell)}_{r_\ell} \le a_I$ then (29) is satisfied with $h = \ell$ and $s_h = r_\ell$, and otherwise, if $a^{(\ell)}_{r_\ell} > a_I$ then (31) is satisfied.

As a last subcase, we must consider $\sigma(\ell) < \ell$. Again we have to distinguish between two cases: if $a^{(\ell)}_{-\sigma(\ell)+\ell+1} \le a^{(\ell+1)}_{r_{\ell+1}}$, we argue exactly as in the above case where $\sigma(\ell) \ge \ell$ and $a^{(\ell)}_1 \le a^{(\ell+1)}_{r_{\ell+1}}$. (We just have to replace $a^{(\ell)}_1$ by $a^{(\ell)}_{-\sigma(\ell)+\ell+1}$ there.) Otherwise, if $a^{(\ell)}_{-\sigma(\ell)+\ell+1} > a^{(\ell+1)}_{r_{\ell+1}}$, we get $b^{(\ell)}_{-\sigma(\ell)+\ell} \ge b^{(\ell+1)}_{r_{\ell+1}}$, because otherwise $a_I < a^{(\ell+1)}_{r_{\ell+1}} < a^{(\ell)}_{-\sigma(\ell)+\ell+1}$ and $b^{(\ell)}_{-\sigma(\ell)+\ell} < b^{(\ell+1)}_{r_{\ell+1}}$, and thus $(a^{(\ell)}_{-\sigma(\ell)+\ell+1}, b^{(\ell+1)}_{r_{\ell+1}})$ is a crossing point with larger x-coordinate than (a_I, d_J), again a contradiction.

Now, if $a^{(\ell)}_{-\sigma(\ell)+\ell} \le a_I$ then (29) is satisfied with $h = \ell$ and $s_h = -\sigma(\ell) + \ell$. On the other hand, if $a^{(\ell)}_{-\sigma(\ell)+\ell} > a_I$ then (30) is satisfied with j replaced by ℓ. In addition we have $\ell < j$. Consequently, we repeat the arguments subsequent to (30) with j replaced by ℓ. In that manner, we may possibly perform several such iterations. However, these iterations must come to an end because $\sigma(1) \ge 1$, and, hence, the conditions (30) cannot be satisfied for $j = 1$.

Proof of (25). We proceed similarly. We first observe that we must have $a_I \le c_{J+1}$, because otherwise we would have $c_{J+1} < a_I$ and by (15d) also $b_{I-1} < d_J < d_{J+1}$, which means that (a_I, d_{J+1}) is a crossing point of T_i and T_{i+1}, contradicting the maximality of (a_I, d_J). Now we distinguish again between the same two cases as in the proof of (24). If $E^{(1)}_1 \le a_I$, then we have the following chain of inequalities:

$$\begin{aligned} b_{I-i+r} &\le b_{I-1} + 1 - i + r \le d_J - i + r \le d_{J+r} - i \\ &\le \tilde{E}^{(i+1)}_2 - i = E^{(i+1)}_2 \le E^{(1)}_2 < f(E^{(1)}_1) \le f(a_I) \le f(c_{J+1}) \le f(c_{J+r}), \end{aligned} \tag{32}$$

as required. (The second inequality in (32) follows from the fact that the rows in (23b) are strictly increasing.) If on the other hand we have $E^{(1)}_1 > a_I$, then let us assume for the purpose of contradiction that (25) does not hold. This implies

$$f(a_I) \le f(c_{J+r}) \le b_{I-i+r} < b_I.$$

Again, this simply means that the point (a_I, b_I) lies outside the ladder region L defined by (2). We are thus in the same situation as in the above proof of (24), which, in the long run, led to a contradiction.

This completes the proof of the theorem.

5 Enumeration of two-rowed arrays

The entries in the determinant in (10) and (11) are all generating functions $\sum z^{|T|/2}$ for two-rowed arrays T. Hence, we have to say how these can be computed. Of course, a "nice" formula cannot be expected in general. There are only two cases in which "nice" formulas exist, the case of the trivial ladder (i.e., $f(x) \equiv b+1$; see (37)), and the case of a ladder determined by a diagonal boundary (i.e., $f(x) = x + D + 1$, for some positive integer D; see (39)). In all other cases one has to be satisfied with answers of recursive nature.

We will describe two approaches to attack this problem. The first leads to an extension of a formula due to Kulkarni [20] (see also [17, Prop. 4]) for the generating function of lattice paths with given starting and end points in a one-sided ladder region. The second extends the alternative to Kulkarni's formula that was proposed in [17, Prop. 5–7]. The first approach has the advantage of producing a formula (see Proposition 5.1 below) that can be compactly stated. The second approach is always at least as efficient as the first, but is by far superior for ladder regions of a particular kind. This is discussed in more detail after the proof of Proposition 5.4.

Proposition 5.1. *Let f be a weakly increasing function $f : [0,a] \to [1, b+1]$ corresponding to a ladder region L by means of (2), as before. Extend f to all integers by setting $f(x) := \alpha_2$ for $x < 0$ and $f(x) := \varepsilon_2 + 1$ for $x > a$. Let $\alpha_1 - 1 < s_{k-1} < \cdots < s_1 < \varepsilon_1$ be a partition of the (integer) interval $[\alpha_1 - 1, \varepsilon_1]$ such that f is constant on each subinterval $[s_i + 1, s_{i-1}]$, $i = k, k-1, \dots, 1$, with $s_k := \alpha_1 - 1$ and $s_0 := \varepsilon_1$. Then the generating function $\sum z^{|T|/2}$ for all two-rowed arrays T of the form (6) and satisfying (5) is given by*

$$\mathrm{GF}(TA(l; (\alpha_1, \alpha_2), (\varepsilon_1, \varepsilon_2); f, d); z^{|\cdot|/2}) \\ = \sum_{\substack{\mathbf{e}+d \geq \mathbf{f} \geq \mathbf{0} \\ e_k - f_k = l}} z^{e_k} \prod_{i=1}^{k} \binom{s_{i-1} - s_i}{e_i - e_{i-1}} \binom{f'(s_{i-1}) - f'(s_i)}{f_i - f_{i-1}}, \quad (33)$$

where $\mathbf{e} = (e_1, e_2, \dots, e_k)$ and $\mathbf{f} = (f_1, f_2, \dots, f_k)$, where, by definition, $e_0 = f_0 = 0$, where $\mathbf{e} + d \geq \mathbf{f} \geq 0$ means $e_i + d \geq f_i \geq 0$, $i = 1, 2, \dots, k$, and where $f'(x)$ agrees with $f(x)$ for $\alpha_1 \leq x < \varepsilon_1$, but where $f'(\alpha_1 - 1) = \alpha_2$ and $f'(\varepsilon_1) = \varepsilon_2 + 1$. (All other values of f' are not needed for the formula (33)).

Proof. Let T be a two-rowed array in $TA(l; (\alpha_1, \alpha_2), (\varepsilon_1, \varepsilon_2); f, d)$, represented as in (6). Suppose that there are e_i entries in the first row of T that are larger than s_i, and that there are f_i entries in the second row of T that are larger than or equal to $f(s_i)$, $i = 1, 2, \dots, k$. Equivalently, we have

$$\varepsilon_1 = s_0 \geq a_1 > \cdots > a_{e_1} > s_1 \geq a_{e_1+1} > \cdots > a_{e_2} > s_2 \\ \geq \cdots > s_{k-1} \geq a_{e_{k-1}+1} > \cdots > a_{e_k} > s_k = \alpha_1 - 1, \quad (34)$$

and

$$f'(s_0) = \varepsilon_2 + 1 > b_1 > \cdots > b_{f_1} \geq f(s_1) > b_{f_1+1} > \cdots > b_{f_2} \geq f(s_2) \\ > \cdots \geq f(s_{k-1}) > b_{f_{k-1}+1} > \cdots > b_{f_k} \geq f'(s_k) = \alpha_2. \quad (35)$$

In particular, we have $e_k - f_k = l$. From (5) it is immediate that we must have $e_i + d \ge f_i \ge 0$. Conversely, given integer vectors $\mathbf{e}$ and $\mathbf{f}$ with $e_i + d \ge f_i \ge 0$ and $e_k - f_k = l$, by (34) and (35) there are

$$\prod_{i=1}^{k} \binom{s_{i-1} - s_i}{e_i - e_{i-1}} \binom{f(s_{i-1}) - f(s_i)}{f_i - f_{i-1}}$$

possible choices for the entries a_i and b_i, $i = 1, 2, \dots$, in the first and second row of a two-rowed array which satisfies (34) and (35), and thus (5). This establishes (33). □ □

Remark. If in Proposition 5.1 we set $l = d = 0$, then we recover Kulkarni's formula [20, Theorem 4] (see also [17, Prop. 4]), because the two-rowed arrays in $TA(0; (\alpha_1, \alpha_2), (\varepsilon_1, \varepsilon_2); f, 0)$ can be interpreted as lattice paths with starting point $(\alpha_1, \alpha_2 - 1)$ and end point $(\varepsilon_1 + 1, \varepsilon_2)$ which stay in the ladder region defined by f.

Now we describe the announced alternative method to compute the generating function $\sum z^{|T|/2}$ for two-rowed arrays T of the form (6) which satisfy (5). For sake of convenience, for $A = (\alpha_1, \alpha_2)$ and $E = (\varepsilon_1, \varepsilon_2)$ as before, $\alpha_1 \le \varepsilon_1$, we introduce the set

$$TA^*(l; A, E; f, d) = TA(l; A, E; f, d) \setminus TA(l; A + (1, 0), E; f, d), \tag{36}$$

which is simply the set of those two-rowed arrays of the given form whose first entry in the first row equals α_1.

This second method is based on the simple facts that are summarized in Propositions 5.2–5.4. The propositions extend in turn Propositions 5–7 in [17]. In the following, all binomial coefficients $\binom{n}{k}$ are understood to be equal to zero if n is negative and k is positive.

Proposition 5.2. *Let L be the trivial ladder determined by the function $f(x) \equiv b+1$ by means of (2). Let $A = (\alpha_1, \alpha_2)$ and $E = (\varepsilon_1, \varepsilon_2)$ be lattice points and l and d arbitrary integers. Then we have*

$$\mathrm{GF}\left(TA(l; A, E; f, d); z^{|\cdot|/2}\right) = \sum_k \binom{\varepsilon_1 - \alpha_1 + 1}{k + l} \binom{\varepsilon_2 - \alpha_2 + 1}{k} z^{k + l/2}, \tag{37}$$

and if $\alpha_1 \le \varepsilon_1$ we have

$$\mathrm{GF}\left(TA^*(l; A, E; f, d); z^{|\cdot|/2}\right) = \sum_k \binom{\varepsilon_1 - \alpha_1}{k + l - 1} \binom{\varepsilon_2 - \alpha_2 + 1}{k} z^{k + l/2}. \tag{38}$$

Proposition 5.3. *Let L_D be a "diagonal" ladder determined by the function $f(x) = x + D + 1$ for an integer D by means of (2). Let d be a nonnegative integer and l*

an integer such that $l+d\ge 0$. Let $A=(\alpha_1,\alpha_2)$ and $E=(\varepsilon_1,\varepsilon_2)$ be lattice points such that $\alpha_1+D+1+l+d\ge\alpha_2$ and $\varepsilon_1+D+1+d\ge\varepsilon_2$. Then we have

$$\operatorname{GF}\big(TA(l;A,E;f,d);z^{|\cdot|/2}\big)=\sum_k\left(\binom{\varepsilon_1-\alpha_1+1}{k+l}\binom{\varepsilon_2-\alpha_2+1}{k}\right.\\ \left.-\binom{\varepsilon_1-\alpha_2+D+1}{k-d-1}\binom{\varepsilon_2-\alpha_1-D+1}{k+l+d+1}\right)z^{k+l/2},\tag{39}$$

and if $\alpha_1\le\varepsilon_1$ we have

$$\operatorname{GF}\big(TA^*(l;A,E;f,d);z^{|\cdot|/2}\big)=\sum_k\left(\binom{\varepsilon_1-\alpha_1}{k+l-1}\binom{\varepsilon_2-\alpha_2+1}{k}\right.\\ \left.-\binom{\varepsilon_1-\alpha_2+D+1}{k-d-1}\binom{\varepsilon_2-\alpha_1-D}{k+l+d}\right)z^{k+l/2}.\tag{40}$$

Proof of Propositions 5.2 and 5.3. Identities (37) and (38) are immediate from the definitions.

To prove identity (41), we note that the number of two-rowed arrays

$$\begin{matrix}\alpha_1\le & a_{-l+1} & a_{-l+2} & \dots & a_0 & a_1 & \dots & a_k & \le\varepsilon_1\\ \alpha_2\le & & & & & b_1 & \dots & b_k & \le\varepsilon_2\end{matrix}\tag{41a}$$

that obey

$$b_i<a_{i+d}+D+1,\quad i=1,2,\dots,k,\tag{41b}$$

is the number of *all* two-rowed arrays of the form (41a) minus those that violate the condition (41b). Clearly, the generating function for the former two-rowed arrays is given by the first term in the sum on the right hand side of (39). We claim that the two-rowed arrays of the form (41a) that violate (41b) are in one-to-one correspondence with two-rowed arrays of the form

$$\begin{matrix}\alpha_2-D\le & & & & & c_1 & \dots & c_{k-d-1} & \le\varepsilon_1\\ \alpha_1+D\le & d_{-l-2d-1} & d_{-l-2d} & \dots & d_0 & d_1 & \dots & d_{k-d-1} & \le\varepsilon_2.\end{matrix}\tag{42}$$

(In particular, if $k\le d$ then there is no two-rowed array of the form (42), in agreement with the fact that there cannot be any two-rowed array of the form (41a) violating (41b) in that case.) The generating function for the two-rowed arrays in (42) is

$$\sum_k\binom{\varepsilon_1-\alpha_2+D+1}{k-d-1}\binom{\varepsilon_2-\alpha_1-D+1}{k+l+d+1}z^{k+l/2},$$

which is exactly the negative of the second term on the right-hand side of (39). This would prove (39). So it remains to construct the one-to-one correspondence.

The correspondence that we are going to describe is gleaned from [18], see also [15, Sec. 13.4] and [16]. Take a two-rowed array of the form (41a) that violates condition (41b), i.e., there is an index i such that $b_i \ge a_{i+d} + D + 1$. Let I be the largest integer with this property. Then map this two-rowed array to

$$\begin{array}{lccccccccc} \alpha_2 - D \le & & (b_1 - D) & \dots\dots\dots\dots & (b_{I-1} - D) & a_{I+d+1} & \dots & a_k & \le \varepsilon_1 \\ \alpha_1 + D \le & (a_{-l+1} + D) & \dots\dots & (a_{I+d} + D) & b_I & \dots\dots\dots\dots\dots\dots & & b_k & \le \varepsilon_2. \end{array}$$

Note that both rows are strictly increasing because of $b_{I-1} - D \le b_{I+1} - D - 2 < a_{I+d+1}$. If $I = 1$, we have to check in addition that $\alpha_2 - D \le a_{d+2}$, which is indeed the case, because

$$a_{d+2} \ge a_{d+1} + 1 \ge \cdots \ge a_{-l+1} + 1 + l + d \ge \alpha_1 + 1 + l + d \ge \alpha_2 - D.$$

Similarly, it can be checked that $b_{I-1} - D \le \varepsilon_1$ if $I = k - d$. It is easy to see that the array is of the form (42).

The inverse of this map is defined in the same way. Take a two-rowed array of the form (42). Let J be the largest integer such that $d_J \ge c_{J+d} + D + 1$, if existent. If there is no such integer, then let $J = -d$. We map this two-rowed array to

$$\begin{array}{lccccccccc} \alpha_1 \le & (d_{-l-2d-1} - D) & \dots\dots\dots\dots\dots\dots & & (d_{J-1} - D) & c_{J+d+1} & \dots & c_{k-d-1} & \le \varepsilon_1 \\ \alpha_2 \le & & (c_1 + D) & \dots & (c_{J+d} + D) & d_J & \dots\dots\dots\dots\dots\dots & d_{k-d-1} & \le \varepsilon_2 \end{array}$$

Since we required $l + d \ge 0$ the entry $d_{J-1} - D$ exists even if $J = -d$. This implies that the two-rowed array we obtained violates condition (41b), since $d_J \ge d_{J-1} + 1 = (d_{J-1} - D) + D + 1$. As above, it can be checked that both rows are strictly increasing, even in the case $J = -d$, and that the array is of the correct form.

Equation (40) is an immediate consequence of (39) and the definition (36) of $TA^*(l; A, E; f, d)$. $\square$

Proposition 5.4. *Let L be an arbitrary ladder given by a function f by means of (2), let $A = (\alpha_1, \alpha_2)$, $E = (\varepsilon_1, \varepsilon_2)$ be lattice points in L, and let d be a nonnegative integer and l an integer such that $l + d \ge 0$. Then for all $x \in [0, a]$ such that $\alpha_2 \le f(x) \le \varepsilon_2 + 1$ we have*

$$\begin{aligned} &\operatorname{GF}\left(TA(l; A, E; f, d); z^{|\cdot|/2}\right) \\ &\qquad = \sum_{j=x+1}^{\varepsilon_1} \operatorname{GF}\left(TA(l+d; A, (j-1, f(x)-1); f, 0); z^{|\cdot|/2}\right) \\ &\qquad\qquad \cdot \operatorname{GF}\left(TA^*(-d; (j, f(x)), E; f, d); z^{|\cdot|/2}\right) \\ &\qquad\quad + \sum_{e=0}^{d} \operatorname{GF}\left(TA(l+d-e; A, (\varepsilon_1, f(x)-1); f, e); z^{|\cdot|/2}\right) \\ &\qquad\qquad \cdot \binom{\varepsilon_2 - f(x) + 1}{d-e} z^{(d-e)/2}. \end{aligned} \tag{43}$$

Proof. We show this recurrence relation by decomposing an array

$$\begin{array}{llllllllll} \alpha_1\le & a_{-l+1} & a_{-l+2} & \dots & a_{-1} & a_0 & a_1 & \dots & a_k & \le\varepsilon_1 \\ \alpha_2\le & & & & & & b_1 & \dots & b_k & \le\varepsilon_2 \end{array} \tag{44}$$

in $TA(l;A,E;f,d)$ — the generating function of which is the left-hand side of (43) — into two parts. Let I be the smallest integer with $b_I\ge f(x)$, or, if all b_I are smaller than $f(x)$, let $I=k+1$. Now we have to distinguish between two cases.

If $I+d<k+1$, we decompose such an array into the array

$$\begin{array}{llllllllllll} \alpha_1\le & a_{-l+1} & a_{-l+2} & \dots & a_{-1} & a_0 & a_1 & \dots & a_{d+1} & \dots & a_{I-1+d} & \le a_{I+d}-1 \\ \alpha_2\le & & & & & & & & b_1 & \dots & b_{I-1} & \le f(x)-1 \end{array}$$

in $TA(l+d;A,\big(a_{I+d}-1,f(x)-1\big);f,0)$, and the array

$$\begin{array}{lllllll} a_{I+d}\le & & & a_{I+d} & \dots & a_k & \le\varepsilon_1 \\ f(x)\le & b_I & \dots & b_{I+d} & \dots & b_k & \le\varepsilon_2 \end{array}$$

in $TA^*(-d;\big(a_{I+d},f(x)\big),E;f,d)$. Clearly, this is a pair of two-rowed arrays enumerated by the first sum in the right hand side of (43), with the summation index j equal to a_{I+d}.

If $I+d\ge k+1$, we decompose (44) into the array

$$\begin{array}{llllllllllll} \alpha_1\le & a_{-l+1} & a_{-l+2} & \dots & a_{-1} & a_0 & a_1 & \dots & a_{k-I+2} & \dots & a_k & \le\varepsilon_1 \\ \alpha_2\le & & & & & & & & b_1 & \dots & b_{I-1} & \le f(x)-1 \end{array}$$

in $TA(l-I+k+1;A,\big(\varepsilon_1,f(x)-1\big);f,d+I-k-1)$, and a single row

$$f(x)\le\quad b_I\quad\dots\quad b_k\quad\le\varepsilon_2.$$

Note that, if $I=k+1$, this row is empty. These pairs are enumerated by the second sum on the right hand side of (43), with the summation index e equal to $d+I-k-1$. □ □

Now, here is the second method for determining $\mathrm{GF}(TA(l;(\alpha_1,\alpha_2),(\varepsilon_1,\varepsilon_2);f,d));z^{|\cdot|/2})$ for any given ladder L of the form (2), with points $A=(\alpha_1,\alpha_2)$ and $E=(\varepsilon_1,\varepsilon_2)$ located inside L: partition the *border* of L, i.e., the set of points $\{(x,f(x)):x\in[0,a]\}$ into horizontal and diagonal pieces, say $L^1,L^2,\dots,L^m$, where $L^i=\{(x,f(x)):x_{i-1}<x\le x_i\}$, for some $-1=x_0<x_1<x_2<\dots<x_m=a$, each L^i being either horizontal or diagonal. Then apply the recurrence (43) in succession with $x=x_{m-1},x_{m-2},\dots,x_1$ and use (37)–(40) to compute all the occurring generating functions.

To give an example, in the case of the ladder of Figure 2 we would choose $m=3$, $x_1=3$, $x_2=7$, $x_3=13$, and the resulting formula reads

$$\begin{aligned}
\operatorname{GF}\left(TA(l;A,E;f,d);z^{|\cdot|/2}\right)&=\sum_{j=8}^{\varepsilon_1}\sum_{k\ge0}z^{k-d/2}\binom{\varepsilon_1-j}{k-d-1}\binom{\varepsilon_2-12}{k}\\
&\quad\cdot\Bigg(\sum_{i=4}^{j-1}\sum_{k_1,k_2\ge0}z^{k_1+k_2+(l+d)/2}\binom{i-\alpha_1}{k_1+l+d}\binom{7-\alpha_2}{k_1}\\
&\qquad\qquad\cdot\left(\binom{j-i-1}{k_2-1}\binom{6}{k_2}-\binom{7-i}{k_2}\binom{j-2}{k_2-1}\right)\\
&\quad+\sum_{k_1\ge0}z^{k_1+(l+d)/2}\binom{j-\alpha_1}{k_1+l+d}\binom{7-\alpha_2}{k_1}\Bigg)\\
&+\sum_{e=0}^{d}z^{(d-e)/2}\binom{\varepsilon_2-12}{d-e}\\
&\quad\cdot\Bigg(\sum_{i=4}^{\varepsilon_1}\sum_{k_1,k_2\ge0}z^{k_1+k_2+(l+d-e)/2}\binom{i-\alpha_1}{k_1+l+d}\binom{7-\alpha_2}{k_1}\\
&\qquad\qquad\cdot\left(\binom{\varepsilon_1-i}{k_2-e-1}\binom{6}{k_2}-\binom{7-i}{k_2}\binom{\varepsilon_1-1}{k_2-e-1}\right)\\
&\quad+\sum_{f=0}^{e}\sum_{k\ge0}z^{k+(l+d+e)/2-f}\binom{\varepsilon_1-\alpha_1+1}{k+l+d-f}\binom{7-\alpha_2}{k}\binom{6}{e-f}\Bigg).
\end{aligned}\tag{45}$$

If L consists of not too many pieces, both methods are feasible methods, see our Example in Section 3. Both methods yield $(2m-1)$-fold sums if the partition of the border consists of horizontal pieces throughout. However, the second method is by far superior in case of long diagonal portions in the border of L, since then Kulkarni's formula involves a lot more summations. For example, when we implemented formula (45) (in *Mathematica*) it was by a factor of 40.000 (!) faster than the corresponding implementation of formula (33). (Indeed, the "simplicity" of the formula (33) in comparison to (45) is deceptive, as (33) involves an 11-fold summation in that case, whereas (45) has only 3-fold, 4-fold, and 5-fold sums.) Of course, in the worst case, when L consists of 1-point pieces throughout, both methods are nothing else than plain counting, and therefore useless. For computation in case of such "fractal" boundaries it is more promising to avoid Theorem 3.1 and instead try to extend the dummy path method in [19] such that it also applies to the enumeration of nonintersecting lattice paths with respect to turns.

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Chapter 4
The h-vector of a ladder determinantal ring cogenerated by 2×2 minors is log-concave*

Abstract

We show that the h-vector of a ladder determinantal ring cogenerated by $M = [u_1 \mid v_1]$ is log-concave. Thus we prove an instance of a conjecture of Stanley, resp. Conca and Herzog.

1 Introduction

Definition 1.1. A sequence of real numbers $a_1, a_2, \ldots, a_n$ is *logarithmically concave*, for short *log-concave*, if $a_{i-1}a_{i+1} \le a_i^2$ for $i \in \{2, 3, \ldots, n-1\}$.

Numerous sequences arising in combinatorics and algebra have, or seem to have this property. In the paper [13] written in 1989, Richard Stanley collected various results on this topic. (For an update see [3].) There he also stated the following conjecture:

Conjecture 1.2. *Let $R = R_0 \oplus R_1 \oplus \ldots$ be a graded (Noetherian) Cohen-Macaulay (or perhaps Gorenstein)* domain *over a field $K = R_0$, which is generated by R_1 and has Krull dimension d. Let $H(R, m) = \dim_K R_m$ be the Hilbert function of R and write*

$$\sum_{m \ge 0} H(R, m) x^m = (1-x)^{-d} \sum_{i=0}^{s} h_i x^i.$$

Then the sequence $h_0, h_1, \ldots, h_s$ is log-concave.

The sequence $h_0, h_1, \ldots, h_s$ is called the *h-vector* of the ring. Orginally the question was to decide whether a given sequence can arise as the h-vector of some ring. In this sense the validity of the conjecture would imply that log-concavity was a necessary condition on the h-vector.

It is now known however [12, 3] that Stanley's conjecture is not true in general. Several natural weakenings have been considered, but are still open. For example, Aldo Conca and Jürgen Herzog conjectured that the h-vector would be log-concave for the special case where R is a ladder determinantal ring. (Note that ladder

*In honour of Miriam Rubey, at the occasion of her second birthday determinantal rings are Cohen-Macaulay, as was shown in [8, Corollary 4.10], but not necessarily Gorenstein.) We will prove the conjecture of Conca and Herzog in the simplest case, i.e., where R is a ladder determinantal ring cogenerated by 2×2 minors, see Corollary 4.6.

In the case of ladder determinantal rings the h-vector has a nice combinatorial interpretation. This follows from work of Abhyankar and Kulkarni [1, 2, 10, 11], Bruns, Conca, Herzog, and Trung [4, 5, 6, 8]. In the following paragraphs, which are taken almost verbatim from [9], we will explain these matters.

2 Ladders, ladder determinantal rings and non-intersecting lattice paths

First we have to introduce the notion of a ladder:

Definition 2.1. Let $\mathbf{X} = (x_{i,j})_{0 \le i \le b, 0 \le j \le a}$ be a $(b+1) \times (a+1)$ matrix of indeterminates. Let $\mathbf{Y} = (y_{i,j})_{0 \le i \le b, 0 \le j \le a}$ be another matrix of the same dimensions, with the property that $y_{i,j} \in \{0, x_{i,j}\}$, and if $y_{i,j} = x_{i,j}$ and $y_{i',j'} = x_{i',j'}$, where $i \le i'$ and $j \le j'$ then $y_{r,s} = x_{r,s}$ for all r and s with $i \le r \le i'$ and $j \le s \le j'$. Such a matrix $\mathbf{Y}$ is called a *ladder*.

A *ladder region* L is a subset of $\mathbb{Z}^2$ with the property that if (i,j) and $(i',j') \in L$, $i \le i'$ and $j \ge j'$ then $(r,s) \in L$ for all $r \in \{i, i+1, \dots, i'\}$ and $s \in \{j', j'+1, \dots, j\}$. Clearly, a ladder region can be described by two weakly increasing functions $\underline{L}$ and $\overline{L}$, such that L is exactly the set of points $\{(i,j) : \underline{L}(i) \le j \le \overline{L}(i)\}$.

We associate with $\mathbf{Y}$ a ladder region $L \subset \mathbb{Z}^2$ via $(j, b-i) \in L$ if and only if $y_{i,j} = x_{i,j}$.

In Figure 1.a an example of a ladder with $a = 8$ and $b = 9$ is shown, the corresponding ladder region is shown in Figure 1.b.

Now we can define the ring we are dealing with:

Definition 2.2. Given a $(b+1) \times (a+1)$ matrix $\mathbf{Y}$ which is a ladder, fix a "bivector" $M = [u_1, u_2, \dots, u_n \mid v_1, v_2, \dots, v_n]$ of integers with $1 \le u_1 < u_2 < \cdots < u_n \le b+1$ and $1 \le v_1 < v_2 < \cdots < v_n \le a+1$. By convention we set $u_{n+1} = b+2$ and $v_{n+1} = a+2$.

Let $K[\mathbf{Y}]$ denote the ring of all polynomials over some field K in the $y_{i,j}$'s, where $0 \le i \le b$ and $0 \le j \le a$. Furthermore, let $I_M(\mathbf{Y})$ be the ideal in $K[\mathbf{Y}]$ that is generated by those $t \times t$ minors of $\mathbf{Y}$ that contain only nonzero entries, whose rows form a subset of the last $u_t - 1$ rows *or* whose columns form a subset of the last $v_t - 1$ columns, $t \in \{1, 2, \dots, n+1\}$. Thus, for $t = n+1$ the rows and columns of minors are unrestricted.

The ideal $I_M(\mathbf{Y})$ is called a *ladder determinantal ideal generated by the minors defined by M*. We call $R_M(\mathbf{Y}) = K[\mathbf{Y}]/I_M(\mathbf{Y})$ the *ladder determinantal ring cogenerated by the minors defined by M*, or, in abuse of language, the *ladder determinantal ring cogenerated by M*.

Note that we could restrict ourselves to the case $u_1 = v_1 = 1$, because all the elements of $\mathbf{Y}$ that are in one of the last $u_1 - 1$ rows or in one of the last $v_1 - 1$ columns are in the ideal.

Next, we introduce the combinatorial objects that will accompany us throughout the rest of this paper:

Definition 2.3. A *two-rowed array of length* k is a pair of strictly increasing sequences of integers, both of length k. A two-rowed array $T = \left(\begin{smallmatrix} a_1 & a_2 & \dots & a_k \\ b_1 & b_2 & \dots & b_k \end{smallmatrix}\right)$ is *bounded* by $A = (A_1, A_2)$ and $E = (E_1, E_2)$, if

$$\begin{aligned} A_1 \;\le\; a_1 < a_2 < \cdots < a_k \;\le\; E_1 - 1 \\ \text{and} \quad A_2 + 1 \;\le\; b_1 < b_2 < \cdots < b_k \;\le\; E_2. \end{aligned}$$

Given *any* subset L of $\mathbb{Z}^2$, we say that the two-rowed array T is *in* L, if $(a_i, b_i) \in L$ for $i \in \{1, 2, \dots, k\}$. By $\mathcal{T}_k^L(A \mapsto E)$ we will denote the set of two-rowed arrays of length k, bounded by A and E which are in L. The *total length* of a family of two-rowed arrays is just the sum of the lengths of its members.

Let $T_1 = \left(\begin{smallmatrix} a_1 & a_2 & \dots & a_k \\ b_1 & b_2 & \dots & b_k \end{smallmatrix}\right)$ and $T_2 = \left(\begin{smallmatrix} x_1 & x_2 & \dots & x_l \\ y_1 & y_2 & \dots & y_l \end{smallmatrix}\right)$ be two-rowed arrays bounded by $A^{(1)} = (A_1^{(1)}, A_2^{(1)})$ and $E^{(1)} = (E_1^{(1)}, E_2^{(1)})$ and $A^{(2)} = (A_1^{(2)}, A_2^{(2)})$ and $E^{(2)} = (E_1^{(2)}, E_2^{(2)})$ respectively. Set $a_{k+1} = E_1^{(1)}$ and $b_0 = A_2^{(1)}$. We say that T_1 and T_2 *intersect* if there are indices I and J such that

$$\begin{aligned} x_J &\le a_I \\ b_{I-1} &\le y_J \end{aligned} \tag{$\times$}$$

where $1 \le I \le k+1$ and $1 \le J \le l$. A family of two-rowed arrays is *non-intersecting* if no two arrays in it intersect.

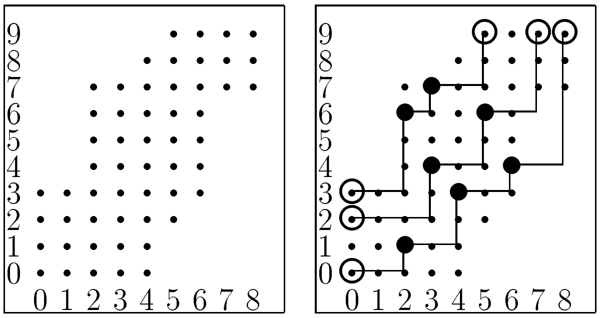
Note that a two-rowed array in $\mathcal{T}_k^L(A \mapsto E)$ can be visualized by a lattice path with east and north steps, that starts in A and terminates in E and has exactly k north-east turns which are all in L: Each pair (a_i, b_i) of a two-rowed array $\left(\begin{smallmatrix} a_1 & a_2 & \dots & a_k \\ b_1 & b_2 & \dots & b_k \end{smallmatrix}\right)$ then corresponds to a north-east turn of the lattice path. It is easy to see that Condition $(\times)$ holds if and only if the lattice paths corresponding to T_1 and T_2 intersect.

For an example see Figure 1.c, where the three two-rowed arrays

$$T^{(1)} = \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}, T^{(2)} = \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix} \text{ and } T^{(3)} = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 4 \end{pmatrix}$$

bounded by $A^{(1)} = (0,3)$, $A^{(2)} = (0,2)$, $A^{(3)} = (0,0)$ and $E^{(1)} = (5,9)$, $E^{(2)} = (7,9)$, $E^{(3)} = (8,9)$ are shown as lattice paths. The points of the ladder-region L are drawn as small dots, the circles indicate the start- and endpoints and the big dots indicate the north-east turns.

$$
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & x_{5,0} & x_{6,0} & x_{7,0} & x_{8,0} \\
0 & 0 & 0 & 0 & x_{4,1} & x_{5,1} & x_{6,1} & x_{7,1} & x_{8,1} \\
0 & 0 & x_{2,2} & x_{3,2} & x_{4,2} & x_{5,2} & x_{6,2} & x_{7,2} & x_{8,2} \\
0 & 0 & x_{2,3} & x_{3,3} & x_{4,3} & x_{5,3} & x_{6,3} & 0 & 0 \\
0 & 0 & x_{2,4} & x_{3,4} & x_{4,4} & x_{5,4} & x_{6,4} & 0 & 0 \\
0 & 0 & x_{2,5} & x_{3,5} & x_{4,5} & x_{5,5} & x_{6,5} & 0 & 0 \\
x_{0,6} & x_{1,6} & x_{2,6} & x_{3,6} & x_{4,6} & x_{5,6} & x_{6,6} & 0 & 0 \\
x_{0,7} & x_{1,7} & x_{2,7} & x_{3,7} & x_{4,7} & x_{5,7} & 0 & 0 & 0 \\
x_{0,8} & x_{1,8} & x_{2,8} & x_{3,8} & x_{4,8} & 0 & 0 & 0 & 0 \\
x_{0,9} & x_{1,9} & x_{2,9} & x_{3,9} & x_{4,9} & 0 & 0 & 0 & 0
\end{array}
$$



a. a ladder with $a = 8$ and $b = 9$

b. the corresponding ladder region

c. a triple of non-intersecting lattice paths in this ladder

Figure 1.

3 A combinatorial interpretation of the h-vector of a ladder determinantal ring

We are now ready to state the theorem which reveals the combinatorial nature of the h-vector of $R_M(\mathbf{Y}) = K[\mathbf{Y}]/I_M(\mathbf{Y})$, the ladder determinantal ring cogenerated by M.

Theorem 3.1. *Let $\mathbf{Y} = (y_{i,j})_{0\le i\le b,\ 0\le j\le a}$ be a ladder and let $M = [u_1, u_2, \ldots, u_n \mid v_1, v_2, \ldots, v_n]$ be a bivector of integers with $1 \le u_1 < u_2 < \cdots < u_n \le a+1$ and $1 \le v_1 < v_2 < \cdots < v_n \le b+1$. For $i \in \{1, 2, \ldots, n\}$ let*

$$
\begin{aligned}
A^{(i)} &= (0, u_{n+1-i} - 1) \\
E^{(i)} &= (a - v_{n+1-i} + 1, b).
\end{aligned}
$$

Let $L^{(n)} = L$ be the ladder region associated with $\mathbf{Y}$ and for $i \in \{1, 2, \ldots, n-1\}$ let

$$
L^{(i)} = \{(x, y) \in L^{(i+1)} : x \le E_1^{(i)}, y \ge A_2^{(i)} \text{ and } (x+1, y-1) \in L^{(i+1)}\}.
$$

Finally, for $i \in \{1, 2, \ldots, n\}$ let

$$
B^{(i)} = \{(x, y) \in L^{(i)} : (x+1, y-1) \notin L^{(i)}\}.
$$

and let d be the cardinality of $\bigcup_{i=1}^{n} B^{(i)}$.

Then, under the assumption that all of the points $A^{(i)}$ and $E^{(i)}$, $i \in \{1, 2, \ldots, n\}$, lie inside the ladder region L, the Hilbert series of the ladder determinantal ring $R_M(\mathbf{Y}) = K[\mathbf{Y}]/I_M(\mathbf{Y})$ equals

$$
\sum_{\ell\ge 0} \dim_K R_M(\mathbf{Y})_\ell \, z^\ell = \frac{\sum_{\ell\ge 0} \left|\mathcal{T}_\ell^L(\mathbf{A} \mapsto \mathbf{E})\right| z^\ell}{(1-z)^d}.
$$

Here, $R_M(\mathbf{Y})_\ell$ denotes the homogeneous component of degree ℓ in $R_M(\mathbf{Y})$ and $\left|\mathcal{T}_\ell^L(\mathbf{A} \mapsto \mathbf{E})\right|$ is the number of non-intersecting families of two-rowed arrays with total length ℓ, such that the i^{th} two-rowed array is bounded by $A^{(i)}$ and $E^{(i)}$ and is in $L^{(i)} \setminus B^{(i)}$ for $i \in \{1, 2, \ldots, n\}$.

The sets $B^{(i)}$, $i \in \{1, 2, \ldots, n\}$ can be visualized as being the lower-right boundary of $L^{(i)}$. Viewed as a path, there are exactly $E_1^{(i)} - A_1^{(i)} + E_2^{(i)} - A_2^{(i)} + 1$ lattice points on $B^{(i)}$, but not all of them are necessarily in L. However, if L is an upper ladder, that is, $(a, 0) \in L$, then this must be the case and we have

$$\begin{aligned} d &= \sum_{i=1}^{n} \left(E_1^{(i)} - A_1^{(i)} + E_2^{(i)} - A_2^{(i)} + 1 \right) \\ &= \sum_{i=1}^{n} \left(a - v_{n+1-i} + 1 + b - u_{n+1-i} + 1 + 1 \right) \\ &= n(a + b + 3) - \sum_{i=1}^{n} \left(u_i + v_i \right), \end{aligned}$$

as in [9].

In Figure 2.a, an example for a ladder region L with $a = 8$ and $b = 9$ is given. The small dots represent elements of L, the circles on the left and on the top of L represent the points $A^{(i)}$ and $E^{(i)}$, $i \in \{1, 2, 3\}$ that are specified by the minor $M = [1, 3, 4 \mid 1, 2, 4]$. The dotted lines indicate the lower boundary of $L^{(i)}$. Note that the point $(4, 9)$ is not an element of L. Therefore, in this example we have

$$d = n(a + b + 3) - \sum_{i=1}^{n} (u_i + v_i) - 1 = 44.$$

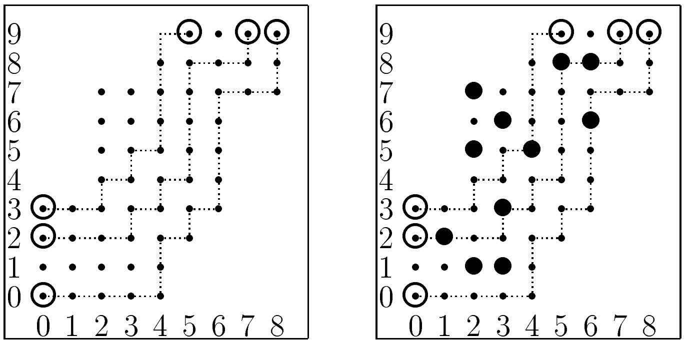
Proof. We will use results of Jürgen Herzog and Ngô Viêt Trung. In Section 4 of [8], ladder determinantal rings are introduced and investigated.

We equip the indeterminates $x_{i,j}$, $i \in \{0, 1, \ldots, b\}$ and $j \in \{0, 1, \ldots, a\}$ with the following partial order:

$$x_{i,j} \leq x_{i',j'} \text{ if } i \geq i' \text{ and } j \leq j'.$$

A t-antichain in this partial order is a family of elements $x_{r_1,s_1}, x_{r_2,s_2}, \ldots, x_{r_t,s_t}$ such that $r_1 < r_2 < \cdots < r_t$ and $s_1 < s_2 < \cdots < s_t$. Thus, a t-antichain corresponds to a sequence $(s_1, b - r_1), (s_2, b - r_2), \ldots, (s_t, b - r_t)$ of t points in the ladder region associated with $\mathbf{Y}$, where each point lies strictly south-east of the previous ones.

Let D_t be the union of the last $u_t - 1$ rows and the last $v_t - 1$ columns of $\mathbf{Y}$. Let $\Delta_M(\mathbf{Y})$ be the simplicial complex whose k-dimensional faces are subsets of elements of $\mathbf{Y}$ of cardinality $k + 1$ which do not contain a t-antichain in D_t for



a. a ladder region with $a = 8$ and $b = 9$

b. a 10 dimensional face of $\Delta_{[1,3,4|1,2,4]}(\mathbf{Y})$

Figure 2.

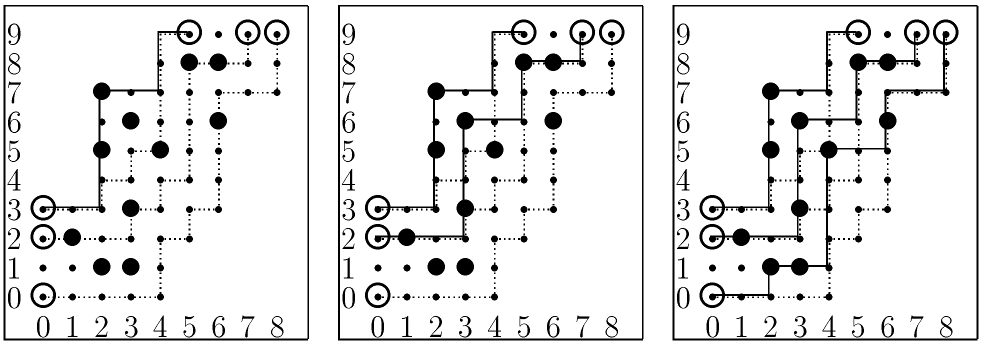


Figure 3. Constructing a family of non-intersecting lattice paths, such that the i^{th} path stays above $L^{(i)}$, $i \in \{1, 2, 3\}$

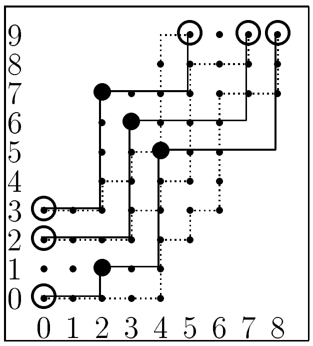


Figure 4. The corresponding family of non-intersecting lattice paths, where the i^{th} path has north-east turns only in $L^{(i)}$ for $i \in \{1, 2, 3\}$

$t \in \{1, 2, \ldots, n+1\}$. Let f_k be the number of k-dimensional faces of $\Delta_M(\mathbf{Y})$ for $k \geq 0$. Then, Corollary 4.3 of [8] states, that

$$\dim_K R_M(\mathbf{Y})_\ell = \sum_{k \geq 0} \binom{\ell - 1}{k} f_k.$$

In the following, we will find an expression for the numbers f_k involving certain families of non-intersecting lattice paths.

In Figure 2.b, a 10-dimensional face of $\Delta_{[1,3,4|1,2,4]}(\mathbf{Y})$ is shown, the elements of the face are indicated by bold dots. We will describe a modification of Viennot's 'light and shadow procedure' (with the sun in the top-left corner) that produces a family of n non-intersecting lattice paths such that the i^{th} path runs from $A^{(i)} = (0, u_{n+1-i})$ to $E^{(i)} = (a - v_{n+1-i}, b)$ and has north-east turns only in $L^{(i)}$, for $i \in \{1, 2, \ldots, n\}$.

Imagine a sun in the top-left corner of the ladder region and a wall along the lower-right border $B^{(1)}$ of $L^{(1)}$. Then each lattice point (r, s) that is either in $B^{(1)}$ or corresponds to an element $x_{s,b-r}$ of the face casts a 'shadow' $\{(x, y) : x \geq r, y \leq s\}$.

The first path starts at $A^{(1)}$, goes along the north-east border of this shadow and terminates in $E^{(1)}$. In the left-most diagram of Figure 3, this is accomplished for the face shown in Figure 2.b.

In the next step, we remove the wall on $B^{(1)}$ and all the elements of the face which correspond to lattice points lying on the first path. Then the procedure is iterated. See Figure 3 for an example. Let P be the resulting family of non-intersecting lattice paths.

Now, for each $i \in \{1, 2, \ldots, n\}$, we remove all elements of the face except those which correspond to north-east turns of the i^{th} path and do not lie on $B^{(i)}$. In the example, $(5, 8)$ is a north-east turn of the second path but lies on $B^{(2)}$, therefore the corresponding element $x_{1,5}$ of the face is removed. On the other hand, $(4, 5)$ lies on $B^{(1)}$, but is a nort-east turn of the third path, so the corresponding element $x_{4,4}$ of the face is kept.

This set of north-east turns defines another family of non-intersecting lattice paths P' that has the property that the i^{th} path has north-east turns only in $L^{(i)}$ for $i \in \{1, 2, \ldots, n\}$.

We now want to count the number of faces of $\Delta_M(\mathbf{Y})$ that reduce under 'light and shadow' to a given family of lattice paths P' with this property. Clearly, P' can be translated into a family P of non-intersecting lattice paths such that the i^{th} path does not go below $B^{(i)}$ for $i \in \{1, 2, \ldots, n\}$. Note that the number of lattice points on such a family P of paths is always equal to d, independently of the given face. Thus, if m is the number of north-east turns of P', there are

$$\binom{d - m}{k + 1 - m}$$

families of non-intersecting lattice paths P that reduce to P'.

Hence, $f_k = \binom{d-m}{k+1-m} \left|\mathcal{T}_\ell^L(\mathbf{A} \mapsto \mathbf{E})\right|$ and we obtain

$$\begin{aligned}\sum_{\ell\geq 0} \dim_K R_M(\mathbf{Y})_\ell \, z^\ell &= \sum_{\ell\geq 0}\left(\sum_{k\geq 0}\binom{\ell-1}{k} f_k\right) z^\ell \\ &= \sum_{\ell\geq 0}\sum_{k\geq 0}\binom{\ell-1}{k}\left(\sum_{m=0}^{k+1}\binom{d-m}{k+1-m}\left|\mathcal{T}_m^L(\mathbf{A}\mapsto\mathbf{E})\right|\right)z^\ell \\ &= \sum_{m\geq 0}\left|\mathcal{T}_m^L(\mathbf{A}\mapsto\mathbf{E})\right|\sum_{\ell\geq 0} z^\ell \sum_{k\geq 0}\binom{\ell-1}{k}\binom{d-m}{d-k-1},\end{aligned}$$

and if we sum the inner sum by means of the Vandermonde summation (see for example [7], Section 5.1, (5.27)),

$$\begin{aligned}\sum_{\ell\geq 0} \dim_K R_M(\mathbf{Y})_\ell \, z^\ell &= \sum_{m\geq 0}\left|\mathcal{T}_m^L(\mathbf{A}\mapsto\mathbf{E})\right|\sum_{\ell\geq 0} z^\ell\binom{d+\ell-m-1}{d-1} \\ &= \frac{\sum_{m\geq 0}\left|\mathcal{T}_m^L(\mathbf{A}\mapsto\mathbf{E})\right| z^m}{(1-z)^d}.\end{aligned}$$

□

4 Log-concavity of the h-vector in the case $M = [u_1 \mid v_1]$

In this paper we will settle Stanley's conjecture when R is a ladder determinantal ring cogenerated by M, where M is just a pair of integers, i.e., $n = 1$. We want to stress, however, that data strongly suggest that Conca and Herzog's conjecture is also true for arbitrary n.

By the preceding theorem, in the case we are going to tackle, the sum $\sum_{i=0}^{s} h_i x^i$ that appears in the conjecture is the generating function $\sum_{k\geq 0}\left|\mathcal{T}_k^L(A\mapsto E)\right| z^k$ of two-rowed arrays bounded by A and E which are in the ladder region L.

As the bounds A and E will not be of any significance throughout the rest of this paper, we will abbreviate $\mathcal{T}_k^L(A\mapsto E)$ to $\mathcal{T}_k^L$. We will show that the h-vector is log-concave by constructing an injection from $\mathcal{T}_{k+1}^L \times \mathcal{T}_{k-1}^L$ into $\mathcal{T}_k^L \times \mathcal{T}_k^L$. This injection will involve some cut and paste operations that we now define:

Definition 4.1. Let A and X be two strictly increasing sequences of integers, such that the length of X is the length of A minus two, i.e., $A = (a_1, a_2, \ldots, a_{k+1})$ and $X = (x_1, x_2, \ldots, x_{k-1})$ for some $k \geq 1$. A *cutting point of A and X* is an index $l \in \{1, 2, \ldots, k\}$ such that

$$\begin{aligned} &a_l < x_l, && (*)\\ \text{and}\quad &x_{l-1} < a_{l+1}, \end{aligned}$$

where we require the inequalities to be satisfied only if all variables are defined. Hence, 1 is a cutting point if $a_1 < x_1$, and k is a cutting point if $x_{k-1} < a_{k+1}$.

The *image of A and X obtained by cutting at l* is

$$\begin{array}{cccccccc} a_1 & a_2 & \dots & a_{l-1} & a_l & \multicolumn{1}{|c}{x_l} & x_{l+1} & \dots & x_{k-1} \\ \hline x_1 & x_2 & \dots & \multicolumn{1}{c|}{x_{l-1}} & a_{l+1} & a_{l+2} & \dots\dots & & a_{k+1} \end{array}$$

Note that both the resulting sequences have length k.

Lemma 4.2. *Let $A = (a_1, a_2, \dots, a_{k+1})$ and $X = (x_1, x_2, \dots, x_{k-1})$ be strictly increasing sequences of integers, such that the length of X is the length of A minus two. Then there exists at least one cutting point of A and X.*

Proof. If $a_l \geq x_l$ for $l \in \{1, 2, \dots, k-1\}$ then $a_{k+1} > a_{k-1} \geq x_{k-1}$ and k is a cutting point. Otherwise, let l be minimal such that $a_l < x_l$. If $l = 1$ then 1 is a cutting point. Otherwise, because of the minimality of l, we have $a_{l+1} > a_{l-1} \geq x_{l-1}$, thus l is a cutting point. $\square$

Definition 4.3. Let $T = (T_1, T_2) \in \mathcal{T}_{k+1} \times \mathcal{T}_{k-1}$ be a pair of two-rowed arrays. Then a *top cutting point of* T is a cutting point of the top rows of T_1 and T_2 and a *bottom cutting point of* T is a cutting point of the bottom rows of T_1 and T_2.

A pair (l, m), where $l, m \in \{1, 2, \dots, k\}$, such that l is a top cutting point and m is a bottom cutting point of T_1 and T_2 is a *cutting point of* T. Cutting the top rows of T at l and the bottom rows at m we obtain the *image of* T. Note that both of the two-rowed arrays in the image have length k. More pictorially, if $l < m$,

$$\begin{array}{cccccccccc} a_1 & \dots\dots & & a_l & x_l & \dots & x_{m-1} & \dots & & x_{k-1} \\ b_1 & \dots\dots & & & b_{l+1} & \dots & b_m & y_m & \dots & y_{k-1} \\ \hline x_1 & \dots & x_{l-1} & a_{l+1} & \dots & & a_m & \dots\dots & & a_{k+1} \\ y_1 & \dots\dots & & y_l & \dots & & y_{m-1} & b_{m+1} & \dots & b_{k+1}, \end{array}$$

and similarly if $l \geq m$.

For $T = (T_1, T_2) \in \mathcal{T}^L_{k+1} \times \mathcal{T}^L_{k-1}$, the pair (l, m) is an *allowed cutting point of* T, if both of the two-rowed arrays in the obtained image are in L.

In Lemma 5.1 we will prove that every pair of two-rowed arrays in $\mathcal{T}^L_{k+1} \times \mathcal{T}^L_{k-1}$ has at least one allowed cutting point. This motivates the following definition:

Definition 4.4. Let $T = (T_1, T_2) \in \mathcal{T}^L_{k+1} \times \mathcal{T}^L_{k-1}$ a pair of two-rowed arrays as before. Consider all allowed cutting points $(\bar{l}, \bar{m})$ of T. Select those with $|\bar{l} - \bar{m}|$ minimal. Among those, let (l, m) be the pair which comes first in the lexicographic order. Then we call (l, m) the *optimal cutting point* of T.

Now we are ready to state our main theorem, which implies that Stanley's conjecture is true, when R is a ladder determinantal ring cogenerated by a pair of integers M:

Theorem 4.5. *Let L be a ladder region. Let $T \in \mathcal{T}_{k+1}^L \times \mathcal{T}_{k-1}^L$. Define $I(T)$ to be the pair of two-rowed arrays obtained by cutting T at its optimal cutting point. Then I is well-defined and an injection from $\mathcal{T}_{k+1}^L \times \mathcal{T}_{k-1}^L$ into $\mathcal{T}_k^L \times \mathcal{T}_k^L$.*

Corollary 4.6. *The h-vector of the ladder determinantal ring cogenerated by $M = [u_1 \mid v_1]$ is log-concave.*

Proof of the corollary. By Theorem 3.1, the h-vector of this ring is equal to the generating function $\sum_{k\geq 0} \left|\mathcal{T}_k^L(A \mapsto E)\right| z^k$ of two-rowed arrays bounded by $A = (0, u_1 - 1)$ and $E = (a - v_1 + 1, b)$ which are in the ladder region L. By the preceding theorem, there is an injection from $\mathcal{T}_{k+1}^L(A \mapsto E) \times \mathcal{T}_{k-1}^L(A \mapsto E)$ into $\mathcal{T}_k^L(A \mapsto E) \times \mathcal{T}_k^L(A \mapsto E)$, thus

$$\left|\mathcal{T}_{k+1}^L(A \mapsto E)\right| \cdot \left|\mathcal{T}_{k-1}^L(A \mapsto E)\right| \leq \left|\mathcal{T}_k^L(A \mapsto E)\right|^2.$$

□

We will split the proof of Theorem 4.5 in two parts. In Section 5 we show that the mapping I is well-defined, that is, for any pair of two-rowed arrays $T \in \mathcal{T}_{k+1}^L \times \mathcal{T}_{k-1}^L$ there is an allowed cutting point. Finally, in Section 6, we show that I is indeed an injection.

5 The mapping I is well-defined

Lemma 5.1. *Let L be a ladder region. Then for every pair of two-rowed arrays in $\mathcal{T}_{k+1}^L \times \mathcal{T}_{k-1}^L$ there is an allowed cutting point (l, m).*

For the proof of this lemma, we have to introduce some more notation: Let $(T_1, T_2) \in \mathcal{T}_{k+1}^L \times \mathcal{T}_{k-1}^L$ with $T_1 = \left(\begin{smallmatrix} a_1 & a_2 & \dots & a_{k+1} \\ b_1 & b_2 & \dots & b_{k+1} \end{smallmatrix}\right)$ and $T_2 = \left(\begin{smallmatrix} x_1 & x_2 & \dots & x_{k-1} \\ y_1 & y_2 & \dots & y_{k-1} \end{smallmatrix}\right)$. We say that Inequality ($\overline{top}$) holds for an interval $[c, d]$ if

$$\overline{L}(a_j) \geq y_{j-1}, \tag{$\overline{top}$}$$

for $j \in [c, d]$. Inequality ($\underline{top}$) holds for an interval $[c, d]$ if

$$\underline{L}(a_j) \leq y_{j-1}, \tag{$\underline{top}$}$$

for $j \in [c, d]$. Similarly, Inequality ($\overline{bottom}$) holds for an interval $[c, d]$ if

$$\overline{L}(x_{j-1}) \geq b_j, \tag{$\overline{bottom}$}$$

for $j \in [c, d]$. Inequality ($\underline{bottom}$) holds for an interval $[c, d]$ if

$$\underline{L}(x_{j-1}) \leq b_j, \tag{$\underline{bottom}$}$$

for $j \in [c, d]$. We say that any of these inequalities holds for a cutting point (l, m) if it holds for the interval $[l + 1, m]$ if $l < m$ and for the interval $[m + 1, l]$ if $m < l$. Clearly, a cutting point (l, m) is allowed if and only if all of these inequalities hold for it.

Most of the work is done by the following lemma:

Lemma 5.2. *Let* $T = (T_1, T_2) \in \mathcal{T}_{k+1}^L \times \mathcal{T}_{k-1}^L$*,* $T_1 = \left(\begin{smallmatrix} a_1 & a_2 & \dots & a_{k+1} \\ b_1 & b_2 & \dots & b_{k+1}\end{smallmatrix}\right)$ *and* $T_2 = \left(\begin{smallmatrix} x_1 & x_2 & \dots & x_{k-1} \\ y_1 & y_2 & \dots & y_{k-1}\end{smallmatrix}\right)$*. Let* $\underline{l}$ *and* $\overline{l}$ *be top cutting points, such that there is no top cutting point in the closed interval* $[\underline{l}+1, \overline{l}-1]$*. Similarly, let* $\underline{m}$ *and* $\overline{m}$ *be bottom cutting points, such that there is no bottom cutting point in the closed interval* $[\underline{m}+1, \overline{m}-1]$*. Then for both of the intervals* $[\underline{l}+1, \overline{l}]$ *and* $[\underline{m}+1, \overline{m}]$*,*

- *either* ($\overline{top}$) *or* ($\overline{bottom}$) *hold,*
- *either* ($\underline{top}$) *or* ($\underline{bottom}$) *hold,*
- *either* ($\overline{top}$) *or* ($\underline{top}$) *hold,*
- *either* ($\overline{bottom}$) *or* ($\underline{bottom}$) *hold.*

Let $l_{min}, l_{max}, m_{min}$ *and* m_{max} *be the minimal and maximal top and bottom cutting points. Then we have*

- ($\overline{top}$) *and* ($\underline{bottom}$) *hold for* $[2, \max(l_{min}, m_{min})]$ *and*
- ($\underline{top}$) *and* ($\overline{bottom}$) *hold for* $[\min(l_{max}, m_{max}), k]$*.*

Proof. Suppose that ($\overline{top}$) does not hold for the interval $[\underline{l}+1, \overline{l}]$. We claim that in that case there is an index $j \in [\underline{l}+1, \overline{l}-1]$ such that $a_j < x_j$: For, by hypothesis there is an index $i \in [\underline{l}+1, \overline{l}]$ such that $\overline{L}(a_i) < y_{i-1}$. We have $\overline{L}(a_i) < y_{i-1} \leq \overline{L}(x_{i-1})$ and because $\overline{L}$ is a weakly increasing function, $a_i < x_{i-1}$. It follows that $a_{i-1} < a_i < x_{i-1} < x_i$. Thus, if $i = \overline{l}$ we choose $j = i - 1$, otherwise $j = i$.

The same statement is true if ($\underline{bottom}$) does not hold for the interval $[\underline{l}+1, \overline{l}]$: In this case there must be an index $i \in [\underline{l}+1, \overline{l}]$ such that $\underline{L}(x_{i-1}) > b_i$. We conclude that $\underline{L}(a_i) \leq b_i < \underline{L}(x_{i-1})$ and thus $a_i < x_{i-1}$.

Next, we will use induction to prove that

$$\begin{aligned} a_l &< x_l \\ \text{and} \quad a_{l+1} &\leq x_{l-1} \end{aligned} \tag{$**$}$$

for $l \in [\underline{l}+1, \overline{l}-1]$. We will first do an induction on l to establish the claim for $l \in [j, \overline{l}-1]$.

We start the induction at $l = j$: Above we already found that $a_j < x_j$. Therefore we must have $a_{j+1} \leq x_{j-1}$, because otherwise j would satisfy ($*$) and hence were a top cutting point.

Now suppose that ($**$) holds for a particular $l < \overline{l} - 1$. Then $a_{l+1} \leq x_{l-1} < x_{l+1}$, and, because there is no top cutting point at $l+1$, we have $a_{l+2} \leq x_l$.

Similarly, to establish ($**$) for $l \in [\underline{l}+1, j]$ we do a reverse induction on l. Suppose that ($**$) holds for a particular $l > \underline{l}+1$. Then $a_{l-1} < a_{l+1} \leq x_{l-1}$, and, because there is no top cutting point at $l-1$, we have $a_l \leq x_{l-2}$.

Thus we obtain

$$\begin{aligned}&\overline{L}(x_{\overline{l}-1}) \geq \overline{L}(x_{\overline{l}-2}) \geq \overline{L}(a_{\overline{l}}) \geq b_{\overline{l}}, \text{ and}\\ &\overline{L}(x_{l-1}) \geq \overline{L}(a_{l+1}) \geq b_{l+1} \geq b_l,\end{aligned}$$

which means that $(\overline{bottom})$ holds for the interval $[\underline{l}+1, \overline{l}]$.

Furthermore,

$$\begin{aligned}&\underline{L}(a_{\underline{l}+1}) \leq \underline{L}(a_{\underline{l}+2}) \leq \underline{L}(x_{\underline{l}}) \leq y_{\underline{l}}, \text{ and}\\ &\underline{L}(a_l) \leq \underline{L}(x_{l-2}) \leq y_{l-2} \leq y_{l-1},\end{aligned}$$

which means that $(\underline{top})$ holds for the interval $[\underline{l}+1, \overline{l}]$.

Next we show that $(\overline{top})$ and $(\underline{bottom})$ hold for the interval $[2, l_{min}]$: Assume that either of these inequalities does not hold for the interval $[2, l_{min}]$ and that $[2, l_{min}]$ does not contain a top cutting point except l_{min}. Then the above reverse induction implies that $a_1 \leq a_3 < x_1$, which means that 1 is a top cutting point. Thus, $l_{min} = 1$ and the interval $[2, l_{min}]$ is empty.

The other assertions are shown in a completely analogous fashion. $\square$

We are now ready to establish Lemma 5.1:

Proof of Lemma 5.1. Let $T = (T_1, T_2) \in \mathcal{T}_{k+1}^L \times \mathcal{T}_{k-1}^L$. By Lemma 4.2 there is at least one cutting point (l, m) of T. Let $l_{min}, l_{max}, m_{min}$ and m_{max} be the minimal and maximal top and bottom cutting points of T as before.

If there is an index j which is a top *and* a bottom cutting point of T, then – trivially – (j, j) is an allowed cutting point. Otherwise, we have to show that there is a cutting point (l, m) for which $(\overline{top})$, $(\underline{top})$, $(\overline{bottom})$, and $(\underline{bottom})$ hold. Suppose that this is not the case.

For the inductive proof which follows, we have to introduce a convenient indexing scheme for the sequence of top and bottom cutting points. Let

$$\begin{aligned}
m_{1,0} &= \max\{m : m < l_{min} \text{ and } m \text{ is a bottom cutting point}\}, \\
m_{i,0} &= \max\{m : m < l_{i-1,1} \text{ and } m \text{ is a bottom cutting point}\} \quad \text{for } i > 1, \\
\text{and} \quad l_{i,0} &= \max\{l : l < m_{i,1} \text{ and } l \text{ is a top cutting point}\} \qquad\qquad\quad \text{for } i \geq 1,
\end{aligned}$$

where $m_{i,j+1}$ is the bottom cutting point directly after $m_{i,j}$, and $l_{i,j+1}$ is the top cutting point directly after $l_{i,j}$. Furthermore, we set $l_{0,1} = l_{min}$.

More pictorially, we have the following sequence of top and bottom cutting points for $i \geq 1$:

$$\cdots < m_{i,0} < l_{i-1,1} < l_{i-1,2} < \cdots < l_{i,0} < m_{i,1} < m_{i,2} < \cdots < m_{i+1,0} < \cdots$$

If $m_{min} > l_{min}$, then $m_{1,0}$ does not exist, of course. Note that there are no bottom cutting points between $l_{i,1}$ and $l_{i+1,0}$, and there are no top cutting points between $m_{i,1}$ and $m_{i+1,0}$.

Suppose first that $m_{min} < l_{min}$. By induction on i, we will show that $(\overline{top})$ and $(\underline{bottom})$ hold for the cutting points $(l_{i-1,1}, m_{i,0})$, where $i \geq 1$. By Lemma 5.2 we know that $(\overline{top})$ and $(\underline{bottom})$ are satisfied for the cutting point $(l_{min}, m_{1,0})$, because $[m_{1,0}+1, l_{min}] \subseteq [2, l_{min}]$. It remains to perform the induction step, which we will divide into five simple steps.

Step 1. *$(\overline{top})$ and $(\underline{bottom})$ hold for the interval $[m_{i,0}+1, l_{i-1,1}]$.* This is just a restatement of the induction hypothesis, i.e., that $(\overline{top})$ and $(\underline{bottom})$ hold for the cutting point $(l_{i-1,1}, m_{i,0})$.

Step 2. *Either $(\overline{bottom})$ or $(\underline{top})$ does not hold for the interval $[m_{i,0}+1, m_{i,1}]$.* Because of Step 1, not both of $(\overline{bottom})$ and $(\underline{top})$ can hold for $(l_{i-1,1}, m_{i,0})$, lest this was an allowed cutting point. Thus either $(\overline{bottom})$ or $(\underline{top})$ does not hold for $[m_{i,0}+1, l_{i-1,0}+1]$. This interval is contained in $[m_{i,0}+1, m_{i,1}]$, thus the inequalities $(\overline{bottom})$ and $(\underline{top})$ cannot hold on this interval either.

Step 3. *$(\overline{top})$ and $(\underline{bottom})$ hold for $[l_{i,0}+1, m_{i,1}]$.* Suppose that $(\overline{bottom})$ does not hold for $[m_{i,0}+1, m_{i,1}]$. Then, by Lemma 5.2 we obtain that $(\overline{top})$ and $(\underline{bottom})$ hold for $[m_{i,0}+1, m_{i,1}]$, because this interval contains no bottom cutting points except $m_{i,1}$. The same is true, if $(\underline{top})$ does not hold for $[m_{i,0}+1, m_{i,1}]$. Because $[l_{i,0}+1, m_{i,1}]$ is a subset of this interval, $(\overline{top})$ and $(\underline{bottom})$ hold for the cutting point $(l_{i,0}, m_{i,1})$, or, equivalently, for the interval $[l_{i,0}+1, m_{i,1}]$.

Step 4. *Either $(\overline{bottom})$ or $(\underline{top})$ does not hold for $[l_{i,0}+1, l_{i,1}]$.* Because of Step 3, not both of $(\overline{bottom})$ and $(\underline{top})$ can hold for the cutting point $(l_{i,0}, m_{i,1})$, nor for the greater interval $[l_{i,0}+1, l_{i,1}]$.

Step 5. *$(\overline{top})$ and $(\underline{bottom})$ hold for $[m_{i+1,0}+1, l_{i,1}]$.* The interval $[l_{i,0}+1, l_{i,1}]$ does not contain a top cutting point except $l_{i,1}$, thus by Lemma 5.2 and Step 4 we see that $(\overline{top})$ and $(\underline{bottom})$ hold. Finally, because $[m_{i+1,0}+1, l_{i,1}] \subset [l_{i,0}+1, l_{i,1}]$, $(\overline{top})$ and $(\underline{bottom})$ hold for the cutting point $(l_{i,1}, m_{i+1,0})$.

If $l_{max} > m_{max}$, then we encounter a contradiction: Let r be such that $m_{r,0} = m_{max}$. We have just shown that $(\overline{top})$ and $(\underline{bottom})$ hold for the cutting point $(l_{r-1,1}, m_{r,0})$. Furthermore, by Lemma 5.2, $(\overline{bottom})$ and $(\underline{top})$ hold for $[m_{r,0}, k]$ and thus also for $(l_{r-1,1}, m_{r,0})$. Hence, this would be an allowed cutting point, contradicting our hypothesis.

If $l_{max} < m_{max}$, let r be such that $l_{r,0} = l_{max}$. By the induction (Step 3) we find that $(\overline{top})$ and $(\underline{bottom})$ hold for the cutting point $(l_{r,0}, m_{r,1})$. Again, because of Lemma 5.2, we know that $(\overline{bottom})$ and $(\underline{top})$ holds for $[l_{r,0}, k]$ and thus also for $(l_{r,0}, m_{r,1})$. Hence, we had an allowed cutting point in this case also.

The case that $m_1 > l_1$ is completely analogous. □

6 The mapping I is an injection

Lemma 6.1. *The mapping I defined above is an injection.*

Proof. Suppose that $I(T) = I(T')$ for $T = (T_1, T_2)$ and $T' = (T'_1, T'_2)$, such that T and T' are elements of $\mathcal{T}^L_{k+1} \times \mathcal{T}^L_{k-1}$. Let (l, m) be the optimal cutting point of T, and let (l', m') be the optimal cutting point of T'.

Observe that we can assume $\min(l, m, l', m') = 1$, because the elements of T and T' with index less than or equal to this minimum retain their position in $I(T)$. Likewise, we can assume that $\max(l, m, l', m') = k$.

Furthermore, we can assume that $l \leq l'$, otherwise we exchange the meaning of T and T'. Thus, we have to consider the following twelve situations:

$$
\begin{array}{rlcccccccl}
(1) & 1 = & l & \leq & l' & \leq & m & \leq & m' & = k \\
(2) & 1 = & l & \leq & l' & \leq & m' & \leq & m & = k \\
(3) & 1 = & l & \leq & m & \leq & l' & \leq & m' & = k \\
(4) & 1 = & l & \leq & m & \leq & m' & \leq & l' & = k \\
(5) & 1 = & l & \leq & m' & \leq & l' & \leq & m & = k \\
(6) & 1 = & l & \leq & m' & \leq & m & \leq & l' & = k \\
(7) & 1 = & m & \leq & l & \leq & l' & \leq & m' & = k \\
(8) & 1 = & m & \leq & l & \leq & m' & \leq & l' & = k \\
(9) & 1 = & m & \leq & m' & \leq & l & \leq & l' & = k \\
(10) & 1 = & m' & \leq & l & \leq & l' & \leq & m & = k \\
(11) & 1 = & m' & \leq & l & \leq & m & \leq & l' & = k \\
(12) & 1 = & m' & \leq & m & \leq & l & \leq & l' & = k
\end{array}
$$

We shall divide these twelve cases into two portions according to whether $l \leq m$ or not.

A: $\mathbf{l \leq m}$

In the Cases (1)–(6), (10) and (11) we have $l \leq m$, thus the pair of two-rowed arrays $T = (T_1, T_2) \in \mathcal{T}^L_{k+1} \times \mathcal{T}^L_{k-1}$ looks like

$$
\begin{array}{l}
a_1 \quad \cdots\cdots \quad a_l \mid a_{l+1} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad a_{k+1} \\
b_1 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad b_m \mid b_{m+1} \quad \cdots\cdots \quad b_{k+1} \\
\hline
x_1 \quad \cdots \quad x_{l-1} \mid x_l \quad \cdots\cdots\cdots\cdots\cdots \quad x_{k-1} \\
y_1 \quad \cdots\cdots\cdots\cdots \quad y_{m-1} \mid y_m \quad \cdots \quad y_{k-1}.
\end{array}
$$

Cutting at (l, m) we obtain $I(T) \in \mathcal{T}^L_k \times \mathcal{T}^L_k$:

$$
\begin{array}{l}
a_1 \quad \cdots\cdots \quad a_l \mid x_l \quad \cdots\cdots\cdots\cdots\cdots \quad x_{k-1} \\
b_1 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad b_m \mid y_m \quad \cdots \quad y_{k-1} \\
\hline
x_1 \quad \cdots \quad x_{l-1} \mid a_{l+1} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad a_{k+1} \\
y_1 \quad \cdots\cdots\cdots\cdots \quad y_{m-1} \mid b_{m+1} \quad \cdots\cdots \quad b_{k+1}.
\end{array}
$$

If $l = 1$, then the top row of the second array in $I(T)$ is $(a_2, a_3, \ldots, a_{k+1})$, if $m = k$, then the bottom row of the first array in $I(T)$ is $(b_1, b_2, \ldots, b_k)$.

Case (1), $\mathbf{1 = l \leq l' \leq m \leq m' = k}$

Given that $I(T) = I(T')$, the pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l}
a_1 \mid x_1 \;\; .. \;\; x_{l'-1} \vdots\, a_{l'+1} \;\; \ldots\ldots\ldots\ldots\ldots\ldots \;\; a_{k+1} \\
b_1 \;\; \ldots\ldots\ldots\ldots\ldots\ldots \;\; b_m \mid y_m \;\; \ldots \;\; y_{k-1} \vdots\, b_{k+1} \\
\hline
\mid a_2 \;\; .. \;\; a_{l'} \vdots\, x_{l'} \;\; \ldots\ldots\ldots\ldots\ldots \;\; x_{k-1} \\
y_1 \;\; \ldots\ldots\ldots\ldots \;\; y_{m-1} \mid b_{m+1} \;\; .. \;\; b_k \;\vdots .
\end{array} \qquad (T')$$

The vertical dots indicate the cut (l', m') which results in $I(T')$. We show that the cutting point $(l, m) = (1, m)$, indicated above by the vertical lines, is in fact an allowed cutting point for T': Cutting at $(1, m)$ yields

$$\begin{array}{l}
a_1 \mid a_2 \;\; \ldots \;\; a_{l'} \vdots\, x_{l'} \;\; \ldots \;\; x_{m-1} \;\; \ldots\ldots \;\; x_{k-1} \\
b_1 \;\; \ldots\ldots\ldots\ldots\ldots\ldots \;\; b_m \mid b_{m+1} \;\; \ldots \;\; b_k \;\vdots \\
\hline
\mid x_1 \;\; .. \;\; x_{l'-1} \vdots\, a_{l'+1} \;\; .. \;\; a_m \;\; \ldots\ldots\ldots\ldots \;\; a_{k+1} \\
y_1 \;\; \ldots\ldots\ldots\ldots \;\; y_{m-1} \mid y_m \;\; \ldots \;\; y_{k-1} \vdots\, b_{k+1} .
\end{array} \qquad (I(T'))$$

Note, that this is the same pair of two-rowed arrays we obtain by cutting T at (l', m'). We have to check that the pair of two-rowed arrays $(I(T'))$ is in the ladder region.

Clearly,

$$(a_2, b_2), (a_3, b_3), \ldots, (a_{l'}, b_{l'}) \text{ and } (x_1, y_1), (a_2, b_2), \ldots, (x_{l'-1}, y_{l'-1})$$

are in the ladder region, because these pairs appear also in T. Furthermore, the pairs

$$\begin{aligned}
&(x_{l'}, b_{l'+1}), (x_{l'+1}, b_{l'+2}), \ldots (x_{m-1}, b_m) \\
\text{and } &(a_{l'+1}, y_{l'}), (a_{l'+2}, y_{l'+1}), \ldots (a_m, y_{m-1})
\end{aligned}$$

appear in $I(T)$ and are therefore in the ladder region, too. All the other pairs, i.e.,

$$\begin{gathered}
(a_1, b_1) \text{ and } (x_m, b_{m+1}), (x_{m+1}, b_{m+2}), \ldots, (x_{k-1}, b_k), \\
(a_{m+1}, y_m), (a_{m+2}, y_{m+1}), \ldots, (a_k, y_{k-1}) \text{ and } (a_{k+1}, b_{k+1}),
\end{gathered}$$

are unaffected by the cut and appear in T'.

Thus we have that (l, m) and (l', m') are allowed cuts for T and T'. We required that (l, m) is optimal for T and that (l', m') is optimal for T', therefore we must have $l = l'$ and $m = m'$.

In all the other cases the reasoning is very similar. Thus we only print the pairs of two-rowed arrays T' and $I(T')$ and leave it to the reader to check that $I(T')$ is in the ladder region.

Case (2), $\mathbf{1 = l \le l' \le m' \le m = k}$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l} a_1 \mid x_1 \;\;..\;\; x_{l'-1} \vdots\, a_{l'+1} \;\cdots\cdots\cdots\; a_{k+1} \\ b_1 \;\cdots\cdots\cdots\; b_{m'} \;\vdots\, y_{m'} \;\;....\;\; y_{k-1} \mid b_{k+1} \\ \hline \mid a_2 \;\;..\;\; a_{l'} \vdots\;\; x_{l'} \;\cdots\cdots\cdots\; x_{k-1} \\ y_1 \;\cdots\cdots\cdots\; y_{m'-1} \vdots\, b_{m'+1} \;\;...\;\; b_k \;\; \mid. \end{array} \qquad (T')$$

Cutting at (l, m) yields

$$\begin{array}{l} a_1 \mid a_2 \;\;....\;\; a_{l'} \;\vdots\, x_{l'} \;\;.....\;\; x_{m'-1} \;\;.......\;\; x_{k-1} \\ b_1 \;\cdots\cdots\cdots\; b_{m'} \;\vdots\, y_{m'} \;\;..\;\; y_{k-1} \mid \\ \hline \mid x_1 \;\;..\;\; x_{l'-1} \vdots\, a_{l'+1} \;\;..\;\; a_{m'} \;\cdots\cdots\; a_{k+1} \\ y_1 \;\cdots\cdots\cdots\; y_{m'-1} \vdots\, b_{m'+1} \;\;...\;\; b_k \mid b_{k+1}. \end{array} \qquad (I(T'))$$

Case (3), $\mathbf{1 = l \le m \le l' \le m' = k}$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l} a_1 \mid x_1 \;\cdots\cdots\cdots\; x_{l'-1} \vdots\, a_{l'+1} \;\cdots\cdots\; a_{k+1} \\ b_1 \;\;.........\;\; b_m \;\mid y_m \;\cdots\cdots\cdots\; y_{k-1} \vdots\, b_{k+1} \\ \hline \mid a_2 \;\cdots\cdots\cdots\; a_{l'} \vdots\;\; x_{l'} \;\;....\;\; x_{k-1} \\ y_1 \;\;..\;\; y_{m-1} \mid b_{m+1} \;\cdots\cdots\cdots\; b_k \;\; \vdots. \end{array} \qquad (T')$$

Cutting at (l, m) yields

$$\begin{array}{l} a_1 \mid a_2 \;\cdots\cdots\cdots\; a_{l'} \;\vdots\, x_{l'} \;\;....\;\; x_{k-1} \\ b_1 \;\;.........\;\; b_m \mid b_{m+1} \;\cdots\cdots\cdots\; b_k \\ \hline \mid x_1 \;\cdots\cdots\cdots\; x_{l'-1} \vdots\, a_{l'+1} \;\;........\;\; a_{k+1} \\ y_1 \;\;..\;\; y_{m-1} \mid y_m \;\cdots\cdots\cdots\; y_{k-1} \vdots\, b_{k+1}. \end{array} \qquad (I(T'))$$

Case (4), $\mathbf{1 = l \leq m \leq m' \leq l' = k}$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l}
a_1 \mid x_1 \;\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\; x_{k-1} \vdots\, a_{k+1} \\
b_1 \;\cdots\cdots\; b_m \mid y_m \;\cdots\; y_{m'-1} \vdots\, b_{m'+1} \;\cdots\cdots\; b_{k+1} \\
\hline
\mid a_2 \;\cdots\cdots\cdots\cdots\cdots\cdots\cdots\; a_k \;\vdots \\
y_1 \;\cdot\cdot\; y_{m-1} \mid b_{m+1} \;\cdot\cdot\; b_{m'} \vdots\, y_{m'} \;\cdots\; y_{k-1}.
\end{array} \qquad (T')$$

Cutting at (l, m) yields

$$\begin{array}{l}
a_1 \mid a_2 \;\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\; a_k \;\vdots \\
b_1 \;\cdots\cdots\; b_m \mid b_{m+1} \;\cdots\; b_{m'} \vdots\, y_{m'} \;\cdots\; y_{k-1} \\
\hline
\mid x_1 \;\cdots\cdots\cdots\cdots\cdots\cdots\cdots\; x_{k-1} \vdots\, a_{k+1} \\
y_1 \;\cdot\cdot\; y_{m-1} \mid y_m \;\cdots\; y_{m'-1} \vdots\, b_{m'+1} \;\cdots\cdots\; b_{k+1}.
\end{array} \qquad (I(T'))$$

Case (5), $\mathbf{1 = l \leq m' \leq l' \leq m = k}$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l}
a_1 \mid x_1 \;\cdots\cdots\cdots\cdots\; x_{l'-1} \vdots\, a_{l'+1} \;\cdots\cdots\; a_{k+1} \\
b_1 \;\cdots\cdots\; b_{m'} \vdots\, y_{m'} \;\cdots\cdots\cdots\cdots\; y_{k-1} \mid b_{k+1} \\
\hline
\mid a_2 \;\cdots\cdots\cdots\cdots\; a_{l'} \vdots\, x_{l'} \;\cdots\; x_{k-1} \\
y_1 \;\cdot\cdot\; y_{m'-1} \vdots\, b_{m'+1} \;\cdots\cdots\cdots\cdots\; b_k \mid.
\end{array} \qquad (T')$$

Cutting at (l, m) yields

$$\begin{array}{l}
a_1 \mid a_2 \;\cdots\cdots\cdots\cdots\; a_{l'} \vdots\, x_{l'} \;\cdot\cdot\; x_{k-1} \\
b_1 \;\cdots\cdots\; b_{m'} \vdots\, y_{m'} \;\cdots\cdots\cdots\cdots\; y_{k-1} \mid \\
\hline
\mid x_1 \;\cdots\cdots\cdots\cdots\; x_{l'-1} \vdots\, a_{l'+1} \;\cdots\; a_{k+1} \\
y_1 \;\cdot\cdot\; y_{m'-1} \vdots\, b_{m'+1} \;\cdots\cdots\cdots\cdots\; b_k \mid b_{k+1}.
\end{array} \qquad (I(T'))$$

Case (6), $\mathbf{1 = l \leq m' \leq m \leq l' = k}$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l}
a_1 \mid x_1 \; \cdots\cdots\cdots\cdots\cdots\cdots \; x_{k-1} \vdots\, a_{k+1} \\
b_1 \; \cdots\cdots \; b_{m'} \vdots\, y_{m'} \; \cdot\cdot \; y_{m-1} \mid b_{m+1} \; \cdots\cdots \; b_{k+1} \\
\hline
\mid a_2 \; \cdots\cdots\cdots\cdots\cdots\cdots \; a_k \; \vdots \\
y_1 \; \cdot\cdot \; y_{m'-1} \vdots\, b_{m'+1} \; \cdots \; b_m \mid \; y_m \; \cdots \; y_{k-1}.
\end{array} \tag{T'}$$

Cutting at (l, m) yields

$$\begin{array}{l}
a_1 \mid a_2 \; \cdots\cdots\cdots\cdots\cdots\cdots \; a_k \; \vdots \\
b_1 \; \cdots\cdots \; b_{m'} \vdots\, y_{m'} \; \cdot\cdot \; y_{m-1} \mid y_m \; \cdots \; y_{k-1} \\
\hline
\mid x_1 \; \cdots\cdots\cdots\cdots\cdots\cdots \; x_{k-1} \vdots\, a_{k+1} \\
y_1 \; \cdot\cdot \; y_{m'-1} \vdots\, b_{m'+1} \; \cdots \; b_m \mid b_{m+1} \; \cdots\cdots \; b_{k+1}.
\end{array} \tag{$I(T')$}$$

Case (10), $\mathbf{1 = m' \leq l \leq l' \leq m = k}$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l}
a_1 \; \cdots\cdots \; a_l \mid x_l \; \cdot\cdot \; x_{l'-1} \vdots\, a_{l'+1} \; \cdots\cdots \; a_{k+1} \\
b_1 \vdots\, y_1 \; \cdots\cdots\cdots\cdots\cdots\cdots \; y_{k-1} \mid b_{k+1} \\
\hline
x_1 \; \cdot\cdot \; x_{l-1} \mid a_{l+1} \; \cdot\cdot \; a_{l'} \vdots\, x_{l'} \; \cdots \; x_{k-1} \\
\vdots\, b_2 \; \cdots\cdots\cdots\cdots\cdots\cdots \; b_k \mid.
\end{array} \tag{T'}$$

Cutting at (l, m) yields

$$\begin{array}{l}
a_1 \; \cdots\cdots \; a_l \mid a_{l+1} \; \cdots \; a_{l'} \vdots\, x_{l'} \; \cdot\cdot \; x_{k-1} \\
b_1 \vdots\, y_1 \; \cdots\cdots\cdots\cdots\cdots\cdots \; y_{k-1} \mid \\
\hline
x_1 \; \cdot\cdot \; x_{l-1} \mid x_l \; \cdots \; x_{l'-1} \vdots\, a_{l'+1} \; \cdots\cdots \; a_{k+1} \\
\vdots\, b_2 \; \cdots\cdots\cdots\cdots\cdots\cdots \; b_k \mid b_{k+1}.
\end{array} \tag{$I(T')$}$$

Case (11), $1 = m' \leq l \leq m \leq l' = k$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l}
a_1 \;\cdots\cdots\; a_l \mid x_l \;\cdots\cdots\cdots\cdots\; x_{k-1} \vdots\, a_{k+1} \\
b_1 \vdots\, y_1 \;\cdots\cdots\cdots\; y_{m-1} \mid b_{m+1} \;\cdots\cdots\; b_{k+1} \\
\hline
x_1 \;\cdot\cdot\; x_{l-1} \mid a_{l+1} \;\cdots\cdots\cdots\; a_k \;\vdots \\
\vdots\, b_2 \;\cdots\cdots\cdots\; b_m \mid y_m \;\cdots\; y_{k-1}.
\end{array} \qquad (T')$$

Cutting at (l, m) yields

$$\begin{array}{l}
a_1 \;\cdots\cdots\; a_l \mid a_{l+1} \;\cdots\cdots\cdots\; a_k \;\vdots \\
b_1 \vdots\, y_1 \;\cdots\cdots\cdots\; y_{m-1} \mid y_m \;\cdots\; y_{k-1} \\
\hline
x_1 \;\cdot\cdot\; x_{l-1} \mid x_l \;\cdots\cdots\cdots\; x_{k-1} \vdots\, a_{k+1} \\
\vdots\, b_2 \;\cdots\cdots\cdots\; b_m \mid b_{m+1} \;\cdots\cdots\; b_{k+1}.
\end{array} \qquad (I(T'))$$

B: $m \leq l$

In the Cases (7)–(9) and (12) we have $m \leq l$, thus the pair of two-rowed arrays $T = (T_1, T_2) \in \mathcal{T}^L_{k+1} \times \mathcal{T}^L_{k-1}$ looks like

$$\begin{array}{l}
a_1 \;\cdots\cdots\cdots\cdots\; a_l \mid a_{l+1} \;\cdots\; a_{k+1} \\
b_1 \;\cdots\cdots\; b_m \mid b_{m+1} \;\cdots\cdots\cdots\; b_{k+1} \\
\hline
x_1 \;\cdots\cdots\cdots\; x_{l-1} \mid x_l \;\cdots\; x_{k-1} \\
y_1 \;\cdots\; y_{m-1} \mid y_m \;\cdots\cdots\cdots\; y_{k-1}.
\end{array}$$

Cutting at (l, m) we obtain $I(T) \in \mathcal{T}^L_k \times \mathcal{T}^L_k$:

$$\begin{array}{l}
a_1 \;\cdots\cdots\cdots\cdots\; a_l \mid x_l \;\cdots\; x_{k-1} \\
b_1 \;\cdots\cdots\; b_m \mid y_m \;\cdots\cdots\cdots\; y_{k-1} \\
\hline
x_1 \;\cdots\cdots\cdots\; x_{l-1} \mid a_{l+1} \;\cdots\; a_{k+1} \\
y_1 \;\cdots\; y_{m-1} \mid b_{m+1} \;\cdots\cdots\cdots\; b_{k+1}.
\end{array}$$

Case (7), $\mathbf{1 = m \leq l \leq l' \leq m' = k}$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l}
a_1 \;\cdots\cdots\; a_l \mid x_l \;\cdot\cdot\; x_{l'-1} \vdots\, a_{l'+1} \;\cdots\cdots\; a_{k+1} \\
b_1 \mid y_1 \;\cdots\cdots\cdots\cdots\cdots\cdots\; y_{k-1} \vdots\, b_{k+1} \\
\hline
x_1 \;\cdot\cdot\; x_{l-1} \mid a_{l+1} \;\cdot\cdot\; a_{l'} \vdots\, x_{l'} \;\cdots\; x_{k-1} \\
\mid b_2 \;\cdots\cdots\cdots\cdots\cdots\cdots\; b_k \;\vdots.
\end{array} \qquad (T')$$

Cutting at (l, m) yields

$$\begin{array}{l}
a_1 \;\cdots\cdots\; a_l \mid a_{l+1} \;\cdots\; a_{l'} \vdots\, x_{l'} \;\cdots\; x_{k-1} \\
b_1 \mid b_2 \;\cdots\cdots\cdots\cdots\cdots\cdots\; b_k \;\vdots \\
\hline
x_1 \;\cdot\cdot\; x_{l-1} \mid x_l \;\cdots\; x_{l'-1} \vdots\, a_{l'+1} \;\cdots\cdots\; a_{k+1} \\
\mid y_1 \;\cdots\cdots\cdots\cdots\cdots\cdots\; y_{k-1} \vdots\, b_{k+1}.
\end{array} \qquad (I(T'))$$

Case (8), $\mathbf{1 = m \leq l \leq m' \leq l' = k}$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l}
a_1 \;\cdots\cdots\; a_l \mid x_l \;\cdots\cdots\cdots\cdots\; x_{k-1} \vdots\, a_{k+1} \\
b_1 \mid y_1 \;\cdots\cdots\cdots\; y_{m'-1} \vdots\, b_{m'+1} \;\cdots\cdots\; b_{k+1} \\
\hline
x_1 \;\cdot\cdot\; x_{l-1} \mid a_{l+1} \;\cdots\cdots\cdots\; a_k \;\vdots \\
\mid b_2 \;\cdots\cdots\cdots\; b_{m'} \vdots\, y_{m'} \;\cdots\; y_{k-1}.
\end{array} \qquad (T')$$

Cutting at (l, m) yields

$$\begin{array}{l}
a_1 \;\cdots\cdots\; a_l \mid a_{l+1} \;\cdots\cdots\cdots\cdots\; a_k \;\vdots \\
b_1 \mid b_2 \;\cdots\cdots\cdots\; b_{m'} \vdots\, y_{m'} \;\cdots\; y_{k-1} \\
\hline
x_1 \;\cdot\cdot\; x_{l-1} \mid x_l \;\cdots\cdots\cdots\cdots\; x_{k-1} \vdots\, a_{k+1} \\
\mid y_1 \;\cdots\cdots\cdots\; y_{m'-1} \vdots\, b_{m'+1} \;\cdots\cdots\; b_{k+1}.
\end{array} \qquad (I(T'))$$

Case (9), $\mathbf{1 = m \leq m' \leq l \leq l' = k}$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l} a_1 \ \cdots\cdots\cdots\cdots\cdots\cdots \ a_l \ | \ x_l \ \cdots\cdots \ x_{k-1} \vdots \ a_{k+1} \\ b_1 \ | \ y_1 \ \cdots \ y_{m'-1} \vdots \ b_{m'+1} \ \cdots\cdots\cdots\cdots\cdots\cdots \ b_{k+1} \\ \hline x_1 \ \cdots\cdots\cdots\cdots\cdots \ x_{l-1} \ | \ a_{l+1} \ \cdot\cdot \ a_k \ \vdots \\ | \ b_2 \ \cdot\cdot \ b_{m'} \vdots \ y_{m'} \ \cdots\cdots\cdots\cdots\cdots \ y_{k-1}. \end{array} \qquad (T')$$

Cutting at (l, m) yields

$$\begin{array}{l} a_1 \ \cdots\cdots\cdots\cdots\cdots\cdots \ a_l \ | \ a_{l+1} \ \cdots \ a_k \ \vdots \\ b_1 \ | \ b_2 \ \cdots\cdots \ b_{m'} \vdots \ y_{m'} \ \cdots\cdots\cdots\cdots\cdots \ y_{k-1} \\ \hline x_1 \ \cdots\cdots\cdots\cdots\cdots \ x_{l-1} \ | \ x_l \ \cdots \ x_{k-1} \vdots \ a_{k+1} \\ | \ y_1 \ \cdot\cdot \ y_{m'-1} \vdots \ b_{m'+1} \ \cdots\cdots\cdots\cdots\cdots\cdots \ b_{k+1}. \end{array} \qquad (I(T'))$$

Case (12), $\mathbf{1 = m' \leq m \leq l \leq l' = k}$

The pair T' can be expressed in terms of the entries of T as follows:

$$\begin{array}{l} a_1 \ \cdots\cdots\cdots\cdots\cdots\cdots \ a_l \ | \ x_l \ \cdots\cdots \ x_{k-1} \vdots \ a_{k+1} \\ b_1 \vdots \ y_1 \ \cdot\cdot \ y_{m-1} \ | \ b_{m+1} \ \cdots\cdots\cdots\cdots\cdots\cdots \ b_{k+1} \\ \hline x_1 \ \cdots\cdots\cdots\cdots\cdots \ x_{l-1} \ | \ a_{l+1} \ \cdot\cdot \ a_k \ \vdots \\ \vdots \ b_2 \ \cdot\cdot \ b_m \ | \ y_m \ \cdots\cdots\cdots\cdots\cdots \ y_{k-1}. \end{array} \qquad (T')$$

Cutting at (l, m) yields

$$\begin{array}{l} a_1 \ \cdots\cdots\cdots\cdots\cdots\cdots \ a_l \ | \ a_{l+1} \ \cdots \ a_k \ \vdots \\ b_1 \vdots \ y_1 \ \cdot\cdot \ y_{m-1} \ | \ y_m \ \cdots\cdots\cdots\cdots\cdots \ y_{k-1} \\ \hline x_1 \ \cdots\cdots\cdots\cdots \ x_{l-1} \ | \ x_l \ \cdots \ x_{k-1} \vdots \ a_{k+1} \\ \vdots \ b_2 \ \cdot\cdot \ b_m \ | \ b_{m+1} \ \cdots\cdots\cdots\cdots\cdots\cdots \ b_{k+1}. \end{array} \qquad (I(T'))$$

$\square$

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