Rhombus Tilings and Plane Partitions

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Introduction

Consider a hexagon with integer sidelengths a, b, c, a, b, c. It is possible to cover such a hexagon with rhombi of sidelength 1 without holes and without overlap (see Figure 0.1). Such a covering is called a *rhombus tiling*. The figure can also be interpreted as a 3–dimensional picture. The tiling then corresponds to stacks of cubes pushed in one corner without invisible holes and the surrounding hexagon corresponds to a containing box with sides $a \times b \times c$. The stacks of cubes are called a *plane partition*.

The results in this thesis can be viewed as generalizations of a result by MacMahon concerning plane partitions. He counted the number of plane partitions in a box with sides $a \times b \times c$ which is also the number of rhombus tilings of a hexagon with sides a, b, c, a, b, c by the correspondence mentioned above (see Equation (1)).

In fact, he gave a more general result, namely, the weighted enumeration of plane partitions in a given box with the weight $q^{\#\text{ cubes}}$. One possible generalization is not to count *all* plane partitions contained in a given box, but only those with certain symmetries (see the list on page 25). MacMahon himself gave a conjecture for the q–enumeration of symmetric plane partitions which correspond to rhombus tilings with a vertical symmetry axis. This conjecture was finally proved in 1978 by Andrews [**1**] and independently in 1979 by Macdonald [**19**]. It is now known that for each of the resulting ten symmetry classes the number of plane partitions contained in that class is given by a "nice" closed formula (see [**3, 34, 30, 26, 25, 21, 16, 4**]). For the case of cyclically symmetric plane partitions, a closed form for the q–enumeration is given in [**22**]. For the case of totally symmetric plane partitions, there is only a conjectured closed formula for a q–enumeration.

The most frequently used approaches to solve enumeration problems on plane partitions and rhombus tilings use either a bijection between rhombus tilings and perfect matchings of their inner dual graph (see Figure 1.2 on page 9 and Figure 1.3 on page 10) or a bijection between rhombus tilings and certain families of nonintersecting lattice paths (see Figure 1.5 on page 13).

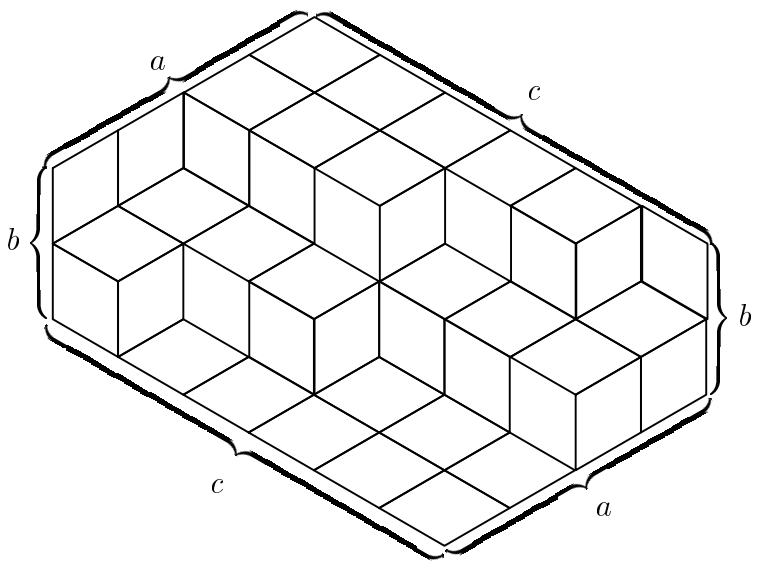


FIGURE 0.1. A rhombus tiling corresponds to a plane partition.

In 1967, Kasteleyn gave a general method of expressing the number of perfect matchings of a planar graph as a determinant or a Pfaffian [**11**]. He used it to find a formula for the number of domino tilings of a rectangle. On the other hand, in 1973, Lindström found a way to express the enumeration of families of nonintersecting lattice paths as determinants ([**18**, Lemma 1], see Lemma 2.8). This method was found independently by Gessel and Viennot in 1985 and applied to the special case of rhombus tilings and plane partitions ([**9**]). The dimension of the matrix in Kasteleyn's determinant equals at least half of the number of vertices in the graph. In comparison, in the case of rhombus tilings the dimension of the matrix of the Lindström–Gessel–Viennot method is approximately one sidelength of the hexagon. This method is used frequently throughout this thesis and most of the work is devoted to evaluating the resulting determinants.

Another possible direction to generalize MacMahon's result about rhombus tilings of a hexagon is to ask for the number of rhombus tilings of other regions. In [**27**], Propp proposed several problems regarding "incomplete" hexagons, i.e., hexagons, where certain triangles are missing. In particular, Problem 4 of [**27**] asks for a formula for the number of rhombus tilings of a regular hexagon, where two of the six central triangles are missing. Up to symmetry, there are only two possibilities. The case when the two triangles form a rhombus has been treated in [**8**]. In the remaining case the two rhombi share a vertex (see Figure 1.1). It turns out that it is also possible to count the number of rhombus tilings if the hexagon has only one symmetry axis and the two triangles are shifted simultaneously along the symmetry axis.

In Chapter 1, the number of these rhombus tilings is given in Theorems 1.1 and 1.2. In Corollary 1.3 we compare this number to the number of rhombus tilings of the complete hexagon if the sidelengths go to infinity simultaneously. It turns out that the ratio of the tiling without the two triangles to the tiling of the complete hexagon can get arbitrarily large.

The main ingredients in the proofs are the bijection between rhombus tilings and perfect matchings of the inner dual graph, the matchings factorization theorem [**5**] and the "identification of factors" method, as explained in [**13**, Sec. 2].

A different way to generalize MacMahon's result is to find weighted enumerations for the other symmetry classes of plane partitions. As mentioned above, for the four symmetry classes without complementation symmetry, there exist weighted enumerations which admit a nice closed form. For the other six cases, a (-1)–weight has been proposed by Kuperberg in [**15**, pp.25/26], where he also gives the result for the case of transpose–complementary plane partitions derived by the method of "forcing planarity". The weight can be described as follows: A plane partition with complementation symmetry contains exactly one half of each orbit with respect to its symmetry class. Now we can define a relative sign by demanding that two plane partitions have different sign if one can be obtained from the other by adding one half of an orbit of cubes and removing the other half of the orbit.

In Chapter 2, I give closed forms for the (-1)–enumeration of transpose–complementary, cyclically symmetric transpose–complementary and totally symmetric self–complementary plane partitions. For symmetric transpose–complementary plane partitions, I obtain only partial results, since in one subcase the enumeration does not seem to have a nice closed form. I give a closed form for the (-1)–enumeration of self–complementary

plane partitions contained in a box with even sidelengths and a conjecture for one case with some odd sidelengths. All these results are stated in Theorems 2.1–2.7 on page 26.

The main ingredient in the proofs is the use of families of nonintersecting lattice paths. The weighted enumeration of these is given by a determinant by the Gessel–Viennot method [**9**]. In some cases this gives only a sum of determinants which can be converted to a single Pfaffian by a theorem of Okada (see Lemma 2.11).

CHAPTER 1

Rhombus tilings of a hexagon with two triangles missing on the symmetry axis

1. Introduction

The interest in rhombus tilings has emerged from the enumeration of plane partitions in a given box. The connection comes from looking at the stacks of cubes of a plane partition and projecting the picture to the plane. Then the box becomes a hexagon, where opposite sides are equal, and the cubes become a rhombus tiling of the hexagon where the rhombi consist of two equilateral triangles (cf. [**6**]). The number of plane partitions in a given box was first computed by MacMahon [**20**, Sec. 429, $q \to 1$, proof in Sec. 494]. Therefore:

The number of all rhombus tilings of a hexagon with sides a, b, c, a, b, c equals

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}=\prod_{i=1}^{a}\frac{(c+i)_b}{(i)_b}. \tag{1}$$

(The symmetric first expression is due to Macdonald.)

In [**27**], Propp proposed several problems regarding "incomplete" hexagons, i.e., hexagons, where certain triangles are missing. In particular, Problem 4 of [**27**] asks for a formula for the number of rhombus tilings of a regular hexagon, where two of the six central triangles are missing. We treat the case of the two triangles lying on the symmetry axis and touching in one vertex (see Figure 1.1). The other case has been solved in [**8**]. We prove the following two theorems.

THEOREM 1.1. *The number of rhombus tilings of a hexagon with sides n, n, $2m$, n, n, $2m$ and two missing triangles on the horizontal symmetry axis sharing the $(s+1)$–th vertex on the axis (counted from the left, see Figure 1.1) equals*

$$\frac{(2m-1)\binom{2m-2}{m-1}\binom{2n-2s}{n-s}\binom{2s}{s}}{\binom{2m+2n}{m+n}}\prod_{i=1}^{n}\frac{(2m+i)_n}{(i)_n}.$$

THEOREM 1.2. *The number of rhombus tilings of a hexagon with sides $n, n, 2m+1$, $n, n, 2m+1$ and two missing triangles on the symmetry axis sharing the s–th vertex on the axis equals*

$$\frac{(2m+1)\binom{2m}{m}\binom{2n-2s}{n-s}\binom{2s-2}{s-1}}{\binom{2m+2n}{m+n}}\prod_{i=1}^{n}\frac{(2m+i+1)_n}{(i)_n}.$$

The following corollary is easily derived using Stirling's approximation formula.

COROLLARY 1.3. *The proportion of rhombus tilings of a hexagon with sides αt, αt, βt, αt, αt, βt and two missing triangles on the horizontal symmetry axis touching the*

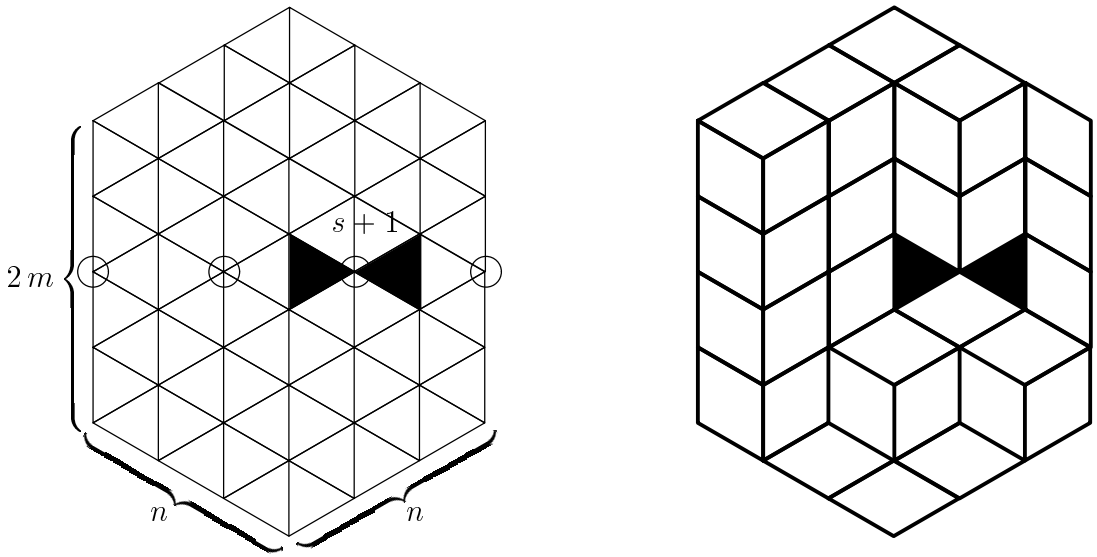


FIGURE 1.1. A hexagon with sides $n, n, 2m, n, n, 2m$ and missing triangles in position $s+1$, where $m = 2, n = 3, s = 2$, and a rhombus tiling.

(γt)-th vertex on the axis in the number of all rhombus tilings of the hexagon with sidelengths $\alpha t, \alpha t, \beta t, \alpha t, \alpha t, \beta t$ (given in (1)) is asymptotically equal to

$$\frac{1}{4\pi}\sqrt{\frac{\beta(2\alpha+\beta)}{\gamma(\alpha-\gamma)}}.$$

This expression can attain arbitrary large values if γ is close to α or 0, i.e. the missing triangles lie near the border of the hexagon. The expression equals $\frac{\sqrt{3}}{2\pi}$ (which is approximately 0.28) for $\alpha = \beta = 2\gamma$, which corresponds to the case of a regular hexagon with two missing triangles touching the center. In comparison, in the other case of Problem 4 of [**27**], the case of a fixed rhombus on the symmetry axis, the analogous proportion must always be smaller than 1 and equals approximately $\frac{1}{3}$ if the central rhombus is missing (see [**8**]).

The rest of the chapter is devoted to the proof of Theorems 1.1 and 1.2. The main ingredients are the matchings factorization theorem by M. Ciucu [**5**], nonintersecting lattice paths, and two determinant evaluations, the latter constituting the most difficult part of the proof. An outline of the proof is given in the next section. The details are filled in in the subsequent sections.

2. Outline of the proofs of Theorems 1.1 and 1.2

Outline of the proof of Theorem 1.1:

Step 1: It suffices to compute the number of rhombus tilings of two regions R^+ and R^- which are roughly the upper and the lower half of the original hexagon (see Section 3).

We use the fact that there is a bijection between rhombus tilings of the hexagon and perfect matchings of the hexagon's 'dual' graph G (see Figure 1.2 for the construction of the dual graph and Figure 1.3a for the correspondence between tilings and matchings).

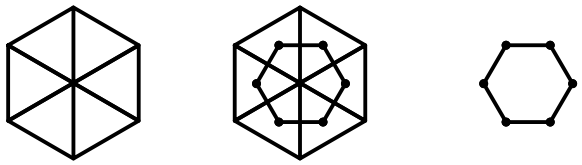


FIGURE 1.2. We place a dot in every bounded region and connect dots corresponding to adjacent regions. The dual graph of the six triangles on the left is the hexagon on the right.

The graph G has reflective symmetry, so the matchings factorization theorem by M. Ciucu [**5**] is applicable (see Lemma 1.4). This theorem expresses the number of perfect matchings of a graph as a power of two times the numbers of perfect matchings of two smaller graphs G^+ and G^- (see Lemma 1.5 and Figure 1.3b), which are roughly the two halves of the original graph G. The remaining task is to count the numbers of perfect matchings of G^+ and G^-.

We use again the correspondence between the rhombus tilings of a region of triangles and the perfect matchings of the dual graph and reduce the problem to counting the rhombus tilings of two regions R^+ and R^- (see Figure 1.4).

Step 2: The numbers of rhombus tilings of R^+ and R^- are certain determinants (see Sections 4 and 5).

The rhombus tilings are in bijection with certain families of nonintersecting lattice paths (see Figures 1.5 and 1.6). Application of the main result of nonintersecting lattice paths expresses the desired numbers as determinants (see Equations (2) and (4)).

Step 3: Evaluation of the determinant corresponding to R^+ (see Section 4).

The determinant corresponding to R^+ is evaluated using a lemma by Krattenthaler (see Equation (3) and Lemma 1.6).

Step 4: Evaluation of the determinant corresponding to R^- (see Section 5).

We take factors out of the determinant, so that we obtain a determinant whose entries are polynomials in m (see the proof of Lemma 1.7). This determinant is evaluated by using the "identification of factors" method, as explained in [**13**, Sec. 2]. The corresponding details are the subject of Sections 6 – 8.

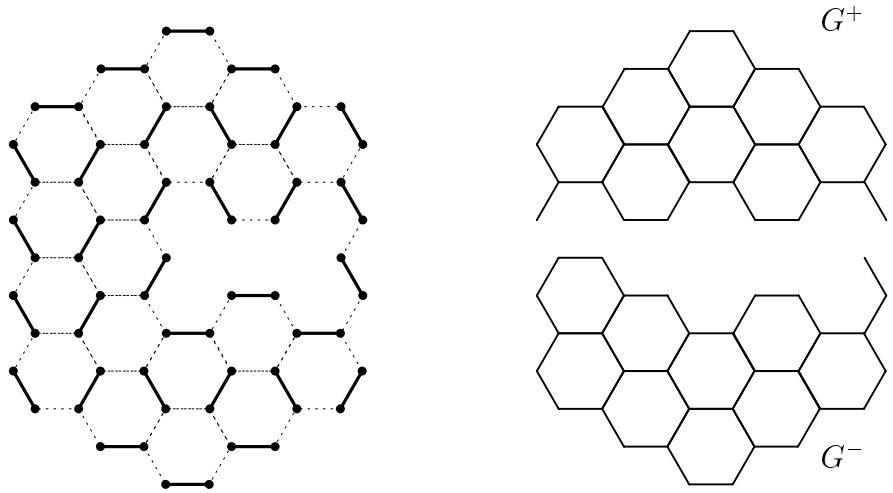
Step 5: A combination of the previous steps proves Theorem 1.1.

We substitute the results of Lemmas 1.6 and 1.7 in Lemma 1.5 and obtain the following expression for the number of rhombus tilings of our original hexagon,

$$2^{\binom{n}{2}-1}\frac{(\mathrm{H}(n))^2\,(2n-2s-1)!!(2s-1)!!}{\mathrm{H}(2n)(n-s)!s!}$$
$$\times\prod_{2\le i\le j\le n}(2m+2j-i)\prod_{k=1}^{n-2}\left(m+k+\frac{1}{2}\right)^{\min(k,n-1-k)}\prod_{k=0}^{n}(m+k)^{\min(k+1,n-k+1)}.$$

Here, $\mathrm{H}(n)$ stands for $\prod_{i=0}^{n-1} i!$. This can easily be transformed to the expression in Theorem 1.1, so the proof of Theorem 1.1 is complete.

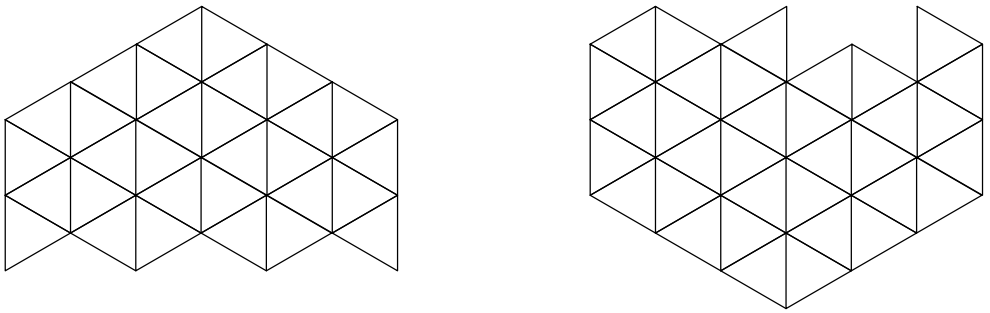
The proof of Theorem 1.2 is given in Section 9. Analogously to the proof of Theorem 1.1, it is enough to count the rhombus tilings of two regions $\widetilde{R^+}$ and $\widetilde{R^-}$ (see



a. The perfect matching corresponding to the rhombus tiling of Figure 1.1.

b. G^+ and G^-.

FIGURE 1.3.



a. R^+, the upper half of the hexagon.

b. R^-, the lower half of the hexagon.

FIGURE 1.4.

Figure 1.7) which are roughly the halves of the original hexagon. This can then be reduced to the determinants already evaluated in the proof of Theorem 1.1.

3. Breaking the hexagon in two parts

We start the proof of Theorem 1.1 by forming the inner dual of the given hexagon (see Figure 1.2). I.e., we replace every triangle by a vertex and connect vertices corresponding to adjacent triangles (see Figure 1.2). Thus, we get a hexagonal graph, whose perfect matchings correspond to rhombus tilings of the original hexagon (see Figure 1.3a).

Now we use a theorem by M. Ciucu (see [**5**]) to, roughly speaking, break the hexagonal graph into two halves. Before we can state it we need a few definitions. First, let H be a graph and assign to each of its edges a number, the weight of the edge. Then the weight of a perfect matching of H is the product of all weights of edges contained in the perfect matching. The weighted enumeration $M(H)$ is just the sum of the weights of all possible perfect matchings. If every edge has weight 1 then $M(H)$ reduces to the number of perfect matchings.

Now we describe how to get the two halves G^+ and G^- from G (see Figure 1.3b). Let G be a planar bipartite graph with reflective symmetry, which splits into two parts after removal of the vertices of the symmetry axis. We can clearly assume that the symmetry axis is the x-axis. Label the vertices of G on the symmetry axis $a_1, b_1, a_2, b_2, \ldots, a_{2l}, b_{2l}$ from left to right. Since G is bipartite, we can colour the vertices of the graph black and white subject to the conditions that a_1 is white and no two adjacent vertices are of the same colour.

Then we delete all edges connecting white a–vertices and black b–vertices to the upper half and all edges connecting black a–vertices and white b–vertices to the lower half and we divide by two all weights of edges lying on the symmetry axis. The graph G splits into two parts G^+ and G^-. Now we can state the matchings factorization theorem from [**5**].

LEMMA 1.4. *Let G be a planar bipartite weighted, symmetric graph, which splits into two parts after removal of the vertices of the symmetry axis. Then*

$$M(G) = 2^{l(G)} M(G^+) M(G^-),$$

where $M(H)$ denotes the weighted count of perfect matchings of the graph H and $G^{\pm}$ denote the upper and lower half of G as described above. $2\,l(G)$ is the number of vertices on the symmetry axis.

We apply Lemma 1.4 to our hexagonal graph, exemplified in Figure 1.3a, with respect to the horizontal symmetry axis. In our case $l(G) = n - 1$. G^+ and G^- are shown in Figure 1.3b.

Thus we get the following lemma.

LEMMA 1.5. *The number of rhombus tilings of a hexagon with sides n, n, $2m$, n, n, $2m$ and two missing triangles on the symmetry axis sharing the $(s+1)$–th vertex on the axis equals*

$$2^{n-1} M(G^+) M(G^-),$$

where G^+ and G^- are formed by the above procedure, as exemplified in Figure 1.3b.

$M(G^+)$ and $M(G^-)$ are computed in the following sections.

4. The enumeration of matchings for the upper half

In this section we evaluate $M(G^+)$. The result is stated in the following lemma.

LEMMA 1.6.

$$M(G^+) = \frac{\mathrm{H}(n) \prod_{2 \le i \le j \le n} (2m + 2j - i)}{\prod_{j=1}^{n} (2j-2)!}.$$

PROOF. We start by expressing $M(G^+)$ as the following determinant.

$$M(G^+) = \det_{1 \le i,j \le n} \left(\binom{m+j-1}{m-j+i} \right). \tag{2}$$

To this end, we convert G^+ back to the corresponding region of triangles, R^+ say (see Figure 1.4a), so that perfect matchings of G^+ correspond bijectively to the rhombus

tilings of R^+. Thus, we have to count rhombus tilings of R^+. The next step is converting rhombus tilings to families of nonintersecting lattice paths, where nonintersecting means that different lattice paths have no common vertices. The reader should consult Figure 1.5, while reading the following passage. Given a rhombus tiling of R^+, the lattice paths start on the centers of upper left diagonal edges (lying on one of the sides of length n). They end on the lower right edges parallel to the starting edges. The paths are generated by connecting the center of the respective edge with the center of the edge lying opposite in the rhombus. This process is iterated using the new edge and the second rhombus it bounds. It terminates on the lower right boundary edges. It is obvious that paths starting at different points have no common vertices, i.e., are nonintersecting. Furthermore, an arbitrary family of nonintersecting paths from the set of the upper left edges to the set of the lower right edges lies completely inside R^+ and can be converted back to a tiling (see Figure 1.5a).

Then we transform the picture to "orthogonal" paths with positive horizontal and negative vertical steps of unit length (see Figure 1.5b,c). Let the starting points of the paths be denoted by $P_1, P_2, \dots, P_n$ and the end points by $Q_1, Q_2, \dots, Q_n$. We can easily write down the coordinates of the starting points and the end points:

$$\begin{aligned} P_i &= (i-1, i+m-1) && \text{for } i = 1, \dots, n, \\ Q_j &= (2j-2, j-1) && \text{for } j = 1, \dots, n. \end{aligned}$$

Next we apply the main result for nonintersecting lattice paths [**9**, Cor.2] (see also [**31**, Theorem 1.2]). We state it for the enumeration of *weighted* nonintersecting lattice paths which we will use in later sections. The weight of a family of paths is the product of the weights of all occurring edges. If each edge is assigned a weight of 1 we have a result for the ordinary enumeration.

The theorem says that the weighted count of families of nonintersecting lattice paths, with path i running from P_i to Q_i, is the determinant of the matrix with (i,j)-entry the weight $\mathcal{P}(P_i \to Q_j)$ of lattice paths running from P_i to Q_j, provided that every two paths $P_i \to Q_j$ and $P_k \to Q_l$ have a common vertex if $i < j$ and $k > l$. It is easily checked that our sets of starting and end points meet the required conditions.

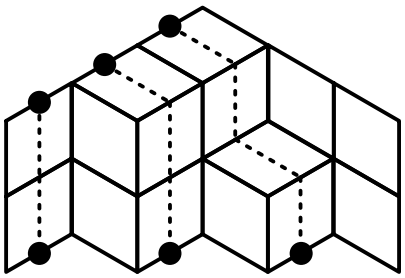
The number of lattice paths with positive horizontal and negative vertical steps from (a,b) to (c,d) equals $\binom{c-a+b-d}{b-d}$. Therefore, the number of families of nonintersecting lattice paths (equivalently, the number of rhombus tilings of R^+) is equal to the following determinant:

$$\det_{1\le i,j\le n}(\mathcal{P}(P_i \to Q_j)) = \det_{1\le i,j\le n}\left(\binom{m+j-1}{m-j+i}\right).$$

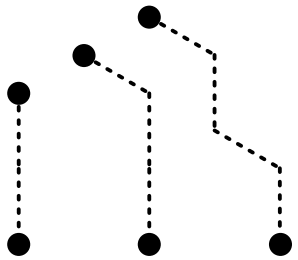
This proves Equation (2).

This determinant can be evaluated with the help of the following determinant identity ([**14**], Lemma 2.2).

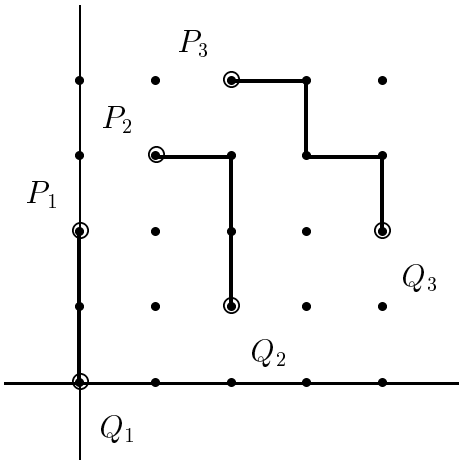
$$\det_{1\le i,j\le n}\big((x_j+a_n)(x_j+a_{n-1})\cdots(x_j+a_{i+1})(x_j+b_i)(x_j+b_{i-1})\dots(x_j+b_2)\big) = \prod_{1\le i<j\le n}(x_i-x_j)\prod_{2\le i\le j\le n}(b_i-a_j). \quad (3)$$



a. A tiling of the upper half of the hexagon and the corresponding lattice path family.



b. The paths isolated.



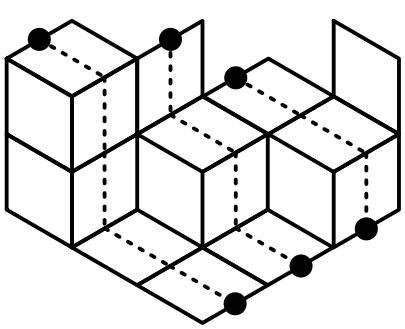
c. The corresponding lattice path family.

FIGURE 1.5.

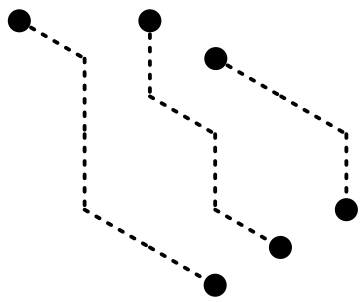
Before we can apply identity (3), we have to transform the expression of Equation (2) in the following way.

$$\begin{aligned} M(G^+) &= \det_{1\le i,j\le n}\left(\binom{m+j-1}{m-j+i}\right) \\ &= \prod_{i=1}^{n}(-2)^{i-1}\prod_{j=1}^{n}\frac{(m+j-1)!}{(n+m-j)!(2j-2)!} \\ &\qquad \times \det_{1\le i,j\le n}\Big((m+i+1-j)(m+i+2-j)\dots(m+n-j) \\ &\qquad\qquad \cdot\left(-j+\frac{i}{2}\right)\left(-j+\frac{i}{2}-\frac{1}{2}\right)\dots(-j+1)\Big). \end{aligned}$$

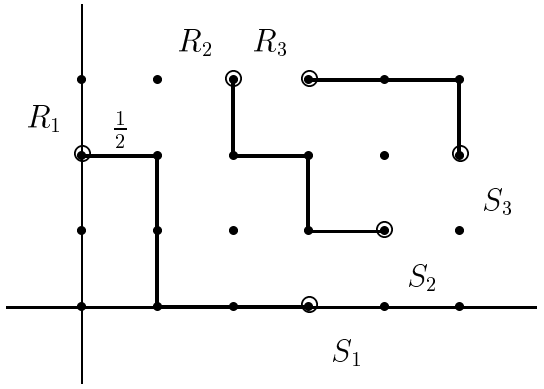
Now we apply the identity (3) with $x_k = -k$, $a_k = m+k$, $b_k = \frac{k}{2}$ and simplify to get the claimed result. □



a. A tiling of the lower half of the hexagon and the corresponding lattice path family.



b. The paths isolated.



c. The corresponding lattice path family.

FIGURE 1.6.

5. The enumeration of matchings for the lower half

In this section we evaluate $M(G^-)$. The result is stated in the following lemma.

LEMMA 1.7.

$$M(G^-) = \frac{2^{\binom{n-1}{2}}\,\mathrm{H}(n)(2n-2s-1)!!(2s-1)!!}{(2n-2s)\prod_{i=1}^{n}(2n+1-2i)!(n-s-1)!s!}$$
$$\times \prod_{k=1}^{n-2}\left(m+k+\frac{1}{2}\right)^{\min(k,n-1-k)} \prod_{k=0}^{n}(m+k)^{\min(k+1,n-k+1)}.$$

PROOF. We start analogously to Section 4 and convert G^- (exemplified in Figure 1.3b) back to a region R^- of triangles (see Figure 1.4b), so that the perfect matchings of G^- correspond bijectively to the rhombus tilings of R^-. However, since G^- contains edges on the symmetry axis of G, which, by Lemma 1.4, has the consequence

that they count with weight $\frac{1}{2}$ in G^-, we are dealing with a weighted count of the rhombus tilings of R^-, where rhombi such as the top-left rhombus in Figure 1.6a count with weight $\frac{1}{2}$. Again, we count the rhombus tilings of R^- by counting the number of nonintersecting lattice path families leading from upper left to lower right edges. The starting and end points can be easily read off Figure 1.6.

The starting points are (see Figure 1.6 and note that the missing triangles at the $(s+1)$–th point of the former symmetry axis make the $(s+1)$–th starting point shift a step to the South-West):

$$R_i = \begin{cases} (2i-2, m+i-1) & \text{for } i \neq s+1 \\ (2s-1, m+s-1) & \text{for } i = s+1. \end{cases}$$

The end points are:

$$S_j = (n+j-1, j-1) \quad \text{for } j = 1, \ldots, n.$$

Now we apply again the main result for nonintersecting lattice paths. The matrix entries are $\mathcal{P}(R_i \to S_j)$. We note that a positive horizontal step starting at R_i, $i \neq s+1$ corresponds to a rhombus of weight $\frac{1}{2}$ in R^-, so paths starting with a horizontal step at R_i, $i \neq s+1$ are counted with weight $\frac{1}{2}$. Therefore, we count paths starting with a horizontal step and paths starting with a vertical step separately.

We get $M(G^-) = \det_{1 \leq i,j \leq n}(A_{ij})$, where

$$A_{ij} = \mathcal{P}(R_i \to S_j) = \begin{cases} \frac{1}{2}\binom{n+m-i}{m+i-j} + \binom{n+m-i}{m+i-1-j} & \text{for } i \neq s+1 \\ \binom{n+m-s}{m+s-j} & \text{for } i = s+1. \end{cases} \tag{4}$$

Since this expression also makes sense for $s = 0$, we can include this case in the following calculations. We take factors out of the rows of $\det(A_{ij})$, so that the remaining entries B_{ij} are polynomials in m and get the following equation

$$\det(A_{ij}) = \frac{(n+m-s)(s+m)}{(2n-2s)\prod_{i=1}^{n}(2n+1-2i)!} \times \det(B_{ij}), \quad s = 0, \ldots, n-1,$$

where

$$B_{ij} = \begin{cases} (n+2+j-2i)_{n-j}(i+m+1-j)_{j-1}\left(m+\frac{n}{2}+\frac{1}{2}-\frac{j}{2}\right) & i \neq s+1 \\ (n+1+j-2s)_{n-j}(s+m+1-j)_{j-1} & i = s+1. \end{cases} \tag{5}$$

Here $(a)_n := a(a+1)(a+2)\ldots(a+n-1)$ is the usual shifted factorial. The entry for $i \neq s+1$ can also be written as

$$B_{ij} = \frac{1}{2}(n+1+j-2i)_{n-j+1}(i+m+1-j)_{j-1} + (n+2+j-2i)_{n-j}(i+m-j)_j. \tag{6}$$

It remains to prove the following identity

$$\det\left(B_{ij}\right) = \frac{2^{\binom{n-1}{2}}\,\mathrm{H}(n)(2n-2s-1)!!(2s-1)!!}{(n-s-1)!s!(m+s)(m+n-s)} \times \prod_{k=1}^{n-2}\left(m+k+\frac{1}{2}\right)^{\min(k,n-1-k)} \prod_{k=0}^{n}(m+k)^{\min(k+1,n-k+1)}. \quad (7)$$

Outline of the proof of Equation (7)*:*
In Section 6 we prove that $\prod_{k=1}^{n-2}\left(m+k+\frac{1}{2}\right)^{\min(k,n-1-k)}$ divides $\det\left(B_{ij}\right)$ as a polynomial in m. In Section 7 we prove that $\prod_{k=0}^{n}(m+k)^{\min(k+1,n-k+1)}$ divides $\det\left(B_{ij}\right)$ as a polynomial in m.

We show this with the help of linear combinations of rows and columns, which vanish, if one of the linear factors is set equal to zero.

In Section 8 we compute the degree of the determinant as a polynomial in m. It is exactly equal to the number of linear factors we have already found to divide the determinant. So we know the determinant up to a constant factor. We compute this constant in Equation (16) by replacing each entry by its leading coefficient and using Vandermonde's determinant formula.

Equation (7) follows immediately from (8), (10) and (16). □

6. The "half-integral" factors of $\det\left(B_{ij}\right)$

In this section we prove the following (see Equation (5) for the definition of B_{ij}).

$$\prod_{k=1}^{n-2}\left(m+k+\frac{1}{2}\right)^{\min(k,n-1-k)} \text{ divides } \det\left(B_{ij}\right) \text{ as a polynomial in } m. \quad (8)$$

We find linear combinations of columns which give zero for $m = -\left(k+\frac{1}{2}\right)$. First, we show that the following linear combination of columns equals zero for $i \neq s+1$, $0 \le l \le k$, $l \ge 2k-n+1$:

$$\sum_{j=0}^{l}\binom{l}{j} B_{i,n+2l-2k-j}\Big|_{m=-k-\frac{1}{2}} = 0. \quad (9)$$

In order to establish this, we break the sum in two parts according to the two summands of B_{ij} in Equation (6) and convert them to hypergeometric form. The left-hand side of (9) becomes

$$(k-2l+i-1-\frac{1}{2}-n)_{n-2k+2l}(2-2k+2l-2i+2n)_{2k-2l} \times {}_2F_1\left[\begin{matrix}-l,-1+2k-2l+2i-2n\\ 2k-2l+i-k-\frac{1}{2}-n\end{matrix};1\right] + (1+2k-2l+i-k-\frac{1}{2}-n)_{-1-2k+2l+n}(2-2k+2l-2i+2n)_{2k-2l} \times {}_2F_1\left[\begin{matrix}-l,-1+2k-2l+2i-2n\\ 1+2k-2l+i-k-\frac{1}{2}-n\end{matrix};1\right].$$

Now, Vandermonde's summation formula

$$ {}_2F_1\left[\begin{matrix}a,-n\\ c\end{matrix};1\right] = \frac{(-a+c)_n}{(c)_n} $$

is applicable to both ${}_2F_1$–series, since $l \geq 0$. It is directly verifiable that the two resulting expressions sum to zero.

It is easily seen that the conditions on l in (9) allow $\min(k+1, n-k)$ possible values for l. Thus, we have $\min(k+1, n-k)$ independent linear combinations of columns which vanish in all coordinates except possibly in the $(s+1)$-th coordinate. (Recall that (9) is valid only for $i \neq s+1$). It is clear that an appropriate combination of two of these linear combinations vanishes in every coordinate. So we have $\min(k, n-k-1)$ independent linear combinations vanishing at $m = -k-\frac{1}{2}$ for $k = 1, \ldots, n-2$, which proves (8).

7. The "integral" factors of $\det(B_{ij})$

In this section we prove the following result (see (5) for the definition of B_{ij}).

$$ \frac{1}{(n+m-s)(s+m)} \prod_{k=0}^{n} (m+k)^{\min(k+1,n-k+1)} \text{divides } \det(B_{ij}) \text{ as a polynomial in } m. \tag{10} $$

We use linear combinations of the rows of B_{ij} that vanish for $m = -k$. Without loss of generality, we can assume that $s \leq \frac{n}{2}$ because both the final result (see Theorem 1.1) and the number of rhombus tilings of the original graph are invariant under the transformation $s \to n-s$.

Most of the factors $(m+k)$ can be taken out directly from the rows of B_{ij}. In fact, it is easily seen that row i is divisible by $(m+1-i+n)_{2i-n-1}$ for $2i \geq n+2$. The product of these terms equals $\prod_{k=1}^{n-1} (m+k)^{\min(k,n-k)}$. The matrix (C_{ij}) which remains after taking out these factors from B_{ij} looks as follows:

$$ C_{ij} = \begin{cases} (n+j+1-2s)_{n-j}(s+m+1-j)_{j-1} & \\ & i = s+1, \\ (n+2+j-2i)_{n-j}(i+m+1-j)_{j-2i+n}(2m+n+1-j) & \\ & i \neq s+1, 2i \geq n+2, \\ (n+2+j-2i)_{n-j}(i+m+1-j)_{j-1}(2m+n+1-j) & \\ & i \neq s+1, 2i < n+2. \end{cases} $$

To finish the proof of (10) we have to find for each value $k = 0, \ldots, n$, $k \neq s$ and $k \neq (n-s)$ one vanishing linear combination of the rows of C_{ij}.

We start with the case $k < s$. We claim that

$$ \sum_{i=k+1}^{s} \left((-1)^{i-k+1} \binom{s-k-1}{i-k-1} \frac{(n+\frac{3}{2}-i)_{i-k-1}(n-i+1)_{i-k-1}}{(s+\frac{1}{2}-i)_{i-k-1}(n-k-i+1)_{i-k-1}} C_{ij}\Big|_{m=-k} \right) $$
$$ + (-1)^{s-k+2} \frac{2(n+\frac{3}{2}-s-1)_{s-k}(n-s+1)_{s-k-1}}{(\frac{1}{2})_{s-k-1}(n-k-s+1)_{s-k-1}} C_{s+1,j}\Big|_{m=-k} = 0. \tag{11} $$

If $-k+s-j<0$ then the terms $(i-k+1-j)_{j-1}$ and $(s-k+1-j)_{j-1}$ (which are factors of $C_{ij}\Big|_{m=-k}$ and $C_{s+1,j}\Big|_{m=-k}$, respectively) are zero for all occurring indices. If $-k+s-j\geq 0$, we reverse the order of summation in the sum and write (11) in hypergeometric form,

$$(-1)^{-k+s}\,\frac{(n+\frac{3}{2}-s)_{s-k-1}\,(1-j-k+s)_{j-1}}{(\frac{1}{2})_{s-k-1}\,(n-k-s+1)_{s-k-1}(2+j+n-2\,s)_{n-j}\,(n-s+1)_{s-k-1}}$$
$$\times\,(-1+j+2k-n)\,{}_3F_2\left[\begin{matrix}\frac{1}{2},n-k+1-s,j+k-s\\ 1+\frac{j}{2}+\frac{n}{2}-s,\frac{3}{2}+\frac{j}{2}+\frac{n}{2}-s\end{matrix};1\right]$$
$$+\,(-1)^{s-k+2}\frac{2(n+\frac{3}{2}-s-1)_{s-k}(n-s+1)_{s-k-1}}{(\frac{1}{2})_{s-k-1}(n-k-s+1)_{s-k-1}}C_{s+1,j}\Big|_{m=-k}=0.$$

Now we can apply the Pfaff–Saalschütz summation formula ([**28**], (2.3.1.3); Appendix (III.2)),

$${}_3F_2\left[\begin{matrix}a,b,-n\\ c,1+a+b-c-n\end{matrix};1\right]=\frac{(-a+c)_n\,(-b+c)_n}{(c)_n\,(-a-b+c)_n},\tag{12}$$

where n is a nonnegative integer.

It is easily verified that the resulting sum of two terms equals zero.

The case $k>n-s$ is quite similar. We claim that

$$\sum_{i=n-k+1}^{s}(-1)^{i-n+k+1}\binom{s-n+k-1}{i-n+k-1}\frac{(n+\frac{3}{2}-i)_{i-n+k-1}(n-i+1)_{i-n+k-1}}{(s+\frac{1}{2}-i)_{i-n+k-1}(k-i+1)_{i-n+k-1}}C_{ij}\Big|_{m=-k}$$

$$-\,(-1)^{s-n+k+2}\frac{2(n+\frac{3}{2}-s-1)_{s-n+k}(n-s+1)_{s-n+k-1}}{(\frac{1}{2})_{s-n+k-1}(k-s+1)_{s-n+k-1}}C_{s+1,j}\Big|_{m=-k}=0.\tag{13}$$

Converting the reversed sum to hypergeometric form gives

$$(-1)^{s+k-n}(-1+j+2k-n)(2+j+n-2\,s)_{n-j}\,(1+n-s)_{-1+k-n+s}$$
$$\times\,\frac{(\frac{3}{2}+n-s)_{-1+k-n+s}(1-j-k+s)_{-1+j}}{(\frac{1}{2})_{-1+k-n+s}(1+k-s)_{-1+k-n+s}}$$
$$\times\,{}_3F_2\left[\begin{matrix}j+k-s,\frac{1}{2},1-k+n-s\\ 1+\frac{j}{2}+\frac{n}{2}-s,\frac{3}{2}+\frac{j}{2}+\frac{n}{2}-s\end{matrix};1\right].$$

Again, the Pfaff–Saalschütz summation formula (12) is applicable because $-(1-k+n-s)$ is a nonnegative integer. It is easily checked that the resulting terms sum to 0.

So our remaining task is the case $s<k<n-s$. For $s<k\leq\frac{n}{2}$ we consider the following linear combination,

$$\sum_{i=k+1}^{\lfloor\frac{n+1}{2}\rfloor}(-4)^{n-i}\frac{(s-i+1)_{i-k-1}(s-n+\frac12)_{n-k-1}}{(2n-2i+1)!(s+\frac12-i)_{i-k-1}(n+1-s)_{-k}}(i-k)_{n+1-2i}C_{ij}\Big|_{m=-k}$$
$$+\sum_{i=\lfloor\frac{n+3}{2}\rfloor}^{\lfloor\frac{n+1+j}{2}\rfloor}(-4)^{n-i}\frac{(s-i+1)_{i-k-1}(s-n+\frac12)_{n-k-1}}{(2n-2i+1)!(s+\frac12-i)_{i-k-1}(n+1-s)_{-k}}C_{ij}\Big|_{m=-k}$$
$$-C_{s+1,j}\Big|_{m=-k}=0. \quad (14)$$

Now, both the term $(i-k)_{n+1-2i}C_{ij}\Big|_{m=-k}$ which is part of the first sum in (14) and the term $C_{ij}\Big|_{m=-k}$ which is part of the second sum in (14) are equal to $(n+2+j-2i)_{n-j}(i-k+1-j)_{j-2i+n}(-2k+n+1-j)$, so we can combine the two sums into one sum of a hypergeometric term.

We distinguish two cases according to the parity of $n-j$. In both cases we reverse the sum and convert it to hypergeometric form. The resulting two hypergeometric series are

$${}_3F_2\left[\begin{matrix}1+l+m,-l-m,\frac12+l-n+s\\ \frac32,1+l-n+s\end{matrix};1\right] \qquad \text{for } n-j=2l,$$
$${}_3F_2\left[\begin{matrix}1+l+m,-1-l-m,\frac12+l-n+s\\ \frac12,1+l-n+s\end{matrix};1\right] \qquad \text{for } n-j=2l+1.$$

So we can use the Pfaff–Saalschütz summation formula (12) again. It is easily verified that in both cases the resulting terms add to zero.

The case $\frac{n}{2}<k<n-s$ is handled similarly. We claim that the following sum equals zero,

$$\sum_{i=n-k+1}^{\lfloor\frac{n+1}{2}\rfloor}(-4)^{n-i}\frac{(s-i+1)_{i-n+k-1}(s-n+\frac12)_{k-1}(i-k)_{n+1-2i}}{(2n-2i+1)!(s+\frac12-i)_{i-n+k-1}(n+1-s)_{-n+k}}C_{ij}\Big|_{m=-k}$$
$$+\sum_{i=\lfloor\frac{n+3}{2}\rfloor}^{\lfloor\frac{n+j+1}{2}\rfloor}(-4)^{n-i}\frac{(s-i+1)_{i-n+k-1}(s-n+\frac12)_{k-1}}{(2n-2i+1)!(s+\frac12-i)_{i-n+k-1}(n+1-s)_{-n+k}}C_{ij}\Big|_{m=-k}$$
$$-(-1)^nC_{s+1,j}\Big|_{m=-k}=0. \quad (15)$$

Again, we write the two sums as one single sum, distinguish two cases according to the parity of $n-j$, and reverse the order of summation. Conversion to hypergeometric form of the resulting sums gives

$${}_3F_2\left[\begin{matrix}1+l-k,-l+k,\frac12+l-n+s\\ \frac32,1+l-n+s\end{matrix};1\right] \qquad \text{for } n-j=2l,$$
$${}_3F_2\left[\begin{matrix}1+l-k,-1-l+k,\frac12+l-n+s\\ \frac12,1+l-n+s\end{matrix};1\right] \qquad \text{for } n-j=2l+1.$$

The Pfaff–Saalschütz summation formula (12) can be applied in both cases. It is easily seen that the results vanish after subtraction of $(-1)^n C_{s+1,j}\Big|_{m=-k}$. Thus (10) is proved.

8. The degree and the leading coefficient

We have to find the degree and the leading coefficient of the determinant $\det(B_{ij})$ as a polynomial in m. The degree of B_{ij} is $j-1$ for $i = s+1$ and j else. Therefore, the degree of $\det(B_{ij})$ is at most $\binom{n+1}{2} - 1$, which is easily seen to be the number of linear factors we have found to divide $\det(B_{ij})$. Therefore $\det(B_{ij})$ is the product of the linear factors and the leading coefficient.

To compute the leading coefficient we look at the leading coefficient of each entry. The leading coefficients of the entries give the matrix D, with

$$D_{ij} = (x_i + n + j)_{n-j}, \text{ where } x_i = \begin{cases} 2-2i & i \neq s+1 \\ 1-2s & i = s+1. \end{cases}$$

This matrix can be transformed by column reduction to (x_i^{n-j}), but this is just the Vandermonde determinant $\prod_{1\le i<j\le n}(x_i - x_j)$. Plugging in the values of the x_i gives the following result.

The leading coefficient of $\det(B_{ij})$ is

$$\frac{2^{\binom{n-1}{2}}\,\mathrm{H}(n)(2n-2s-1)!!(2s-1)!!}{(n-s-1)!s!}. \tag{16}$$

(8), (10) and (16) immediately give Equation (7). Thus the proof of Lemma 1.7 is complete. □

9. Proof of Theorem 1.2

If the side divided by the symmetry axis has odd length, the position s of the missing triangles ranges from 1 to n (see Figure 1.7). We can form the inner dual graph and denote it by $\widetilde{G}$. Now we can proceed analogously to Section 3, break the graph in two parts $\widetilde{G^+}$ and $\widetilde{G^-}$ with the help of the matchings factorization theorem (see Lemma 1.4). We convert $\widetilde{G^+}$ and $\widetilde{G^-}$ back to regions $\widetilde{R^+}$ and $\widetilde{R^-}$ of triangles and have to count rhombus tilings again. $\widetilde{R^+}$ and $\widetilde{R^-}$ are shown in an example in Figure 1.7.

Thus, we have

$$M(\widetilde{G}) = 2^{n-1} M(\widetilde{R^+}) M(\widetilde{R^-}). \tag{17}$$

Now we reduce the evaluation of $M(\widetilde{R^+})$ to the evaluation of $M(R^+)$, which we have already done in Lemma 1.6. $M(R^+)$ and $M(\widetilde{R^+})$ are related in the following way. The tiles of the upper half R^+ of the hexagon with sides $n, n, 2m, n, n, 2m$ (as exemplified in Figure 1.1) sharing an edge with the border of length m are enforced as shown in Figure 1.8 (the forced tiles are shaded). After removal of these tiles we are left with the upper half $\widetilde{R^+}$ of the hexagon with sides $n-1, n-1, 2m+1, n-1, n-1, 2m+1$. Thus, we have $M(R^+(n,m)) = M(\widetilde{R^+}(n-1,m))$ and Lemma 1.6 implies directly the

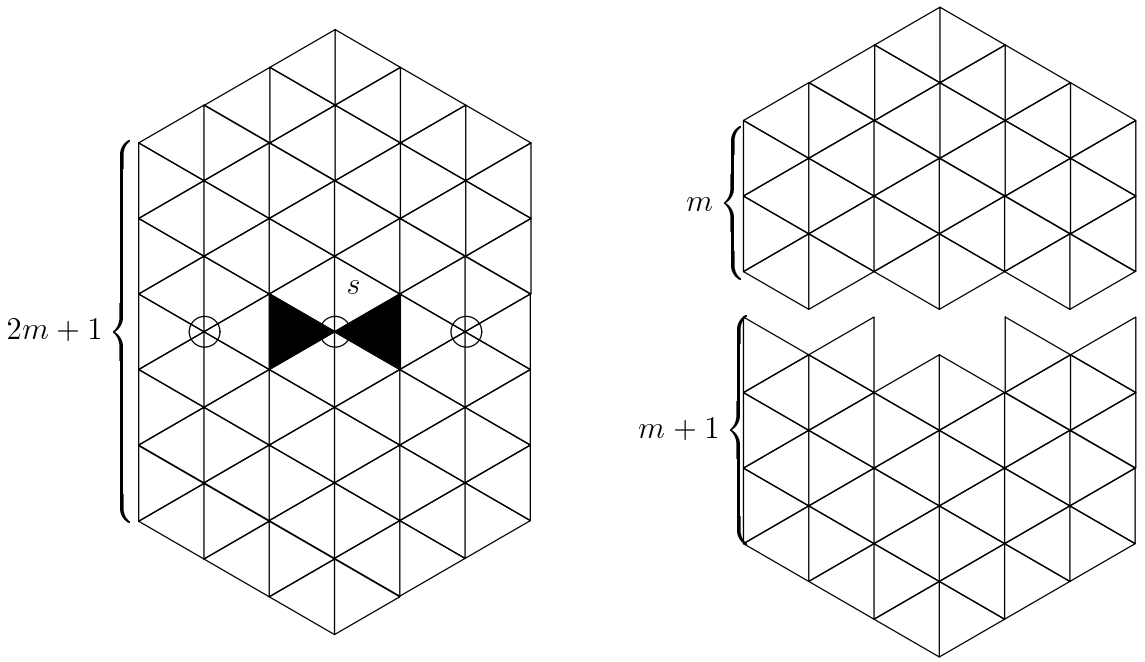


FIGURE 1.7. The hexagon in the case of odd sidelength and the two halves $\widetilde{R^+}$ and $\widetilde{R^-}$. $m = 2$, $n = 3$, $s = 2$.

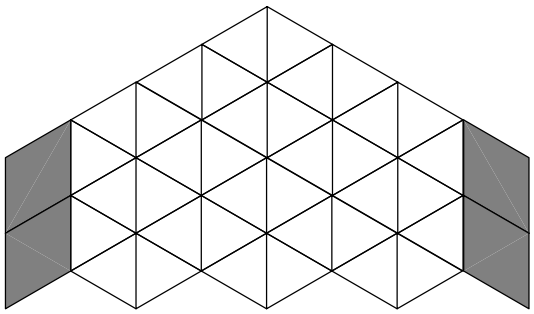


FIGURE 1.8. $M(\widetilde{R^+}(3,2))$ equals $M(R^+(2,2))$.

following result:

$$M(\widetilde{R^+}) = \frac{\mathrm{H}(n+1)\prod_{2\le i\le j\le n+1}(2m+2j-i)}{\prod_{j=1}^{n+1}(2j-2)!}. \tag{18}$$

The lower half $\widetilde{R^-}$ can be turned into a determinant in a manner analogous to Section 5 (see Figure 1.9).

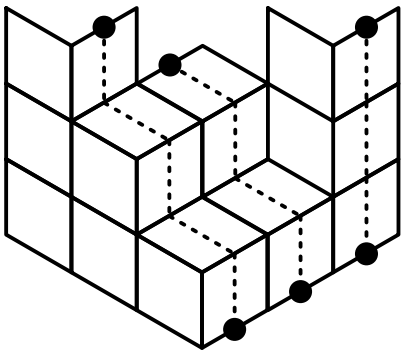


FIGURE 1.9. A tiling and lattice paths for $\widetilde{R^-}$.

The starting and end points and the resulting determinant equal

$$\widetilde{R_i} = \begin{cases} (2i-1, i+m) & \text{for } i \neq s \\ (2s-2, s+m-1) & \text{for } i = s \end{cases}$$

$$\widetilde{S}_j = (n+j-1, j-1)$$

$$\widetilde{A_{ij}} = \begin{cases} \frac{1}{2}\binom{n+m-i}{m+i-j+1} + \binom{n+m-i}{m+i-j} & \text{for } i \neq s \\ \binom{n+m-s+1}{m+s-j} & \text{for } i = s. \end{cases}$$

Since the original problem and the claimed final result are invariant under $n+1-s \to s$, we can assume $s \neq n$. (The case $n = s = 1$ is trivial to check.) Then

$$\begin{aligned} \widetilde{A_{nj}} &= \frac{1}{2}\binom{m}{m+n-j+1} + \binom{m}{m+n-j} \\ &= \begin{cases} 1 & \text{for } j = n, \\ 0 & \text{else,} \end{cases} \end{aligned}$$

since m is a nonnegative integer. It is easily seen that

$$\widetilde{A_{ij}}(n, m, s) = A_{ij}(n-1, m+1, s-1) \quad \text{for } i, j < n,$$

where A_{ij} is defined in Equation (4). We expand $\det_{1 \le i,j \le n}(\widetilde{A_{ij}}(n, m, s))$ with respect to row n and get

$$\det_{1 \le i,j \le n}(\widetilde{A_{ij}}(n, m, s)) = \det_{1 \le i,j \le n-1}(A_{ij}(n-1, m+1, s-1)),$$

Hence,

$$M(\widetilde{R^-}(n, m, s)) = M(R^-(n-1, m+1, s-1)).$$

Thus, Lemma 1.7 yields

$$M(\widetilde{R^-}) = \frac{2^{\binom{n-2}{2}-1}\,\mathrm{H}(n-1)(2n-2s-1)!!(2s-3)!!}{(n-s)!(s-1)!\prod_{i=0}^{n-2}(2i+1)!} \times \prod_{k=1}^{n-3}\left(m+1+k+\frac{1}{2}\right)^{\min(k,n-2-k)} \prod_{k=0}^{n-1}(m+1+k)^{\min(k+1,n-k)} \quad (19)$$

Now we substitute the results of Equations (18) and (19) in (17). We get

$$M(\widetilde{G}) = 2^{n-1}M(\widetilde{R^+})M(\widetilde{R^-}) \quad (20)$$
$$= \frac{2^{\binom{n-1}{2}-1}\,\mathrm{H}(n-1)\,\mathrm{H}(n+1)(2n-2s-1)!!(2s-3)!!}{(n-s)!(s-1)!\prod_{j=0}^{n}(2j)!\prod_{i=0}^{n-2}(2i+1)!} \times \prod_{k=2}^{n-2}\left(m+k+\frac{1}{2}\right)^{\min(k-1,n-k-1)} \prod_{k=1}^{n}(m+k)^{\min(k,n-k+1)} \prod_{2\le i\le j\le n+1}(2m+2j-i),$$

which can easily be transformed to the expression in Theorem 1.2. □

CHAPTER 2

(-1)–enumeration of plane partitions with complementation symmetry

1. Introduction

A plane partition P can be defined as a finite set of points (i, j, k) with $i, j, k > 0$ and if $(i, j, k) \in P$ and $1 \leq i' \leq i$, $1 \leq j' \leq j$, $1 \leq k' \leq k$ then $(i', j', k') \in P$. We interpret these points as midpoints of cubes and represent a plane partition by stacks of cubes (see Figure 2.1). If we have $i \leq a$, $j \leq b$ and $k \leq c$ for all cubes of the plane partition, we say that the plane partition is contained in a box with sidelengths a, b, c.

Plane partitions were first introduced by MacMahon. One of his main results is the following [**20**, Sec. 429, $q \to 1$, proof in Sec. 494]:

The number of all plane partitions contained in a box with sidelengths a, b, c equals

$$B(a, b, c) = \prod_{i=1}^{a} \frac{(c+i)_b}{(i)_b}, \tag{21}$$

where $(a)_n := a(a+1)(a+2)\dots(a+n-1)$ is the usual shifted factorial.
A plane partition can have several kinds of symmetries which we now list.
A plane partition P is called

- *symmetric* if whenever $(i, j, k) \in P$ then also $(j, i, k) \in P$,
- *cyclically symmetric* if whenever $(i, j, k) \in P$ then also $(j, k, i) \in P$,
- *totally symmetric* if it is both symmetric and cyclically symmetric.

A plane partition P contained in the box $a \times b \times c$ is called

- *self–complementary* if whenever $(i, j, k) \in P$ then $(a+1-i, b+1-j, c+1-k) \notin P$ for $1 \leq i \leq a$, $1 \leq j \leq b$, $1 \leq k \leq c$.

A plane partition P contained in the box $a \times a \times c$ is called

- *transpose–complementary* if whenever $(i, j, k) \in P$ then $(a+1-j, a+1-i, c+1-k) \notin P$ for $1 \leq i \leq a$, $1 \leq j \leq a$, $1 \leq k \leq c$ (see Figure 2.1).

The various combinations of these symmetries lead to ten symmetry classes (cf. [**30**]).

It is known that for each of the ten classes the number of plane partitions contained in that class is given by a "nice" closed formula (see [**1, 3, 34, 30, 26, 25, 21, 16, 4**]). Additionally, the four symmetry classes without complementation admit a natural q–enumeration. In the case of no symmetry this is also a result of MacMahon. The weight is just $q^{\#\text{cubes}}$. If one counts symmetric plane partitions or cyclically symmetric plane partitions with respect to this weight then one obtains nice closed formulas as well (see [**1, 2, 22**]). The second weight which has been considered for the symmetry classes without complementation is $q^{\#\text{orbits}}$. (Here, we mean orbits with respect to the symmetries of the applicable class.) Aside of the case of no symmetries (clearly, in this case the weights $q^{\#\text{cubes}}$ and $q^{\#\text{orbits}}$ are identical), there exist nice closed formulas for

the enumeration of symmetric plane partitions (see [**1**]) and, conjecturally, for totally symmetric plane partitions (see [**30**]).

Amazingly, upon setting $q = -1$ in these enumerations we get the *plain enumeration* of plane partitions with complementation symmetry (see [**32, 33**]). If we consider the plane partitions in a symmetry class with complementation symmetry, then there seems to be no natural q–enumeration (except in the case of self–complementary plane partitions [**30**]). In particular, counting these plane partitions with respect to $q^{\#\text{cubes}}$ gives nothing new because this statistic is constant for all the plane partitions in the symmetry class. (Obviously, counting with respect to $q^{\#\text{orbits}}$ makes no sense as well.) However, a natural (-1)–enumeration for plane partitions with complementation symmetry has been recently proposed by Kuperberg (cf. [**15**, pp.25/26]).

This (-1)–enumeration is defined as follows: A plane partition with complementation symmetry contains exactly one half of each orbit. Let a move consist of removing one half of an orbit and adding the other half. Two plane partitions are connected either by an odd or by an even number of moves, so it is possible to define a relative sign. The sign becomes absolute if we assign a certain plane partition the weight 1 (see Figure 2.15 for cyclically symmetric self–complementary, cyclically symmetric transpose–complementary and totally symmetric self–complementary plane partitions; see Figure 2.2 for (symmetric) transpose–complementary plane partitions and Figure 2.18 for self–complementary plane partitions).

For example, in the case of transpose–complementary plane partition this can be realized by counting the number $n(P)$ of cubes contained in the upper half of the plane partition P and doing the enumeration $\sum_P (-1)^{n(P)}$.

In [**15**], Kuperberg conjectures that this (-1)–enumeration has a nice expression for all the six symmetry classes with complementation symmetry. He gives the result for the case of transpose–complementary plane partitions derived by the method of "forcing planarity". This result is stated below in Theorem 2.1 and proved in a different way. Kuperberg has also found a proof for the case of cyclically symmetric self–complementary plane partitions [**17**] which is reproduced in [**7**]. The same method could be used to prove Theorem 2.6 below except for the sign.

The main purpose of this chapter is to prove Kuperberg's conjecture in almost all other cases. We determine the expressions for the (-1)–enumeration of cyclically symmetric transpose–complementary plane partitions and totally symmetric self–complementary plane partitions. We get partial results depending on the parity of the sidelengths for symmetric transpose–complementary plane partitions and self–complementary plane partitions. All these results are stated in Theorems 2.1–2.7 below. In fact, as it turns out, Kuperberg's conjecture is only partially "true," because in the case of symmetric transpose–complementary plane partitions there is one case in which there is apparently no compact expression for the (-1)–enumeration.

We now present these results.

THEOREM 2.1 (Kuperberg). *The enumeration of transpose–complementary plane partitions in a box with sides $a \times a \times 2b$ with weight $(-1)^{n(P)}$ equals 0 for b odd and a even and*

$$\prod_{j=1}^{\lceil a/2 \rceil - 1} \frac{(\lfloor b/2 \rfloor + j)(a-j)_b}{(j)_{b+1}}$$

else, where $(a)_n$ *denotes the shifted factorial* $a(a+1)\dots(a+n-1)$ *and* $n(P)$ *is the number of cubes in the plane partition* P *contained in the upper half of the box (cf. the explanation of the weight in the paragraphs preceding the theorem).*

THEOREM 2.2. *The enumeration of symmetric transpose–complementary plane partitions in a box with sides* $2\alpha \times 2\alpha \times 2b$ *with weight* $(-1)^{n(P)}$ *equals*

$$\begin{cases} \dfrac{\prod_{k=1}^{\frac{\alpha}{2}}(b+2k)_{\alpha-1}}{\prod_{k=1}^{\frac{\alpha}{2}}(2k)_{\alpha-1}} & \text{for } \alpha \text{ even and } b \text{ even} \\ \dfrac{\prod_{k=1}^{\frac{\alpha-1}{2}}(b+2k)_{\alpha}}{\prod_{k=1}^{\frac{\alpha-1}{2}}(2k)_{\alpha}} & \text{for } \alpha \text{ odd and } b \text{ even} \\ 0 & \text{for } b \text{ odd.} \end{cases}$$

Here, $n(P)$ *denotes the number of cubes in* P *contained in the upper right quarter of the box (cf. the explanation of the weight in the paragraphs preceding Theorem 2.1).*

THEOREM 2.3. *The enumeration of symmetric transpose–complementary plane partitions in a box with sides* $(2\alpha+1) \times (2\alpha+1) \times 2b$ *with weight* $(-1)^{n(P)}$ *has the form*

$$\begin{cases} \displaystyle\prod_{k=1}^{\alpha/2}(b/2+k)_{\alpha/2+1}(b+2\alpha+2)\cdot p_1(b) & \text{for } \alpha \text{ even and } b \text{ even} \\ \displaystyle\prod_{k=1}^{\alpha/2}((b-1)/2+k)_{\alpha/2+1}(b-1)\cdot p_2(b) & \text{for } \alpha \text{ even and } b \text{ odd} \\ \displaystyle\prod_{i=1}^{(\alpha+1)/2}((\alpha+b-1)/2-i+2)_{2i-1}\cdot p_3(b) & \text{for } \alpha \text{ odd and } b \text{ even} \\ \displaystyle\prod_{i=1}^{(\alpha+1)/2}((b+\alpha)/2-i+1)_{2i-1}\cdot p_4(b) & \text{for } \alpha \text{ odd and } b \text{ odd.} \end{cases}$$

Here, p_1 *and* p_2 *are polynomials of degree* $(\alpha/2)^2$, p_3 *and* p_4 *are polynomials of degree* $(\alpha^2-1)/4$ *and* $n(P)$ *denotes the number of cubes in* P *contained in the upper right quarter of the box (cf. the explanation of the weight in the paragraphs preceding Theorem 2.1).*

THEOREM 2.4. *The enumeration of cyclically symmetric transpose–complementary plane partitions in a box with sides* $2\alpha \times 2\alpha \times 2\alpha$ *with weight* $(-1)^{n(P)}$ *equals*

$$\begin{cases} \left(\displaystyle\prod_{k=1}^{(\alpha-1)/2}\frac{(6k-2)!}{(2k+\alpha-1)!}\right)^2 & \text{for } \alpha \text{ odd,} \\ 0 & \text{else,} \end{cases}$$

where $n(P)$ *is the number of cubes in* P *contained in the upper right eighth of the box (cf. the explanation of the weight in the paragraphs preceding Theorem 2.1).*

THEOREM 2.5. *The enumeration of totally symmetric self–complementary plane partitions in a box with sides* $2\alpha \times 2\alpha \times 2\alpha$ *with weight* $(-1)^{n(P)}$ *equals*

$$\begin{cases} \displaystyle\prod_{k=1}^{(\alpha-1)/2} \frac{(6k-2)!}{(2k+\alpha-1)!} & \text{for } \alpha \text{ odd,} \\ 0 & \text{else.} \end{cases}$$

Here, $n(P)$ *is the number of half orbits contained in the plane partition* P *and not contained in the plane partition shown in Figure 2.15, (cf. the explanation of the weight in the paragraphs preceding Theorem 2.1).*

This is also the number of vertically symmetric alternating sign matrices of size $(\alpha+2)$.

THEOREM 2.6. *For even* a, b, c*, the enumeration of self–complementary plane partitions in a box with sides* $a \times b \times c$ *with weight* $(-1)^{n(P)}$ *equals*

$$B\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right),$$

where $B(a,b,c)$ *is defined in Equation* (21).
Here, $n(P)$ *is the weight explained in the paragraphs preceding Theorem 2.1.*

THEOREM 2.7 (Kuperberg). *The* (-1)*–enumeration of cyclically symmetric self–complementary plane partitions in a box with sides* $2\alpha \times 2\alpha \times 2\alpha$ *with weight* $(-1)^{n(P)}$ *is the square root of the ordinary enumeration, that is*

$$\pm \prod_{k=0}^{\alpha-1} \frac{(3k+1)!}{(\alpha+k)!}. \tag{22}$$

This is also the number of alternating sign matrices (see [**37**]), the number of totally symmetric self–complementary plane partitions [**4**] and the number of descending plane partitions. Results for small values of α suggest that the sign is $+1$ for all α.

Thus, the only cases that are still open are the case of symmetric transpose–complementary plane partitions in a box with two odd sides (in which case no nice formula seems to exist) and the case of self–complementary plane partitions in a box with at least one odd side. Here, the enumeration seems to have a nice closed form. The case a even and b and c odd is stated in the following conjecture.

CONJECTURE 1. For a even and b, c odd, the enumeration of self–complementary plane partitions in a box with sides $a \times b \times c$ with weight $(-1)^{n(P)}$ equals up to sign

$$\begin{aligned}
&B\left(\tfrac{a}{4}, \tfrac{b+1}{4}, \tfrac{c+1}{4}\right)^2 B\left(\tfrac{a}{4}, \tfrac{b-3}{4}, \tfrac{c+1}{4}\right) B\left(\tfrac{a}{4}, \tfrac{b+1}{4}, \tfrac{c-3}{4}\right) \\
&\qquad \text{for } a \equiv 0 \pmod 4 \text{ and } b \equiv c \equiv 3 \pmod 4, \\
&B\left(\tfrac{a}{4}, \tfrac{b-1}{4}, \tfrac{c-1}{4}\right)^2 B\left(\tfrac{a}{4}, \tfrac{b+3}{4}, \tfrac{c-1}{4}\right) B\left(\tfrac{a}{4}, \tfrac{b-1}{4}, \tfrac{c+3}{4}\right) \\
&\qquad \text{for } a \equiv 0 \pmod 4 \text{ and } b \equiv c \equiv 1 \pmod 4, \\
&B\left(\tfrac{a-2}{4}, \tfrac{b+1}{4}, \tfrac{c+1}{4}\right)^2 B\left(\tfrac{a+2}{4}, \tfrac{b-3}{4}, \tfrac{c+1}{4}\right) B\left(\tfrac{a+2}{4}, \tfrac{b+1}{4}, \tfrac{c-3}{4}\right) \\
&\qquad \text{for } a \equiv 2 \pmod 4 \text{ and } b \equiv c \equiv 3 \pmod 4, \\
&B\left(\tfrac{a+2}{4}, \tfrac{b-1}{4}, \tfrac{c-1}{4}\right)^2 B\left(\tfrac{a-2}{4}, \tfrac{b+3}{4}, \tfrac{c-1}{4}\right) B\left(\tfrac{a-2}{4}, \tfrac{b-1}{4}, \tfrac{c+3}{4}\right) \\
&\qquad \text{for } a \equiv 2 \pmod 4 \text{ and } b \equiv c \equiv 1 \pmod 4, \\
&B\left(\tfrac{a}{4}, \tfrac{b-1}{4}, \tfrac{c+1}{4}\right)^2 B\left(\tfrac{a}{4}, \tfrac{b-1}{4}, \tfrac{c+1}{4}\right) B\left(\tfrac{a}{4}, \tfrac{b+3}{4}, \tfrac{c-3}{4}\right) \\
&\qquad \text{for } a \equiv 0 \pmod 4,\ b \equiv 1 \pmod 4,\ c \equiv 3 \pmod 4, \\
&B\left(\tfrac{a}{4}, \tfrac{b+1}{4}, \tfrac{c-1}{4}\right)^2 B\left(\tfrac{a}{4}, \tfrac{b+1}{4}, \tfrac{c-1}{4}\right) B\left(\tfrac{a}{4}, \tfrac{b-3}{4}, \tfrac{c+3}{4}\right) \\
&\qquad \text{for } a \equiv 0 \pmod 4,\ b \equiv 3 \pmod 4,\ c \equiv 1 \pmod 4, \\
&\qquad 0 \text{ for } a \equiv 2 \pmod 4 \text{ and } b \not\equiv c \pmod 4,
\end{aligned}$$

where $B(a, b, c)$ is defined in Equation (21).

The remaining case are the self–complementary plane partitions with a odd and b and c even. Also there, the (-1)–enumeration seems to have a nice closed form but we did not bother to work out precise conjectures.

We prove Theorems 2.1–2.6 by adjusting a well-known bijection between plane partitions and families of nonintersecting lattice paths. In the cases of transpose–complementary plane partitions, cyclically symmetric transpose–complementary plane partitions and totally symmetric self–complementary plane partitions, these path families can be enumerated by a determinant given by the Gessel–Viennot method.

In the case of symmetric transpose–complementary and self–complementary plane partitions the path families can be enumerated by a sum of minors that can be expressed as a Pfaffian by a theorem of Okada (see Lemma 2.11).

The resulting determinants and Pfaffians are then shown to be equal to the expressions given in the theorems.

Theorem 2.1 is proved in Section 2. Theorem 2.2 is proved in Sections 3, 4 and 5. Theorem 2.3 is proved in Section 6. Theorem 2.4 is proved in Section 7. Theorem 2.5 is proved in Section 8. Theorem 2.6 is proved in Section 9.

2. Transpose–complementary plane partitions

The aim of this section is to compute the enumeration of transpose–complementary plane partitions contained in a given box with weight $(-1)^{n(P)}$, where $n(P)$ is the number of cubes of P in the upper half of the box (cf. the paragraphs preceding Theorem 2.1). By the definition of transpose–complementary, the box must have sidelengths of the form $a \times a \times 2b$.

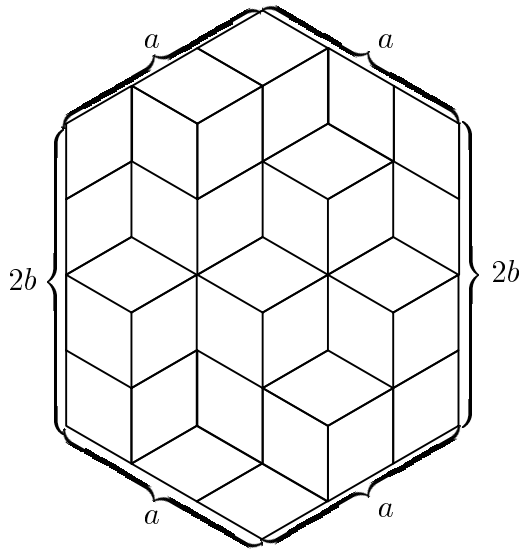


FIGURE 2.1. A transpose–complementary plane partition.

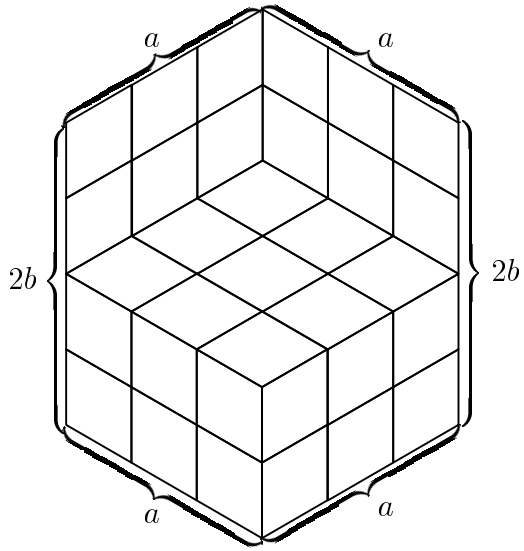
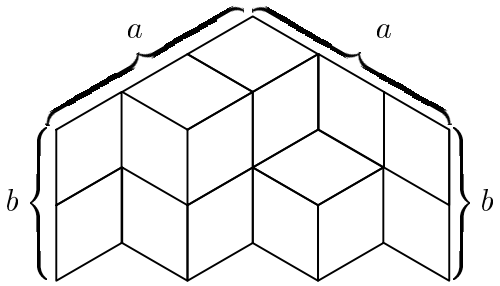
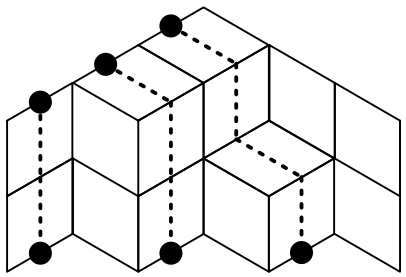


FIGURE 2.2. A (symmetric) transpose–complementary plane partition with weight 1.

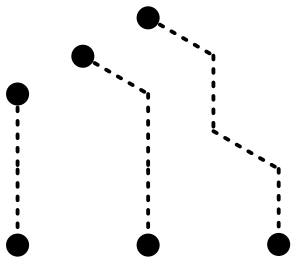
Step 1: From plane partitions to rhombus tilings
The first step is using a well–known bijection between plane partitions and rhombus tilings to convert the problem to a tiling enumeration problem. (In this thesis by rhombus we always mean a rhombus consisting of two equilateral triangles of sidelength 1.) The rhombus tiling of a hexagon with sides $a, a, 2b, a, a, 2b$ corresponding to a given plane partition is simply a projection of the 3–dimensional stack of cubes to the plane (see Figure 2.1). It is easy to see that transpose–complementary plane partitions correspond to rhombus tilings with a horizontal symmetry axis.
Step 2: From rhombus tilings to families of nonintersecting lattice paths.
We use a variant of a well-known translation of rhombus tilings to families of nonintersecting lattice paths. We start with a rhombus tiling with a horizontal symmetry axis



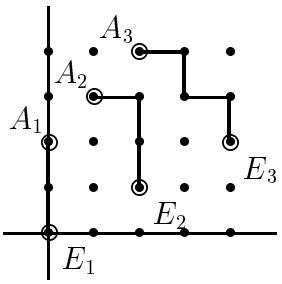
a. The upper half of the tiling.



b. The corresponding path family.



c. The path family isolated.



d. The path family made orthogonal.

FIGURE 2.3.

(see Figure 2.1). Clearly, the symmetry axis must be covered by horizontal rhombi. Since the tiling is symmetric we need only consider the rhombi lying above the symmetry axis (see Figure 2.3a). We mark the midpoints of the edges along the upper left side of length a and of the parallel edges on the zig-zag line (see Figure 2.3b). We connect these points by paths which follow the rhombi of the tiling as illustrated in Figure 2.3b. Clearly, the resulting paths are nonintersecting, i.e., no two paths have a common vertex. If we slightly distort the underlying lattice, we get orthogonal paths with south and east steps (see Figure 2.3d).

We can introduce a coordinate system in a way such that the coordinates of the starting points A_i and end points E_j are

$$A_i = (i-1, b+i-1), \qquad i = 1, \ldots, a, \tag{23}$$

$$E_j = (2j-2, j-1), \qquad j = 1, \ldots, a. \tag{24}$$

We have to find a description of the weight $(-1)^{n(P)}$ in terms of the path families. A horizontal rhombus in the tiling can be viewed as the top square of a stack of cubes in the plane partition interpretation. Therefore, it is assigned the weight $(-1)^n$, where n is the number of cubes (i, j, k) with $k > b$ below the square. In terms of paths this is the same as assigning to a path p the weight $(-1)^{\text{area1}(p)}$ where area1(p) is the area between the path p and the horizontal line containing the lowest point of the path. We want to use instead the weight $(-1)^{\text{area2}(p)}$ where area2(p) is the area between the path

p and the x–axis to be able to apply Lemma 2.8 below. When we do this, we make an overall error of $(-1)^{1^2+2^2+\cdots+(a-1)^2} = (-1)^{a(a-1)/2}$. Hence, we have to count families of nonintersecting lattice paths with starting points A_i (see (23)), end points E_j (see (24)) and weight $(-1)^{\mathrm{area2}(p)}$ and multiply the result by $(-1)^{a(a-1)/2}$.

Step 3: From lattice paths to a determinant

By the main theorem on nonintersecting lattice paths (see [**18**, Lemma 1] or [**9**, Theorem 1]) the weighted count of such families of paths can be expressed as a determinant. We employ the notation $\mathcal{P}(A_i \to E_j)$ for the weighted count of paths from A_i to E_j and $\mathcal{P}(\mathbf{A} \to \mathbf{E}, \text{nonint.})$ for the weighted count of families of nonintersecting lattice paths with the ith path running from A_i to E_i, $i = 1, 2, \ldots, n$. Note that the weight of a path is the product of the weights of its steps. Now we can state the theorem for paths with south and east steps in the integer lattice.

LEMMA 2.8. *Let $A_1, A_2, \ldots, A_n, E_1, E_2, \ldots, E_n$ be integer points meeting the following condition: Any path from A_i to E_l has a common vertex with any path from A_j to E_k for any i, j, k, l with $i < j$ and $k < l$.*

Then we have

$$\mathcal{P}(\mathbf{A} \to \mathbf{E}, \text{nonint.}) = \det_{1 \le i,j \le n} (\mathcal{P}(A_i \to E_j)). \tag{25}$$

This is still applicable if some of the points are isolated, i.e., unconnected to any other point.

The lemma is clearly applicable to the enumeration problem formulated in Step 2 since area2 is the sum of the areas between each horizontal step and the x–axis. Now we have to determine the matrix entries $\mathcal{P}(A_i \to E_j)$ with A_i and E_j as in (23) and (24) and the weight $(-1)^{\mathrm{area2}(p)}$. It is well-known that the enumeration of paths from (a, b) to (c, d) with weight $q^{\mathrm{area1}(p)}$ is the q–binomial coefficient $\begin{bmatrix} c-a+b-d \\ c-a \end{bmatrix}_q$. The q–binomial coefficient (see [**29**, p. 26] for further information) is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{j=n-k+1}^{n}(1-q^j)}{\prod_{j=1}^{k}(1-q^j)}.$$

Although it is not obvious from this definition, the q–binomial coefficient is a polynomial in q. So it makes sense to put $q = -1$. Since we want to use $(-1)^{\mathrm{area2}(p)}$ there is an additional factor of $(-1)^{(j-1)(2j-1-i)}$. In summary, the matrix entries are

$$M_{ij} = (-1)^{(j-1)(i-1)} \begin{bmatrix} b+j-1 \\ 2j-i-1 \end{bmatrix}_{-1}.$$

Writing $M = (M_{ij}), 1 \le i, j \le a$, the remaining task is to compute $(-1)^{a(a-1)/2} \det M$.

Step 4: Evaluation of a useful determinant

For the evaluation of $\det M$, we make use of the following determinant lemma by Krattenthaler [**12**, Lemma 2.2]:

LEMMA 2.9.

$$\det_{1 \le i,j \le n} ((X_j + A_n) \ldots (X_j + A_{i+1})(X_j + B_i) \ldots (X_j + B_2)) = \prod_{1 \le i < j \le n} (X_i - X_j) \prod_{2 \le i \le j \le n} (B_i - A_j).$$

□

This lemma implies the following determinant evaluation, which is crucial in this section.

LEMMA 2.10.

$$\det_{1\le i,j\le\alpha}\left(\binom{\beta+j}{2j-i-\gamma}\right)=\prod_{j=1}^{\alpha}\frac{(\beta+j)!(j-1)!(2\beta+\gamma+j+1)_{j-1}}{(2j-1-\gamma)!(\beta+\gamma+j-1)!}.$$

PROOF. We start by taking out factors of the determinant:

$$\det_{1\le i,j\le\alpha}\left(\binom{\beta+j}{2j-i-\gamma}\right)=\prod_{j=1}^{\alpha}\frac{(\beta+j)!}{(2j-1-\gamma)!(\beta+\gamma+\alpha-j)!}(-2)^{\binom{\alpha}{2}}$$
$$\times\det_{1\le i,j\le\alpha}\left((-j+\tfrac{\gamma+1}{2})\dots(-j+\tfrac{\gamma+i-1}{2})(-j+\alpha+\beta+\gamma)\dots(-j+\beta+\gamma+i+1)\right).$$

Now we are in the position to apply Lemma 2.9 with $n=\alpha$, $X_k=-k$, $A_k=\beta+\gamma+k$, $B_k=\frac{\gamma+k-1}{2}$. After a little simplification we get the claimed result. □

Step 5: Evaluation of det M

We want to evaluate $(-1)^{a(a-1)/2}\det M$, where

$$M_{ij}=(-1)^{(i-1)(j-1)}\begin{bmatrix}b+j-1\\2j-i-1\end{bmatrix}_{-1},\quad 1\le i,j\le a.$$

It is easy to verify that

$$\begin{bmatrix}n\\k\end{bmatrix}_{-1}=\begin{cases}0 & n \text{ even, } k \text{ odd},\\ \binom{\lfloor n/2\rfloor}{\lfloor k/2\rfloor} & \text{else.}\end{cases}\tag{26}$$

So $M_{ij}=0$ for $b+j$ odd and i even. We split the problem into several cases according to the parities of a and b.

Case 1: b even.

In this case we have $M_{ij}=0$ for i even, j odd. If we rearrange the rows and columns of M so that the even–numbered ones come before the odd–numbered ones then we get a block matrix of the form $\begin{pmatrix}A & 0\\ * & B\end{pmatrix}$, where A is a $\lfloor\frac{a}{2}\rfloor\times\lfloor\frac{a}{2}\rfloor$–matrix with $A_{ij}=-\binom{\frac{b}{2}+j-1}{2j-i-1}$ and B is a $\lceil\frac{a}{2}\rceil\times\lceil\frac{a}{2}\rceil$–matrix with $B_{ij}=\binom{\frac{b}{2}+j-1}{2j-i-1}$. Clearly, $\det M$ is now the product of the determinants of A and B. Therefore, we have

$$(-1)^{\frac{a(a-1)}{2}}\det M=(-1)^{\frac{a(a-1)}{2}}\det_{1\le i,j\le\lfloor\frac{a}{2}\rfloor}\left(-\binom{\frac{b}{2}+j-1}{2j-i-1}\right)\det_{1\le i,j\le\lceil\frac{a}{2}\rceil}\left(\binom{\frac{b}{2}+j-1}{2j-i-1}\right)$$
$$=\det_{1\le i,j\le\lfloor\frac{a}{2}\rfloor}\left(\binom{\frac{b}{2}+j-1}{2j-i-1}\right)\det_{1\le i,j\le\lceil\frac{a}{2}\rceil}\left(\binom{\frac{b}{2}+j-1}{2j-i-1}\right).$$

For the first determinant we use Lemma 2.10 with $\beta=b/2-1$, $\gamma=1$ and $\alpha=\lfloor a/2\rfloor$. For the second determinant we use Lemma 2.10 with $\beta=b/2-1$, $\gamma=1$ and $\alpha=\lceil a/2\rceil$.

It is a routine computation to check that the product of these two determinants can be written as

$$\prod_{j=1}^{\lceil a/2\rceil-1} \frac{(b/2+j)(a-j)_b}{(j)_{b+1}} \tag{27}$$

which agrees with the claimed expression in Theorem 2.1.

Case 2: *b* odd and *a* even.

In this case, we have $M_{ij}=0$ for i,j even. Again, rearranging rows and columns of M according to parity as before yields a block form $\begin{pmatrix} 0 & B \\ A & * \end{pmatrix}$, where A is an $\frac{a}{2}\times\frac{a}{2}$–matrix with $A_{ij}=\binom{(b-1)/2+j}{2j-i}$ and B is an $\frac{a}{2}\times\frac{a}{2}$–matrix with $B_{ij}=\binom{(b-1)/2+j-1}{2j-i-2}$. Now, $\det M$ is the product of the determinants of A and B times $(-1)^{\frac{a}{2}}$. Since the first column of B is obviously zero, $\det B$ and thus the entire weighted enumeration is equal to zero in this case.

Case 3: *b* odd and *a* odd. It is easy to see that in this case

$$M_{i1}=\begin{bmatrix} b \\ 1-i \end{bmatrix}_{-1}=\begin{cases} 1 & \text{for } i=1 \\ 0 & \text{else.} \end{cases}$$

We expand $\det M$ along the first column and get $\det \widetilde{M}$ where $\widetilde{M}$ is the $(a-1)\times(a-1)$–matrix with $\widetilde{M}_{ij}=M_{i+1,j+1}$. It is easy to check that $\widetilde{M}_{2i-1,2j-1}=0$. Again, we rearrange the rows and columns of $\widetilde{M}$ such that the even-numbered ones come before the odd-numbered ones and get a block matrix of the form

$$\begin{pmatrix} * & B \\ A & 0 \end{pmatrix},$$

where A is an $\frac{a-1}{2}\times\frac{a-1}{2}$–matrix with $A_{ij}=M_{2i,2j+1}=\binom{(b-1)/2+j}{2j-i}$ and B is an $\frac{a-1}{2}\times\frac{a-1}{2}$–matrix with $B_{ij}=M_{2i+1,2j}=\binom{(b-1)/2+j}{2j-i-1}$. Therefore, $(-1)^{a(a-1)/2}\det\widetilde{M}$ is the product of the determinants of A and B times $(-1)^{a(a-1)/2+(a-1)/2}$, i.e., we have to evaluate

$$(-1)^{a(a-1)/2+(a-1)/2}\det_{1\le i,j\le(a-1)/2}\left(\binom{(b-1)/2+j}{2j-i}\right)\det_{1\le i,j\le(a-1)/2}\left(\binom{(b-1)/2+j}{2j-i-1}\right).$$

This is done by using Lemma 2.10 with $\alpha=(a-1)/2$, $\beta=(b-1)/2$, $\gamma=0$ and $\alpha=(a-1)/2$, $\beta=(b-1)/2$, $\gamma=1$, respectively.

We get after little simplification

$$\prod_{j=1}^{(a-1)/2} \frac{((b-1)/2+j)(a-j)_b}{(j)_{b+1}}$$

which again agrees with the expression in Theorem 2.1. Thus Theorem 2.1 is proved.

3. Symmetric transpose–complementary plane partitions, I

In Sections 3–6 we carry out the (-1)–enumeration for symmetric transpose–complementary plane partitions contained in a given box, i.e., we count each half orbit of cubes contained in the upper half with -1. For example, the plane partition in Figure 2.4 has 10 cubes in the upper half but only 7 half orbits. Its weight is therefore $(-1)^7=-1$.

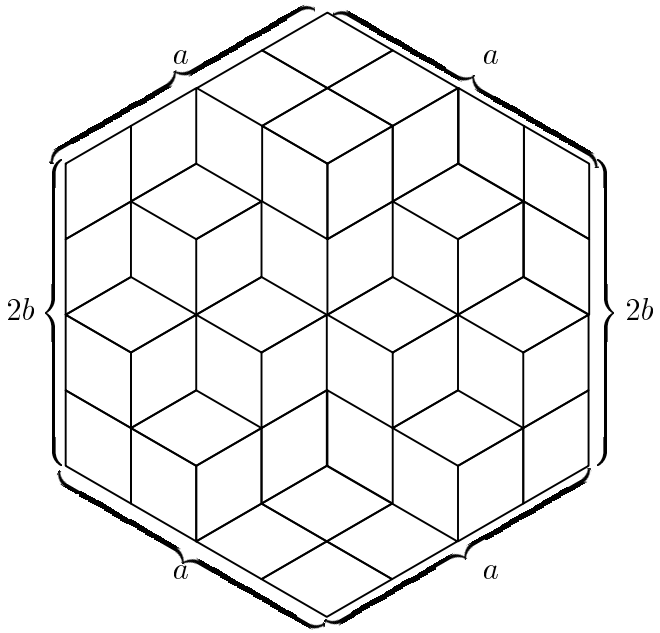


FIGURE 2.4. A symmetric transpose–complementary plane partition.

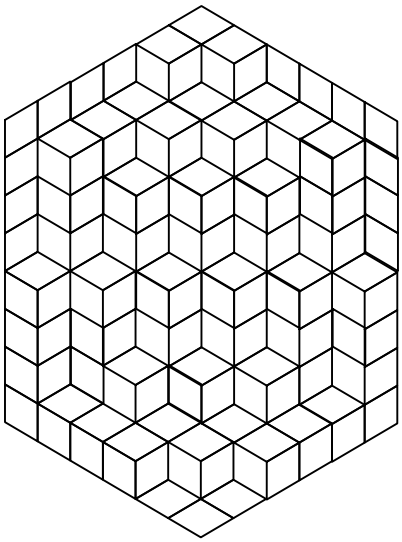


FIGURE 2.5. A symmetric transpose–complementary plane partition.

On the other hand, the "half-full" plane partition in Figure 2.2 containing exactly the cubes (i, j, k) with $k \leq b$ is counted with weight 1. An alternative way to state this is counting each cube in the upper right quarter with -1.

Symmetric transpose–complementary plane partitions are contained in boxes with sidelengths $a \times a \times 2b$. In this section we treat the case $a = 2\alpha$, α is even. The case of α being odd is done in Section 5. For the case $a = 2\alpha + 1$ see Section 6.

For the remainder of this section we assume $a = 2\alpha$ and that α is even.

Step 1: From plane partitions to rhombus tilings

Again, we start by converting the plane partitions into rhombus tilings by projecting them to the plane. We get rhombus tilings of a hexagon with sidelengths 2α, 2α,

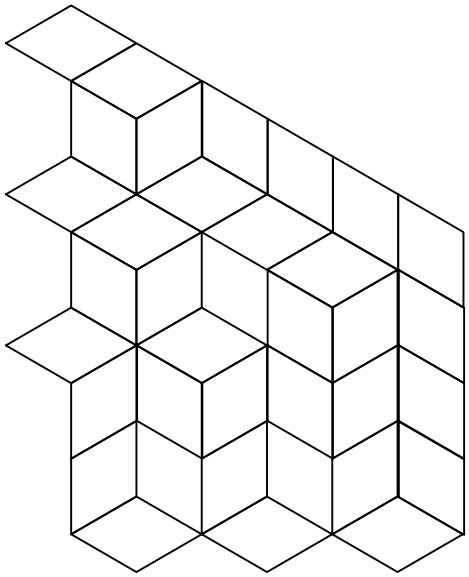


FIGURE 2.6. The upper right quarter of the plane partition in Figure 2.5.

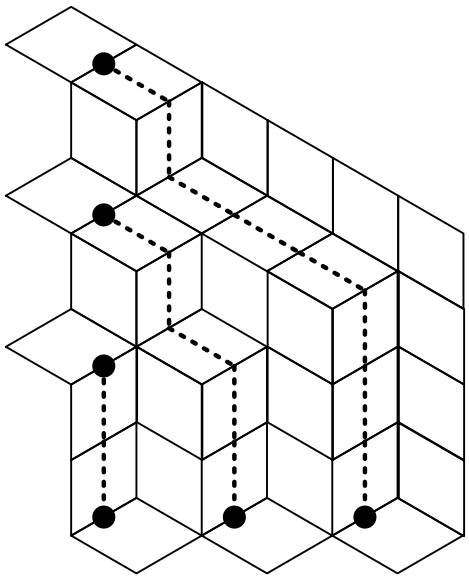


FIGURE 2.7. The corresponding paths.

$2b$, 2α, 2α, $2b$ which have a vertical and a horizontal symmetry axis. These symmetry conditions imply that the corresponding rhombus tiling is determined by its upper right quarter. As in the previous section, in the tiling interpretation the horizontal axis is completely covered by horizontal rhombi.

Step 2: From rhombus tilings to families of nonintersecting lattice paths. As before, we can convert each tiling to a family of nonintersecting lattice paths (see Figures 2.7 and 2.8). Here, the starting points may vary since the horizontal rhombi on the vertical axis can be in different places.

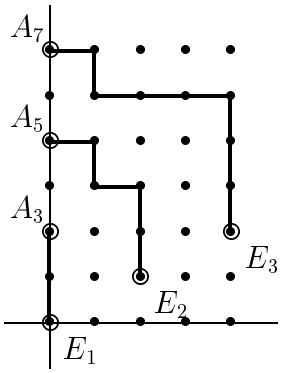


FIGURE 2.8. The path family made orthogonal. Note that $k_1 = 3$, $k_2 = 5$, $k_3 = 7$.

With a suitable coordinate system, the end points are

$$E_j = (2j-2, j-1),\ j = 1, \ldots, \alpha, \tag{28}$$

while the starting points are α points among the points

$$A_i = (0, i-1),\ i = 1, \ldots, \alpha + b. \tag{29}$$

As before, the horizontal steps of the paths correspond to horizontal rhombi in the tiling interpretation which in turn correspond to the squares on the top of a stack of cubes. Each horizontal step should therefore carry the weight $(-1)^n$ where n is the number of cubes (i, j, k) below the corresponding top square and in the upper half i.e. $k > b$. This leads to the weight area1(p) for each path p (the area below the path and above the lowest point of the path). However, this weight does not include the horizontal rhombi on the vertical symmetry axis, because the paths have no horizontal steps there. We can remedy this by assigning the additional factor $(-1)^i$ to a path starting at A_i.

In summary, the weight of a path p starting at A_i is $(-1)^{i+\text{area1}(p)}$. Since our points A_i and E_j have even x–coordinates, area1(p) and area2(p) have the same parity in this case for all occurring paths. Therefore, we can use Lemma 2.8 for the enumeration of lattice path families for every fixed choice of starting points.

To make this precise, let $G_{ij} = \mathcal{P}(A_i \to E_j)$ be the weighted enumeration of all paths running from A_i to E_j, $1 \le i \le \alpha + b$ and $1 \le j \le \alpha$ with weight $(-1)^{i+\text{area1}(p)}$. Then $G_{ij} = (-1)^i \left[\begin{smallmatrix} i+j-2 \\ 2j-2 \end{smallmatrix}\right]_{-1}$. For each path family we have to choose for the starting points α indices from $1, \ldots, \alpha + b$, say $k_1 < k_2 < \cdots < k_\alpha$. By Equation (25) the enumeration of lattice path families with these starting points equals

$$\det_{1 \le i,j \le \alpha} (\mathcal{P}(A_{k_i} \to E_j)) = \det_{1 \le i,j \le \alpha} (G_{k_i,j}).$$

Therefore, the weighted count of all path families starting at α points among the A_i's, $1 \le i \le \alpha + b$ and ending at the points E_j, $1 \le j \le \alpha$ equals

$$\sum_{1 \le k_1 < \cdots < k_\alpha \le \alpha + b} \det_{1 \le i,j \le \alpha} (G_{k_i,j}).$$

Step 3: The minor-summation formula

Our ingredient is a theorem by Okada [**23**, Theorem 3] (cf. also [**31**, Theorem 3.1]) which we use to express the sum in form of a Pfaffian. Recall that the Pfaffian of a skew–symmetric $2n \times 2n$–matrix M is defined as

$$\operatorname{Pf} M = \sum_m \operatorname{sgn} m \prod_{\substack{\{i,j\}\in m \\ i<j}} M_{ij},$$

where the sum runs over all $m = \{\{m_1, m_2\}, \{m_3, m_4\}, \ldots, \{m_{2n-1}, m_{2n}\}\}$ with the conditions $\{m_1, \ldots, m_{2n}\} = \{1, \ldots, 2n\}$, $m_{2k-1} < m_{2k}$ and $m_1 < m_3 < \cdots < m_{2n-1}$. The term $\operatorname{sgn} m$ is the sign of the permutation $m_1 m_2 m_3 \ldots m_{2n}$.

Specifically, $(\operatorname{Pf} M)^2 = \det M$.

Our way of stating Okada's theorem is taken from [**24**, Corollary 3.2].

LEMMA 2.11. *Suppose that $n \le p$ and n is even. Let $T = (t_{ik})$ be a $p \times n$ matrix and $A = (a_{kl})$ be a $p \times p$ skew-symmetric matrix. Then we have*

$$\sum_{1\le k_1<\cdots<k_n\le p} \operatorname{Pf}\left(A^{k_1,\ldots,k_n}_{k_1,\ldots,k_n}\right) \det(T_{k_1,\ldots,k_n}) = \operatorname{Pf}({}^tTAT),$$

where ${}^tT$ denotes the transpose of the matrix T, $T_{k_1,\ldots,k_n}$ is the matrix composed of the rows of T with indices $k_1, \ldots, k_n$ and $A^{k_1,\ldots,k_n}_{k_1,\ldots,k_n}$ is the matrix composed of the rows and columns of A with indices $k_1, \ldots, k_n$.

The specialization $a_{ij} = \operatorname{sgn}(j - i)$ together with the fact that $\operatorname{Pf} A = 1$ for this matrix gives the following:

LEMMA 2.12. *Suppose that $n \le p$ and n is even. Let $T = (t_{ik})$ be a $p \times n$ matrix and $A = (a_{kl})$ be a $p \times p$ skew-symmetric matrix with $a_{kl} = \operatorname{sgn}(l - k)$. Then we have*

$$\sum_{1\le k_1<\cdots<k_n\le p} \det(T_{k_1,\ldots,k_n}) = \operatorname{Pf}({}^tTAT),$$

where ${}^tT$ denotes the transpose of the matrix T, $T_{k_1,\ldots,k_n}$ is the matrix composed of the rows of T with indices $k_1, \ldots, k_n$ and $\operatorname{Pf}(M)$ denotes the Pfaffian of a skew–symmetric matrix M.

The sum obtained in Step 2 can be evaluated using Lemma 2.12 with $n = \alpha$, $p = \alpha + b$ and $T = G$. Here we use the assumption that α is even.

We get the following expression for our (-1)–enumeration:

$$\operatorname{Pf}_{1\le i,j\le\alpha}\left(\sum_{r=1}^{\alpha+b}\sum_{l=1}^{\alpha+b} G_{li}G_{rj}\operatorname{sgn}(r-l)\right),$$

where

$$G_{ij} = (-1)^i \begin{bmatrix} i+j-2 \\ 2j-2 \end{bmatrix}_{-1} = (-1)^i \binom{\lfloor \frac{i+j-2}{2} \rfloor}{j-1}.$$

We set

$$M_{ij}(\alpha, b) := \sum_{r=1}^{\alpha+b}\sum_{l=1}^{\alpha+b} G_{li}G_{rj}\operatorname{sgn}(r-l) \quad \text{for } 1 \le i, j \le \alpha. \tag{30}$$

With $M = (M_{ij})_{i,j=1}^{\alpha}$, the (-1)–enumeration is equal to $\operatorname{Pf} M$ by the lemma above. Since M is skew-symmetric, we have $\operatorname{Pf} M = \pm\sqrt{\det M}$. So our object is to prove the following lemma.

LEMMA 2.13. *With M defined as in* (30)*, we have*

$$\det M = \begin{cases} \left(\dfrac{\prod_{k=1}^{\alpha/2}(b+2k)_{\alpha-1}}{\prod_{k=1}^{\alpha/2}(2k)_{\alpha-1}}\right)^2 & \text{for } b \text{ even} \\ 0 & \text{else.} \end{cases}$$

The proof of the lemma is given in Section 4.

For α even and b even, the entries of the Pfaffian are polynomials in b, so the result of the enumeration is continuous in b. Therefore, in order to determine the sign of $\operatorname{Pf} M$, it suffices to determine the sign of the enumeration for $b = 0$. Trivially, this is 1. This is clearly the sign chosen in the statement of Theorem 2.2.

4. The proof of Lemma 2.13

First, we consider the case that b is odd. We have $G_{l1} = (-1)^l$, and since the sum $\sum_{l=1}^{\alpha+b}(-1)^l \operatorname{sgn}(r-l) = 0$ for all $1 \le r \le \alpha+b$ we have $M_{1j} = 0$ for all j. Therefore $\det M = 0$.

Now we assume that b is even. We claim that $M_{2i,2j} = 0$. To see this, consider the inner sum in the definition of M_{ij}. Most summands cancel with a neighbouring summand since $(-1)^l$ changes sign and $\binom{\lfloor \frac{l+i-2}{2} \rfloor}{i-1}$ has the same value. The remaining terms are those where l is close to r or close to the summation limits:

$$\begin{aligned} M_{2i,2j}(\alpha,b) = &\sum_{k=1}^{\frac{\alpha+b}{2}-1} (-1)\binom{i-1+k}{2i-1}\binom{j-1+k}{2j-1} \\ &+ \sum_{k=1}^{\frac{\alpha+b}{2}-1} (-1)(-1)\binom{i-1+k}{2i-1}\binom{j-1+k}{2j-1} \\ &+ \sum_{r=1}^{\alpha+b-1} (-1)^r(-1)\binom{i-1+\frac{\alpha+b}{2}}{2i-1}\binom{j-1+\lfloor \frac{r}{2} \rfloor}{2j-1}. \end{aligned} \tag{31}$$

The first sum on the right-hand side corresponds to $r = 2k+1, l = 2k$, the second sum corresponds to $r = 2k, l = 2k+1$, and the third to $l = \alpha+b$. The first two sums clearly cancel each other. The last sum vanishes because the $(r=1)$–term is zero and the other summands cancel pairwise.

Hence, if we reorder M such that even–numbered rows and columns come before odd–numbered ones, we get a block matrix of the form

$$\begin{pmatrix} 0 & \widetilde{M} \\ -\widetilde{M} & N \end{pmatrix},$$

where $\widetilde{M}$ is an $\frac{\alpha}{2} \times \frac{\alpha}{2}$–matrix with $\widetilde{M}_{ij} = M_{2i,2j-1}$. Therefore, we have $\det M = (\det \widetilde{M})^2$. Using the argument described after (31), we can get rid of one of the sums in $\widetilde{M}_{ij}$ and

write it as

$$\widetilde{M}_{ij} = M_{2i,2j-1} = \sum_{k=1}^{\frac{\alpha+b}{2}} \binom{k+j-2}{2j-2}\binom{k+i-2}{2i-2} \quad \text{for } 1 \le i,j \le \frac{\alpha}{2}. \tag{32}$$

The (i,j)–entry of the matrix $\widetilde{M}$ is clearly a polynomial in b of degree $2i+2j-3$. It follows that the determinant of $\widetilde{M}$ is a polynomial in b of degree at most $\frac{\alpha}{2}(\alpha-1)$. We will find a closed form for this polynomial in three steps.

Step 1: $\prod_{j=1}^{\frac{\alpha}{2}} \left(\frac{\alpha+b}{2} - j + 1\right)_{2j-1}$ divides $\det \widetilde{M}$.

We apply Zeilberger's algorithm [**35, 36**] to the sum (32) representing $\widetilde{M}_{ij}$ and get

$$(j+i-1)\widetilde{M}_{ij} + 2(2j+2i-1)\widetilde{M}_{i+1,j} = \frac{(\alpha+b)\left(\frac{\alpha+b}{2}-i+1\right)_{2i-1}\left(\frac{\alpha+b}{2}-j+1\right)_{2j-1}}{(2i)!(2j-2)!}. \tag{33}$$

Therefore:
If $(\frac{\alpha+b}{2} - j + 1)_{2j-1}$ divides $\widetilde{M}_{ij}$ then $(\frac{\alpha+b}{2} - j + 1)_{2j-1}$ divides $\widetilde{M}_{i+1,j}$.

Since $\widetilde{M}_{1j} = \frac{\left(\frac{\alpha+b}{2}-j+1\right)_{2j-1}}{(2j-1)!}$, the jth column of $\widetilde{M}$ is divisible by $\left(\frac{\alpha+b}{2} - j + 1\right)_{2j-1}$ for $j = 1, \dots \frac{\alpha}{2}$. It follows immediately that

$$\prod_{j=1}^{\alpha/2} (\tfrac{\alpha+b}{2} - j + 1)_{2j-1} \text{ divides } \det \widetilde{M}.$$

Step 2: $\prod_{t=1}^{\frac{\alpha}{2}-1} \left(\frac{\alpha+b}{2} - t + \frac{1}{2}\right)_{2t}$ divides $\det \widetilde{M}$.

We prove this by showing that

$$(\tfrac{\alpha+b}{2} - t + \tfrac{1}{2})_{2t} \text{ divides } \widetilde{M}_{t+1,j} + \sum_{s=1}^{t} (-1)^{s-1} \frac{\binom{2s-1}{s}}{(2s-1)2^{4s-1}} \widetilde{M}_{t+1-s,j}$$

for all j. The sum equals

$$M_{2t+2,2j-1} + \sum_{k=1}^{(\alpha+b)/2} \binom{k+j-2}{2j-2} \sum_{s=1}^{t} (-1)^{s-1} \frac{\binom{2s-1}{s}}{(2s-1)2^{4s-1}} \binom{k+t-s-1}{2t-2s}. \tag{34}$$

We reverse the order of summation of the innermost sum and convert it to hypergeometric form, i.e., we want to write it in the form

$${}_rF_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; z\right] = \sum_{n \ge 0} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{(b_1)_n (b_2)_n \dots (b_s)_n} \frac{z^n}{n!}$$

This generates an additional summand that has to be subtracted. To be precise, we get for the inner sum

$$\sum_{s=1}^{t} \binom{2s-1}{s} \frac{(-1)^{s-1}}{(2s-1)2^{4s-1}} \binom{t-s+k-1}{2t-2s} = \\ = \frac{(-1)^t (t)_t}{2^{4t-1}(1-2t)(1)_t} {}_3F_2\left[\begin{matrix} k, 1-k, -t \\ \frac{3}{2}-t, \frac{1}{2} \end{matrix}; 1\right] - \binom{k+t-1}{2t}. \tag{35}$$

The last term cancels exactly with the summand generated by $M_{2t+2,2j-1}$. Now we can apply the Pfaff–Saalschütz summation formula ([**28**], (2.3.1.3); Appendix (III.2)),

$$ {}_3F_2\left[\begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix}; 1\right] = \frac{(-a+c)_n\,(-b+c)_n}{(c)_n\,(-a-b+c)_n}, \tag{36} $$

where n is a nonnegative integer.

Expression (34) becomes a single sum:

$$ \sum_{k=1}^{(\alpha+b)/2} \binom{k+j-2}{2j-2} \frac{(k-1/2)(k-t+1/2)_{2t-1}}{(2t)!}. $$

The remaining sum can be evaluated by the Gosper algorithm [**10**]. It simplifies to

$$ \binom{\frac{\alpha+b}{2}+j-2}{2j-2} \frac{\left(\frac{\alpha+b}{2}-t+\frac{1}{2}\right)_{2t}\left(\frac{\alpha+b}{2}+j-1\right)}{(2t)!(2t+2j-1)}. $$

This is clearly divisible by $\left(\frac{\alpha+b}{2}-t+\frac{1}{2}\right)_{2t}$ viewed as polynomial in b.

Step 3: The degree and the leading term of the determinant.

In the previous two steps we have found a polynomial in b of degree $\sum_{j=1}^{\frac{\alpha}{2}}(2j-1)+\sum_{t=1}^{\frac{\alpha}{2}-1}2t=\alpha^2/2-\alpha/2$ which divides the determinant. The latter number is exactly the maximal possible degree of the determinant. Therefore, we know the determinant up to a factor which is independent of b. The factors we have found can be written as $\prod_{k=1}^{\frac{\alpha}{2}}(b+2k)_{\alpha-1}$. It is clear that in the original problem there is only one plane partition for $b=0$ (the empty plane partition). So, $\det \widetilde{M}(\alpha,0)=\pm 1$. This proves that $\det \widetilde{M}=\pm\prod_{k=1}^{\alpha/2}\frac{(b+2k)_{\alpha-1}}{(2k)_{\alpha-1}}$, as was claimed.

5. Symmetric transpose–complementary plane partitions, II

In this section, we treat the case $a=2\alpha$ and α odd. We can convert the plane partitions to families of nonintersecting lattice paths as described in Section 3. We have to enumerate path families starting at α points among $A_i=(0,i-1)$, $i=1,\dots,\alpha+b$ and ending at the points $E_j=(2j-2,j-1)$, $j=1,\dots,\alpha$, where the weight of a path p from A_i to E_j is $(-1)^{i+\mathrm{area1}(p)}$. (As in Section 3, area1 has the same parity as area2, so we can use area1 with Lemma 2.8.) We also know from Section 3 how to proceed from here in the case of an even number of fixed end points. So we just add a dummy path, i.e., a point $A_{\alpha+b+1}=E_{\alpha+1}=(2\alpha,\alpha)$ which is not connected to the other points. Clearly $\mathcal{P}(A_{\alpha+b+1}\to E_{\alpha+1})=1$ and $\mathcal{P}(A_i\to E_{\alpha+1})=\mathcal{P}(A_{\alpha+b+1}\to E_j)=0$ for $i\neq\alpha+b+1$ and $j\neq\alpha+1$.

As before, because of Lemmas 2.8 and 2.12, the enumeration of nonintersecting lattice paths starting at $\alpha+1$ points among the points A_i, $1\le i\le\alpha+b+1$, and ending at the points E_j, $1\le j\le\alpha+1$, equals $\mathrm{Pf}_{1\le i,j\le\alpha+1}\left(\sum_{r=1}^{\alpha+b+1}\sum_{l=1}^{\alpha+b+1}G_{li}G_{rj}\,\mathrm{sgn}(r-l)\right)$, where

$$ G_{ij}=\mathcal{P}(A_i\to E_j)=\begin{cases} 1 & \text{for } i=\alpha+b+1 \text{ and } j=\alpha+1 \\ (-1)^i\left[\begin{smallmatrix} i+j-2 \\ 2j-2 \end{smallmatrix}\right]_{-1} & \text{for } i\neq\alpha+b+1 \text{ and } j\neq\alpha+1 \\ 0 & \text{else.} \end{cases} $$

So we have to evaluate

$$\sqrt{\det_{1\le i,j\le \alpha+1}(M_{ij})},$$

where $M_{ij} = \sum_{r=1}^{\alpha+b+1}\sum_{l=1}^{\alpha+b+1} G_{li}G_{rj}\operatorname{sgn}(r-l)$. The right sign of the square root is easily found by the fact that the enumeration is 1 for $b=0$ and the result must be a continuous function in b.

Case 1: α, b odd It is a routine calculation to verify that $M_{2i-1,2j-1} = 0$ (cf. the computation in Equation (31)). After reordering the rows and columns of M such that the even–numbered rows and columns come before the odd–numbered ones we have a skew–symmetric block matrix. Therefore, the Pfaffian of M equals the determinant of one of the blocks up to sign. To be precise, we have to evaluate

$$\det_{1\le i,j\le (\alpha+1)/2}(M_{2i,2j-1}).$$

It is readily seen that $M_{\alpha+1,2j-1} = 0$ for all j.

Therefore, the final result is 0 in this case.

Case 2: α odd, b even

Similar to earlier calculations we see that $M_{2i-1,2j-1} = 0$. After reordering of rows and columns of the matrix according to parity to put it in block form, we obtain $\det_{1\le i,j\le(\alpha+1)/2}(M_{2i,2j-1})$ for our (-1)–enumeration, up to sign. It is easily seen that

$$M_{2i,1} = \begin{cases} 1 & \text{for } 2i = \alpha+1,\\ 0 & \text{else.}\end{cases}$$

Expansion of $\det_{1\le i,j\le(\alpha+1)/2}(M_{2i,2j-1})$ with respect to the first column gives

$$\det_{1\le i,j\le(\alpha-1)/2}\left(\widetilde{M}_{ij}\right)$$

where $\widetilde{M}_{ij} = M_{2i,2j+1}$.

It is a routine calculation to verify that

$$\widetilde{M}_{ij} = -\sum_{k=1}^{(\alpha+b-1)/2}\binom{k+i-1}{2i-1}\binom{k+j-1}{2j-1}.$$

We show analogously to Lemma 2.13 that

$$\det\widetilde{M} = \pm\frac{\prod_{k=1}^{\frac{\alpha-1}{2}}(b+2k)_\alpha}{\prod_{k=1}^{\frac{\alpha-1}{2}}(2k)_\alpha}.$$

The proofs of the following steps are analogous to the corresponding steps in Section 4.

Step 1': $\prod_{j=1}^{(\alpha-1)/2}((\alpha+b+1)/2-j)_{2j}$ divides $\det\widetilde{M}$.

Step 2': $\prod_{t=0}^{(\alpha-1)/2-1}((\alpha+b)/2-t)_{2t+1}$ divides $\det\widetilde{M}$.

The appropriate linear combination here is

$$\widetilde{M}_{t+1,j} + \sum_{s=1}^{t}(-1)^{s-1}\binom{2s-1}{s}\frac{\widetilde{M}_{t+1-s,j}}{(2s-1)2^{4s-1}}.$$

Step 3': The degree and the leading coefficient

It is easy to check that the maximal degree of the determinant equals the number of

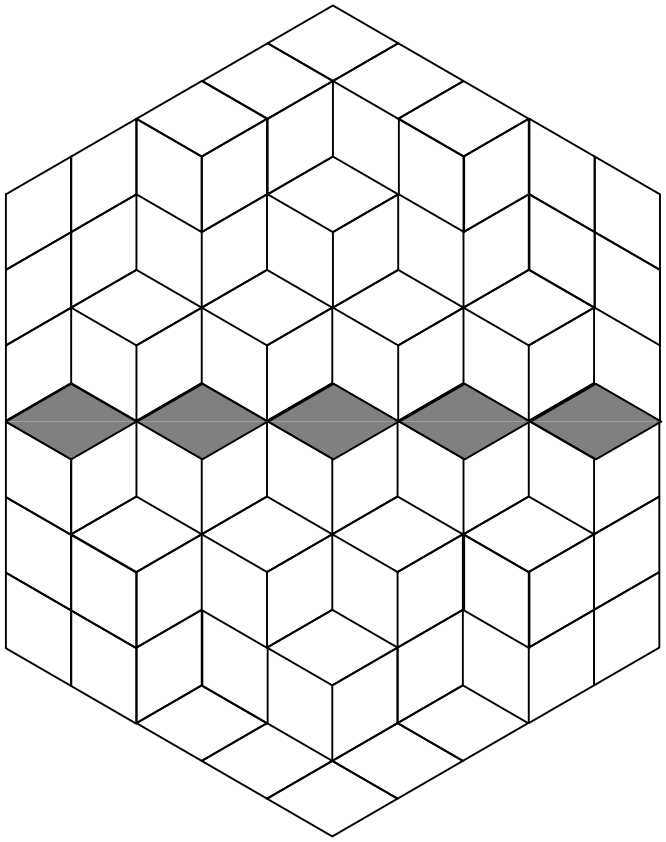


FIGURE 2.9. A symmetric transpose–complementary plane partition with $a = 2\alpha + 1$.

factors already found. As noted before, the enumeration equals 1 for $b = 0$. Therefore, the constant term of the polynomial is ± 1 for the determinant and 1 in the final result.

It remains to show that $\prod_{j=1}^{(\alpha-1)/2}((\alpha+b+1)/2-j)_{2j}\prod_{t=0}^{(\alpha-1)/2-1}((\alpha+b)/2-t)_{2t+1}$ is a constant multiple of $\prod_{k=1}^{(\alpha-1)/2}(b+2k)_\alpha$, which is readily verified. Thus Theorem 2.2 is proved.

6. Symmetric transpose–complementary plane partitions, III

In this section we treat the case $a = 2\alpha+1$ (see Figure 2.9). We can still express the (-1)–enumeration as a Pfaffian whose entries are polynomials in b but the determinant does not factor completely. The first two steps of Section 3 are completely analogous:

Step 1: From plane partitions to rhombus tilings

Again, we start by converting the plane partitions to rhombus tilings by projecting them to the plane. We obtain rhombus tilings of a hexagon with sidelengths $2\alpha + 1$, $2\alpha + 1$, $2b$, $2\alpha + 1$, $2\alpha + 1$, $2b$ which have a vertical and a horizontal symmetry axis. These symmetry conditions imply that the corresponding rhombus tiling is determined by its upper right quarter.

Step 2: From rhombus tilings to families of nonintersecting lattice paths.

As before, we can convert each tiling to a family of nonintersecting lattice paths (see Figure 2.10). Again, the starting points may vary since the horizontal rhombi on the vertical axis can be in different places.

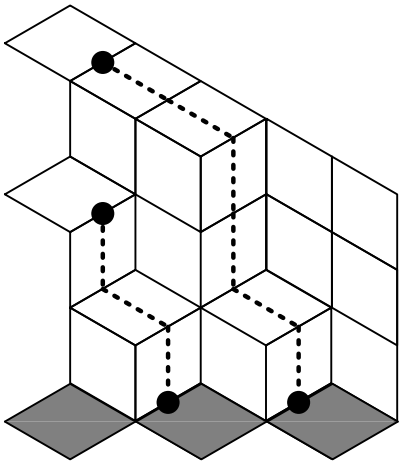


FIGURE 2.10. The corresponding paths.

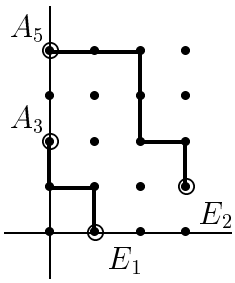


FIGURE 2.11. The path family made orthogonal.

The starting points and end points are only slightly different from the ones in the case $a = 2\alpha$:

With a suitable coordinate system, the end points are

$$E_j = (2j-1, j-1),\ j = 1, \dots, \alpha, \tag{37}$$

while the starting points are α points among the points

$$A_i = (0, i-1),\ i = 1, \dots, \alpha + b. \tag{38}$$

The weight of a path p starting at A_i is $(-1)^{i+\mathrm{area1}(p)}$ (cf. Step 2 of Section 3). Since area1(p) can also be thought of as the area left of the path (and therefore as weight of the vertical steps), we can use Lemma 2.8 again.

To make this precise, let $G_{ij} = \mathcal{P}(A_i \to E_j)$ be the weighted enumeration of all paths running from A_i to E_j, $1 \le i \le \alpha + b$ and $1 \le j \le \alpha$. Then $G_{ij} = (-1)^i \begin{bmatrix} i+j-1 \\ 2j-1 \end{bmatrix}_{-1}$. For each path family we have to choose for the starting points α indices from $1, \dots, \alpha+b$, say $k_1 < k_2 < \cdots < k_\alpha$. By Equation (25) the enumeration of lattice path families with these starting points equals

$$\det_{1 \le i,j \le \alpha} (\mathcal{P}(A_{k_i} \to E_j)) = \det_{1 \le i,j \le \alpha} (G_{k_i,j}).$$

Therefore, the weighted count of all path families starting at α points among the A_i's, $1 \le i \le \alpha + b$ and ending at the points E_j, $1 \le j \le \alpha$ equals

$$\sum_{1 \le k_1 < \cdots < k_\alpha \le \alpha+b} \det_{1 \le i,j \le \alpha} (G_{k_i,j}). \tag{39}$$

Step 3: Application of the minor–summation formula
Case 1: α even, b odd.
If α is even, the minor–summation formula (see Lemma 2.12) is directly applicable. Therefore, the weighted count equals $\mathrm{Pf}_{1 \le i,j \le \alpha}(M_{ij})$, where

$$M_{ij} = \sum_{r=1}^{\alpha+b} \sum_{l=1}^{\alpha+b} (-1)^l \begin{bmatrix} l+i-1 \\ 2i-1 \end{bmatrix}_{-1} (-1)^r \begin{bmatrix} r+j-1 \\ 2j-1 \end{bmatrix}_{-1} \operatorname{sgn}(r-l). \tag{40}$$

First, we express M_{ij} as a single sum. We can add the summands corresponding to $r = 0$ and $l = 0$ because they are zero. In M_{ij} the term $\begin{bmatrix} l+i-1 \\ 2i-1 \end{bmatrix}_{-1}$ is zero if $l + i$ is odd, so the sum really runs only over l with the same parity as i. So, let i_0 be 0 for i even and 1 for i odd. It is now clear that we can take the sum over l of the form $l = i_0 + 2k$ where k runs from 0 to $(\alpha + b - 1)/2$. This gives $\begin{bmatrix} l+i-1 \\ 2i-1 \end{bmatrix}_{-1} = \binom{k+\lfloor (i-1)/2 \rfloor}{i-1}$. Analogously, we choose $j_0 \in \{0, 1\}$ with the same parity as j and write $r = j_0 + 2u$. We get

$$\begin{aligned}
M_{ij} &= \sum_{k=0}^{\frac{\alpha+b-1}{2}} \sum_{u=0}^{\frac{\alpha+b-1}{2}} (-1)^{i+j} \binom{k + \left\lfloor \frac{i-1}{2} \right\rfloor}{i-1} \binom{u + \left\lfloor \frac{j-1}{2} \right\rfloor}{j-1} \operatorname{sgn}(2u + j_0 - 2k - i_0) \\
&= \sum_{k=0}^{\frac{\alpha+b-1}{2}} (-1)^{i+j} \binom{k + \left\lfloor \frac{i-1}{2} \right\rfloor}{i-1} \left(\sum_{u=0}^{k-1} -\binom{u + \left\lfloor \frac{j-1}{2} \right\rfloor}{j-1} \right. \\
&\qquad \left. + \binom{k + \left\lfloor \frac{j-1}{2} \right\rfloor}{j-1} \operatorname{sgn}(j_0 - i_0) + \sum_{u=k+1}^{\frac{\alpha+b-1}{2}} \binom{u + \left\lfloor \frac{j-1}{2} \right\rfloor}{j-1} \right) \\
&= \sum_{k=0}^{\frac{\alpha+b-1}{2}} (-1)^{i+j} \binom{k + \left\lfloor \frac{i-1}{2} \right\rfloor}{i-1} \left(-\binom{k + \left\lfloor \frac{j-1}{2} \right\rfloor}{j} + \binom{k + \left\lfloor \frac{j-1}{2} \right\rfloor}{j-1} \operatorname{sgn}(j_0 - i_0) \right. \\
&\qquad \left. + \binom{\frac{\alpha+b+1}{2} + \left\lfloor \frac{j-1}{2} \right\rfloor}{j} - \binom{k+1 + \left\lfloor \frac{j-1}{2} \right\rfloor}{j} \right)
\end{aligned}$$

The summand is clearly a polynomial in b and k, so after summation up to $(\alpha+b-1)/2$ we get a polynomial in b. So the Pfaffian of M is again a polynomial in b. We will find several factors but this time the determinant is not fully factorizable.

If both indices are even, M_{ij} can be written in closed form (the sum with one occurrence of k is easy, the remaining terms can be done with Zeilberger's algorithm). Therefore:

$$M_{2i,2j} = \binom{(\alpha+b-1)/2 + j}{2j} \binom{(\alpha+b-1)/2 + i}{2i} \frac{j-i}{j+i}. \tag{41}$$

We show the following result:

$$\det(M(\alpha,b)) = \Big((b-1)\prod_{k=1}^{\alpha/2}((b-1)/2+k)_{\alpha/2+1} \times (\text{polynomial of degree } (\alpha/2)^2-1)\Big)^2.$$

We will do so by taking the factors $((\alpha+b-1)/2-i+1)_{2i}$ out of row $2i$ and the factors $((\alpha+b-1)/2-j+1)_{2j}$ out of column $2j$.

We have already seen the the entry can be written as

$$M_{2i,j} = \sum_{k=0}^{(\alpha+b-1)/2} (k-i+1)_{2i-1}p(k),$$

where p is a polynomial.

The term $(k-i+1)_{2i-1}$ is zero for $-i+1 \le k \le i-1$. This holds for all occurring k if $-i \le (\alpha+b-1)/2 \le i-1$. Therefore $((\alpha+b-1)/2-i+1)_{2i}$ divides $M_{2i,2j}$.

By the skewsymmetry of M the analogous result holds for the columns. Equation (41) ensures that there are enough factors at crossings of evenindexed rows and columns.

We have now the factors

$$\left(\prod_{i=1}^{\alpha/2}((\alpha+b-1)/2-i+1)_{2i}\right)^2$$

which are easily seen to be the same as

$$\left(\prod_{k=1}^{\alpha/2}((b-1)/2+k)_{\alpha/2+1}\right)^2.$$

It remains to find the factor $(b-1)^2$.

Let $b=1$. Equation (39) reduces to the sum of minors obtained by deleting one row of the $(\alpha+1)\times\alpha$–matrix G with $G_{ij} = (-1)^i \begin{bmatrix} i+j-1 \\ 2j-1 \end{bmatrix}_{-1}$. This sum equals the determinant of the $(\alpha+1)\times(\alpha+1)$–matrix $\widetilde{G}$ obtained from G by appending the column vector $\begin{pmatrix} 1 \\ -1 \\ \vdots \\ -1 \\ 1 \end{pmatrix}$. Since $\widetilde{G}_{2i,2j-1} = 0$ (by (26)), it is enough to show that $\det\left(\widetilde{G}_{2i,2j}\right) = 0$. This follows from the fact that the first column contains only 1's and the last column only -1's.

Since the entries of the skew-symmetric matrix are polynomials in b the factor $(b-1)$ must occur twice in the determinant. The degree of the remaining polynomial can be found by comparing the degree of the product with the degrees of the entries.

Case 2: α even, b even.

We denote the (i,j)–entry in this case $\widetilde{M_{ij}}$ because it will be a different polynomial in b.

Starting from Equation (40) we get by a calculation analogous to the case b odd:

$$\widetilde{M}_{ij} = \sum_{k=0}^{(\alpha+b)/2-1} (-1)^{i+j} \binom{k+\lfloor i/2\rfloor}{i-1} \left(-\binom{k+\lfloor j/2\rfloor}{j} - \binom{k+\lfloor j/2\rfloor}{j-1} \operatorname{sgn}(j_0 - i_0) + \binom{(\alpha+b)/2+\lfloor j/2\rfloor}{j} - \binom{k+1+\lfloor j/2\rfloor}{j} \right)$$

It can now easily be checked that the substitution $b \to -b-2\alpha-1$ changes $\widetilde{M}_{ij}(\alpha, b))$ to $(-1)^{i+j} M_{ij}(\alpha, b)$. The statement in Theorem 2.3 is just the analogous substitution.

This settles the case α even.

Case 3: α odd, b even.

As before, we add a dummy path. That is, we add a point $A_{\alpha+b+1} = E_{\alpha+1}$ which is disconnected from all the other points. If we write $\widetilde{G}_{ij}$ for the enumeration of paths from A_i to E_j we get

$$\widetilde{G}_{ij} = \begin{cases} G_{ij} & \text{for } i \le \alpha+b, j \le \alpha, \\ 1 & \text{for } i = \alpha+b+1, j = \alpha+1, \\ 0 & \text{else.} \end{cases}$$

Now, we can apply the minor–summation formula (see Lemma 2.12). Therefore, the weighted count equals $\operatorname{Pf}_{1\le i,j\le \alpha+1}(M'_{ij})$, where

$$M'_{ij} = \sum_{r=1}^{\alpha+b+1} \sum_{l=1}^{\alpha+b+1} \widetilde{G}_{li} \widetilde{G}_{rj} \operatorname{sgn}(r-l).$$

We have

$$M'_{ij} = \begin{cases} M_{ij} & \text{for } i \ne \alpha+1 \text{ and } j \ne \alpha+1, \\ -(-1)^j \binom{(\alpha+b+1)/2+\lfloor (j-1)/2\rfloor}{j} & \text{for } i = \alpha+1, \\ (-1)^i \binom{(\alpha+b+1)/2+\lfloor (i-1)/2\rfloor}{i} & \text{for } j = \alpha+1. \end{cases}$$

From previous results it is now easily seen that $((\alpha+b-1)/2-i+2)_{2i-1}$ divides $M'_{2i-1,j}$ for $i = 1, \ldots, (\alpha+1)/2$, similarly for the columns. We have to check that we can take enough factors out at the crossings of rows and columns. Using Zeilberger's algorithm again, we get

$$M'_{2i-1,2j-1} = \binom{(\alpha+b-1)/2+j}{2j-1} \binom{(\alpha+b-1)/2+i}{2i-1} \frac{j-i}{i+j-1}.$$

Therefore, the product $\left(\prod_{i=1}^{(\alpha+1)/2} ((\alpha+b-1)/2-i+2)_{2i-1}\right)^2$ divides the determinant.

This is easily seen to be the same as $\left(\prod_{k=1}^{(\alpha+1)/2} (b/2+k)_{(\alpha+1)/2}\right)^2$. Therefore, the enumeration has the form

$$\prod_{k=1}^{(\alpha+1)/2} (b/2+k)_{(\alpha+1)/2} \cdot (\text{polynomial of degree } (\alpha^2-1)/4\ ). \tag{42}$$

Case 4: α odd, b odd. Analogously to the previous case, we get

$$M''_{ij} = \begin{cases} \widetilde{M}_{ij} & \text{for } i \neq \alpha+1 \text{ and } j \neq \alpha+1, \\ -(-1)^j \binom{(\alpha+b)/2 + \lfloor j/2 \rfloor}{j} & \text{for } i = \alpha+1, \\ (-1)^i \binom{(\alpha+b)/2 + \lfloor i/2 \rfloor}{i} & \text{for } j = \alpha+1. \end{cases}$$

This is a polynomial in b. If we replace b with $-b-2\alpha-1$ we get

$$\begin{cases} (-1)^{i+j} M_{ij} & \text{for } i \neq \alpha+1 \text{ and } j \neq \alpha+1, \\ -(-1)^j (-1)^j \binom{(b+\alpha-1)/2 + \lceil j/2 \rceil}{j} & \text{for } i = \alpha+1, \\ (-1)^i (-1)^i \binom{(b+\alpha-1)/2 + \lceil i/2 \rceil}{i} & \text{for } j = \alpha+1, \end{cases}$$

which is clearly equal to $(-1)^{i+j} M'_{ij}$.

Replacing b with $-b-2\alpha-1$ in Equation (42) yields the desired result.

This finishes the proof of Theorem 2.3. The remaining polynomials seem to be irreducible in general.

7. Cyclically symmetric transpose–complementary plane partitions

In this section we treat the case of cyclically symmetric transpose–complementary plane partitions. These plane partitions are contained in boxes with sidelengths $2\alpha \times 2\alpha \times 2\alpha$. If we view such a plane partition as a rhombus tiling, it has a horizontal symmetry axis (because of being transpose–complementary). The cyclic symmetry gives two more symmetry axes (see Figure 2.12). The weight is $(-1)^{n(P)}$, where $n(P)$ is the number of half orbits in the plane partition not contained in the plane partition shown in Figure 2.15. The orbits consist either of two cubes or of six cubes. The former ones consist of cubes with coordinates of the form (x, x, x) and it is easy to see that these cubes are part of a cyclically symmetric transpose–complementary plane partition if and only if $1 \leq x \leq \alpha$, so they do not contribute to the weight. The orbits consisting of 6 cubes have 3 cubes in the plane partition and 3 cubes outside. Therefore, it is enough to look at one of the six regions of the rhombus tiling (this corresponds to one of the 8 big cubes of sidelength α partitioning the box containing the plane partition). We choose the upper right region (see Figure 2.13). The weight is now simply $(-1)^n$ where n is the number of cubes in this region.

As before, we can convert the rhombus tilings to families of nonintersecting lattice paths. With a suitable coordinate system the coordinates of the starting points A_i and the end points E_j are

$$A_i = (i, 2i) \qquad i = 1, \ldots, \alpha-1, \tag{43}$$

$$E_j = (2j, j) \qquad j = 1, \ldots, \alpha-1. \tag{44}$$

The weight is again $(-1)^{\text{area1}(p)}$. We want to use Lemma 2.8, so we have to use area2 instead of area1. This gives the sign $(-1)^{j(2j-i)}$ in each entry and a global sign of $(-1)^{\sum_{k=1}^{\alpha-1} k^2}$. According to Lemma 2.8 and the paragraph after it, our (-1)–enumeration

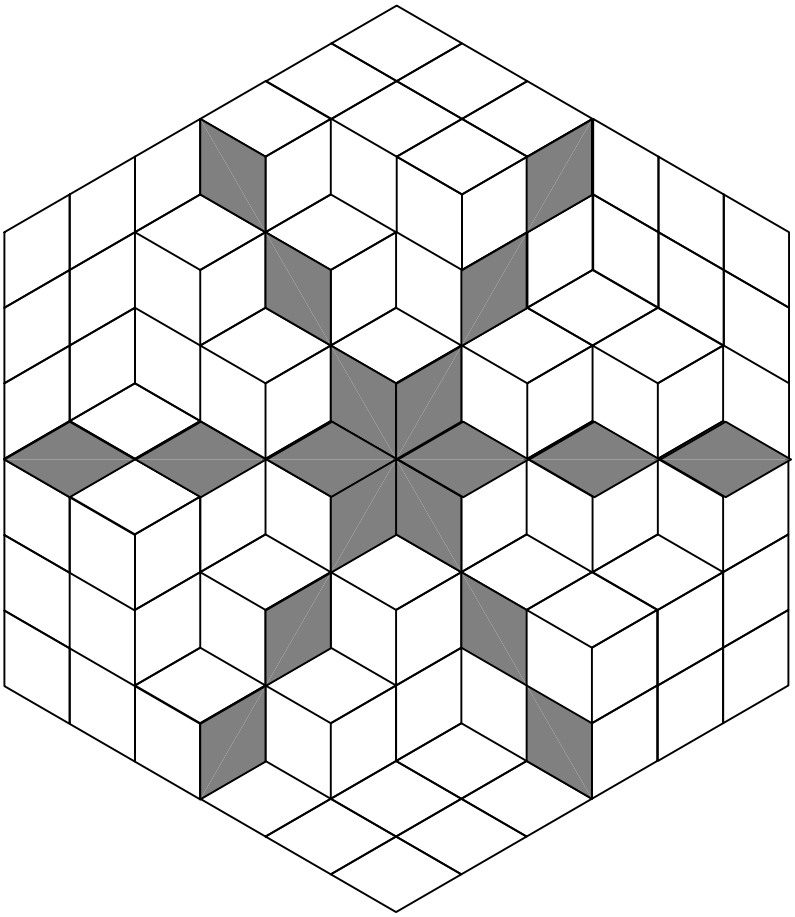


FIGURE 2.12. A cyclically symmetric transpose–complementary plane partition.

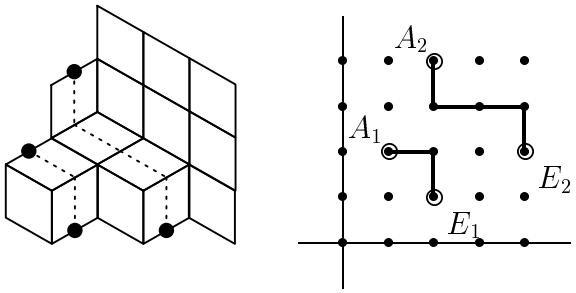


FIGURE 2.13. A path family corresponding to a cyclically symmetric transpose–complementary plane partition.

is given by

$$(-1)^{\sum_{k=1}^{\alpha-1} k^2} \det_{1\le i,j\le \alpha-1} \left((-1)^{j(2j-i)} \begin{bmatrix} i+j \\ 2j-i \end{bmatrix}_{-1} \right).$$

The (-1)–binomial coefficient is 0 for i and j odd (cf. (26)), so the sign $(-1)^{j(2j-i)}$ can be dropped. Now we reorder rows and columns so that odd indices come before even indices. The arising matrix has a zero block in the upper left corner. For even α, this block immediately forces the determinant to be zero. For odd α, we see that the

determinant is the product of two identical determinants times $(-1)^{(\alpha-1)/2}$. The sign cancels exactly with $(-1)^{\sum_{k=1}^{\alpha-1} k^2}$. Explicitly, the (-1)–enumeration reduces to

$$\det_{1\le i,j\le(\alpha-1)/2}\left(\begin{bmatrix}2i+2j-1\\4j-2i+1\end{bmatrix}_{-1}\right)^2 = \det_{1\le i,j\le(\alpha-1)/2}\left(\binom{i+j-1}{2j-i}\right)^2 = \det_{0\le i,j\le(\alpha-3)/2}\left(\binom{i+j+1}{2i-j}\right)^2. \quad (45)$$

The determinant on the right-hand side is the case $\mu = 1, n = (\alpha-1)/2$ of the following identity from [**21**]:

$$\det_{0\le i,j\le n-1}\left(\binom{\mu+i+j}{2i-j}\right) = (-1)^{\chi(n\equiv 3(4))}2^{\binom{n}{2}}\prod_{i=1}^{n-1}\frac{(\mu+i+1)_{\lfloor(i+1)/2\rfloor}(-\mu-3n+i+3/2)_{\lfloor i/2\rfloor}}{(i)_i}. \quad (46)$$

It is a routine computation to verify that the square of the right hand side agrees with the expression in Theorem 2.4.

8. Totally symmetric self–complementary plane partitions

In this section we do the (-1)–enumeration of totally symmetric self–complementary plane partitions. The weight is $(-1)^n$ where n is the number of half orbits in the plane partition which are not in the plane partition shown in Figure 2.15. The corresponding rhombus tilings have six symmetry axes dividing the hexagon in 12 parts (see Figure 2.14). It is enough to consider the tiling of one of them (see Figure 2.16). Clearly, the hexagon must have sidelengths of the form $2\alpha \times 2\alpha \times 2\alpha$. We use again nonintersecting lattice paths. With a suitable coordinate system the starting points are $\alpha - 1$ points among A_i and the end points are E_j, where

$$A_i = (i, i) \quad i = 1, \dots, 2\alpha - 2, \quad (47)$$

$$E_j = (2j, j) \quad j = 1, \dots, \alpha - 1. \quad (48)$$

The appropriate weight of a path from A_i to E_j is $(-1)^{\text{area2}(p)+i(i+1)/2}$ where area2 is the area between the path and the x–axis and the factor $(-1)^{i(i+1)/2}$ accounts for the rhombi on the symmetry axis. The enumeration of paths from A_i to E_j with this weight is thus

$$T_{ij} = \begin{bmatrix} j \\ i-j \end{bmatrix}_{-1} (-1)^{j(2j-i)}(-1)^{i(i+1)/2}.$$

This weight gives the (-1)–enumeration up to a global sign. If we assign the plane partition in Figure 2.15 the weight 1, this global sign equals $(-1)^{(\alpha-1)/2}$ for odd α. Now we distinguish between two cases according to the parity of α.

Case 1: α odd For odd α the number of endpoints – $(\alpha - 1)$ – is even. Therefore, Lemma 2.12 is applicable. The (-1)–enumeration is $(-1)^{(\alpha-1)/2}\operatorname{Pf} M$, where

$$M_{ij} = \sum_{k=1}^{2\alpha-2}\sum_{l=1}^{2\alpha-2} T_{ki}\operatorname{sgn}(l-k)T_{lj} \quad 1 \le i, j \le \alpha - 1.$$

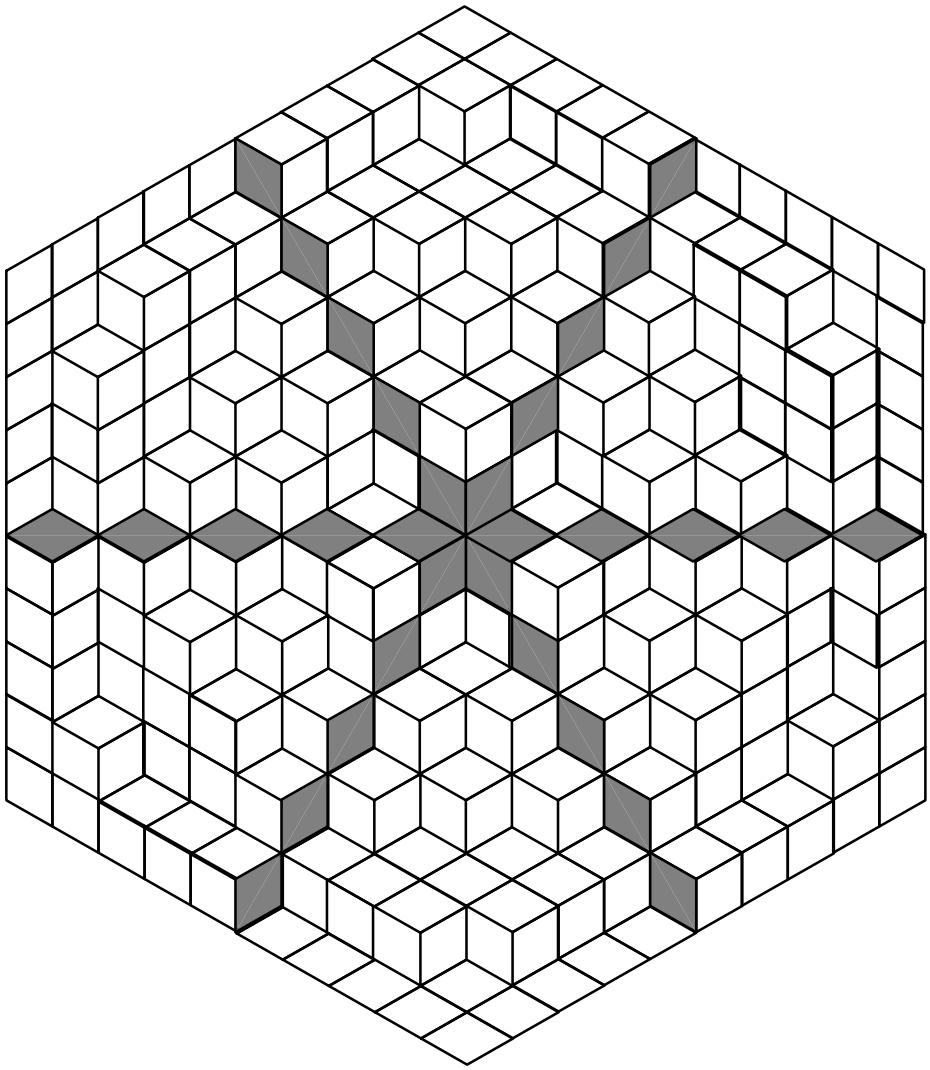


FIGURE 2.14. A totally symmetric self–complementary plane partition.

We claim that $M_{2i-1,2j-1} = 0$. This is proved by splitting the double sums according to the cases $k = 2r, l = 2s$, $k = 2r, l = 2s-1$, $k = 2r-1, l = 2s-1$ and $k = 2r-1, l = 2s$

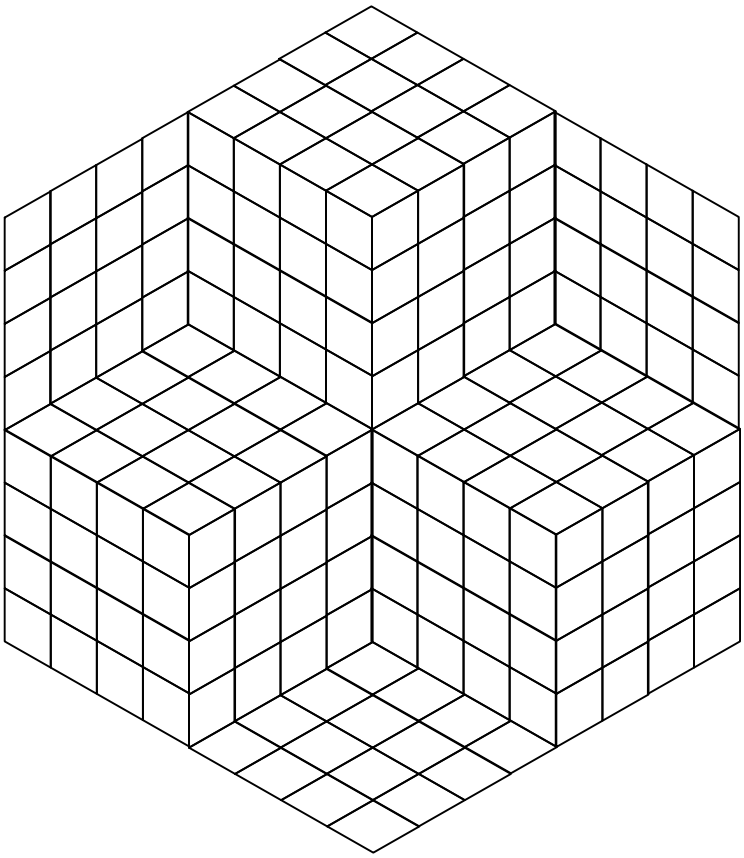


FIGURE 2.15. A totally (cyclically) symmetric self–complementary plane partition with weight 1.

into four double sums. Using Equation (26) we get

$$\begin{aligned}
M_{2i-1,2j-1} &= \sum_{r=1}^{\alpha-1}\sum_{s=1}^{\alpha-1}\binom{i-1}{r-i}\binom{j-1}{s-j}(-1)^{r+s}\operatorname{sgn}(s-r)\\
&\quad+\sum_{r=1}^{\alpha-1}\sum_{s=1}^{\alpha-1}\binom{i-1}{r-i}\binom{j-1}{s-j}(-1)^{r+s-1}\operatorname{sgn}(2s-2r-1)\\
&\quad+\sum_{r=1}^{\alpha-1}\sum_{s=1}^{\alpha-1}\binom{i-1}{r-i}\binom{j-1}{s-j}(-1)^{r+s}\operatorname{sgn}(s-r)\\
&\quad+\sum_{r=1}^{\alpha-1}\sum_{s=1}^{\alpha-1}\binom{i-1}{r-i}\binom{j-1}{s-j}(-1)^{r+s-1}\operatorname{sgn}(2s-2r+1)\\
&=\sum_{r=1}^{\alpha-1}\binom{i-1}{r-i}\binom{j-1}{r-j}(-1)^{r+r-1}\operatorname{sgn}(2r-2r+1)\\
&\quad+\sum_{r=1}^{\alpha-1}\binom{i-1}{r-i}\binom{j-1}{r-j}(-1)^{r+r-1}\operatorname{sgn}(2r-2r-1)\\
&=0
\end{aligned}$$

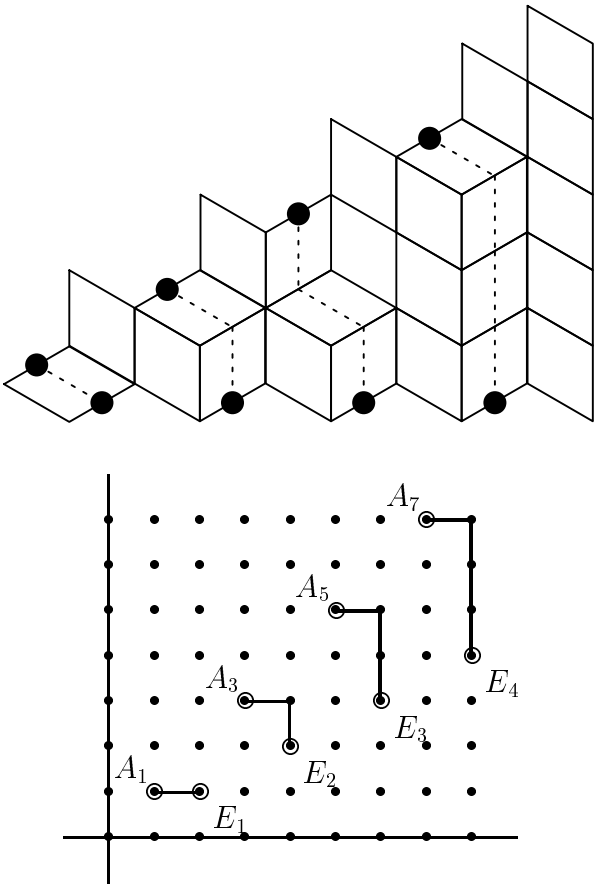


FIGURE 2.16. A path family corresponding to a totally symmetric self–complementary plane partition.

In the last two steps we have used the fact that the first two sums and the last two sums cancel each other except for the terms with $r = s$ in the second and fourth sum. These remaining terms cancel each other completely.

Now we can reorder the rows and columns of M such that the even-numbered ones come before the odd-numbered ones. We call this new matrix M'. We have $\operatorname{Pf} M = (-1)^{(\alpha-1)(\alpha+1)/8} \operatorname{Pf} M'$. Since M' is skew-symmetric, we get a block matrix of the form

$$M' = \begin{pmatrix} * & A \\ -{}^tA & 0 \end{pmatrix},$$

where A is an $(\alpha-1)/2 \times (\alpha-1)/2$–matrix with $A_{ij} = M_{2i,2j-1}$.

It follows from the definition of the Pfaffian that $\operatorname{Pf} M' = (-1)^{(\alpha-1)(\alpha-3)/8} \det A$. We want to evaluate $(-1)^{(\alpha-1)/2+(\alpha-1)(\alpha+1)/8+(\alpha-1)(\alpha-3)/8} \det A = \det A$

Now we simplify A_{ij}.

$$\begin{aligned}
A_{ij} &= \sum_{k,l=1}^{2\alpha-2} \begin{bmatrix} 2i \\ k-2i \end{bmatrix}_{-1} \begin{bmatrix} 2j-1 \\ l-2j+1 \end{bmatrix}_{-1} (-1)^l (-1)^{k(k+1)/2+l(l+1)/2} \operatorname{sgn}(l-k) \\
&\overset{k=2r}{=} \sum_{l=1}^{2\alpha-2} \sum_{r=1}^{\alpha-1} \binom{i}{r-i} \binom{j-1}{\lfloor (l+1)/2 - j \rfloor} (-1)^r (-1)^{l(l-1)/2} \operatorname{sgn}(l-2r) \\
&= \sum_{r=1}^{\alpha-1} \sum_{l=1}^{2r-1} \binom{i}{r-i} \binom{j-1}{\lfloor (l+1)/2 - j \rfloor} (-1)^{r+l(l-1)/2} (-1) \\
&\quad + \sum_{r=1}^{\alpha-1} \sum_{l=2r+1}^{2\alpha-2} \binom{i}{r-i} \binom{j-1}{\lfloor (l+1)/2 - j \rfloor} (-1)^{r+l(l-1)/2} \\
&\overset{(*)}{=} \sum_{r=1}^{\alpha-1} \binom{i}{r-i} \binom{j-1}{r-j} \\
&= \binom{i+j-1}{2j-i-1}
\end{aligned}$$

At the step $(*)$ we have used the fact that two summands for $l = 2s-1$ and $l = 2s$ cancel each other. It remains to evaluate

$$\det_{1 \le i,j \le (\alpha-1)/2} \left(\binom{i+j-1}{2j-i-1} \right) = \det_{0 \le i,j \le (\alpha-3)/2} \left(\binom{i+j+1}{2j-i} \right).$$

This is just Equation (46) with i, j interchanged $n = (\alpha-1)/2$ and $\mu = 1$. Therefore, the weighted count equals

$$(-1)^{\chi((\alpha-1)/2 \equiv 3(4))} 2^{\binom{(\alpha-1)/2}{2}} \prod_{i=1}^{(\alpha-1)/2-1} \frac{(i+2)_{\lfloor (i+1)/2 \rfloor} (-3(\alpha-1)/2 + i + 1/2)_{\lfloor i/2 \rfloor}}{(i)_i}$$

which is easily seen to be the expression claimed in Theorem 2.5.

Case 2: α even Since $\alpha - 1$ is odd now, we need an additional path. Therefore, we use the following starting and end points:

$$A_i = (i, i) \quad i = 0, \dots, 2\alpha - 2, \tag{49}$$

$$E_j = (2j, j) \quad j = 0, \dots, \alpha - 1. \tag{50}$$

Again, the weighted enumeration of the paths from A_i to E_j is

$$T_{ij} = \begin{bmatrix} j \\ i-j \end{bmatrix}_{-1} (-1)^{j(2j-i)} (-1)^{i(i+1)/2}.$$

By Lemma 2.12, we have to evaluate $\operatorname{Pf} M$ where M is the $\alpha \times \alpha$–matrix with entries

$$M_{ij} = \sum_{k=0}^{2\alpha-2} \sum_{l=0}^{2\alpha-2} T_{ki} \operatorname{sgn}(l-k) T_{lj}.$$

We show that $\operatorname{Pf} M = 0$ by showing that $M_{0j} = 0$ for $0 \le j \le \alpha - 1$. We use the fact that

$$T_{k0} = \begin{bmatrix} 0 \\ k \end{bmatrix}_{-1} (-1)^{k(k+1)/2} = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{else.} \end{cases}$$

Therefore,

$$M_{0j} = \sum_{l=0}^{2\alpha-2} \sum_{k=0}^{2\alpha-2} T_{k0} \operatorname{sgn}(l-k) T_{lj} = \sum_{l=1}^{2\alpha-2} T_{lj}.$$

Let $j = 2u$ be even with $u \neq 0$ ($j = 0$ is trivial):

$$\begin{aligned} M_{0j} &= \sum_{l=1}^{2\alpha-2} \begin{bmatrix} j \\ l-j \end{bmatrix}_{-1} (-1)^{lj} (-1)^{l(l+1)/2} \\ &= \sum_{l=1}^{2\alpha-2} \begin{bmatrix} 2u \\ l-2u \end{bmatrix}_{-1} (-1)^{l(l+1)/2} \\ &= \sum_{r=1}^{\alpha-1} \binom{u}{r-u} (-1)^r \\ &= \sum_{s=0}^{u} \binom{u}{s} (-1)^{s+u} \\ &= 0 \end{aligned}$$

Now, let $j = 2u+1$ be odd:

$$\begin{aligned} M_{0j} &= \sum_{l=1}^{2\alpha-2} \begin{bmatrix} j \\ l-j \end{bmatrix}_{-1} (-1)^{lj} (-1)^{l(l+1)/2} \\ &= \sum_{l=1}^{2\alpha-2} \begin{bmatrix} 2u+1 \\ l-2u-1 \end{bmatrix}_{-1} (-1)^{l(l-1)/2} \\ &= \sum_{r=1}^{\alpha-1} \binom{u}{r-1-u} (-1)^r + \sum_{r=0}^{\alpha-2} \binom{u}{r-u} (-1)^r \\ &= \sum_{s=0}^{u} \binom{u}{s} (-1)^{s+u+1} + \sum_{s=0}^{u} \binom{u}{s} (-1)^{s+u} \\ &= 0 \end{aligned}$$

Thus, Theorem 2.5 is proved.

9. Self–complementary plane partitions

In this section we do the (-1)–enumeration for self–complementary plane partitions contained in boxes with *even sidelengths*. These plane partitions correspond to rhombus tilings with 180° rotational symmetry, see Figure 2.17 for an example. The weight is $(-1)^{n(P)}$ where $n(P)$ counts all half orbits in the plane partition P that are not in the half–full plane partition (cf. Figure 2.18). For example, the plane partition in Figure 2.17 has weight $(-1)^4 = 1$.

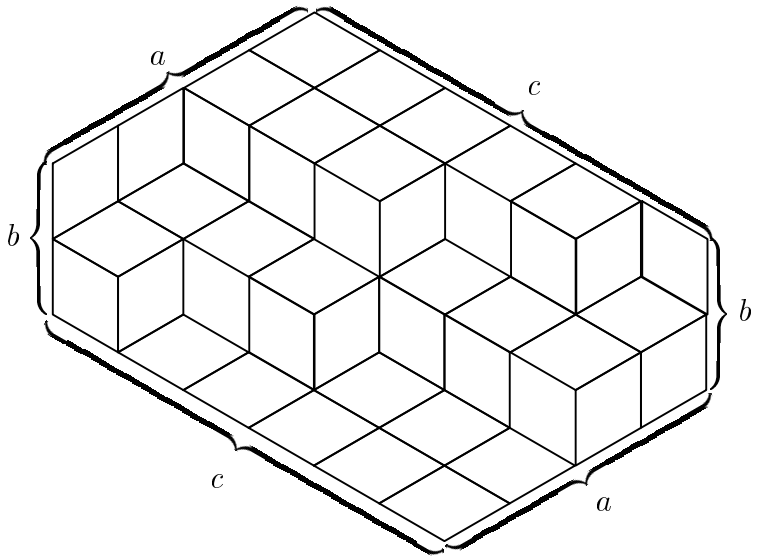


FIGURE 2.17. A self–complementary plane partition.

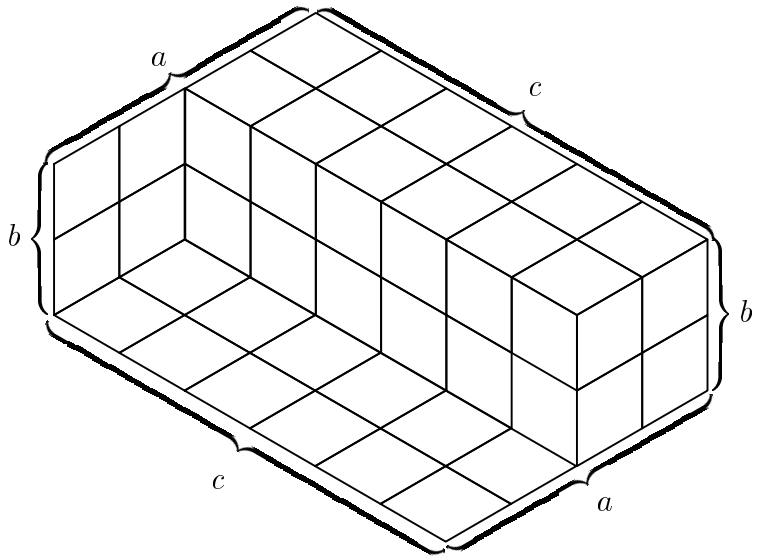


FIGURE 2.18. A self–complementary plane partition with weight 1.

The tiling is clearly determined by one half of the hexagon. Similarly to the previous cases, we find a bijection with families of nonintersecting lattice paths (see Figure 2.19). Without loss of generality we assume that $b \leq c$. The result turns out to be symmetric in b and c, so we can drop this condition in the statement of Theorem 2.6.

Write x for $(c-b)/2$. The starting points of the orthogonal version of our lattice paths are

$$A_i = (i-1, b+i-1) \quad \text{for } i = 1, \ldots, a.$$

The end points are a points chosen symmetrically among

$$E_j = (x+j-1, j-1) \quad \text{for } j = 1, \ldots, a+b.$$

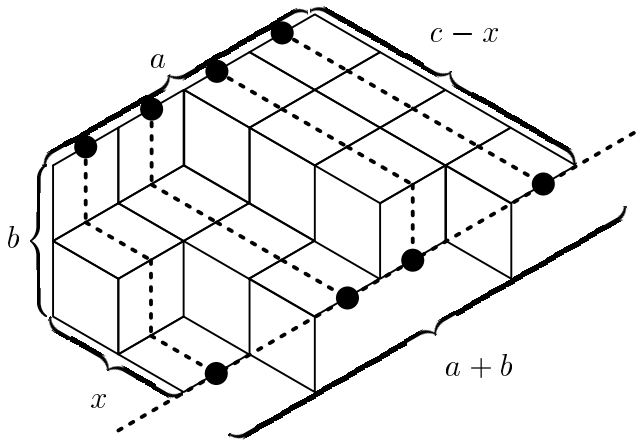


FIGURE 2.19. The paths for self–complementary plane partitions. ($x = \frac{c-b}{2}$)

We claim that for a path from A_i to E_j we can use the weight $(-1)^{\text{area2}}$ multiplied by $(-1)^j$ if $j \le \frac{a+b}{2}$. (area2 is the area between the path and the x–axis.) This can be expressed as a product of weights of individual steps, so Lemma 2.8 is applicable. We have to check that the weight changes sign if we replace a half orbit with the complementary half orbit. If one of the affected cubes is completely inside the half shown in Figure 2.19, area2 changes by one. If the two affected cubes are on the border of the figure, two symmetric endpoints, say E_j and $E_{a+b+1-j}$, are changed to E_{j+1} and E_{a+b-j} or vice versa. It is easily checked that in this case area2 changes by $j + (a+b-j)$ which is even. The factor $(-1)^j$ becomes $(-1)^{j+1}$ which gives the desired sign change. It is straightforward to check that this weight equals $(-1)^{1+\cdots+a/2+x(0+\cdots+(a/2-1))+(c-x)((a+b-1)+\cdots+(a+b-a/2))}$ for the plane partition in Figure 2.18, so we have to multiply the path enumeration by the global sign $(-1)^{a(a+2)/8+xa/2}$.

Now define S to be an $a \times (a+b)$–matrix with

$$S_{ij} = \begin{cases} (-1)^{(x+j-i)(j-1)} \begin{bmatrix} b+x \\ x+j-i \end{bmatrix}_{-1} (-1)^j & \text{for } 1 \le j \le (a+b)/2, \\ (-1)^{(x+j-i)(j-1)} \begin{bmatrix} b+x \\ x+j-i \end{bmatrix}_{-1} & \text{for } (a+b)/2+1 \le j \le a+b. \end{cases}$$

Observe that S_{ij} is the weighted enumeration of lattice paths from A_i to E_j with the weight described above. By Lemma 2.8 applied to all sets of fixed end points, the enumeration can be expressed as

$$\sum_{1 \le k_1 < \cdots < k_{a/2} \le (a+b)/2} \det\left(S_{k_1}, \ldots, S_{k_{a/2}}, S_{a+b+1-k_{a/2}}, \ldots, S_{a+b+1-k_1}\right),$$

where S_j is the jth column of S.

We can express this sum as a single Pfaffian using the following lemma which is a simple consequence of Lemma 2.11:

LEMMA 2.14. *Let S be a $2m \times 2n$–matrix with $m \le n$ and S^* be the matrix*

$$(S_1, \ldots, S_n, S_{2n}, \ldots, S_{n+1})$$

where S_j denotes the jth column of S. Let A be the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then the following identity holds:

$$\sum_{1\le k_1<\cdots<k_m\le n} \det(S_{k_1},\dots,S_{k_m},S_{2n+1-k_m},\dots,S_{2n+1-k_1}) = \mathrm{Pf}(S^*A({}^tS^*)).$$

PROOF. The proof follows from Lemma 2.11 with $A = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $T = {}^tS^*$. The sign of $\mathrm{Pf}\left(A^{k_1,\dots,k_m,k_1+n,\dots,k_m+n}_{k_1,\dots,k_m,k_1+n,\dots,k_m+n}\right)$ cancels exactly with the sign obtained from the reordering of the columns of S in the determinant. □

In our case $2m = a$, $2n = a+b$ and

$$S^*_{ij} = (-1)^{(x+j-i)(j-1)} \begin{bmatrix} b+x \\ x+j-i \end{bmatrix}_{-1} (-1)^j \quad \text{for } 1\le j\le (a+b)/2,$$

$$S^*_{i(j+(a+b)/2)} = (-1)^{(x-i)j} \begin{bmatrix} b+x \\ -a-1+j+i \end{bmatrix}_{-1} \quad \text{for } 1\le j\le (a+b)/2.$$

It remains to determine the Pfaffian of the $a\times a$–matrix $M = S^*A({}^tS^*)$. We distinguish between two cases according to the parity of x.

Case 1: x odd In this case the entry M_{ij} of the $a\times a$–matrix $M = S^*A({}^tS^*)$ can be written as

$$M_{ij} = \sum_{k=1}^{(a+b)/2} (-1)^{k(1+i+j)+1} \left((-1)^i \binom{(b+x-1)/2}{\lfloor (b+i-k)/2\rfloor} \binom{(b+x-1)/2}{\lfloor (k+j-1-a)/2\rfloor} - (-1)^j \binom{(b+x-1)/2}{\lfloor (b+j-k)/2\rfloor} \binom{(b+x-1)/2}{\lfloor (k+i-1-a)/2\rfloor} \right).$$

The sum can be split into two parts according to even and odd summation indices, the sum over odd indices is reversed and then combined with the other sum. This gives

$$\sum_{l=1}^{(a+b)/2} \left((-1)^j \binom{(b+x-1)/2}{b/2-l+\lfloor j/2\rfloor} \binom{(b+x-1)/2}{l-a/2+\lfloor (i-1)/2\rfloor} - (-1)^i \binom{(b+x-1)/2}{\lfloor (b+i)/2\rfloor - l} \binom{(b+x-1)/2}{l+\lfloor (j-1)/2\rfloor - a/2} \right).$$

By the Chu–Vandermonde summation formula this equals

$$M_{ij} = (-1)^j \binom{b+x-1}{(b-a)/2+\lfloor j/2\rfloor+\lfloor (i-1)/2\rfloor} - (-1)^i \binom{b+x-1}{(b-a)/2+\lfloor i/2\rfloor+\lfloor (j-1)/2\rfloor}.$$

It is easily seen that $M_{2i,2j} = 0$. As before, we reorder the rows and columns so that even–indexed ones come before odd–indexed ones. We thus obtain a block matrix. The Pfaffian of M equals the determinant of the lower left block matrix.

So we have to evaluate

$$\det_{1\le i,j\le a/2} M_{2i-1,2j} = \det_{1\le i,j\le a/2} \left(\binom{b+x}{(b-a)/2+i+j-1} \right).$$

This is done by taking $\frac{(x+(b+a)/2-j+1)_{(b-a)/2+j}}{(b/2+j-1)!}$ out of the jth column, $j = 1, 2, \ldots, a/2$, and $(-1)^{i-1}$ out of the ith row, $i = 1, 2, \ldots, a/2$, then applying Lemma 2.9 with $X_j = j$, $B_k = -x-(b+a)/2+k-2$ and $A_k = (b-a)/2+k-1$. This gives the desired result multiplied by $(-1)^{1+\cdots+(a/2-1)}$, which cancels exactly with the global sign.

Case 2: x even In this case the entry M_{ij} of the $a \times a$–matrix $M = S^* A({}^t S^*)$ can be written as

$$M_{ij} = \sum_{k=1}^{(a+b)/2} (-1)^{k(1+i+j)} \left((-1)^i \begin{bmatrix} b+x \\ b+i-k \end{bmatrix}_{-1} \begin{bmatrix} b+x \\ k+j-1-a \end{bmatrix}_{-1} - (-1)^j \begin{bmatrix} b+x \\ b+j-k \end{bmatrix}_{-1} \begin{bmatrix} b+x \\ k+i-1-a \end{bmatrix}_{-1} \right).$$

Here we have some vanishing entries. If i and j have the same parity then in each product of two (-1)–binomial coefficients one of them is zero. So as before we reorder even–indexed rows and columns before odd–indexed rows and columns and get a block matrix. By a calculation analogous to the previous subcase we get for the lower left block:

$$M_{2i-1,2j} = -\binom{b+x}{(b-a)/2+i+j-1}.$$

Up to the sign $(-1)^{a/2}$ this is exactly the same determinant as in the previous subcase. Since this sign is the difference in the global signs generated by the change in parity of x, Theorem 2.6 is proved.

If not all sidelengths are even, we can express the (-1)–enumeration as a Pfaffian in a similar way. We find experimentally that the result has again a nice product formula for small values of a, b, c (see the conjecture on page 28), but the matrix does not contain blocks of zeros in these cases, so the analogous method does not work.

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Lebenslauf

Ich wurde am 23. März 1976 in Wien als Tochter von Christine Eisenkölbl geb. Schilly (Volksschullehrerin) und Johann Eisenkölbl (praktischer Arzt) geboren. Ich absolvierte die Volksschule in Wien 11, Florian–Hedorfer–Straße (1982–1986) und die Mittelschule am Akademischen Gymnasium, Wien 1, (1986–1994). Dort habe ich auch im Juni 1994 mit ausgezeichnetem Erfolg maturiert. In den Jahren 1992, 1993, 1994 habe ich an der Internationalen Mathematischen Olympiade teilgenommen und habe dabei 1992 und 1993 jeweils eine Bronzemedaille und dann 1994 die Goldmedaille gewonnen. Im Wintersemester 1994/95 Beginn des Studiums der Mathematik und der Physik an der Universität Wien. Im April 1998 habe ich das Diplom in Mathematik mit dem Mag. rer. nat. abgeschlossen (Diplomarbeit: „Abzählung von Matchings in speziellen Graphen“). Seit Juli 1998 bin ich als Universitätsassistentin am Institut für Mathematik der Universität Wien beschäftigt. Im Jahr 1998 habe ich den Würdigungspreis des BM für Wissenschaft, Forschung und Verkehr erhalten und im Jahr 1999 den Studienpreis der Österreichischen Mathematischen Gesellschaft.