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"Rhombus tilings and electrostatics"

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To Anne, and Ruth, whose letters I sorely miss.

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SUMMARY

For over one hundred years a great many mathematicians have studied rhombus tilings of hexagons in one form or another. The story begins with Major Percy Alexander MacMahon in 1916 who was interested in objects known as boxed plane partitions (these are equivalent to rhombus tilings), and since then beautiful enumerative formulas corresponding to a total of ten different symmetry classes of tilings of hexagons have been established. The methods by which these results were derived have varied over the years, however more recently the tools and techniques used to approach such counting problems have often relied on some wonderful bijections between tilings and other families of combinatorial objects such as non-intersecting lattice paths and dimer coverings (or perfect matchings). It is perhaps safe to say that with respect to the field of combinatorics this area is well-studied and, by now, considered classical.

Arguably less well-understood is the effect that punctures in the interior of a hexagon have on the ways in which such a holey region can be tiled. For the past twenty years or so various mathematicians have embarked on quests to uncover formulas that count tilings of regions containing holes, often employing combinatorial techniques arising from the bijections described above. This has resulted in the discovery of a wide range of seemingly disparate formulas, each one pertaining to a particular family of holes.

Although a completely general formula has yet to reveal itself, many of those that have already been found exhibit, asymptotically speaking, some fascinating behaviour. It appears to be the case that on a large scale, when the holes are fixed and lie far apart from each other within tilings of the entire plane, the effect that the holes have on each other with respect to the tiles that surround them is governed by precisely the same physical principle that determines the total electric potential energy between an ensemble of static, electrically charged point particles in two dimensions. In other words, the interaction between the holes in the discrete rhombus tiling model is apparently governed by a two dimensional version of Coulomb's law for electrostatics. This relationship, which places rhombus tilings of regions that contain holes firmly at the boundary between pure mathematics and statistical physics, was first conjectured by Mihai Ciucu in 2008. Almost a decade later, despite having been confirmed for a number of different families of holes, Ciucu's conjecture ultimately remains wide open.

This thesis is a collection of my own investigations into this area over the past three and a half years which resulted in the discovery of a number of exact formulas that count tilings of regions containing different families of holes, found using a variety of established combinatorial methods coupled with a great deal of computational experimentation and guesswork. Where possible these formulas have been analysed asymptotically and in each case confirm Ciucu's original conjecture. In some instances closely related and well-known physical phenomena – chiefly, the method of images – emerge directly from the mathematics, providing yet more evidence that rhombus tilings with holes discretely model two dimensional electrostatics very well indeed.

ZUSAMMENFASSUNG

Seit über hundert Jahren wurden von zahlreichen Mathematikern Rhombuspflasterungen von Sechsecken in der einen oder anderen Form studiert. Die Geschichte begann 1916 mit Major Percy Alexander MacMahon, der sich für ebene Partitionen, die man einer Schachtel einschreiben kann, interessierte (diese Objekte sind äquivalent zu Rhombuspflasterungen) und setzte sich fort mit dem Beweis eleganter Abzählformeln für die insgesamt zehn verschiedenen Symmetrieklassen von Rhombusflasterungen eines Sechsecks. Die Methoden, mit welchen diese Resultate erzielt wurden, variierten über die Jahre. Allerdings lagen in der jüngeren Vergangenheit den Techniken, mit denen derartige Abzählprobleme angegangen wurden, oftmals wunderbare Bijektionen zwischen Pflasterungen und anderen Familien von kombinatorischen Objekten zugrunde, wie zum Beispiel nichtkreuzende Gitterpunktwege, Dimer-Überdeckungen oder perfekte Paarungen. Man darf mit Sicherheit behaupten, dass dieses Themenfeld in der Kombinatorik viel Aufmerksamkeit erfahren hat und mittlerweile als klassisches Teilgebiet angesehen wird.

Weniger gut verstanden wird indessen der Effekt, den Löcher im Inneren eines Sechseckss auf die Anzahl der möglichen Pflasterungen solch eines löchrigen Sechsecks hat. In den vergangenen zwanzig Jahren widmeten sich unterschiedliche Mathematiker der Aufgabe Formeln zu finden, welche Pflasterungen von Regionen abzählen, die Löcher enthalten – oft unter Verwendung der oben angeführten kombinatorischen Techniken und der zugrundeliegenden Bijektionen. Das Resultat war die Entdeckung eines breiten Spektrums an anscheinend unvereinbaren Formeln, von denen jede eine ganz spezielle Art von Löchern behandelt.

Obwohl noch keine große einheitliche Formel gefunden werden konnte, zeigen viele der bisher bekannten Formeln ein faszinierendes asymptotisches Verhalten. Es scheint im Großen der Fall zu sein, dass der Effekt, den Löcher mit fixierten Positionen, die weit auseinander in der Ebene liegen, in Bezug auf die Rhomben, die sie umgeben, aufeinander haben, genau denselben physikalischen Gesetzen unterliegt, die die gesamte potentielle elektrische Energie eines Systems von elektrostatischen Punktladungen in zwei Dimensionen bestimmen. In anderen Worten, die Wechselwirkung zwischen Löchern im diskreten Modell der Rhombuspflasterungen entspricht offenbar einer zweidimensionalen Version des Coulombschen Gesetzes aus der Elektrostatik. Diese Beziehung, die Rhombuspflasterungen von Regionen mit Löchern direkt an der Grenze zwischen abstrakter Mathematik und statistischer Physik ansiedelt, wurde zum ersten Mal 2008 von Mihai Ciucu vermutet. Wenngleich sie für eine Anzahl verschiedener Familien von Löchern verifiziert werden konnte, bleibt Ciucus Vermutung auch fast zehn Jahre später ein völlig offenes Problem.

Diese Dissertation vereint meine eigenen Untersuchungen in diesem Forschungsgebiet der letzten dreieinhalb Jahre, welche zur Entdeckung mehrerer exakter Abzählresultate für Pflasterungen von Regionen mit unterschiedlichen Familien von Löchern geführt haben. Meine Resultate wurden unter Verwendung einer Vielzahl von etablierten kombinatorischen Methoden gepaart mit computergestütztem Experimentieren und Raten erhalten. Wenn möglich wurden diese Formeln auch auf ihr asymptotisches Verhalten hin analysiert und bestätigen ausnahmslos Ciucus ursprüngliche Vermutung. In manchen Fällen ergeben sich bekannte physikalische Phänomene – hauptsächlich das Prinzip der Spiegelladung – direkt aus der Mathematik, was weitere Evidenz dafür liefert, dass Rhombuspflasterungen mit Löchern in der Tat ein ausgezeichnetes diskretes Modell für die Elektrostatik sind.

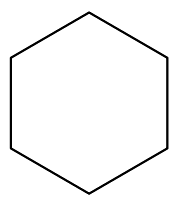
CHAPTER
ONE

INTRODUCTION

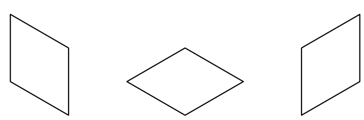
This introductory chapter is intended to give a flavour of what this thesis is about and why I think the mathematics it contains is interesting. For this reason mathematical rigour and proper referencing have taken something of a back seat until subsequent chapters in favour of giving a general sense of the overall ideas and themes that run throughout this piece of work.

1.1. How do I tile thee? Let me count the ways

Imagine I draw you a hexagon:



and give you a set of identical diamond shaped tiles:



and ask you to arrange the tiles inside the hexagon so that they do not overlap and they cover its interior. How many different arrangements of tiles, or *rhombus tilings*,[1] can you come up with?

The answer, of course, depends on how large the hexagon is (or, if you like, how small the tiles are). In fact what is most important is the size of the tiles *relative* to the size of the hexagon, so it makes sense to think of the lengths of the sides of the hexagon in units corresponding to the length of one side of a rhombus tile. There are, for example, exactly 2 different rhombus tilings of a hexagon where all the side lengths of the hexagon are equal to the length of the side of each rhombus (see Figure 1), whereas the number of possible tilings of a hexagon of side length 2 is 20 (see Figure 2).

For small hexagons the answer is relatively easy to figure out – the number of combinations is low enough that they can all be drawn by hand – but what about hexagons of side length 5? Or 14? Or even 42? What if, instead of a *regular hexagon* (where all sides are the same length), I draw you a *semi-regular hexagon*, where only pairs of opposite sides have the same length?

[1]A diamond is also called a rhombus.

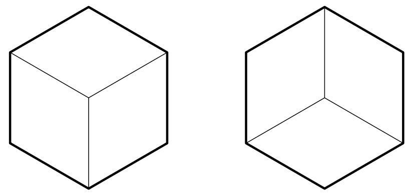


FIGURE 1. The 2 rhombus tilings of a hexagon of side length 1.

As the size of the hexagon grows the brute-force approach (that is, simply trying out all possibilities and counting them) quickly becomes untenable; a hexagon of side length 16, for example, has

$$1333238967268838729644960699395260091156640392837595395230823030619103588479705246924800$$

different tilings. To put this into context an estimate of the number of atoms in the observable universe is, according to Wikipedia, roughly 10^{80} (a 1 followed by 80 zeros):

$$100000000000000000000000000000000000000000000000000000000000000000000000000000000.$$

It would be much easier, then, to have a formula that calculates the number of tilings of a semi-regular hexagon of a given size, rather than having to actually draw every possible tiling. Luckily for us such a formula already exists – it was discovered by Major Percy Alexander MacMahon and published in his book *Combinatory Analysis* in 1916.[2]

1.2. Holeyness

Over the past 100 years since their introduction by MacMahon as boxed plane partitions, rhombus tilings have been studied in one form or another by many different people and a great number of questions concerning their *enumeration* have already been answered. More recently, however, much attention has been focused on what happens if we cut holes into the interior of the hexagon, thus creating a so-called *holey hexagon*.

[2]It is probably important to point out that MacMahon was interested not in tilings but in other structures known as *boxed plane partitions*, however it turns out that these two different sets of objects are equivalent.

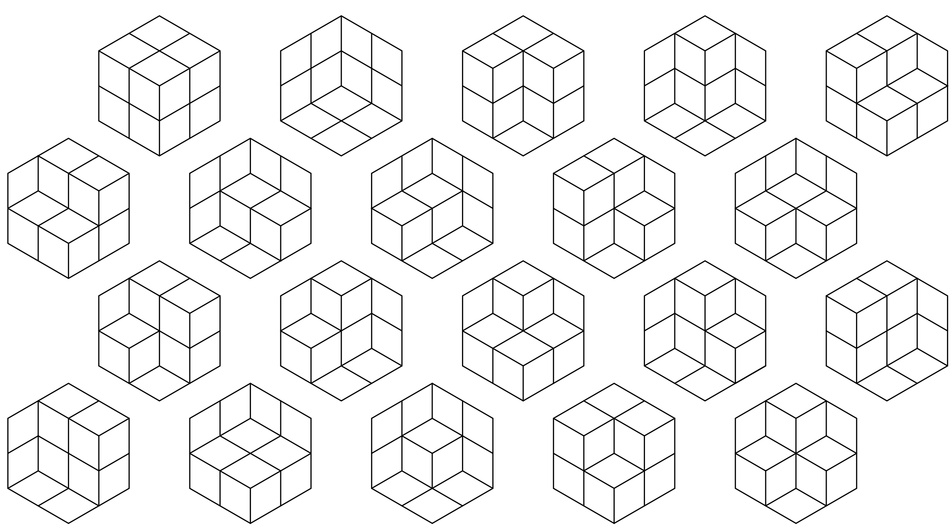
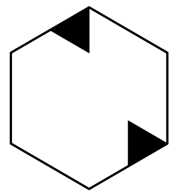
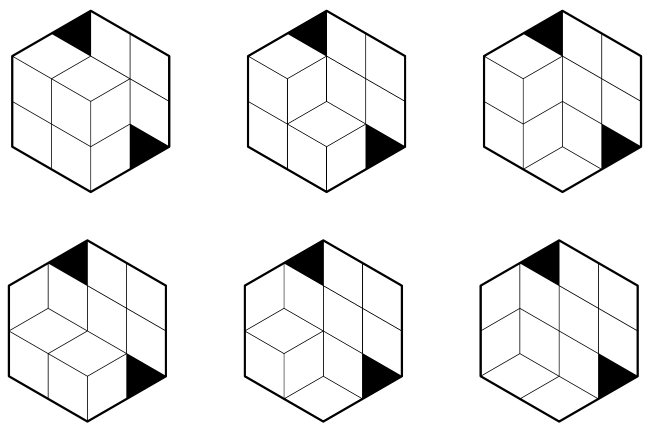


FIGURE 2. The 20 rhombus tilings of a regular hexagon of side length 2.

Suppose now that I give you the following holey hexagon of side length 2 that has holes that share an edge with its outer boundary:

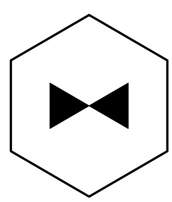


and ask you again for the number of possible tilings of this shape. You would undoubtedly come back after a little time and show me no more than six possible arrangements:

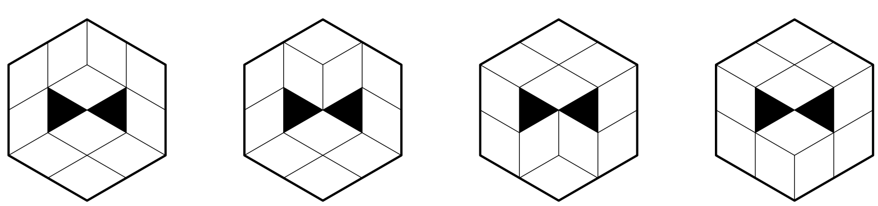


.

Compare this with the number of tilings of the same size hexagon that contains no holes (an *unholey hexagon*, or just a hexagon, again see Figures 1 and 2). Puncturing the interior clearly has an effect on the number of ways we can tile the shape, and it seems the locations of these holes also play an important role. If we place the same two unit triangular holes at different points inside the hexagon, forming, say, a little holey bow-tie:



then the number of possible tilings diminishes further:



.

If we want to numerically measure and compare the effect of placing the holes in the centre with that of placing them on the boundary then for each set of holes we can divide the number of tilings of the holey hexagon by the number of tilings of the hexagon without any holes. For the holes that sit on the edge of the boundary we have

$$\frac{\text{Number of tilings of hexagon with boundary holes}}{\text{Number of tilings of unholey hexagon}} = \frac{6}{20} = 0.3,$$

whereas for the holes that are in the centre

$$\frac{\text{Number of tilings of hexagon with a holey bow-tie}}{\text{Number of tilings of unholey hexagon}} = \frac{4}{20} = 0.2.$$

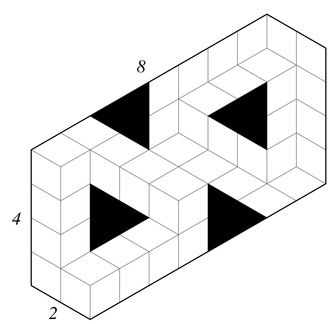
That 0.3 is larger than 0.2 reflects the fact that placing the holes at different locations inside the same size hexagon affects the possible arrangements of tiles that surround them in differing ways.

This thesis is pre-occupied with looking at the effects that different types of holes have on the ways in which we can tile different types of hexagons. This involves in the first instance finding formulas that count tilings exactly, and then analysing roughly how these formulas behave as the sides of the hexagon become increasingly large and the distances between the holes grow. In slightly more mathematical terms I would say that this thesis is really a collection of results I have found over the past three and a half years concerning

the exact and asymptotic enumeration of rhombus tilings of holey hexagons.

1.3. The bigger picture

Why is this interesting? What is it about these tiling problems that is alluring enough to compel people to study them? For myself, having always enjoyed counting things,[3] the various ways in which you can go about trying to find such formulas are intriguing enough. The following rhombus tiling:



has at least two different representations (see Figure 3) and in choosing the lens through which we look at tilings we are able to choose between two distinct methods by which it is possible to count their number. In Chapter 3 I describe one such method using *families of non-intersecting lattice paths* (due to both Lindström and Gessel and Viennot), while in Chapter 6 I describe a different method using *dimer coverings* or *perfect matchings* due to Kasteleyn. Each approach has its own merits and drawbacks, but it turns out, in fact, that the most general result I have been able to obtain relies on marrying them both together.[4]

There are further, and arguably less esoteric, reasons why one might be interested in looking at these sorts of tiling problems. It seems as though on a large scale – that is, when the hexagons are infinitely big and the holes lie far away from each other – the effect the holes have on the ways in which we can tile the space around them is proportionally dependent on the distances between them. This dependence, conjectured by Mihai Ciucu in 2008, appears to mimic a well-known physical law

[3]That is, enumerative combinatorics.

[4]In holey matrimony.

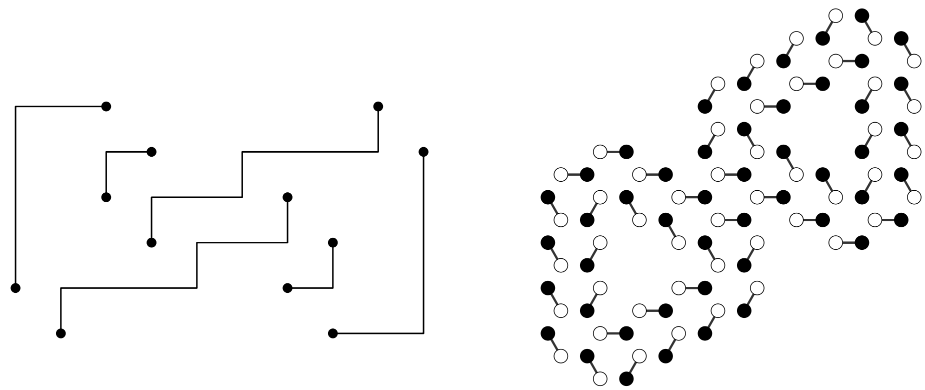


FIGURE 3. Two different representations of a rhombus tiling.

called *Coulomb's law* which governs the potential electric energy of arrangements of static electrically charged particles in two dimensions.

To the best of my knowledge Ciucu's conjecture remains just that – a conjecture. Over the past few years it has certainly been shown to hold for a number of different classes of holes, however we have thus far been denied a full proof. Its implications are quite profound; if it were proved to be true in general it would mean that rhombus tilings of regions with holes model two dimensional electrostatics very well. In one of his papers Ciucu himself discusses how his conjecture potentially yields very strong *mathematical evidence* for empirically observed *physical laws*.

1.4. Organisation of this thesis

In the following chapters the grammatical person switches from the first person singular to the first person plural as we wander together down a certain avenue of established research, pausing here and there to look at one or two results I have found.

Chapter 2 begins with some preliminary concepts and ideas and also establishes our notation. Here we will also see the context in which these tiling problems first arose as well as an explicit description of the conjectured link with statistical physics.

In Chapter 3 we set out one well-known method for enumerating rhombus tilings via families of non-intersecting lattice paths which is then used in Chapter 4 to derive enumerative formulas that count tilings belonging to three different symmetry classes of hexagons that contain sets of horizontally collinear holes.

The focus of Chapter 5 is the asymptotic behaviour of the formulas from Chapter 4 as the boundaries of the hexagonal regions become infinite and the distances between the holes grow large. Here we will also see other physical principles emerging from the mathematical analysis, providing yet more evidence in support of Ciucu's electrostatic conjecture.

We will then approach rhombus tilings from a completely different angle in Chapter 6, marrying techniques developed by Kasteleyn together with lattice path enumeration in order to obtain an enumerative formula that counts tilings for a vast class of holes. This class includes a huge number of existing results as well as enumerating tilings of regions with holes that have until now never been considered. This final result could potentially lead to the most general proof of Ciucu's conjecture to date.

CHAPTER
TWO

PRELIMINARIES

In this chapter we discuss the origins of rhombus tilings as boxed plane partitions and describe the apparent relationship between tilings of regions with holes and well-known physical principles. We also provide some basic definitions and establish notation.

2.1. Partitions and plane partitions

A *partition* of a non-negative integer n is a representation of n as a sum of positive integers

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

where the order of the parts is not important.[5] Whenever we express a positive integer in this way we assume with complete generality that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. Often it is convenient to denote a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$.

Example 2.1.1

The five partitions of 4 are $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$, since

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1.$$

Leibniz was apparently the first to consider problems related to the partitions of integers in his doctoral dissertation [**60**], although many might claim that the study of partitions actually began a little later with Euler who showed that if $p(n)$ denotes the number of partitions of n then

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1-x^j}.$$

The proof of this generating function identity is really quite lovely – it follows from re-writing each factor $(1-x^j)^{-1}$ in the product as an infinite series[6] and then considering the coefficients of x^n in the expansion of this product.

Regardless of who first studied them partitions have captured the interest of many eminent mathematicians since their introduction, the most famous from the twentieth century being perhaps Hardy and Ramanujan[7] who established (amongst many other things) the asymptotic behaviour of $p(n)$.

Another mathematician who was particularly enamoured with partitions was Alfred Young [**75**] who chose to represent them pictorially as rows of boxes stacked upon one another, where the

[5] We define the partition of 0 to be the *empty partition*, that is, the partition consisting of a sum of zero positive integers.

[6] That is,

$$\frac{1}{1-x^j} = \sum_{n\geq 0} (x^j)^n.$$

[7] Famous in the conventional sense of the word (that is, not only famous within certain mathematical fields), since they were the subjects of both the 2007 play entitled *A disappearing number* and the 2016 film entitled *The man who knew infinity*.

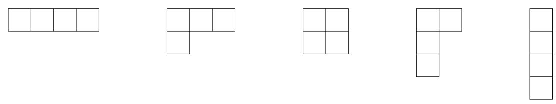


FIGURE 4. The Young diagrams of $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$.

number of boxes in each row corresponds to each part of the partition.[8] These depictions are referred to as *Young diagrams* (see Figure 4), and if the boxes are replaced by circles then they are known as *Ferrers diagrams*. They crop up in many different areas of mathematics, perhaps most famously in the representation theory of the symmetric group on n letters, denoted $\mathfrak{S}_n$.[9]

The starting point of this thesis is the two dimensional analogue of a partition introduced at the beginning of the twentieth century by Major Percy Alexander MacMahon [**63**]. A *plane partition* $\pi = (\pi_{i,j})_{i,j\geq 1}$ is an array of positive integers

$$\begin{array}{ll} \pi_{1,1} & \cdots\cdots\cdots \pi_{1,\lambda_1} \\ \pi_{2,1} & \cdots\cdots \pi_{2,\lambda_2} \\ \vdots & \quad\quad \iddots \\ \pi_{k,1} & \cdots \pi_{k,\lambda_k} \end{array}$$

that are weakly decreasing along rows and down columns, that is,

$$\pi_{i,j} \geq \pi_{i+1,j}$$

and

$$\pi_{i,j} \geq \pi_{i,j+1}$$

for all i, j. The $\pi_{i,j}$s are referred to as the *parts* of the plane partition. If the sum over the parts $\sum_{i,j} \pi_{i,j}$ is equal to n then we say that π is a plane partition of n, and write $|\pi| = n$. We also define the empty plane partition of 0 to be the plane partition consisting of no parts. The row lengths of π are clearly indexed (from top to bottom) by the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, thus a plane partition of *shape* λ can be thought of as the Young diagram corresponding to λ in which the boxes have been filled with positive integers that are weakly decreasing along rows and down columns.

EXAMPLE 2.1.2

Three different plane partitions of 64:

$$\begin{array}{ccccc} 8 & 7 & 5 & 5 & 3 \\ 6 & 6 & 4 & 3 & 1 \\ 5 & 3 & 2 & 2 & \\ 3 & 1 & & & \end{array} \qquad \begin{array}{cccc} 8 & 8 & 6 & 5 \\ 8 & 7 & 4 & \\ 6 & 4 & 3 & \\ 5 & & & \end{array} \qquad \begin{array}{cccc} 8 & 7 & 5 & 4 \\ 6 & 5 & 4 & 3 \\ 4 & 4 & 3 & 1 \\ 4 & 4 & 2 & \end{array}$$

In [**64**] MacMahon gave a wonderful product formula for the generating function for the number of plane partitions of n (denoted $pp(n)$):

$$\sum_{n=0}^{\infty} pp(n)x^n = \prod_{j=1}^{\infty} \frac{1}{(1-x^j)^j} \tag{2.1}$$

(compare this with Euler's formula above, it is striking how similar they are). His investigations, however, did not stop there and he went on to conjecture (and in some cases prove) a number of beautiful product formulas that enumerate certain classes of plane partitions. The first of these we will consider shall be referred to as (a, b, c)*-boxed plane partitions*.

[8] In the literature these are drawn in one of three ways depending on the French/English/Russian sympathies of the author. We shall draw them in the English fashion – with each row of length λ_i above that of length λ_{i+1}.

[9] The conjugacy classes of $\mathfrak{S}_n$ can be labelled by the partitions of n and it turns out that Young diagrams happen to index the irreducible representations of $\mathfrak{S}_n$ over the complex numbers $\mathbb{C}$.

Definition 2.1.3

An (a,b,c)-*boxed plane partition* π is a plane partition consisting of rows and columns that contain at most a and c entries (respectively), where each entry is at most b.

Example 2.1.4

Three different $(5,8,4)$-boxed plane partitions of 64:

$$\begin{array}{ccccc} 8 & 7 & 5 & 5 & 3 \\ 6 & 6 & 4 & 3 & 1 \\ 5 & 3 & 2 & 2 & \\ 3 & 1 & & & \end{array} \qquad \begin{array}{cccc} 8 & 8 & 6 & 5 \\ 8 & 7 & 4 & \\ 6 & 4 & 3 & \\ 5 & & & \end{array} \qquad \begin{array}{cccc} 8 & 7 & 5 & 4 \\ 6 & 5 & 4 & 3 \\ 4 & 4 & 3 & 1 \\ 4 & 4 & 2 & \end{array}$$

These objects are named in this way because we have placed constraints on three dimensions: the maximum row length; the maximum column length; and the maximum size of each entry. If a plane partition π satisfies these constraints for given a, b, and c then we say that π *fits inside* an (a,b,c)-box.

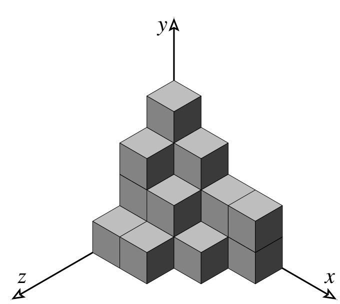
MacMahon was interested in enumerating precisely how many plane partitions fit inside an (a,b,c)-box for given a, b, and c. Before we state his answer to this question note that our current representation of plane partitions is not completely satisfactory since we cannot really see the box within the array. The plane partitions in Example 2.1.4 each fit inside an infinite family of boxes, indeed all of them fit inside an (a,b,c)-box for $a \geq 5, b \geq 8$ and $c \geq 4$. It turns out that we can represent plane partitions (and thus boxed plane partitions) as three dimensional structures whose two dimensional pictorial representation (when viewed from a particular angle) also depicts the constraints that arise from the dimensions of the box.

2.2. Piles of cubes

Given a plane partition π, we may replace each part $\pi_{i,j}$ with a stack of (three dimensional) unit cubes equal in height to $\pi_{i,j}$. Every plane partition gives rise in this way to a three dimensional object (a pile of unit cubes) which we are then free to align with three co-ordinate axes in the fashion depicted in the following example.

Example 2.2.1

$$\begin{array}{cccc} 4 & 3 & 2 & 2 \\ 3 & 2 & 1 & \\ 1 & 1 & & \end{array}$$



On the left hand side we have a plane partition of 19 with shape $(4,3,2)$ and on the right we have one particular view of its three dimensional representation as a pile of unit cubes. The spine of the stack of cubes corresponding to $\pi_{1,1}$ is always aligned along the y-axis, with the base of the spine coinciding with the origin.

Remark 2.2.2

It is worth observing that while the object on the left of Example 2.2.1 *is* a plane partition according to our earlier definition, the one on the right is simply a two dimensional image that depicts the corresponding pile of unit cubes *when viewed from a particular perspective*.

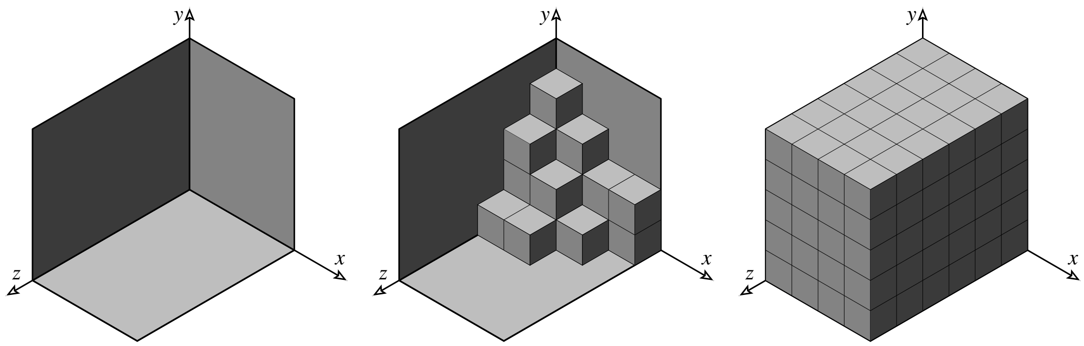


FIGURE 5. Three plane partitions that each fit inside a $4 \times 5 \times 6$ box.

In the absence (within the pages of this thesis) of a three dimensional medium with which we can represent these objects we shall make the convention that we *always* view piles of cubes from this angle. This is not without certain benefits – it is easy to see that due to the monotonicity of their rows and columns no two distinct piles of cubes will look the same from this fixed vantage point.

In this three dimensional representation a plane partition that fits inside an (a, b, c)-box is one that is contained inside the box that has width a (along the x-axis), height b (along the y-axis), and length c (along the z-axis). From this perspective the number of (a, b, c)-boxed plane partitions can be thought of as the number of ways to fill an empty $a \times b \times c$ box with n unit cubes (subject to some filling criteria), where $n \in \{0, 1, 2, \ldots, abc\}$. Figure 5 shows a collection of three plane partitions sat inside an $4 \times 5 \times 6$ box.

REMARK 2.2.3

When we depict the box in three dimensions we omit the top, front left, and front right sides so that we can see the cubes within it.

THEOREM 2.2.4 – MACMAHON'S BOX FORMULA [**64**]

The number of plane partitions that fit inside an $a \times b \times c$ box is

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

REMARK 2.2.5

Technically speaking MacMahon actually gave a product formula for the generating function $\sum_\pi q^{|\pi|}$ over all partitions that fit inside an $a \times b \times c$ box. His original result was written in a rather long-winded fashion, however in 1995 Macdonald [**62**] expressed it in the following way:

$$\sum_\pi q^{|\pi|} = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}},$$

from which we can derive Theorem 2.2.4 by letting $q \to 1$. Note also that if we send a, b, and c to infinity we easily recover the generating function for all plane partitions from (2.1)

$$\sum_\pi q^{|\pi|} = \prod_{j=1}^{\infty}\frac{1}{(1-q^j)^j}.$$

2.3. Symmetries of boxed plane partitions

MacMahon also discussed [**64**, Sections 425-428 and 509] the different families of plane partitions that are invariant under the action of sub-groups of $\mathfrak{S}_3$, which leads to four inequivalent symmetry classes. Later Stanley [**72**] extended a different operation on boxed plane partitions (namely, that of *complementation* which was first introduced by Mills, Robin and Rumsey [**65**] for descending plane partitions – see Section 2.5), and this operation together with the operations that arise from the symmetric group takes the number of inequivalent symmetry classes of boxed plane partitions to ten (see [**56**] for a summary of Stanley's sizeable contribution to this area).

The approach that both Stanley and MacMahon took in order to study symmetry classes of these objects was to co-ordinatise them. We may encode each unit cube that comprises a plane partition as a triple (d_1, d_2, d_3) of positive integers, where d_1, d_2, and d_3 are the unit distances, from the origin, along the x, y, and z axes respectively of the corner that lies closest to our viewpoint. Hence any plane partition π gives rise to a set $P(\pi) \subset \mathbb{N}^3$, so in order for π to fit inside an $a \times b \times c$ box each point $(d_1, d_2, d_3) \in P(\pi)$ must satisfy $d_1 \leq a$, $d_2 \leq b$, and $d_3 \leq c$. We denote the box that contains them $\overline{B}(a,b,c) = [a] \times [b] \times [c]$, where $[a] = \{1, 2, \ldots, a\}$, thus $P(\pi) \subseteq \overline{B}(a,b,c)$.

We define an action θ of $\mathfrak{S}_3$ on $\mathbb{N}^3$ to be the function $\theta : \mathfrak{S}_3 \times \mathbb{N}^3 \to \mathbb{N}^3$ given by

$$\theta(\sigma, d_1, d_2, d_3) := (d_{\sigma(1)}, d_{\sigma(2)}, d_{\sigma(3)}).$$

For convenience we denote by $\sigma \circ X$ the set

$$\{(d_{\sigma(1)}, d_{\sigma(2)}, d_{\sigma(3)}) : (d_1, d_2, d_3) \in X\}$$

and we say that a boxed plane partition is *invariant* under the action of a sub-group G of $\mathfrak{S}_3$ if

$$\sigma \circ P(\pi) = P(\pi) \tag{2.2}$$

for all $\sigma \in G$.

Remark 2.3.1

Note that we also require the box that contains $P(\pi)$ to be invariant under these operations, so different symmetry classes force constraints on the sizes of the boxes that contain the plane partitions.

The elements of $\mathfrak{S}_3$ with which we will be chiefly concerned are the identity permutation,[10] denoted $\mathbb{1}$, and the transposition that swaps 1 and 3, $\sigma = (1\ 3)$. All boxed plane partitions are clearly invariant under $\mathbb{1}$, thus the number of boxed plane partitions for fixed positive integers a, b, and c that are invariant under the action of the trivial group is given by MacMahon's box formula (Theorem 2.2.4). The plane partitions that are invariant under the sub-group consisting of the identity and the transposition $\sigma = (1\ 3)$ are discussed in the following section.

2.4. Symmetric plane partitions

For a plane partition $\pi = (\pi_{i,j})_{i,j \geq 1}$, the plane partition π^* that corresponds to the set of points $(1\ 3) \circ P(\pi)$ may be characterised in the following way: π^* is obtained by reflecting the parts of π about its main diagonal (thus we have $\pi^* = (\pi_{j,i})_{i,j \geq 1}$). We call such an operation *transposition* and π^* is referred to as the *transpose* of π.

Example 2.4.1

$$\begin{matrix} 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 & \\ 3 & 3 & & & \\ 1 & & & & \end{matrix} \qquad \begin{matrix} 4 & 3 & 3 & 1 \\ 4 & 3 & 3 & \\ 3 & 2 & & \\ 2 & 1 & & \\ 1 & & & \end{matrix}$$

[10] This is the permutation $\sigma \in \mathfrak{S}_n$ that maps each element to itself.

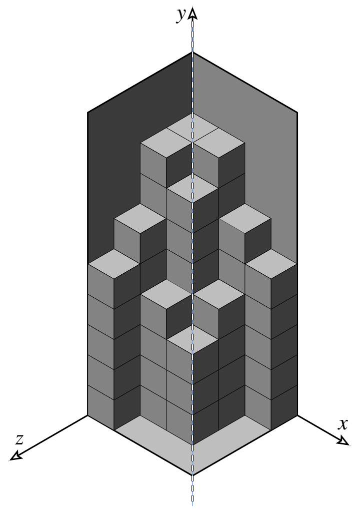


FIGURE 6. A symmetric plane partition in a $4 \times 8 \times 4$ box.

A plane partition together with its transpose.

It follows that plane partitions that are invariant under $\sigma = (1\ 3)$ are all those satisfying $\pi = \pi^*$, that is, those that are equal to their transpose. We also require that the box containing them must be invariant under $(1\ 3)$, thus $c = a$. Plane partitions $P(\pi) \subseteq \overline{B}(a, b, a)$ for which

$$(1\ 3) \circ P(\pi) = P(\pi)$$

are referred to as *symmetric boxed plane partitions*.

We have already encountered an example of a symmetric boxed plane partition, namely

$$\begin{array}{cccc} 8 & 8 & 6 & 5 \\ 8 & 7 & 4 & \\ 6 & 4 & 3 & \\ 5 & & & \end{array}$$

from Example 2.1.2, which may be placed inside an $a \times b \times a$ box for $a \geq 4, b \geq 8$. The symmetry becomes even more apparent when it is viewed as a pile of cubes, since symmetry about the main diagonal in π corresponds to reflective symmetry in the hyperplane $x = z$ (see Figure 6). Of course from our specified viewpoint this hyperplane appears as a straight vertical line (in Figure 6 this is marked in white).

MacMahon [**64**, Section 509] conjectured a formula for the weighted enumeration of symmetric plane partitions that fit inside an $a \times b \times a$ box. In 1978, over 60 years later, Andrews [**4**] was able to prove it. Around 1970 Gordon [**43**] independently proved the Bender-Knuth conjecture [**5**] (although this was actually published several years later), which for a special case gives the ordinary enumeration of symmetric plane partitions.[11] Further refinements and alternative proofs have since been given by Macdonald [**62**, pp. 83–85], Proctor [**67**, Proposition 7.3], Krattenthaler [**51**], and Fischer [**34**].

[11]The equivalence of MacMahon's conjecture and that of Bender and Knuth was proved by Andrews in [**3**].

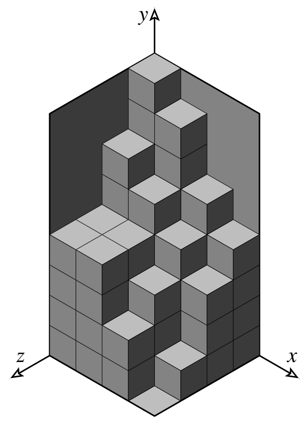


FIGURE 7. A transpose-complementary plane partition in a $4 \times 8 \times 4$ box.

THEOREM 2.4.2 – ANDREWS [4], GORDON [43]

The number of symmetric plane partitions that fit inside an $a \times b \times a$ box is

$$\prod_{i=1}^{a} \frac{2i+b-1}{2i-1} \prod_{1 \le i < j \le a} \frac{i+j+b-1}{i+j-1}.$$

2.5. Transpose-complementary plane partitions

We define now a further operation on plane partitions – that of *complementation* – which was first introduced by Mills, Robin and Rumsey [65] for descending plane partitions and then extended to all boxed plane partitions by Stanley in [72]. Consider the co-ordinatised plane partition $P(\pi) \subseteq \overline{B}(a,b,c)$ and let ϕ be the function from $\mathbb{N}^3$ to itself given by

$$\phi((d_1, d_2, d_3)) := (a+1-d_1, b+1-d_2, c+1-d_3).$$

If a plane partition fits inside an $a \times b \times c$ box then its *complement* is the set of points

$$P'(\pi) := \{\phi((d_1, d_2, d_3)) : (d_1, d_2, d_3) \in \overline{B}(a,b,c) \setminus P(\pi)\}.$$

We denote by π^c the plane partition corresponding to $P'(\pi)$. A plane partition satisfying $\pi^* = \pi^c$ is called *transpose-complementary*. Note that this symmetry class forces the dimensions of the box so that $a = c$ and $b = 2m$ for some positive integer m.

EXAMPLE 2.5.1

If we place the right-most plane partition from Example 2.1.4

$$\begin{matrix} 8 & 7 & 5 & 4 \\ 6 & 5 & 4 & 3 \\ 4 & 4 & 3 & 1 \\ 4 & 4 & 2 & \end{matrix}$$

into the box $\overline{B}(4,8,4)$ then its complement is

$$\begin{matrix} 8 & 6 & 4 & 4 \\ 7 & 5 & 4 & 4 \\ 5 & 4 & 3 & 2 \\ 4 & 3 & 1 & \end{matrix}$$

which is clearly equal to the transpose of the original plane partition.

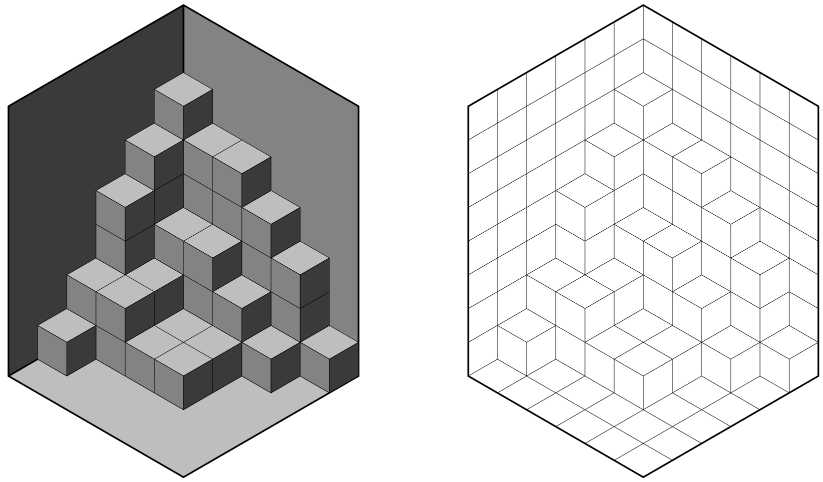


FIGURE 8. A $(6, 8, 6)$-boxed plane partition and a rhombus tiling of $H_{6,8,6}$.

REMARK 2.5.2

In the three dimensional unit cube representation of the upper-most plane partition in the preceding example (see Figure 7), note that if we were to imagine the space around the plane partition to be solid and then remove the original pile of cubes, then by rotating the box we would obtain precisely the plane partition with which we began. This holds more generally for all transpose-complementary plane partitions.

Transpose-complementary plane partitions that fit inside $\overline{B}(a, 2b, a)$ may be characterised in the following way: the anti-diagonal elements are all b; the entries above the anti-diagonal are weakly decreasing along rows and down columns; and the entries below the anti-diagonal satisfy

$$\pi_{i,j} = 2b - \pi_{a+1-j,a+1-i}.$$

The number of transpose-complementary boxed plane partitions for positive integers a and b was first given by Proctor in 1984.

THEOREM 2.5.3 – PROCTOR [**67**]

The number of transpose-complementary $(a, 2b, a)$-boxed plane partitions is

$$\binom{a+b-1}{a-1} \prod_{i=1}^{a-2} \prod_{j=i}^{a-2} \frac{2b+i+j+1}{i+j+1}.$$

2.6. Ceci n'est pas une pile

After MacMahon it seems that plane partitions and the theory surrounding them lay dormant for a number of years. They were revived in the late 1960s by Gordon and Houten [**44**,**45**], whose papers were quickly followed[12] by a two-part survey due to Stanley [**71**] and Bender and Knuth's article [**5**] mentioned above. A flurry of intensive investigation followed and many interesting results were discovered, however until 1989 these objects were always considered as arrays of positive integers that could be expressed in terms of sets of co-ordinates belonging to the three dimensional lattice $\mathbb{N}^3$. Everything changed, however, with the publication of David and Tomei's sweet article [**25**] entitled *The problem of the calissons*.[13]

[12]Relatively speaking, given the time it takes to actually publish results in mathematics.

[13]A calisson is a certain kind of French confectionary.

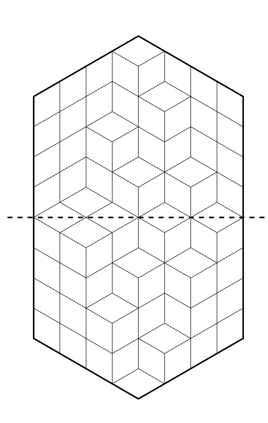
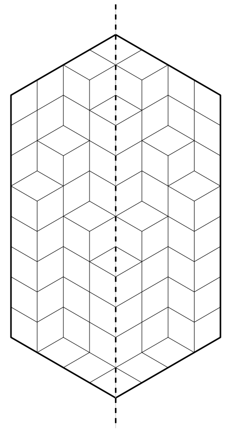



FIGURE 9. Vertically and horizontally symmetric tilings of $H_{4,8,4}$.

Consider the three dimensional representation of a boxed plane partition pictured on the left in Figure 8. As discussed in Remark 2.2.2 this diagram is *not* a plane partition in box, rather it is a two dimensional *depiction* of a three dimensional object, viewed from a particular angle. The figure is a composition of tessellating unit rhombi that are oriented in one of three ways, clustered together in the middle and surrounded by three larger shapes that depict the sides of the box. These outer shapes are arranged in such a way that the external boundary forms a large hexagon whose side lengths correspond precisely to the dimensions of the box. It is also easy to see how the three interior shapes that are not rhombi may each be constructed from homogeneous sets of unit rhombi.

This was the remarkable observation made in [**25**] (although plane partitions are not mentioned at all in this paper): by seeing the depiction of the boxed plane partition on the left of Figure 8 for what it really is (an arrangement of two dimensional shapes) we are afforded an entirely new representation of an (a, b, c)-boxed plane partition as a *rhombus tiling* of a *semi-regular hexagon* with side lengths a, b, c, a, b, c going clockwise from the south-west side, denoted $H_{a,b,c}$ (see Figure 8, right). Letting $M(H_{a,b,c})$ denote the number of rhombus tilings of $H_{a,b,c}$ then leads to a re-statement of MacMahon's box formula.

THEOREM 2.6.1 – MACMAHON [**64**]

The number of rhombus tilings of the hexagon $H_{a,b,c}$ is

$$M(H_{a,b,c}) = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

Let us look again at a symmetric boxed plane partition through this new lens (see Figure 9, right). The symmetry of these objects is clear – they are rhombus tilings that are symmetric about the vertical symmetry axis of $H_{a,b,a}$. If we denote by $M_{|}(H_{a,b,a})$ the number of *vertically symmetric rhombus tilings* of $H_{a,b,a}$ then we may re-state Theorem 2.4.2.

THEOREM 2.6.2 – ANDREWS [**4**], GORDON [**43**]

The number of vertically symmetric rhombus tilings of $H_{a,b,a}$ is

$$M_{|}(H_{a,b,a}) = \prod_{i=1}^{a} \frac{2i+b-1}{2i-1} \prod_{1\le i<j\le a} \frac{i+j+b-1}{i+j-1}.$$

What of the other symmetry class considered in the previous section? The description of transpose-complementary boxed plane partitions is relatively complicated – one has to imagine the

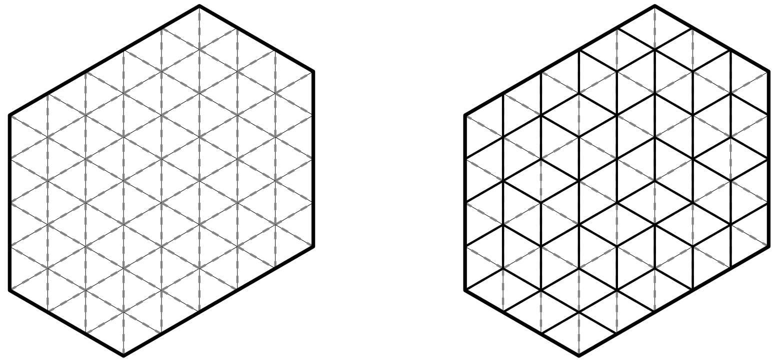


FIGURE 10. The hexagon $H_{3,4,5}$ and a rhombus tiling of the same region.

complement of a boxed plane partition and rotate it in their mind's eye (see Remark 2.5.2). However by changing our perspective and viewing such boxed plane partitions as tilings the symmetry becomes obvious: transpose-complementary boxed plane partitions correspond to rhombus tilings of a hexagon that are invariant under reflection about the horizontal symmetry axis (see Figure 9, left). If we denote by $M_{-}(H_{a,2b,a})$ the number of *horizontally symmetric rhombus tilings* of $H_{a,2b,a}$ then Theorem 2.5.3 may be re-stated as follows.

THEOREM 2.6.3 – PROCTOR [**67**]

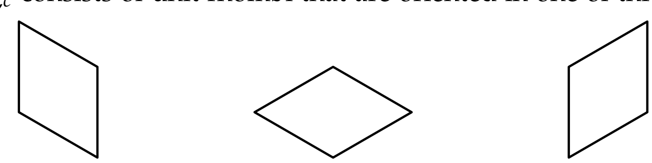
The number of horizontally symmetric rhombus tilings of $H_{a,2b,a}$ is

$$M_{-}(H_{a,2b,a}) = \binom{a+b-1}{a-1} \prod_{i=1}^{a-2} \prod_{j=i}^{a-2} \frac{2b+i+j+1}{i+j+1}.$$

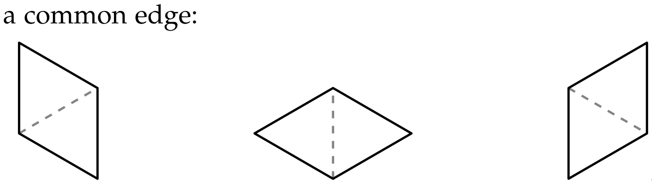
2.7. The triangular lattice

The main focus of this thesis is rhombus tilings of certain shapes and we shall see in subsequent chapters various ways in which one may approach their enumeration. Before doing so, however, we consider how one may obtain a rhombus tiling in a way that circumvents the plane partition construction described in the previous sections, thus allowing us to consider more general regions than simply semi-regular hexagons.

Any tiling of $H_{a,b,c}$ consists of unit rhombi that are oriented in one of three ways:



which we shall refer to as *left leaning*, *horizontal*, and *right leaning* (from left to right). Each unit rhombus above is formed by joining together two unit equilateral triangles (one left pointing, one right pointing) along a common edge:



.

It is also true that the semi-regular hexagon $H_{a,b,c}$ is composed of left and right pointing unit triangles ($(ab+bc+ac)$-many in each class), thus a rhombus tiling of $H_{a,b,c}$ is obtained by partitioning the set of unit triangles contained within it into pairs, each consisting of one left and one right pointing triangle that share an edge (see Figure 10). Clearly $H_{a,b,c}$ is a sub-region of the *unit triangular*

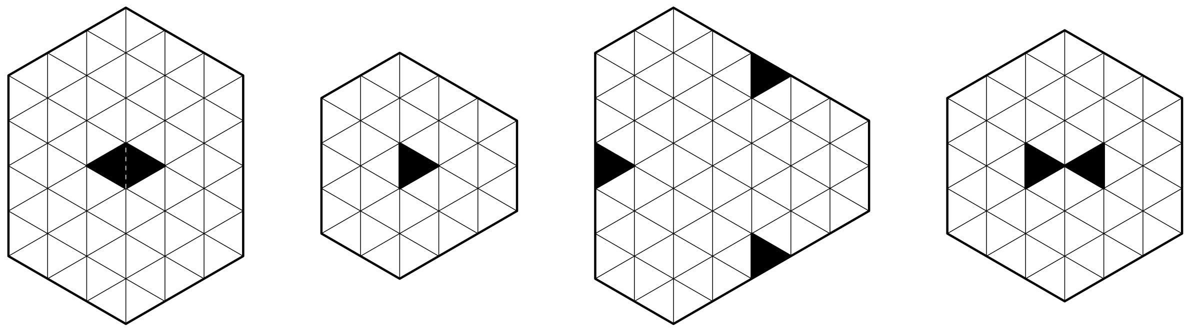


FIGURE 11. Holey hexagons proposed by Propp in [**68**, **69**].

lattice $\mathscr{T}$ (that is, the lattice consisting of unit equilateral triangles, drawn so that one of the families of lattice lines is vertical) and we write $H_{a,b,c} \subset \mathscr{T}$.

This set-up prompts two natural questions that lead to a generalisation of these hexagonal regions:

what if we remove some set of unit triangles contained in the interior of $H_{a,b,c}$*?*

That is, if $H_{a,b,c} \setminus T$ denotes the hexagon $H_{a,b,c}$ with some set of unit triangles T removed from its interior then

what is the formula that gives $M(H_{a,b,c} \setminus T)$ *and how does it compare to* $M(H_{a,b,c})$*?*

REMARK 2.7.1

Removing triangles from the interior of a hexagon may be seen as puncturing the hexagon in a way that leaves behind a set of holes. Thus if $T \neq \emptyset$ then we often refer to $H_{a,b,c} \setminus T$ as a hexagon that *contains holes*, or a *holey hexagon* (where the set T is the set of unit triangles corresponding to the holes). If $T = \emptyset$ then $H_{a,b,c} \setminus T = H_{a,b,c}$ is simply a hexagon. From now on we shall abuse our notation, denoting by H the hexagon $H_{a,b,c}$ for non-specific a, b, c.

It could be argued that it was Propp [**68**] who, in his manuscript entitled *Twenty open problems in enumeration of matchings*, first ignited an interest in tilings of holey hexagons (Kuperberg [**59**] had already shown how rhombus tilings correspond to perfect matchings on the hexagonal lattice and we shall discuss this particular viewpoint in Chapter 6). Propp's list of problems was extended and presented together with a number of their solutions in [**69**], of which five relate directly to the enumeration of *lozenge tilings*[14] of holey hexagons.

Examples of the five different types of holey hexagons that Propp proposed are pictured in Figure 11. The mathematicians who set about enumerating tilings of these regions include Ciucu [**7**, **18**], Eisenkölbl [**27**, **28**], Fischer [**32**], Fulmek [**36**, **37**], Gessel and Helfgott [**38**], Krattenthaler [**18**, **36**, **37**, **57**], and Okada [**57**]. These articles, however, are really just the tip of the iceberg; over the past twenty years a huge variety of holey hexagons and their symmetry classes have been studied. There is a plethora of further theorems and conjectures in this area, many of them extending the classes described in [**69**] (see, for example, [**8**, **9**, **13**, **15–17**, **19**, **21–23**, **29**, **33**, **49**, **52**, **54**]).

REMARK 2.7.2

It is worth noting that although a large number of different families of holes and symmetry classes have already been covered in the literature, as far as the author is aware there seems to be no single result that unifies these different enumerations.

[14] In Propp's paper (and elsewhere in the literature) a rhombus is sometimes referred to as a *lozenge*, however in order to avoid any confusion between diamond shaped tiles and demulcent containing medicated throat sweets we shall use the term rhombus tiling, or simply *tiling*, exclusively from now on.

2.8. An open conjecture

Aside from posing an interesting and complicated enumerative problem there are other good reasons to study tilings of holey hexagons. It seems that when we take a step back and look at the bigger picture[15] the behaviour of the holes in terms of the effect that they have on the ways in which large regions can be tiled parallels very well-known and established two dimensional physical phenomena. In order to discuss this further we make the following definition.

DEFINITION 2.8.1

For a fixed set of holes T that lie inside a region $\mathcal{R}_n$ with outer boundary parametrised by n, the *correlation function* (or *interaction*) of the holes (if it exists) is given by

$$\omega_{\mathcal{R}}(T) := \lim_{n\to\infty} \frac{M(\mathcal{R}_n \setminus T)}{M(\mathcal{R}_n)}.$$

The existence of the limit in the above definition depends on every region in the sequences

$$\mathcal{R}_1, \mathcal{R}_2, \ldots$$

and

$$\mathcal{R}_1 \setminus T, \mathcal{R}_2 \setminus T, \ldots$$

being *tileable*. We say that a region $\mathcal{R}_n \setminus T \subset \mathscr{T}$ is tileable if $M(\mathcal{R}_n \setminus T) > 0$, that is, if unit rhombi can cover its interior in such a way that every unit triangle contained in $\mathcal{R}_n \setminus T$ is covered completely by a rhombus exactly once. Note that a necessary (but not sufficient) condition on the tileability of a region is that it must comprise equinumerous sets of left and right pointing unit triangles.

REMARK 2.8.2

All of the different regions we will consider in this thesis are tileable, but tileability, it seems, is not enough to guarantee that the limit in the correlation function is finite or non-zero. In subsequent chapters we will see how tilings are also subject to certain *boundary effects* – whether the correlation function vanishes, is finite and non-vanishing, or infinite depends quite delicately on the rate at which the boundaries of the region approach infinity.

The interaction may be seen as a function of the holes that measures their effect on the tilings that surround them as the size of the region in which they are contained grows infinitely large. Clearly we can obtain different regions of the plane as the limit of certain finite shapes – for example by replacing $\mathcal{R}_n$ with $H_{an,bn,cn}$ we obtain the entire plane as a limit of finite hexagons as $n \to \infty$.

A hole t in H is created by removing some set of unit triangles from its interior. We assign a statistic[16] to t which we denote $q(t)$, defined so that if t is comprised of L-many left pointing unit triangles and R-many right pointing ones then

$$q(t) := R - L.$$

REMARK 2.8.3

If we remove a set of unit triangles T from a tileable region $\mathcal{R}_n$ then in order for $\mathcal{R}_n \setminus T$ to also be tileable we require that

$$\sum_{t\in T} q(t) = 0$$

(note that this condition is necessary but not sufficient in order to obtain a tiling of $\mathcal{R}_n \setminus T$). On the other hand, if

$$\sum_{t} q(t) = s > 0$$

[15] Quite literally.

[16] By which we mean a function that associates an integer value to each hole.

and the holes are embedded inside tilings of the surface of some torus T_n obtained by identifying parallel sides of a rhombus $\mathcal{R}_n$ then the limit in Definition 2.8.1 does not exist. This calls for an extension of the definition of the correlation function which Ciucu [**11**] gives inductively as

$$\omega_{\mathcal{R}}(T) := \lim_{r\to\infty} \frac{r^{\frac{1}{2s}}\omega_{\mathcal{R}}(T,t')}{\sqrt{C}}$$

where t' is a left pointing unit triangle placed at the boundary of $\mathcal{R}_n$. The value of C is determined by the correlation function of t' and t_0 (the right pointing unit triangle at the centre of $\mathcal{R}_n$):

$$\omega(t_0,t') \sim C\cdot r^{-1/2},$$

where r is the Euclidean distance between t' and t_0. This inductive definition is sufficient for all types of holes, since if

$$\sum_t q(t) = s < 0$$

we can simply reflect the entire region in a vertical straight line.

Having defined the correlation function for holes within tilings we may now state a conjecture made by Ciucu in 2008.

Conjecture 2.8.4 – Ciucu [**11**]

Let $T = \{t_1, t_2, \ldots, t_\alpha\}$ be a fixed set of holes contained in $H_{n,n,n}$, separated by Euclidean distances that are proportional to some real $\tau > 0$. For large τ we have

$$\omega_H(T) \sim \prod_{t_i\in T} C_{t_i} \prod_{1\le i<j\le\alpha} \mathrm{d}(t_i,t_j)^{\frac{1}{2}q(t_i)q(t_j)},$$

where C_{t_i} is a constant dependent on the hole t_i and $\mathrm{d}(t_i,t_j)$ is the Euclidean distance between the holes t_i and t_j.

This conjecture remains wide open, though it has been proved for a small number of different classes of holes (see for example [**10, 26, 40, 42**], and also Ciucu's result [**12**] for tilings embedded on the torus). A full proof of Conjecture 2.8.4 is thus desirable, not least because it also incorporates an analogous conjecture to that of Fisher and Stephenson [**35**] (first published in 1963, this result was settled as recently as two years ago by Dubédat [**26**] using a very different approach to the one outlined within this thesis).[17]

2.9. Let's get physical

For reasons that will soon become clear we will make a brief sojourn into the world of physics and statistical mechanics. All of the material presented in this section can be found in much greater, beautifully clear detail in the first two volumes of Feynman's lectures on physics [**30, 31**].[18]

Imagine we have a *point particle* Q, which is a particle that consists simply of *electrons* or *protons* and hence has practicably negligible mass,[19] fixed somewhere in three dimensional space. We say that Q carries a *charge*, $\mathrm{ch}(Q)$, dependent on the number of electrons and protons that comprise it. If Q consists of more electrons than protons then its overall charge is negative; if it has more protons

[17] It should also be noted that it is in this paper that Dubédat also prove Ciucu's conjecture for the case where T consists of a set of unit triangular holes.

[18] A really wonderful reference for so many ideas in physics, all three volumes are available for free at `http://www.feynmanlectures.caltech.edu/` courtesy of the California Institute of Technology.

[19] The question of what an electron might be is an interesting one. The current thinking is that an electron is some sort of object that behaves simultaneously like a particle and a wave – a wavicle – however for our purposes we shall simply assume it to be a sub-atomic particle that has a charge of negative electrical energy, while a proton is a sub-atomic particle that has a charge of positive electrical energy.

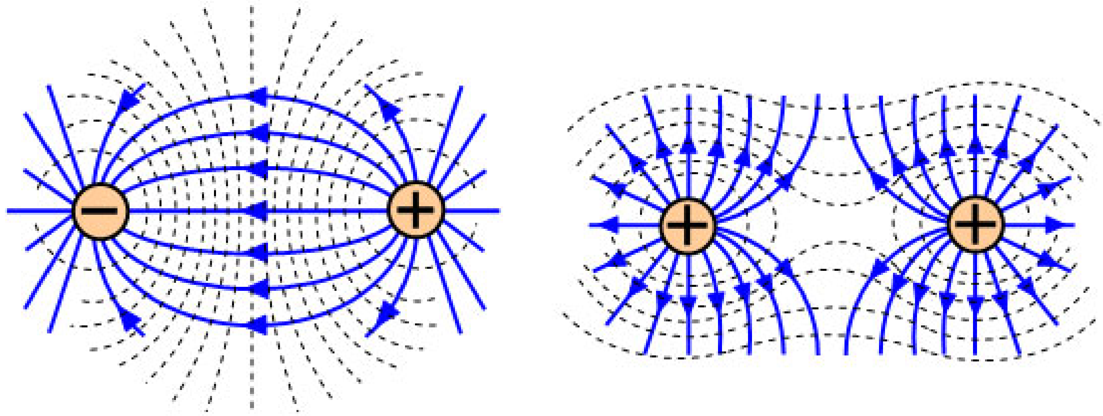


FIGURE 12. The electric fields induced by points of differing charges (left), and a pair of points of positive charge (right).

than electrons the charge is positive; otherwise Q is said to have zero charge. In the following we shall assume that whenever we speak of charge-carrying particles their charge is non-zero and we refer to them simply as *point charges*.

The point charge Q induces what is referred to as an *electric field* in the space that surrounds it. Placing another point charge Q' within this space induces a new electric field that feels the force of the field produced by Q. We can calculate the (signed) magnitude of this force using *Coulomb's law*:

$$F_{Q,Q'} = k_c \frac{\mathrm{ch}(Q)\,\mathrm{ch}(Q')}{r^2},$$

where

$$k_c = \frac{\mu_0 c_0^2}{4\pi}$$

is *Coulomb's constant*[20] and $r = \mathrm{d}(Q, Q')$ is the Euclidean distance between Q and Q'. If the charges of Q and Q' are either both positive or both negative then this force is positive and the charges repel each other, otherwise the force between them is negative and thus attractive. Figure 12 shows two different types of electric fields produced by a pair of point charges.[21]

In overcoming the force between Q and Q' a certain amount of work is done by the electric field, thus we can consider the notion of the *electrostatic potential energy* of Q and Q', denoted $U_{Q,Q'}$. Since

$$-F_{Q,Q'} = \frac{d}{dr} U_{Q,Q'}$$

it follows that the electrostatic potential energy of Q and Q' is

$$k_c \frac{\mathrm{ch}(Q)\,\mathrm{ch}(Q')}{r}.$$

REMARK 2.9.1

The study of electrical forces in systems of charges where all the point charges are *static* (that is, their position is fixed and independent of time) is known as *electrostatics*. Coulomb's law has been shown via experimentation to correctly give the electric force in this case, however if the point charges are permitted to move freely then part of the electric force is also dependent

[20] Here c_0 is the speed of light in a vacuum (299792458 m/s) and μ_0 the vacuum permeability (equal to $4\pi \times 10^{-7} Hm^{-1}$, where H stands for Henry, the unit of electrical inductance). Note that from now on π assumes the value of the ordinary mathematical constant – the ratio of the diameter of a circle to its circumference.

[21] This image was gratefully taken with the author's permission from `http://electricity-automation.com/en/electricity/6`.

on their motion (this is called the *magnetic force*). The forces felt by the fields produced by freely moving point charges are thus *electromagnetic* and are described completely by *Maxwell's equations*,[a] however since the waters quickly muddy once we introduce motion into the picture we shall leave the study of electromagnetism to the authorities (see Feynman [**31**] for example) and concern ourselves with the simpler case of electrostatics.

[a]A set of partial differential equations that form the foundation of classical electromagnetism, quantum field theory, classical optics, and electric circuits.

Consider now an *ensemble of charges* $Q = \{Q_1, Q_2, \ldots, Q_{n-1}\}$ where $n > 2$. When we add a new point charge to the system, say Q_n, it induces an electric field that feels a force from each of the other charges. In order to calculate the total force that acts on the field produced by adding Q_n we can apply what is known as the *superposition principle*:

for all linear systems the net response at a given place and time caused by two or more stimuli is the sum of the responses that would have been caused by each stimulus individually.

We have

$$F_{Q \cup Q_n} = \sum_{i=1}^{n-1} F_{Q_i, Q_n}.$$

If we want to calculate the *total electrostatic potential energy* of this system then we calculate first the electrostatic potential energy between Q_1 and Q_2, and then add the point charges $Q_3, \ldots, Q_n$ to the system one at a time, calculating the potential energy each time a new point charge is added. The total electrostatic potential energy of $Q = \{Q_1, Q_2, \ldots, Q_n\}$ is thus

$$\sum_{1 \leq i < j \leq n} U_{Q_i, Q_j}.$$

For the remainder of this section we will focus on ensembles of point charges that all lie on one plane. In this case the electric force between two static charges Q_i and Q_j is equal to

$$\frac{k_c}{2} \frac{\mathrm{ch}(Q_i)\,\mathrm{ch}(Q_j)}{\mathrm{d}(Q_i, Q_j)}$$

(this is *Coulomb's law in two dimensions*, see [**31**, Chapter 4, Section 2 and Chapter 5, Section 5]), thus the total electrostatic potential energy of the ensemble of charges Q is

$$U_Q = \frac{k_c}{2} \sum_{1 \leq i < j \leq n} \mathrm{ch}(Q_i)\,\mathrm{ch}(Q_j)(-\ln(\mathrm{d}(Q_i, Q_j))). \tag{2.3}$$

There are many good reasons why one might want to calculate the total potential energy of systems of attractive/repulsive charges. One of the central tenets of classical statistical mechanics is *Boltzmann's law* which in the context of electrostatics states that if we have a set of point charges Q in space that induce an electric field then the probability of finding them arranged at a set of mutual distances $\{\mathrm{d}(Q_i, Q_j) : Q_i, Q_j \in Q\}$ is proportional to

$$\exp\left(-U_Q/\kappa\mathbb{T}\right), \tag{2.4}$$

where κ is *Boltzmann's constant*[22] and $\mathbb{T}$ is absolute temperature (see [**30**, Chapter 40, Section 3]). Substituting (2.3) into (2.4) gives this probability as

$$\exp\left(k_c/\kappa\mathbb{T}\right) \prod_{1 \leq i < j \leq n} \mathrm{d}(Q_i, Q_j)^{\frac{1}{2}\,\mathrm{ch}(Q_i)\,\mathrm{ch}(Q_j)} \tag{2.5}$$

[22]Typically stated as $1.3806485279 \times 10^{-23} J \cdot K^{-1}$, where $J \cdot K^{-1}$ stands for Joules per unit of temperature in Kelvin.

Let us return to Ciucu's conjecture from the previous section. If the set of holes $T = \{t_1, t_2, \ldots, t_n\} \subset \mathscr{T}$ satisfy $q(t) \neq 0$ for all $t \in T$ then according to Conjecture 2.8.4 the correlation function $\omega_H(T)$ is given by

$$\prod_{t \in T} C_t \prod_{1 \leq i < j \leq n} \mathrm{d}(t_i, t_j)^{\frac{1}{2} q(t_i) q(t_j)} \tag{2.6}$$

Compare (2.5) and (2.6) – if we suppose that $q(t)$ is equivalent to the charge $\mathrm{ch}(Q)$ of a point particle then these two equations agree up to some multiplicative constant that is independent of the distances between holes or point charges. In other words, if Ciucu's conjecture were to be proved completely we could conclude that the interaction of holes in tilings of the plane by unit rhombi models two dimensional electrostatics very well indeed.[23]

Remark 2.9.2

Perhaps a word should be said regarding the multiplicative constants in (2.5) and (2.6). Although Ciucu does not give a specific value for this constant in general, from a physics perspective the potential discrepancy is not particularly important – what matters is the fact that both expressions are proportionally dependent on the distances between the holes. If we suppose that Q is a set of point charges at mutual distances

$$\mathbf{d} = \{\mathrm{d}(Q_i, Q_j) : Q_i, Q_j \in Q\}$$

then the probability of finding them at a set of mutual distances

$$\mathbf{d}' = \{\mathrm{d}'(Q_i, Q_j) : Q_i, Q_j \in Q\}$$

relative to $\mathbf{d}$ is equal to

$$\frac{P(\mathbf{d}')}{P(\mathbf{d})} = \exp\left(\frac{k_c}{2\kappa\mathbb{T}}\left(\sum_{1 \leq i < j \leq |Q|} \mathrm{ch}(Q_i)\,\mathrm{ch}(Q_j)\ln(\mathrm{d}'(Q_i, Q_j)) - \sum_{1 \leq i < j \leq |Q|} \mathrm{ch}(Q_i)\,\mathrm{ch}(Q_j)\ln(\mathrm{d}(Q_i, Q_j))\right)\right). \tag{2.7}$$

In the rhombus tiling model consider the family $\mathcal{S}$ of all possible arrangements of the set of holes $t_1, t_2, \ldots, t_n$ in the plane where the size and charge of each hole is fixed. Suppose $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ are both elements of $\mathcal{S}$ (so $\mathcal{A}$ is the set of holes $t_1, t_2, \ldots, t_n$ at one set of locations, $\mathcal{B}$ the same set of holes potentially positioned elsewhere, with $q(A_i) = q(B_i) = q(t_i)$ for all i). In **[10]** Ciucu points out that we can define a probability distribution on $\mathcal{S}$ by requiring the ratio of the probabilities $P(\mathcal{A})$ and $P(\mathcal{B})$ to satisfy

$$\frac{P(\mathcal{A})}{P(\mathcal{B})} := \lim_{n \to \infty} \frac{M(H_{n,n,n} \setminus \mathcal{A})}{M(H_{n,n,n} \setminus \mathcal{B}))} = \frac{\omega_H(\mathcal{A})}{\omega_H(\mathcal{B})}.$$

By substituting the conjectured result for the correlation function into the right hand side above we obtain

$$\exp\left(\frac{1}{2}\left(\sum_{1 \leq i < j \leq n} q(A_i)q(A_j)\ln(\mathrm{d}(A_i, A_j)) - \sum_{1 \leq i < j \leq n} q(B_i)q(B_j)\ln(\mathrm{d}(B_i, B_j))\right)\right),$$

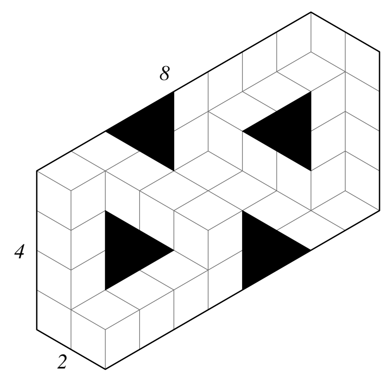
which, if we assume the holes $t_1, t_2, \ldots, t_n$ correspond to the point charges $Q_1, Q_2, \ldots, Q_n$ and let $\mathrm{ch}(Q_i) = q(t_i)$, is precisely (2.7) with $\mathbb{T} = k_c/\kappa$.

[23] It should be noted that Ciucu recently upgraded his conjecture to reflect different arrangements of holes near various types of boundaries **[14]**, hence the tiling model is currently conjectured to model two dimensional steady state heat flows between sources and sinks in a uniform block of material. The same physical laws (that is, Maxwell's equations) govern each of these systems, hence we shall remain throughout this thesis in the realm of electrostatics.

CHAPTER
THREE

NON-INTERSECTING LATTICE PATHS

As we have already seen every boxed plane partition can be visualised as a pile of cubes which is depicted in two dimensions as a rhombus tiling of a hexagon H. Once we begin to remove regions from the interior of H and look at rhombus tilings of the shape that remains this pile-of-cubes interpretation breaks down and we must relax our three dimensional perspective somewhat. Indeed, once we begin to introduce holes into the interior it becomes possible to construct rhombus tilings of regions with holes that assume some quite wonderful Escher-like qualities:



Holey hexagons are more general than boxed plane partitions and this generalisation comes, of course, at a price: it is no longer clear exactly how applicable the tools and techniques that rely on the three dimensional interpretation are when we generalise in this way. As one door closes, however, many more open and in this chapter we shall see how tilings of holey hexagons can be translated into families of non-intersecting lattice paths, which in turn may be counted by evaluating determinants of certain matrices. In order to understand this process we will first examine a bijection that has in some sense been subsumed into the folklore of rhombus tilings and plane partitions.

3.1. A classical bijection

Take a rhombus tiling of $H \setminus T$ and place start (and end, respectively) points at the mid-points of the south-west (north-east) sides of the unit rhombi that lie along the south-west (north-east) edge of H. Apply the same procedure to those rhombi that lie along the north-east (south-west) edges of any holes that lie within its interior. We label the set of start points $S_{H\setminus T}$ and the set of end points $E_{H\setminus T}$.

From a start point $s \in S_{H\setminus T}$ we may construct a *path across unit rhombi* by travelling from one side of a rhombus to the mid-point of its opposite parallel side, and then repeating this process across every rhombus we encounter until our path meets with some end point $e \in E_{H\setminus T}$. By constructing

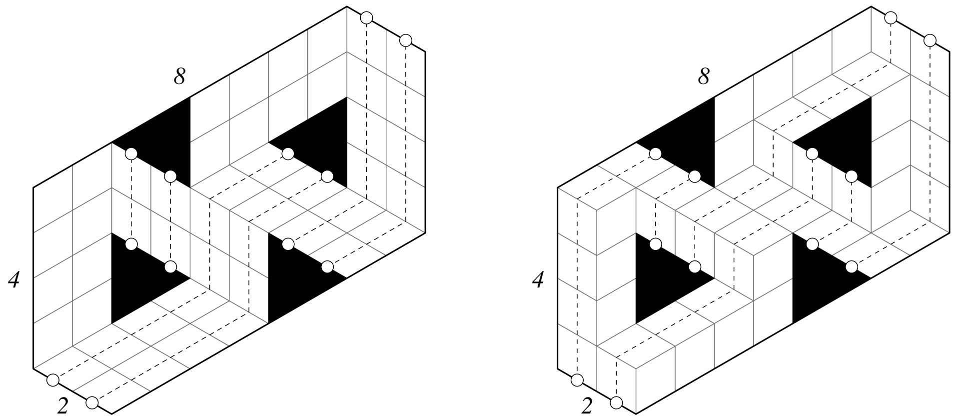


Figure 13. Two families of non-intersecting paths across rhombi.

such a path for each start point in $S_{H\setminus T}$ we obtain a *family of non-intersecting paths across unit rhombi*[24] that corresponds to a particular rhombus tiling of $H \setminus T$. It follows that the set of rhombus tilings of $H \setminus T$ may be represented as a set of families of non-intersecting paths across unit rhombi, where every path traverses rhombi that are oriented in one of two ways (see Figure 13). Moreover it is easy to see that every rhombus contained in $H \setminus T$ that is oriented in one of these two directions is traversed by such a path, hence a family of paths beginning at S and ending at E determines a tiling completely. These paths across rhombi may in turn be translated into non-intersecting lattice paths consisting of unit north and east steps on $\mathbb{Z}_{a,c} \times \mathbb{Z}_{a,b}$, where

$$\mathbb{Z}_{p,q} := \{x + y/2 : x \in \mathbb{Z}, y \equiv p + q - 1 \pmod 2\}.$$

In order to state this bijection explicitly, however, we must first introduce some notation so that we can specify each unit triangle contained in H.

Remark 3.1.1

On reading the following section one could be forgiven for thinking that it is perhaps far more complicated than it needs to be. There is, however, good reason for the following set-up since in later chapters we will need to be able to specifically refer to each unit triangle contained in $H \setminus T$.

3.2. Labelling the interior of a hexagon

The triangular lattice $\mathscr{T}$ consists of three infinite families of lines L_+, L_-, L_∞, where in each family every line has the same gradient: L_+ consists of a set of lines in the polar direction[25] $\pi/6$ with neighbouring lines separated by a distance of $\sqrt{3}$ in the horizontal direction; L_- is the family of lines in the polar direction $-\pi/6$ with neighbouring lines separated by a unit distance along the lines in L_+; and L_∞ consists of a family of vertical lines that intersect all points where the lines in L_- and L_+ intersect (in Figure 14 the families L_+, L_-, and L_∞ are coloured blue, green, and red respectively).

We centre a hexagon $H \subset \mathscr{T}$ by placing an origin O at the point at which two specific lines meet: one connecting the north west corner of H with the south east, the other connecting the north and south corners (this point is indicated by the black dot in Figure 14). Let h denote the horizontal line that intersects O. Each line contained in the interior of H can be labelled according to the signed distance of its intersection with h from O (those intersections that lie to the right have a distance

[24] Within this context *non-intersecting* means that no two paths traverse a common rhombus.

[25] That is, the anti-clockwise angle from a line directed to the east.

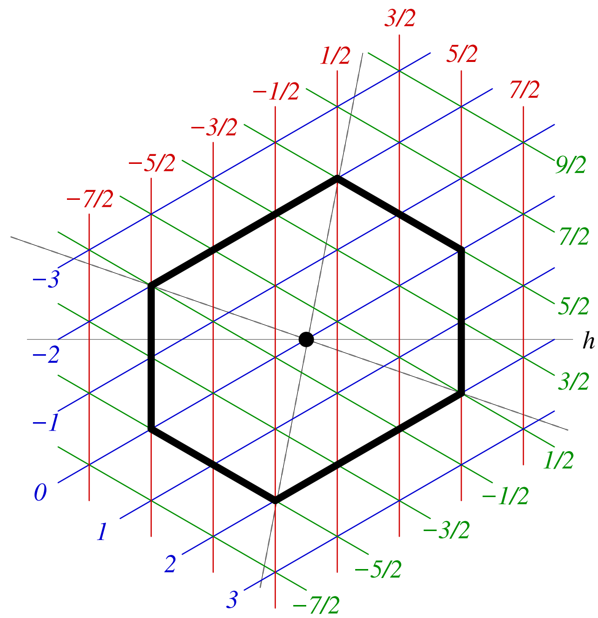


FIGURE 14. Labelling the interior of $H_{2,2,3}$.

treated with a positive sign, those to the left negative). A line will intersect h at Euclidean distance $d \cdot (\sqrt{3}/2)$ along h from O (where $d \in \frac{1}{2}\mathbb{Z}$). The lines that belong to the families L_- or L_+ are labelled $d/2$, while those that belong to L_∞ are simply labelled d. The hexagon $H \subset \mathscr{T}$ is thus the sub-region of $\mathscr{T}$ enclosed by the lines labelled $\pm\frac{b+c}{2} \in L_-$, $\pm\frac{a+b}{2} \in L_+$, and $\pm\frac{a+c}{2} \in L_\infty$. It follows that every unit triangle contained in H may be described by a triple $(u, v, w) \in \mathbb{Z}_{b+1,c} \times \mathbb{Z}_{a+1,b} \times \mathbb{Z}_{a,c+1}$ where u is the label of a line belonging to L_-, so

$$u \in \{-\frac{b+c}{2}, 1 - \frac{b+c}{2}, \ldots, \frac{b+c}{2}\},$$

v is the label of a line belonging to L_+, so

$$v \in \{-\frac{a+b}{2}, 1 - \frac{a+b}{2}, \ldots, \frac{a+b}{2}\},$$

and w is the label of a line belonging to L_∞, so

$$w \in \{-\frac{a+c}{2}, 1 - \frac{a+c}{2}, \ldots, \frac{a+c}{2}\}.$$

Given a holey hexagon $H \setminus T$, let $H^T_\triangleleft$ denote the set of left pointing unit triangles contained in $H \setminus T$ and $H^T_\triangleright$ the right pointing ones. As discussed earlier, a rhombus tiling of $H \setminus T$ is obtained by joining together pairs of unit triangles that share precisely one edge, which is equivalent to partitioning the set $H \setminus T$ into pairs $(\triangleleft, \triangleright)$ where $\triangleleft = (u, v, w)$ belongs to $H^T_\triangleleft$, $\triangleright = (u', v', w')$ belongs to $H^T_\triangleright$, and one of the following conditions hold:

(i) $u' = u + 1$, $v' = v$, and $w' = w - 1$;
(ii) $u' = u$, $v' = v + 1$, and $w' = w - 1$;
(iii) $u' = u + 1$, $v' = v + 1$, and $w' = w$.

A pair satisfying the first condition above constitutes a left leaning rhombus while a right leaning rhombus is comprised of a pair satisfying the second condition. Horizontal rhombi then correspond to pairs of unit triangles satisfying the third condition (see Figure 15).

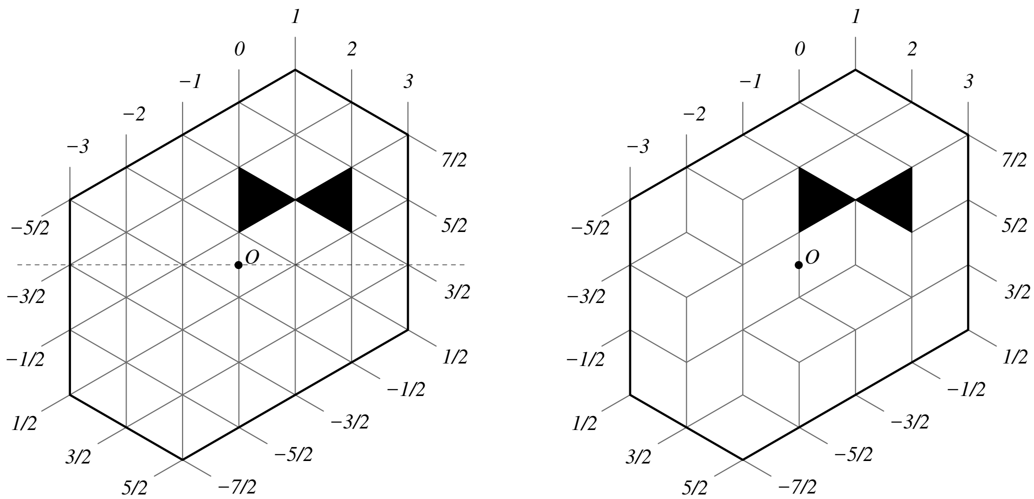


FIGURE 15. The hexagonal region $H_{2,3,4} \setminus \{(\frac{3}{2}, -\frac{1}{2}, 0), (\frac{3}{2}, -\frac{1}{2}, 2)\}$ and one of its tilings.

3.3. From rhombus tilings to families of non-intersecting lattice paths

We have already seen how rhombus tilings of $H \setminus T$ give rise to families of non-intersecting paths across unit rhombi. According to our convention regarding start and end points the unit rhombi that are traversed by these paths are either horizontal or left leaning, therefore the pairs of unit triangles that comprise these paths are those that satisfy conditions (i) and (iii) in the previous section.

Let us identify the set of start points of our paths across rhombi by the left pointing unit triangles on whose south-west edges these start points lie, thus $S_{H\setminus T} := S_H \cup S_T$ where

$$S_H := \{(-\tfrac{b+c}{2}, \tfrac{b-a}{2} + i - 1, i - \tfrac{a+c}{2}) \in H^T_{\triangleleft} : 1 \leq i \leq a\}$$

denotes the set of triangles that lie along the south-west edge of H and

$$S_T := \{(u, v, w) \in H^T_{\triangleleft} \setminus S_H : (u, v+1, w-1) \notin H \setminus T\}$$

those that lie along the north-east edge of any holes in its interior. In a similar way we identify the end points $E_{H\setminus T}$ by the right pointing unit triangles on whose north-east edges the end points lie, thus $E_{H\setminus T} := E_H \cup E_T$ where

$$E_H := \{(\tfrac{b+c}{2}, j - \tfrac{a+b}{2}, \tfrac{c-a}{2} + j - 1) \in H^T_{\triangleright} : 1 \leq j \leq a\}$$

corresponds to those points in $E_{H\setminus T}$ that lie along the north-east boundary of H and

$$E_T := \{(u, v, w) \in H^T_{\triangleright} \setminus E_H : (u, v-1, w+1) \notin H \setminus T\}$$

is the set of unit triangles corresponding to the points in $E_{H\setminus T}$ that lie along the south-west edge of any holes in its interior.

A path across rhombi from a point in $S_{H\setminus T}$ to a point in $E_{H\setminus T}$ consisting of p-many horizontal and q-many left leaning rhombi may be written as a tuple $(R_1, \ldots, R_{p+q})$ of pairs of unit triangles $R_i := (\triangleleft_i, \triangleright_i)$ that correspond to either left leaning or horizontal rhombi, where $\triangleleft_1 \in S_{H\setminus T}$, $\triangleright_{p+q} \in E_{H\setminus T}$, and the north-east side of R_i coincides with the south-west side of R_{i+1} for $1 \leq i \leq p+q-1$ (that is, the first co-ordinate of $\triangleright_i$ agrees with that of $\triangleleft_{i+1}$).

Consider the function $\psi : \mathbb{Z}_{b+1,c} \times \mathbb{Z}_{a+1,b} \times \mathbb{Z}_{a,c+1} \to \mathbb{Z}_{a,c} \times \mathbb{Z}_{a,b}$ given by

$$\psi((u, v, w)) := (\tfrac{1}{2}(u + v + w), \tfrac{1}{2}(u - v - w)).$$

For a horizontal rhombus R_i, if $\triangleleft_i = (u, v, w)$ then $\triangleright_i = (u+1, v+1, w)$, thus

$$\psi(\triangleleft_i) = (\tfrac{1}{2}(u + v + w), \tfrac{1}{2}(u - v - w)),$$

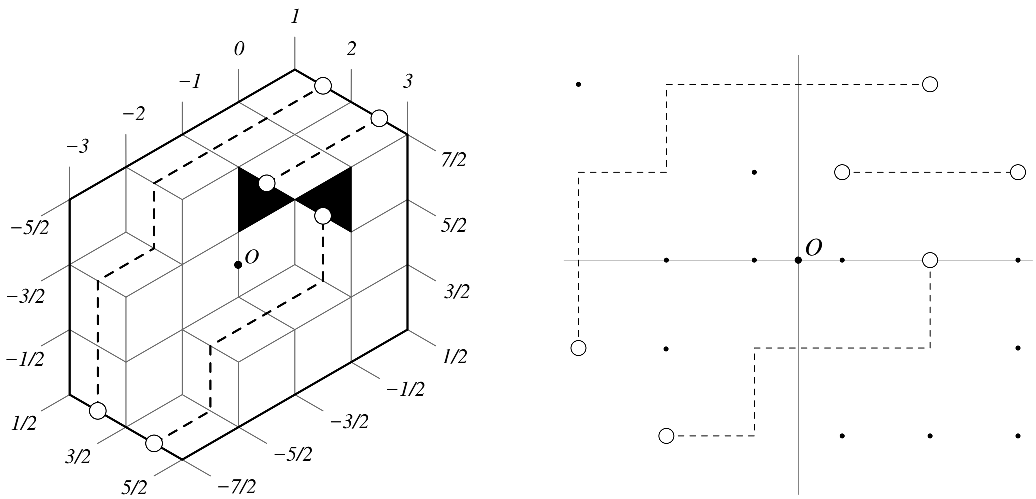


FIGURE 16. A family of non-intersecting lattice paths across unit rhombi and the corresponding paths on $(\mathbb{Z} + \frac{1}{2}) \times \mathbb{Z}$.

and

$$\psi(\triangleright_i) = (\tfrac{1}{2}(u + v + w) + 1, \tfrac{1}{2}(u - v - w)).$$

Clearly ψ maps horizontal rhombi to a pair of co-ordinates in $\mathbb{Z}_{a,c} \times \mathbb{Z}_{a,b}$ that describe an east unit step beginning at $\psi(\triangleleft_i)$ and ending at $\psi(\triangleright_i)$.

In a similar way it can be shown that if R_i is instead a left leaning rhombus then ψ maps R_i to a pair of co-ordinates that describe a north unit step.[26] Furthermore, for $\triangleright_i \in R_i$ and $\triangleleft_{i+1} \in R_{i+1}$, we have

$$\psi(\triangleright_i) = \psi(\triangleleft_{i+1}),$$

hence under ψ a path across rhombi corresponds to a sequence of co-ordinates that encode a *lattice path* on $\mathbb{Z}_{a,c} \times \mathbb{Z}_{a,b}$ that begins at

$$\psi(\triangleleft_1) = (x, y),$$

ends at

$$\psi(\triangleright_{p+q}) = (x + p, y + q),$$

and consists of p-many east and q-many north unit steps.

Applying ψ to every path across rhombi obtained from a tiling of $H \setminus T$ yields a family of lattice paths beginning at

$$\psi(S_{H\setminus T}) := \{\psi(\triangleleft) : \triangleleft \in S_{H\setminus T}\}$$

and ending at

$$\psi(E_{H\setminus T}) := \{\psi(\triangleright) : \triangleright \in E_{H\setminus T}\}.$$

Since in any tiling of $H \setminus T$ no two distinct paths across rhombi will traverse a common rhombus, it follows that no two distinct lattice paths in this family will intersect at a common vertex in $\mathbb{Z}_{a,c} \times \mathbb{Z}_{a,b}$. Such a family of lattice paths is referred to as *non-intersecting*.

The number of tilings of $H \setminus T$ is then the number of families of non-intersecting lattice paths that begin at $\psi(S_{H\setminus T})$ and end at $\psi(E_{H\setminus T})$, and from now on we shall use $S_{H\setminus T}$ and $E_{H\setminus T}$ to denote these sets of points (respectively). An example of a tiling together with its corresponding family of non-intersecting lattice paths may be found in Figure 16.

[26]Similarly ψ maps a right leaning rhombus to a single point.

3.4. The lattice path matrix

We now have a precise method by which we can translate each rhombus tiling of $H \setminus T$ into a distinct family of non-intersecting lattice paths on $\mathbb{Z}_{a,c} \times \mathbb{Z}_{a,b}$ and it is also clear how one might recover a tiling from a set of paths.[27] Enumerating tilings of $H \setminus T$ is therefore equivalent to enumerating the number of families of non-intersecting lattice paths that begin at $S := S_{H\setminus T}$ and end at $E := E_{H\setminus T}$, and this allows us to borrow a particular counting result from the theory of lattice path enumeration.

Remark 3.4.1

The bijection is really quite simple and one could argue that we have over-complicated the relationship to a certain degree. As already mentioned in Remark 3.1.1, in subsequent chapters we will rely not only on this labelling of the interior of the hexagon, but also the function ψ to determine precisely the points in the (half) integer lattice that correspond to the unit triangles in the interior of $H \setminus T$.

Suppose $|S| = k$ and let us fix a labelling of the points in S from s_1 to s_k (we label the points in E in a similar fashion). Within any family of non-intersecting paths that begin at S and end at E each start point s_i will be joined to a distinct end point e_j, thus we describe the connectivity of the start and end points in terms of a permutation on k letters. For some $\sigma \in \mathfrak{S}_k$ let $N(S, E_\sigma)$ denote the total number of families of non-intersecting lattice paths in which each point s_i is connected to $e_{\sigma(i)}$ (for some permutations this could very well be zero).

Let $\mathcal{P}(s \to e)$ denote the number of lattice paths that begin at the point s and end at the point e. We define the *lattice path matrix* corresponding to S and E, denoted $P_{S,E}$, in the following way:

$$P_{S,E} := (\mathcal{P}(s_i \to e_j))_{1\leq i,j\leq k}.$$

The following theorem enables us to express the sum over signed families of non-intersecting paths between S and E as the determinant of such a matrix.

Theorem 3.4.2 – Lindström [61], Gessel and Viennot [39]

For sets of labelled start and end points S and E we have

$$\sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}(\sigma) N(S, E_\sigma) = \pm \det(P_{S,E}),$$

where $\operatorname{sgn}(\sigma) = (-1)^{inv(\sigma)}$ is the signature of σ ($inv(\sigma)$ denotes the number of inversions of σ, that is, the number of pairs $i < j$ such that $\sigma(i) > \sigma(j)$).

Remark 3.4.3

The above result should perhaps also be attributed to Karlin and McGregor, whose earlier result [**48**] was quite vehemently shown by Karlin in [**47**] to be equivalent to that of Gessel and Viennot [**39**] (who were chiefly concerned with lattice path enumeration, whereas Lindström [**61**] was motivated by problems arising in matroid theory). In fact in the 1980s the result of Lindström was independently re-discovered by scientists from three separate communities, however Krattenthaler [**55**, Footnote 5] pointed out that it was Gessel and Viennot who gave it in its most general form (the theorem stated above is a specialisation) and thus proposed that it be named the Lindström-Gessel-Viennot Theorem.

Theorem 3.4.2 is immensely powerful, indeed many of the results found in the following pages arise from explicitly evaluating determinants of lattice path matrices that correspond to rhombus tilings of certain regions. Before we delve, Theorem 3.4.2 in hand, into the enumeration of tilings of

[27] Replace each north step with a left leaning rhombus, each east step with a horizontal one. The outer boundary of the hexagon is determined by the number of left leaning (vertical sides) and horizontal (north-west and south-east sides) rhombi in a single path, the remaining space is tiled by right leaning rhombi.

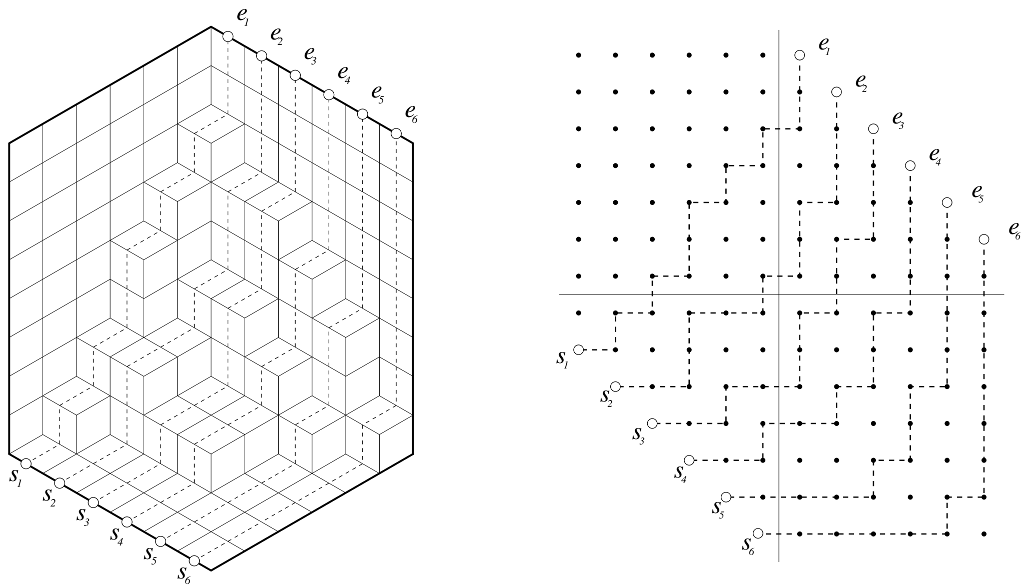


FIGURE 17. A tiling of $H_{6,8,6}$ and its corresponding family of non-intersecting lattice paths.

holey hexagons let us first examine a little further how this theorem fits within the theory of tilings of ordinary (that is, unholey) hexagons.

Consider the hexagon H. Every rhombus tiling of this hexagon corresponds to a family of non-intersecting lattice paths that begin at the set of labelled points $S := \{s_1, s_2, \ldots, s_a\}$, where

$$s_i = \left(i - \tfrac{a+c+1}{2}, \tfrac{a-b+1}{2} - i\right)$$

and end at the set of labelled points $E := \{e_1, e_2, \ldots, e_a\}$, where

$$e_j = \left(j + \tfrac{c-a-1}{2}, \tfrac{b+a+1}{2} - j\right)$$

for $1 \leq i, j \leq a$. Clearly the only permutation $\sigma \in \mathfrak{S}_a$ that gives rise to a family of non-intersecting lattice paths between S and E is the identity permutation $\sigma = \mathbb{1}$ (see Figure 17, for example, where $a = 6$, $b = 8$, and $c = 6$) so according to the Lindström-Gessel-Viennot Theorem

$$N(S, E_{\mathbb{1}}) = \pm \det(P_{S,E}).$$

The number of lattice paths that start at the point (x_1, y_1) and end at the point (x_2, y_2) is given by a simple *binomial coefficient*:[28]

$$\binom{(x_2 - x_1) + (y_2 - y_1)}{x_2 - x_1},$$

thus the (i, j)-entries of the lattice path matrix

$$P_{S,E} = (\mathcal{P}(s_i \to e_j))_{1 \leq i,j \leq a}$$

are given by

$$\mathcal{P}(s_i \to e_j) = \binom{b+c}{c+j-i}.$$

[28]We always interpret binomial coefficients in the natural way, that is

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & 0 \leq k \leq n, \\ 0 & \textit{otherwise}. \end{cases}$$

Since $N(S, E_{\mathbb{1}})$ is equal in this case to the total number of tilings of H, we have the following explicit determinant evaluation for a family of matrices:

$$\det\left(P_{S,E}\right) = \pm \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

Remark 3.4.4

The sum on the left hand side in Theorem 3.4.2,

$$\sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}(\sigma) N(S, E_\sigma),$$

counts what we call *signed families of non-intersecting lattice paths*, since each summand is the size of a set of families of non-intersecting lattice paths in which each family is counted with a sign that corresponds to the connectivity of the k-many labelled start and end points. In the case where the sign of every non-zero term is consistent (as in the preceding example) the number of signed non-intersecting lattice paths from S to E agrees (up to a factor of ± 1) with the number of non-intersecting lattice paths from S to E.

CHAPTER
FOUR

COLLINEAR HOLES

Throughout this chapter we will be concerned with hexagons that have both horizontal and vertical reflective symmetry, thus for the time being we shall denote by H the hexagon with sides of length $a, 2b, a, a, 2b, a$ (going clockwise from the south-west side). The regions we shall remove from the interior of H will be horizontally collinear equilateral triangles of side length 2, arranged symmetrically about the horizontal symmetry axis of H (see Figure 18). We call these 2-holes and throughout we shall assume that T indexes equinumerous left and right pointing 2-holes. All of the regions we will consider in this chapter are tileable.

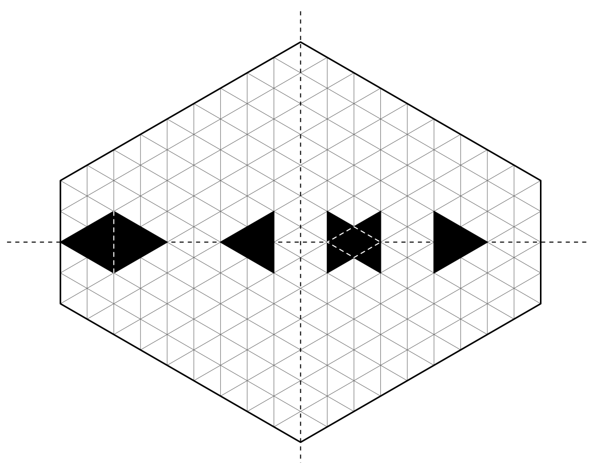


FIGURE 18. The hexagon $H_{9,4,9}$ containing a set of 2-holes.

REMARK 4.0.1

Observe that in order to study $2k$-holes for *any* positive integer k it suffices to consider arrangements of contiguous 2-holes, since k-many 2-holes that have the same orientation and are positioned so that no rhombus can fit between them forces a local arrangement of fixed rhombi (see Figure 19). This is equivalent to creating a hole of side length $2k$ and we say that the k-many 2-holes positioned in this way *induce* a $2k$-hole.

We label each 2-hole according to its directed lattice distance from the centre of H along the horizontal symmetry axis of H. The set T that indexes the 2-holes in H may thus be written as the pair (L, R) of equinumerous sets of integers that are all of the same parity (coinciding with the parity of a), where L is the set of labels of the vertical lattice lines that intersect the sides of the left pointing 2-holes (similarly for R and the right pointing 2-holes). We shall denote the

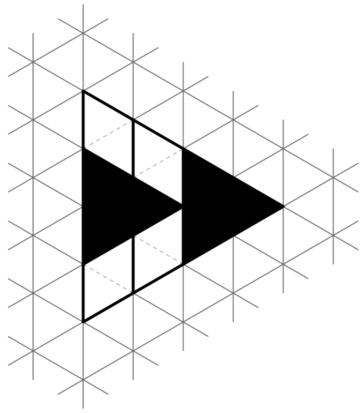


FIGURE 19. Two 2-holes that induce a 4-hole.

hexagon H containing a set of such holes by $H \setminus (L, R)$. The example in Figure 18 is hence denoted $H_{9,4,9} \setminus (\{-7, -1, 3\}, \{-7, 1, 5\})$.

In order to enumerate tilings of these holey hexagons we will make use of a theorem of Ciucu [**6**] that allows us to express the number of tilings of $H \setminus (L, R)$ as a product of the (weighted) count of tilings of smaller sub-regions contained within it.

4.1. Ciucu's Factorisation Theorem

Let us begin by dividing $H \setminus (L, R)$ into two smaller sub-regions by cutting along the zig-zag path that proceeds just below its horizontal symmetry axis. We denote the upper region $\widehat{H} \setminus (L, R)$, the lower $\check{H} \setminus (L, R)$.

Until now we have not considered the notion of the *weight* $w(\mathcal{T})$ of a tiling $\mathcal{T}$, however it is quite reasonable to attach weights (taken from some commutative ring) to rhombi that occupy certain positions inside some region $\mathcal{R}$. We then define the weight of a tiling to be the product of the weights of the tiles that comprise it

$$w(\mathcal{T}) := \prod_{t \in \mathcal{T}} w(t),$$

thus summing over all such products gives the weighted count of tilings of $\mathcal{R}$ (note that the ordinary enumeration of tilings is obtained by setting the weight of every tile to 1).

When we count tilings of $\check{H} \setminus (L, R)$ we attach a weight of 1 to every rhombus, thus the sum over all such weighted tilings, $M(\check{H} \setminus (L, R))$, of this region corresponds to the number of horizontally symmetric tilings of $H \setminus (L, R)$. When we count tilings of $\widehat{H} \setminus (L, R)$, on the other hand, we will attach a weight of 1 to all rhombi with the exception of those horizontal ones that lie within the *nooks* of the bottom zig-zag boundary,[29] which will instead be counted with weight $1/2$ (tilings of the regions under consideration here may contain at most $a - |L \cup R|$ such tiles, positioned where the circles lie in Figure 20). We denote by $M_w(\widehat{H} \setminus (L, R))$ this weighted count of tilings of $\widehat{H} \setminus (L, R)$.

THEOREM 4.1.1 – CIUCU'S FACTORISATION THEOREM [**6**]

The number of ordinary tilings of $H \setminus (L, R)$ is given by

$$M(H \setminus (L, R)) = 2^{a - |L \cup R|} M(\check{H} \setminus (L, R)) M_w(\widehat{H} \setminus (L, R)).$$

The problem of calculating $M(H \setminus (L, R))$ is thus reduced to counting the (weighted) tilings of the sub-regions $\check{H} \setminus (L, R)$ and $\widehat{H} \setminus (L, R)$, to which we can easily apply the theorem of Lindström, Gessel and Viennot from Section 3.4.

[29] By this we mean the sections of the zig-zag boundary in which a horizontal rhombus fits entirely.

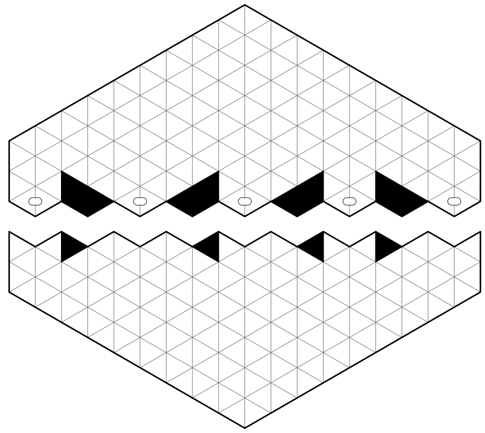


FIGURE 20. The sub-regions $\widehat{H}_{9,4,9} \setminus (\{-1,3\},\{-7,5\})$ and $\check{H}_{9,4,9} \setminus (\{-1,3\},\{-7,5\})$.

REMARK 4.1.2

Ciucu's Factorisation Theorem is in fact far more general than it appears here; the original result pertains to enumerating perfect matchings of any planar graph that has a reflective symmetry axis, with the caveat that removing the vertices that lie along this axis decomposes the graph into two disconnected pieces. Later on we shall see how tilings are related to perfect matchings.

4.2. The reflection method

We begin by identifying the start and end points of the families of non-intersecting lattice paths that correspond to tilings of $H \setminus (L, R)$, where $L = \{l_1, l_2, \ldots, l_k\}$ and $R = \{r_1, r_2, \ldots, r_k\}$ and we suppose without loss of generality that

$$l_1 < l_2 < \cdots < l_k$$

and

$$r_1 < r_2 < \cdots < r_k.$$

In order to make our lives a little easier later on we first rotate $H \setminus (L, R)$ by $\pi/3$ radians (in the anti-clockwise direction), so that the start points and end points that do not border interior holes are placed along the sides of length $2b$. Applying the recipe described in the previous chapter to obtain start and end points on the (half) integer lattice, it follows that the families of non-intersecting lattice paths that correspond to tilings of $H \setminus (L, R)$ begin at the set of points[30]

$$S := \{s_1, s_2, \ldots, s_{2b+2k}\},$$

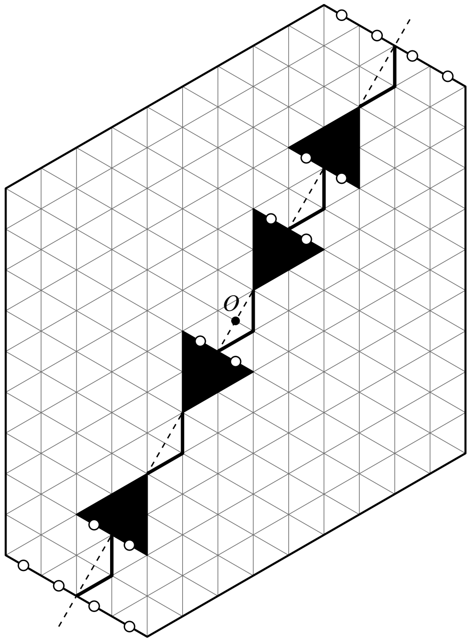
where

$$s_i = \begin{cases} \left(i - \frac{a}{2} - \frac{1}{2}, \frac{1}{2} - \frac{a}{2} - i\right) & 1 \le i \le b, \\ \left(\frac{1+l_{i-b}}{2}, \frac{l_{i-b}-1}{2}\right) & b+1 \le i \le b+k, \\ \left(b + k - \frac{a}{2} + \frac{1}{2} - i, i - b - k - \frac{a}{2} - \frac{1}{2}\right) & b+k+1 \le i \le 2b+k, \\ \left(\frac{l_{i-2b-k}-1}{2}, \frac{1+l_{i-2b-k}}{2}\right) & 2b+k+1 \le i \le 2b+2k, \end{cases}$$

and end at the set of points

$$E := \{e_1, e_2, \ldots, e_{2b+2k}\},$$

[30]From now on we will always assume our start and end points are labelled.



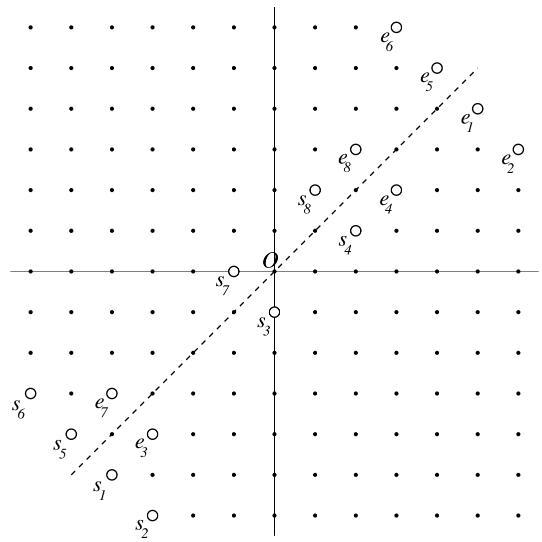


FIGURE 21. The start and end points for families of lattice paths that correspond to tilings of $H_{9,4,9} \setminus (\{-1,3\},\{-7,5\})$.

where

$$e_j = \begin{cases} \left(j-\frac{1}{2}+\frac{a}{2}, \frac{a}{2}+\frac{1}{2}-j\right) & 1 \le j \le b, \\ \left(\frac{1+r_{j-b}}{2}, \frac{r_{j-b}-1}{2}\right) & b+1 \le j \le b+k, \\ \left(b+k+\frac{a}{2}+\frac{1}{2}-j, j-b-k+\frac{a}{2}-\frac{1}{2}\right) & b+k+1 \le j \le 2b+k, \\ \left(\frac{r_{j-2b-k}-1}{2}, \frac{1+r_{j-2b-k}}{2}\right) & 2b+k+1 \le j \le 2b+2k. \end{cases}$$

Sets of such start and end points are shown in Figure 21.

REMARK 4.2.1

The set of start points $s_1, s_2, \ldots, s_{b+k}$ and $e_1, e_2, \ldots, e_{b+k}$ as defined above are all of the start and end points that lie below the horizontal symmetry axis of $H \setminus (L,R)$, while the remaining ones are the reflections of these points in the line $y = x$.

Within this set-up tilings of $\check{H} \setminus (L,R)$ correspond to those families of non-intersecting lattice paths where each path begins at a point in

$$\check{S} := \{s_1, s_2, \ldots, s_{b+k}\},$$

and ends at a point in

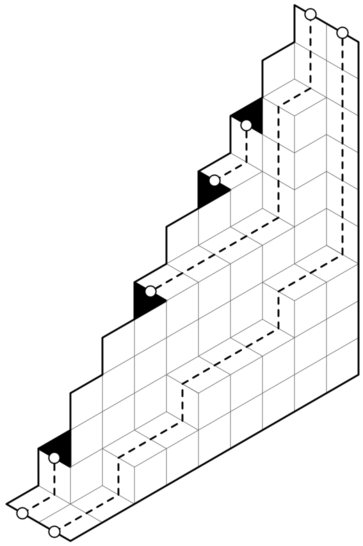
$$\check{E} := \{e_1, e_2, \ldots, e_{b+k}\},$$

and where none of the paths in each family *intersect the line* $y = x$ (see Figure 22).[31] In order to enumerate such paths we employ a technique that is often referred to as *André's reflection method*.[32]

Consider the set of all paths that start at the point (a,b) and end at the point (c,d), denoted $\mathscr{P}((a,b) \to (c,d))$. This set consists of those paths that intersect $y = x$, say $\mathscr{P}^*$, and those that do not. Take a path in $\mathscr{P}^*$ and let p denote the last point at which our path intersects $y = x$.

[31] This includes, of course, paths that touch the line but do not cross it – no touching!

[32] This method is best known for providing an elegant solution to the ballot problem which was originally solved by André using a different approach **[2]**, to whom Feller then falsely attributed the reflection method in his book entitled *An introduction to probability theory and its applications*.



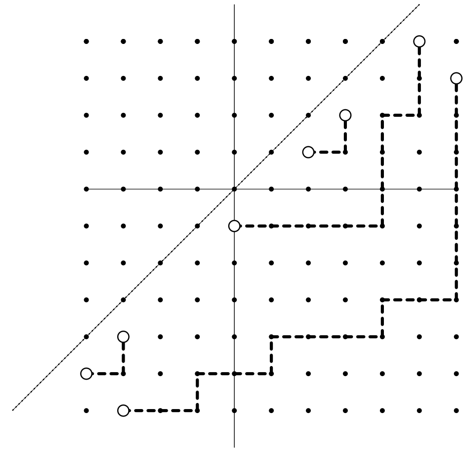


FIGURE 22. A tiling of $\check{H} \setminus (\{-1,3\},\{-7,5\})$ and the corresponding set of lattice paths.

By reflecting the portion of the path that runs from (a,b) to p in $y = x$ we obtain a path from (b,a) to (c,d). A moment's thought convinces us that by doing this to every path in $\mathscr{P}^*$ we obtain $\mathscr{P}((b,a) \to (c,d))$, that is, the set of all lattice paths from (b,a) to (c,d).

The number of paths from (a,b) to (c,d) that do not intersect $y = x$ is thus

$$\check{\mathcal{P}}((a,b) \to (c,d)) = \mathcal{P}((a,b) \to (c,d)) - \mathcal{P}((b,a) \to (c,d)),$$

that is,

$$\check{\mathcal{P}}((a,b) \to (c,d)) = \binom{c+d-a-b}{c-a} - \binom{c+d-b-a}{c-b}.$$

It is straightforward to see how there is precisely one permutation $\sigma \in \mathfrak{S}_{b+k}$ connecting each start point s_i in $\check{S}$ to the end point $e_{\sigma(i)} \in \check{E}$ that gives rise to families of non-intersecting paths where no path touches $y = x$. Combining this fact with the above formula in light of Theorem 3.4.2 yields the following proposition.

PROPOSITION 4.2.2

The total number of horizontally symmetric tilings of $\check{H} \setminus (L,R)$ is given by

$$M(\check{H} \setminus (L,R)) = |\det(P_{\check{S},\check{E}})|,$$

where the matrix $P_{\check{S},\check{E}} := (\check{P}_{i,j})_{1 \le i,j \le b+k}$ has entries given by

$$\check{P}_{i,j} = \begin{cases} \binom{2a}{a-i+j} - \binom{2a}{a-j-i+1} & 1 \le i,j \le b, \\ \frac{2i-1}{a+r_{j-b}+1}\binom{a+r_{j-b}+1}{a/2+r_{j-b}/2-i+1} & 1 \le i \le b, b+1 \le j \le b+k, \\ \frac{2j-1}{a-l_{i-b}+1}\binom{a-l_{i-b}+1}{a/2-l_{i-b}/2-j+1} & b+1 \le i \le b+k, 1 \le j \le b, \\ \frac{1}{r_{j-b}-l_{i-b}+1}\binom{r_{j-b}-l_{i-b}+1}{r_{j-b}/2-l_{i-b}/2} & b+1 \le i,j \le b+k. \end{cases}$$

4.3. Weighted paths

We have so far established a way of evaluating $M(\check{H} \setminus (L,R))$, but there is a further term on the right hand side of the equation in Theorem 4.1.1 that requires our attention:

$$2^{a-|L\cup R|} M_w(\widehat{H} \setminus (L,R)). \tag{4.1}$$

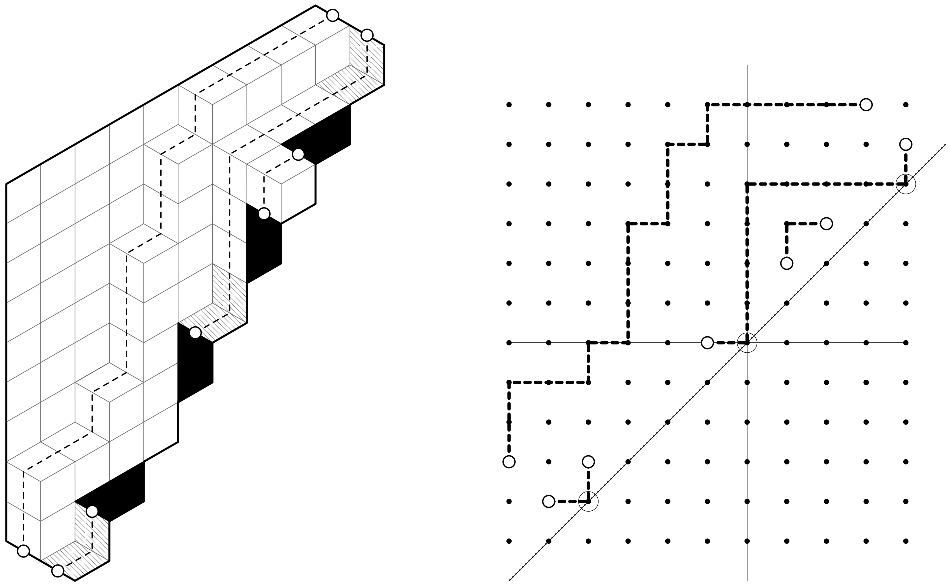


FIGURE 23. A tiling of $\widehat{H} \setminus (\{-1,3\},\{-7,5\}$ that has weight 2^3 and the corresponding set of lattice paths.

The expression on the right is a weighted count over tilings where the weight of each tiling $\mathcal{T}$ is

$$\left(\frac{1}{2}\right)^{a-|L\cup R|-t(\mathcal{T})} = \left(\frac{1}{2}\right)^{a-|L\cup R|} 2^{t(\mathcal{T})},$$

(here $t(\mathcal{T})$ is the number of nooks along the zig-zag boundary of $\widehat{H} \setminus (L,R)$ that are not occupied by a horizontal rhombus). The factor above cancels with the power of 2 in (4.1), thus (4.1) equals the sum over weighted tilings of $\widehat{H} \setminus (L,R)$ where the weight of each tiling $\mathcal{T}$ is given by $2^{t(\mathcal{T})}$.

Ordinary tilings of $\widehat{H} \setminus (L,R)$ correspond to families of paths that begin at

$$\widehat{S} := \{\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{b+k}),$$

(where $\hat{s}_i = s_{b+k+i}$), end at

$$\widehat{E} := \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{b+k}\}$$

(where $\hat{e}_j = e_{b+k+j}$) and do not intersect the line $y = x - 1$,[33] however in order to obtain a determinantal expression for (4.1) we also need to encode the weight of each tiling within this path construction.

A nook occupied by two non-horizontal tiles corresponds in our path representation to a point at which a path touches the line $y = x$, thus if a tiling has weight $2^{t(\mathcal{T})}$ then the corresponding family of paths must have weight $2^{t(P_{\mathcal{T}})}$, where $t(P_{\mathcal{T}})$ is the number of points on $y = x$ that are touched by a path p in the family of paths $P_{\mathcal{T}}$ corresponding to $\mathcal{T}$ (see Figure 23).

Suppose that each unit north or east step p_s in p has weight 1, unless it is a step of the form $(x, x-1) \to (x, x)$ to which we assign a weight of 2. The weight $w(p)$ of a path p is defined to be the product

$$w(p) := \prod_{p_s \in p} w(p_s)$$

and the weight of a family of paths $w(P_{\mathcal{T}})$ is thus

$$w(P_{\mathcal{T}}) := \prod_{p \in P_{\mathcal{T}}} w(p).$$

[33]This means that they can touch but do not cross the line $y = x$.

Clearly under this construction a family of paths $P_{\mathcal{T}}$ from $\widehat{S}$ to $\widehat{E}$ that touches $y = x$ at $t(P_{\mathcal{T}})$ points has weight $2^{t(P_{\mathcal{T}})}$. It is also easy to see that there is only ever precisely one permutation mapping $\widehat{S}$ to $\widehat{E}$ that gives rise to a family of non-intersecting paths that do not intersect $y = x - 1$, thus according to Theorem 3.4.2

$$2^{a-|L\cup R|} M_w(\widehat{H} \setminus (L,R)) = |\det(P_{\widehat{S},\widehat{E}})|,$$

where the (i,j)-entries of $P_{\widehat{S},\widehat{E}}$ are given by the function

$$\widehat{\mathcal{P}}(\hat{s}_i \to \hat{e}_j)$$

that counts the number of paths from $\hat{s}_i$ to $\hat{e}_j$ weighted in the manner described above.

We can interpret such weighted paths combinatorially. Suppose a path p that starts at s_p and ends at e_p touches $y = x$ at $t(p)$ many points. Imagine we reflect any of the following portions of p in the main diagonal: those sections that are contained between touching points; the segment of path from the right-most touching point to e_p. There are $2^{t(p)}$ ways in which we can do this, and a moment's thought convinces us that by applying such a procedure to *every* path from s_p to e_p we obtain *all* paths from s_p to e_p together with *all* paths from s_p to the reflection of the point e_p in $y = x$. It follows that

$$\widehat{\mathcal{P}}((a,b) \to (c,d)) = \mathcal{P}((a,b) \to (c,d)) + \mathcal{P}((a,b) \to (d,c)),$$

that is,

$$\widehat{\mathcal{P}}((a,b) \to (c,d)) = \binom{c+d-b-a}{c-a} + \binom{c+d-b-a}{d-a}.$$

PROPOSITION 4.3.1

For weighted tilings of $\widehat{H} \setminus (L,R)$ we have

$$2^{a-|L|-|R|} M_w(\widehat{H} \setminus (L,R)) = |\det(P_{\widehat{S},\widehat{E}})|,$$

where the matrix $P_{\widehat{S},\widehat{E}} = (\widehat{P}_{i,j})_{1\le i,j\le b+k}$ has entries given by

$$\widehat{P}_{i,j} = \begin{cases} \binom{2a}{a+j-i} + \binom{2a}{a+j+i-1} & 1 \le i,j \le b, \\ \binom{a+r_{j-b}+1}{a/2+r_{j-b}/2+i} & 1 \le i \le b, b+1 \le j \le b+k, \\ \binom{a-l_{i-b}+1}{a/2-l_{i-b}/2+j} & b+1 \le i \le b+k, 1 \le j \le b, \\ \binom{r_{j-b}-l_{i-b}+1}{r_{j-b}/2-l_{i-b}/2+1} & b+1 \le i,j \le b+k. \end{cases}$$

4.4. Two exact formulas

Now that we can express $M(H \setminus (L,R))$ as the product of determinants of two different lattice path matrices

$$M(H \setminus (L,R)) = |\det(P_{\widehat{S},\widehat{E}}) \cdot \det(P_{\check{S},\check{E}})| \tag{4.2}$$

our goal is to evaluate these determinants explicitly. We do this by expressing each matrix as a product of lower and upper triangular matrices, from which the determinant can be easily deduced.

THEOREM 4.4.1 – TG [40]

The matrix $P_{\breve{S},\breve{E}}$ *has LU-decomposition*

$$P_{\breve{S},\breve{E}} = \breve{L} \cdot \breve{U}$$

in which $\breve{L} = (\breve{L}_{i,j})_{1\le i,j\le b+k}$ *has entries given by*

$$\breve{L}_{i,j} = \begin{cases} \breve{A}(a,i,j) & 1 \le j \le i \le b, \\ \breve{B}(a,l_{i-b},j) & b+1 \le i \le b+k, 1 \le j \le b, \\ \breve{E}(a,i,j) & b+1 \le j \le i \le b+k, \\ 0 & \textit{otherwise}, \end{cases}$$

and the matrix $\breve{U} = (\breve{U}_{i,j})_{1\le i,j\le b+k}$ *has entries given by*

$$\breve{U}_{i,j} = \begin{cases} \breve{C}(a,i,j) & 1 \le i \le j \le b, \\ \breve{D}(a,i,r_{j-b}) & 1 \le i \le b, b+1 \le j \le b+k, \\ \breve{F}(a,i,j) & b+1 \le i \le j \le b+k, \\ 0 & \textit{otherwise}, \end{cases}$$

where

$$\breve{A}(a,i,j) = \frac{\Gamma(2i)\Gamma(a+1)\Gamma(i+j-1)\Gamma(2j+a)}{\Gamma(2i-1)\Gamma(2j)\Gamma(i-j+1)\Gamma(j-i+a+1)\Gamma(i+j+a)},$$

$$\breve{B}(a,l,j) = \frac{(-1)^{j+1}\Gamma(j+a-1)\Gamma(2j+a)\Gamma(a-l+1)\Gamma(j+\frac{l}{2}+\frac{a}{2}-1)}{2\,\Gamma(j)\Gamma(2j+2a-2)\Gamma(\frac{a}{2}-\frac{l}{2}+1)\Gamma(\frac{l}{2}+\frac{a}{2})\Gamma(j-\frac{l}{2}+\frac{a}{2}+1)},$$

$$\breve{C}(a,i,j) = \frac{\Gamma(2j)\Gamma(a+1)\Gamma(i+j-1)\Gamma(2i+2a-1)}{\Gamma(2j-1)\Gamma(j-i+1)\Gamma(2i+a-1)\Gamma(i-j+a+1)\Gamma(i+j+a)},$$

$$\breve{D}(a,i,r) = \frac{(-1)^{i+1}\Gamma(2i+1)\Gamma(i+a)\Gamma(a+r+1)\Gamma(i+\frac{a}{2}-\frac{r}{2}-1)}{2\,\Gamma(2i+a-1)\Gamma(i+1)\Gamma(\frac{a}{2}-\frac{r}{2})\Gamma(\frac{a}{2}+\frac{r}{2}+1)\Gamma(i+\frac{a}{2}+\frac{r}{2}+1)},$$

and $\breve{E}(a,i,j), \breve{F}(a,i,j)$ *are matrix entries satisfying*

$$\breve{P}_{i,j} = \sum_{s=1}^{b} \breve{B}(a,l_{i-b},s)\breve{D}(a,s,r_{j-b}) + \sum_{t=b+1}^{\min(i,j)} \breve{E}(a,i,t)\breve{F}(a,t,j)$$

for $b+1 \le i,j \le b+k$.

REMARK 4.4.2

The entries of $\breve{L}$ and $\breve{U}$ in the theorem above are expressed in terms of the *gamma function*, defined for complex z with positive real part to be

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx.$$

In the above theorem the argument of every gamma function is a positive integer (since a, l, and r have the same parity). For any positive integer n it can be shown that $\Gamma(n) = (n-1)!$.

REMARK 4.4.3

The formulas for the LU-decomposition in the above theorem (and, indeed, for every other LU-decomposition found in this thesis) were discovered through a great deal of brute-force enumeration, guesswork, and computational experimentation. An invaluable resource when one finds themselves on the hunt for exact formulas is the software package `RATE` (the german word for *guess*) developed by Christian Krattenthaler (and available at `http://www.mat.univie.ac.at/~kratt/rate/rate.html`).

Proof. The equality involving the entries $\breve{E}(a,i,j)$ and $\breve{F}(a,i,j)$ follows immediately from expressing $P_{\breve{S},\breve{E}}$ as a product of lower and upper triangular matrices, thus proving the above theorem amounts to showing that the following equalities hold:

(i) $\sum_{s=1}^{\min(i,j)} \breve{A}(a,i,s)\breve{C}(a,s,j) = \binom{2a}{a-i+j} - \binom{2a}{a-j-i+1}$;
(ii) $\sum_{s=1}^{i} \breve{A}(a,i,s)\breve{D}(a,s,r) = \frac{2i-1}{r+a+1}\binom{r+a+1}{r/2-i+a/2+1}$;
(iii) $\sum_{s=1}^{j} \breve{B}(a,l,s)\breve{C}(a,s,j) = \frac{2j-1}{a-l+1}\binom{a-l+1}{a/2-l/2-j+1}$.

In order to prove an equality of the form

$$\sum_{k=1}^{n} \alpha(n,k) = \Omega(n,k)$$

we simply need to find a recurrence that satisfies both sides and then check that the equation holds under initial conditions. In **[77]** and **[76]** Zeilberger outlines an efficient algorithm for uncovering such recurrences for terminating hypergeometric series, an efficient computer implementation of which is described by Paule and Schorn in **[66]**. This is what we shall use to prove the three equalities above.

Since in the first equality we have

$$\breve{A}(a,i,s)\breve{C}(a,s,j) = \breve{A}(a,j,s)\breve{C}(a,s,i),$$

we may assume without loss of generality that $\min(i,j) = i$. According to Zeilberger's algorithm the sum

$$\sum_{s=1}^{i} \breve{A}(a,i,s)\breve{C}(a,s,j)$$

satisfies the following recurrence relation with respect to i:

$$\begin{aligned}(i+j-a-1)(a-i+j)\sum_{s=1}^{i} \breve{A}(a,i,s)\breve{C}(a,s,j)\\ +2(a(a+1)-i(i+1)+j(j-1))\sum_{s=1}^{i+1} \breve{A}(a,i+1,s)\breve{C}(a,s,j)\\ -(a+i-j+2)(a+i+j+1)\sum_{s=1}^{i+2} \breve{A}(a,i+2,s)\breve{C}(a,s,j) = 0.\end{aligned}$$

The expression on the right hand side of the equation in (i) satisfies precisely the same recurrence, thus the equality holds in general once it has been verified under initial conditions.

Zeilberger's algorithm shows that the sum in the second equality satisfies

$$\begin{aligned}(2i+1)(2i-a-r-2)\sum_{s=1}^{i} \breve{A}(a,i,s)\breve{D}(a,s,r)\\ +(2i-1)(a+2i+r+2)\sum_{s=1}^{i+1} \breve{A}(a,i+1,s)\breve{D}(a,s,r) = 0,\end{aligned} \quad (4.3)$$

as does the right hand side, so once again the equality holds in general once we have checked it is true for $i = 1$ and $i = 2$.

Replacing $\breve{A}(a,i,s), \breve{D}(a,s,r), i$, and r with $\breve{C}(a,s,j), \breve{B}(a,l,s), j$, and $-l$ respectively in (4.3) yields a recurrence satisfied by the left hand side of the third equality. The right hand side of the equation in (iii) also satisfies the same recurrence, thus after confirming that this equality holds for $j = 1$ and $j = 2$ we are done. ☐

Theorem 4.4.1 immediately gives rise to the following corollary.

COROLLARY 4.4.4 – TG [40]

The number of tilings of $\check{H} \setminus (L, R)$ *is*

$$M(\check{H} \setminus (L,R)) = \binom{a+b-1}{a-1} \prod_{i=1}^{a-2} \prod_{j=i}^{a-2} \frac{2b+i+j+1}{i+j+1} \cdot |\det(\check{Q}_{L,R})|,$$

where $\check{Q}_{L,R} = (\check{Q}_{l,r})_{l \in L, r \in R}$ *is the matrix with entries given by*

$$\check{Q}_{l,r} = \frac{1}{r-l+1} \binom{r-l+1}{r/2 - l/2} - \sum_{s=1}^{b} \check{B}(a,l,s)\check{D}(a,s,r).$$

PROOF. As the matrix $\check{L}$ is lower triangular and $\check{L}_{i,i} = 1$ for $1 \leq i \leq b$ we have

$$\det(P_{\check{S},\check{E}}) = \left(\prod_{s=1}^{b} \check{U}_{s,s} \right) \det(\check{L}^* \cdot \check{U}^*),$$

where $\check{L}^* = (\check{L}_{b+i,b+j})_{1 \leq i,j \leq k}$ and $\check{U}^* = (\check{U}_{b+i,b+j})_{1 \leq i,j \leq k}$. It turns out that the product over s above is in fact equal to Proctor's formula[34] from Theorem 2.6.3 and since the entries of the matrix on the right are given by

$$\sum_{t=b+1}^{\min(i,j)} \check{E}(a, b+i, t)\check{F}(a, t, b+j)$$

for $1 \leq i, j \leq k$, which according to Theorem 4.4.1 is equal to

$$\check{P}_{b+i,b+j} - \sum_{s=1}^{b} \check{B}(a, l_i, s)\check{D}(a, s, r_j),$$

this completes the proof. □

A result very similar to Theorem 4.4.1 also holds for the matrix $P_{\widehat{S},\widehat{E}}$.

THEOREM 4.4.5 – TG [40]

The matrix $P_{\widehat{S},\widehat{E}}$ *defined above has LU-decomposition*

$$P_{\widehat{S},\widehat{E}} = \widehat{L} \cdot \widehat{U},$$

where $\widehat{L} = (\widehat{L}_{i,j})_{1 \leq i,j \leq b+k}$ *is given by*

$$\widehat{L}_{i,j} = \begin{cases} \widehat{A}(a,i,j) & 1 \leq j \leq i \leq b, \\ \widehat{B}(a, l_{i-b}, j) & b+1 \leq i \leq b+k, 1 \leq j \leq b, \\ \widehat{E}(a,i,j) & b+1 \leq j \leq i \leq b+k, \\ 0 & \textit{otherwise}, \end{cases}$$

and $\widehat{U} = (\widehat{U}_{i,j})_{1 \leq i,j \leq b+k}$ *is given by*

$$\widehat{U}_{i,j} = \begin{cases} \widehat{C}(a,i,j) & 1 \leq i \leq j \leq b, \\ \widehat{D}(a,i,r_{j-b}) & 1 \leq i \leq b, b+1 \leq j \leq m+k, \\ \widehat{F}(a,i,j) & b+1 \leq i \leq j \leq b+k, \\ 0 & \textit{otherwise}, \end{cases}$$

[34] Indeed this makes sense, for if $L = R = \emptyset$ then the number of tilings of $\check{H} \setminus (L, R)$ is simply the number of horizontally symmetric tilings of H.

where

$$\widehat{A}(a,i,j) = \frac{\Gamma(a+1)\Gamma(i+j-1)\Gamma(2j+a)}{\Gamma(2j-1)\Gamma(i-j+1)\Gamma(j-i+a+1)\Gamma(i+j+a)},$$

$$\widehat{B}(a,l,j) = \frac{(-1)^{j+1}\Gamma(j+a)\Gamma(2j+a)\Gamma(a-l+2)\Gamma(j+\frac{l}{2}+\frac{a}{2}-1)}{\Gamma(j)\Gamma(2j+2a)\Gamma(\frac{a}{2}-\frac{l}{2}+1)\Gamma(\frac{l}{2}+\frac{a}{2})\Gamma(j-\frac{l}{2}+\frac{a}{2}+1)},$$

$$\widehat{C}(a,i,j) = \frac{\Gamma(a+1)\Gamma(i+j-1)\Gamma(2i+2a)}{\Gamma(j-i+1)\Gamma(2i+a-1)\Gamma(i-j+a+1)\Gamma(i+j+a)},$$

$$\widehat{D}(a,i,r) = \frac{(-1)^{i+1}\Gamma(2i-1)\Gamma(i+a)\Gamma(a+r+2)\Gamma(i+\frac{a}{2}-\frac{r}{2}-1)}{\Gamma(i)\Gamma(2i+a-1)\Gamma(\frac{a}{2}-\frac{r}{2})\Gamma(\frac{a}{2}+\frac{r}{2}+1)\Gamma(i+\frac{a}{2}+\frac{r}{2}+1)},$$

and $\widehat{E}(a,i,j)$ *and* $\widehat{F}(a,i,j)$ *are matrix entries satisfying*

$$\widehat{P}_{i,j} = \sum_{s=1}^{b} \widehat{B}(a,l_{i-b},s)\widehat{D}(a,s,r_{j-b}) + \sum_{t=b+1}^{\min(i,j)} \widehat{E}(a,i,t)\widehat{F}(a,t,j)$$

for $b+1 \le i,j \le b+k$.

The proof of this theorem is, somewhat unsurprisingly, almost identical to that of Theorem 4.4.1 and as such has been deferred to Appendix A along with the proof of the following corollary.

Corollary 4.4.6 – TG [**42**]

For tilings of $\widehat{H} \setminus (L,R)$ *we have*

$$2^{a-|L|-|R|} M_w(\widehat{H} \setminus (L,R)) = \prod_{i=1}^{a} \frac{2i+2b-1}{2i-1} \prod_{1\le i<j\le a} \frac{i+j+2b-1}{i+j-1} \cdot |\det(\widehat{Q}_{L,R})|,$$

where $\widehat{Q}_{L,R} = (\widehat{Q}_{l,r})_{l\in L, r\in R}$ *is the matrix with entries given by*

$$\widehat{Q}_{l,r} = \binom{r-l+1}{r/2-l/2+1} - \sum_{s=1}^{b} \widehat{B}(a,l,s)\widehat{D}(a,s,r).$$

Comparing the formula above with Theorem 2.6.2 we see that the product part (by which we mean the pre-factor in front of the determinant) is precisely the formula conjectured by MacMahon (proved by Andrews and Gordon) that counts vertically symmetric tilings of H.[35] Thus combining Corollary 4.4.4 and Corollary 4.4.6 by way of (4.2) yields the following theorem.

Theorem 4.4.7 – TG [**42**]

The number of tilings of $H \setminus (L,R)$ *is*

$$M(H \setminus (L,R)) = M_{-}(H) \cdot M_{|}(H) \cdot |\det(\check{Q}_{L,R}) \cdot \det(\widehat{Q}_{L,R})|,$$

that is,

$$M(H \setminus (L,R)) = M_{-}(H \setminus (L,R)) \cdot M_{|}(H) \cdot |\det(\widehat{Q}_{L,R})|.$$

This theorem, which under a certain specialisation reduces to a case that is also covered by a different result of Ciucu [**9**], is useful because it reduces the problem of calculating the number of tilings of $H_{a,2b,a}$ containing a set of horizontally collinear holes of any even side length into two determinant evaluations where the size of the matrices involved is dependent on the number of 2-holes and not the size of the entire region. This will be especially useful when we come to considering the asymptotics of these formulas, since obtaining the correlation function of the holes

[35] This is certainly very surprising – although they are equal in number it is not clear at all how the strangely weighted tilings of $\widehat{H}$ (the number of which is obtained by setting $L = R = \emptyset$ in Corollary 4.4.6) are related to vertically symmetric tilings of H.

reduces, via the above theorem, to studying the asymptotics of the entries of $\widehat{Q}_{L,R}$ and $\check{Q}_{L,R}$. To see this note that if $L = R = \emptyset$ then Theorem 4.4.7 yields

$$M(H) = M_{-}(H) \cdot M_{|}(H), \tag{4.4}$$

(this is also a special case of Theorem 4.4.8 below) thus the correlation of the holes[36] is equal to

$$\omega_H(L,R) = \lim_{a,b\to\infty} \left(|\det(\check{Q}_{L,R}) \cdot \det(\widehat{Q}_{L,R})| \right).$$

We shall return to this expression in the following chapter.

The factorisation (4.4) is certainly curious. Of course it may be gleaned from directly studying the enumeration formulas for the different symmetry classes, but it turns out that much more is true.

Theorem 4.4.8 – Ciucu and Krattenthaler [22]

Suppose $r > 0$ for all $r \in R$ and $L = \{-r : r \in R\}$. The number of tilings of $H \setminus (L,R)$ has the following factorisation

$$M(H \setminus (L,R)) = M_{-}(H \setminus (L,R)) \cdot M_{|}(H \setminus (L,R)).$$

The conditions on L and R in the above theorem correspond to a class of holey hexagons where the horizontally collinear holes are symmetric about the horizontal *and* vertical symmetry axes of H, and all left pointing holes lie to the left of all of the right pointing ones (see Figure 24). Theorem 4.4.7 and Theorem 4.4.8 together then yield the following lemma, generalising the enumerative result given by Ciucu and Krattenthaler in **[20]**.

Lemma 4.4.9 – TG

If $r > 0$ for all $r \in R$ and $L = \{-r : r \in R\}$ then the number of vertically symmetric tilings of $H \setminus (L,R)$ is

$$M_{|}(H \setminus (L,R)) = M_{|}(H) \cdot |\det(\widehat{Q}_{L,R})|.$$

Remark 4.4.10

Clearly if $|R| = 1$ then the above lemma gives an equivalent expression to the exact formula in **[20]**.

The proof of Theorem 4.4.8 that appears in **[22]** relies on manipulating a matrix whose Pfaffian (see Definition 4.5.1 below) counts vertically symmetric tilings of $H \setminus (L,R)$ in such a way that we see this Pfaffian as the square root of the product of determinants of two identical smaller matrices. This smaller matrix turns out to be precisely the lattice path matrix $P_{\widehat{S},\widehat{E}}$ defined above for this class of holes.

For ordinary hexagons that do not contain holes there are many similar such factorisation results (see **[72]**), however it is often the case that the proofs follow from manipulating the formulas themselves (or certain polynomials related to the families of tilings), rather than establishing explicit bijections between sets of tilings. Stanley briefly touches on this at the end of **[72]**, where he alludes to an alternative bijective proof to the one he provides which nonetheless relies on something called the *Littlewood-Richardson rule*.[37]

[36]This was defined in Section 2.8, see Definition 2.8.1.

[37]The Littlewood-Richardson rule is a combinatorial description of the coefficients that arise when the product of two Schur functions is decomposed into a linear combination of other Schur functions.

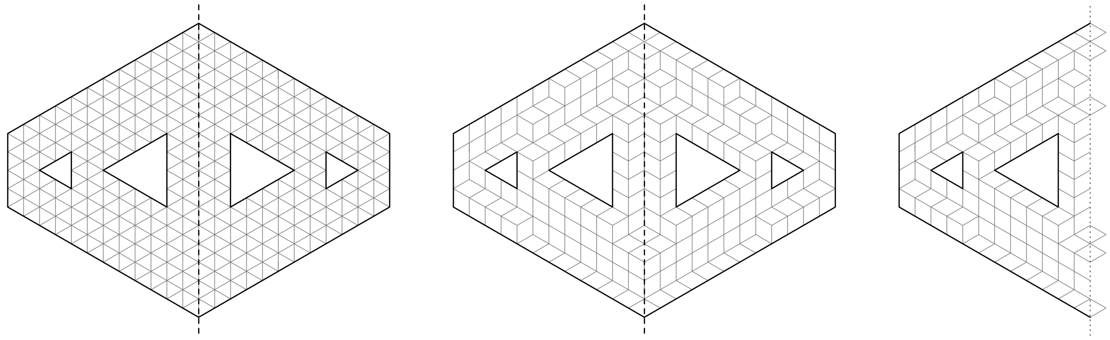


FIGURE 24. The hexagon $H_{12,4,12} \setminus (\{-8,-4,-2\},\{2,4,8\})$, a vertically symmetric tiling of the entire hexagon, and the corresponding tiling of the half hexagon $V_{12,4,12} \setminus \{-8,-4,-2\}$.

OPEN PROBLEM 4.4.11

Is it possible to prove Theorem 4.4.8 bijectively without employing the Littlewood-Richardson rule? That is, if T_- and $T_|$ denote the sets of horizontally (and vertically, respectively) symmetric tilings of $H \setminus (L,R)$, then what is the operation on *tilings* that bijectively maps

$$T_| \times T_- \longleftrightarrow T,$$

where T is the set of all tilings of $H \setminus (L,R)$?

Such a proof would be a wonderful find, even for the case $L = R = \emptyset$. This seemingly innocuous problem appears to be really quite difficult to solve and a great many hours can be whiled away endlessly trying to transform sets of pairs of symmetric tilings into sets of tilings and back again. A map certainly exists – we have a proof of this theorem that is based on matrix manipulation – however it would be far more satisfying to see the relationship directly within the tilings themselves.

A clue to unearthing such a bijection could perhaps lie in the fact that if the holes described in Theorem 4.4.8 are flipped (that is, if $r < 0$ for all $r \in R$ and $L = \{-r : r \in R\}$ – see Figure 25), then we can no longer express $M(H \setminus (L,R))$ as a straightforward product of $M_|(H \setminus (L,R))$ and $M_-(H \setminus (L,R))$. We will establish this analogue in the following sections.

4.5. Vertically symmetric tilings

Consider now a vertically symmetric tiling of the hexagon $H \setminus (L,R)$ where $r < 0$ for all $r \in R$ and $L = \{-r : r \in R\}$ (we shall assume this is the case for the remainder of this section). This corresponds to a tiling of the left half of $H \setminus (L,R)$, bounded on the right by a vertical free boundary[38] that coincides with the vertical symmetry axis of $H \setminus (L,R)$. We shall denote this sub-region $V \setminus R$ (see Figure 25). Unless otherwise specified we shall assume that $|R| = k$.

Tilings of $V \setminus R$ may again be translated into families of non-intersecting lattice paths as in the previous sections, however in doing so it is clear that the families of lattice paths we need to enumerate are those in which every point that arises from a hole is an end point for a path, while

[38] This is a vertical boundary across which unit rhombi are permitted to protrude half way.

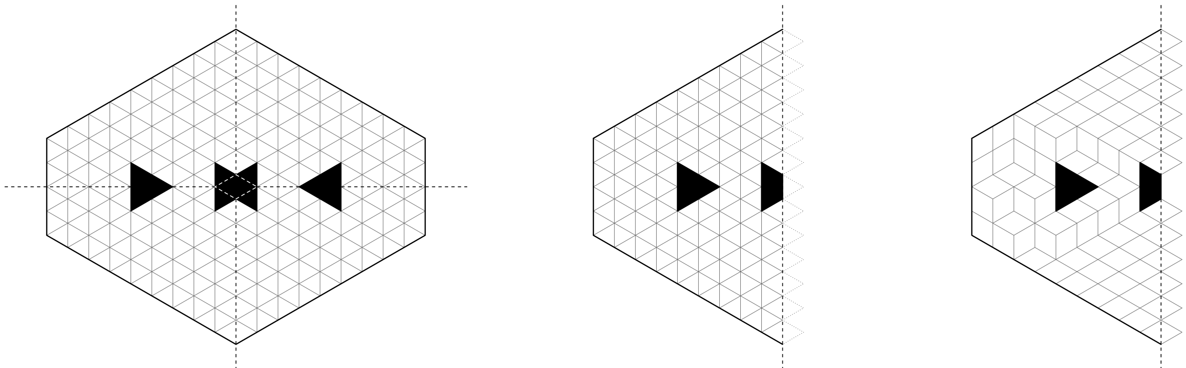


FIGURE 25. The hexagon $H_{9,4,9} \setminus (\{1,5\},\{-5,-1\})$, the half hexagon $V_{9,4,9} \setminus \{-5,-1\}$ together with a tiling of this region.

the remaining $2b-2k$ paths end at some point on the line $y=-x$.[39] Figures 26 and 27 illustrate two different tilings that correspond to families of non-intersecting paths with differing end points.

Enumerating this symmetry class of tilings, then, appears to be quite a complicated task, however by borrowing another result from the world of non-intersecting lattice paths we can reduce such a calculation to evaluating the Pfaffian of a single *skew-symmetric matrix*.[40]

DEFINITION 4.5.1

The *Pfaffian* of a $2n \times 2n$ skew-symmetric matrix $A = (A_{i,j})_{1\le i,j\le 2n}$ is

$$\mathrm{Pf}(A) := \sum_{\sigma\in\mathcal{M}_{2n}} \mathrm{sgn}_{\mathcal{M}}(\sigma) \prod_{\substack{i<j \\ \text{matched in } \sigma}} A_{i,j},$$

where $\mathcal{M}_{2n}$ denotes the set of perfect matchings on the vertices $\{1,2,\dots,2n\}$, and the signature of a matching, $\mathrm{sgn}_{\mathcal{M}}(\sigma)$, is $(-1)^{\mathrm{cr}(\sigma)}$, where $\mathrm{cr}(\sigma)$ is the number of *crossings* of σ.

REMARK 4.5.2

In the above definition $\mathcal{M}_{2n}$ is the set of perfect (permutation) matchings, which is a partition of the set $\{1,2,\dots,2n\}$ into pairs. If σ is a perfect matching then a crossing is a quadruple of elements $i<j<k<l$ such that i is matched with k and j is matched with l. It is a well-known fact that

$$\mathrm{Pf}(A)^2 = \det(A).$$

We now state a theorem that can be found in **[20]** which is a trivial extension of an older result of Stembridge **[73]** and can easily be applied to the families of paths in which we are currently interested.

THEOREM 4.5.3 – STEMBRIDGE **[73]**, CIUCU AND KRATTENTHALER**[20]**

Let $S := \{S_1,S_2,\dots,S_p\}$, $E' := \{E'_1,E'_2,\dots,E'_q\}$ and $E := \{E_1,E_2,\dots\}$ be finite sets of lattice points on the integer lattice $\mathbb{Z}^2$ with $p+q$ even. Then

$$\mathrm{Pf}\begin{pmatrix} X & Y \\ -Y^t & 0 \end{pmatrix} = (-1)^{\binom{q}{2}} \sum_{\sigma\in\mathfrak{S}_p} \mathrm{sgn}(\sigma) N(S_\sigma, E', E),$$

[39] With respect to tilings, these end points that are allowed to vary along this line correspond to positions along the free boundary that are not occupied by horizontal rhombi.

[40] That is, a matrix A satisfying $A^T = -A$.

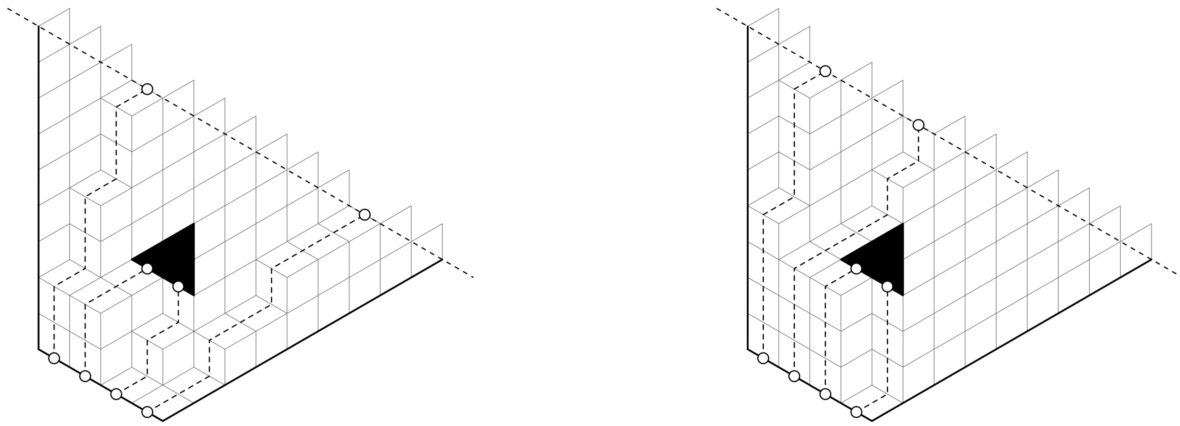


FIGURE 26. Two tilings of $V \setminus \{-5\}$ with differing end points.

where $N(S_\sigma, E', E)$ is the number of families of non-intersecting lattice paths where the start point $S_{\sigma(k)}$ is connected to E'_k for $k = 1, 2, \ldots, q$, and to E_{j_k} for $k = q+1, q+2, \ldots, p$ with the indices satisfying $j_{q+1} < j_{q+2} < \cdots < j_p$. The matrix $X = (x_{i,j})_{1\le i,j\le p}$ has entries given by

$$x_{i,j} = \sum_{1\le u<v} (\mathcal{P}(S_i \to E_u)\mathcal{P}(S_j \to E_v) - \mathcal{P}(S_i \to E_v)\mathcal{P}(S_j \to E_u)),$$

while $Y = (y_{i,j})_{1\le i\le p, 1\le j\le q}$ has entries given by

$$y_{i,j} = \mathcal{P}(S_i \to E'_j).$$

By replacing S with the set

$$\{s_1, s_2, \ldots, s_{2b}\},$$

where

$$s_i = (b - i - \frac{a}{2} + \frac{1}{2}, i - \frac{a}{2} - b - \frac{1}{2}),$$

letting E' be the set of end points

$$\{e'_1, e'_2, \ldots, e'_{2k}\},$$

where

$$e'_j = \begin{cases} (\frac{r_j+1}{2}, \frac{r_j-1}{2}) & 1 \le j \le k, \\ (\frac{r_{j-k}-1}{2}, \frac{r_{j-k}+1}{2}) & k+1 \le j \le 2k, \end{cases}$$

and E be the set

$$\{e_1, e_2, \ldots, e_{2b+a}\},$$

where

$$e_j = (j - b - \tfrac{a}{2} - \frac{1}{2}, b + \tfrac{a}{2} + \frac{1}{2} - j),$$

in Theorem 4.5.3 we see that the sum over signed families of non-intersecting paths that correspond to tilings of $V \setminus R$ is given (up to sign) by the Pfaffian of a certain matrix.

REMARK 4.5.4

If the start and end points S, E, E' in Theorem 4.5.3 satisfy a condition that goes by the name of *D-compatibility*[a] we recover the original result of Stembridge **[73]**, where the expression on the right hand side counts all families of non-intersecting lattice paths from S to E', E. The set of points with which we are concerned clearly do not satisfy this condition, nonetheless it is easily shown that in our situation the non-zero terms in the sum on the right hand side

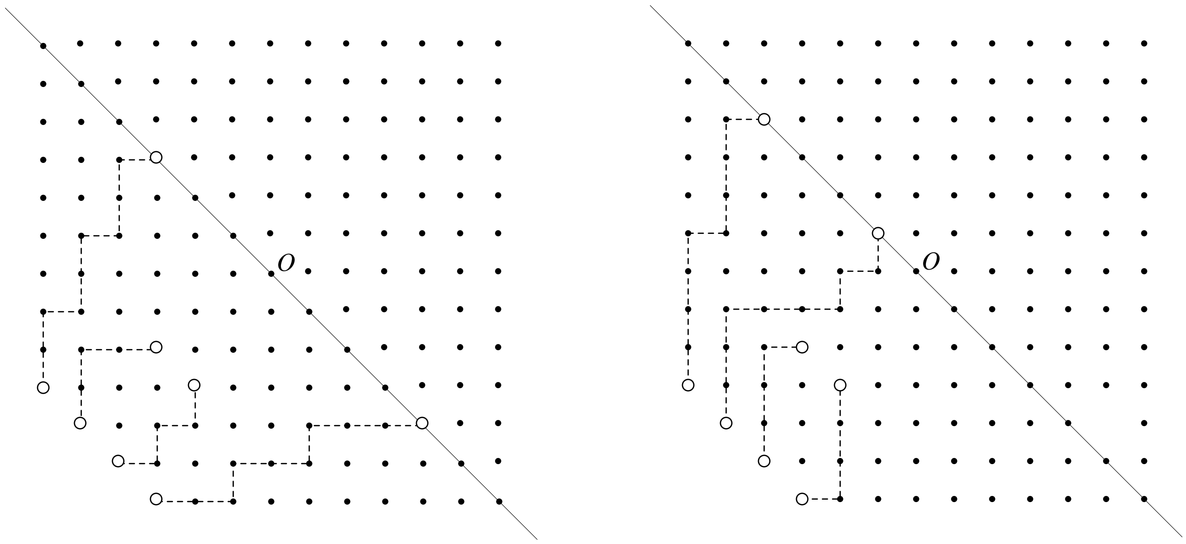


FIGURE 27. The two families of non-intersecting paths corresponding to the tilings in Figure 26.

of the equation in Theorem 4.5.3 all have the same sign, hence it can still be used to count tilings of $V \setminus R$.

[a] Two sets of totally ordered points $S = \{s_1, s_2, \dots, s_p\}$ and $E = \{e_1, e_2, \dots, e_p\}$ are *D-compatible* if for $i < j$ and $k < l$, every path from s_i to e_l intersects every path from s_j to e_k at a vertex.

PROPOSITION 4.5.5

The number of vertically symmetric tilings of $H \setminus (L, R)$ is given by the Pfaffian of the skew-symmetric matrix $V = (V_{i,j})_{1 \le i,j \le 2b+2k}$, where

$$V_{i,j} = \begin{cases} \sum_{s=i-j+1}^{j-i} \binom{2a}{a+s}, & 1 \le i < j \le 2b, \\ \binom{a+r_{j-2b}}{a/2+r_{j-2b}/2-b+i}, & 1 \le i \le 2b, 2b+1 \le j \le 2b+k, \\ \binom{a+r_{j-2b-k}}{a/2+r_{j-2b-k}/2-b-1+i}, & 1 \le i \le 2b, 2b+k+1 \le j \le 2b+2k, \\ 0, & 2b+1 \le i < j \le 2b+2k. \end{cases}$$

REMARK 4.5.6

Throughout this thesis we interpret sums according to the following standard convention:

$$\sum_{p=q}^{r-1} \mathrm{Expr}(p) = \begin{cases} \sum_{p=q}^{r-1} \mathrm{Expr}(p), & r > q, \\ 0, & r = q, \\ -\sum_{p=r}^{q-1} \mathrm{Expr}(p), & r < q. \end{cases}$$

PROOF. According to Theorem 4.5.3 we must show that for $1 \le i, j \le 2b$, each entry $V_{i,j}$ is equal to

$$\sum_{1 \le u < v \le 2b+a} \mathcal{P}(s_i \to e_u)\mathcal{P}(s_j \to e_v) - \mathcal{P}(s_i \to e_v)\mathcal{P}(s_j \to e_u),$$

which, when we consider the above definitions of S and E, may be written as

$$\sum_{s=1}^{2b+a} \sum_{t=1}^{2b+a} \left(\binom{a}{a-s+i}\binom{a}{s+t-j} - \binom{a}{a-s+j}\binom{a}{s+t-i} \right).$$

We then apply the *Chu-Vandermonde convolution*[41] to the above expression in order to obtain

$$\sum_{t=1}^{2b+a}\left(\binom{2a}{a+t+i-j}-\binom{2a}{a+t+j-i}\right)=\sum_{s=i-j+1}^{j-i}\binom{2a}{a+s}.$$

The remaining non-zero (i,j)-entries of V quite clearly count paths between the points s_i and e'_{j-2b} as defined previously, thus the proof is complete. □

Our goal now is to evaluate the Pfaffian of the matrix V and in order to do so we first establish an extension of a lemma concerning skew-symmetric matrices that was originally due to Gordon [**43**].

4.6. Extending Gordon's lemma

It turns out that evaluating the Pfaffian of V (which is a matrix of size $2b+2k$) from the previous section becomes much simpler once we apply a lemma that reduces the calculation of the Pfaffian to evaluating the determinant of a matrix that is half the size of V. In an approach similar to that taken by Ciucu and Krattenthaler in [**22**] the following lemma is an extension of the one that may be found in Gordon's article [**43**].

LEMMA 4.6.1 – TG [**42**]

For a positive integer m and a non-negative integer l, let A be the $(2m+2l)\times(2m+2l)$ skew-symmetric matrix of the form

$$A=\begin{pmatrix} X & Y \\ -Y^t & Z\end{pmatrix},$$

for which the following properties hold:

(i) X is a skew-symmetric $2m\times 2m$ matrix such that $X=(x_{j-i})_{1\le i,j\le 2m}$ (and for positive α, $x_{-\alpha}=-x_\alpha$);

(ii) $Z=(z_{i,j})_{1\le i,j\le 2l}$ is a matrix satisfying $z_{i,j}+z_{i+l,j}+z_{i,j+l}=0$ for $1\le i,j\le l$, and $z_{j,i}=-z_{i,j}$;

(iii) $Y=(y_{i,j})_{1\le i\le 2m,1\le j\le 2l}$ is a matrix for which

$$y_{i,j}=\begin{cases} y_{2m-i,j}, & 1\le i\le m, 1\le j\le l,\\ y_{2m+1-i,j-l}, & 1\le i\le 2m, l+1\le j\le 2l.\end{cases}$$

Then

$$\operatorname{Pf}(A)=(-1)^{\binom{l}{2}}\det(B),$$

where B is an $(m+l)\times(m+l)$ matrix of the form

$$B=\begin{pmatrix} \widehat{X} & \widehat{Y}_1 \\ \widehat{Y}_2 & \widehat{Z}\end{pmatrix},$$

the block matrices of which are defined by

$$(\widehat{X})_{i,j}=x_{i+j-1}+x_{i+j-3}+\cdots+x_{|i-j|+1} \qquad \text{for } 1\le i,j\le m,$$

$$(\widehat{Y}_1)_{i,j}=\sum_{s=0}^{i-1}(y_{m+1-i+2s,j}-y_{m+i-2s,j}) \qquad \text{for } 1\le i\le m \text{ and } 1\le j\le l,$$

$$(\widehat{Y}_2)_{i,j}=\sum_{s=0}^{j-1}(y_{j+m-2s,i}+y_{m+1-j+2s,i}) \qquad \text{for } 1\le i\le l \text{ and } 1\le j\le m,$$

$$(\widehat{Z})_{i,j}=z_{i,j+l}+z_{i+l,j+l} \qquad \text{for } 1\le i,j\le l.$$

---

[41]This states that for non-negative integers m,n,r we have

$$\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}=\binom{m+n}{r}.$$

Proof. Beginning with A, construct a new matrix $A' = (A'_{i,j})_{1\le i,j\le 2m+2l}$ by simultaneously replacing the i-th row of A by the sum

$$\sum_{s=0}^{m-i}(\text{row } i+2s \text{ of A}),$$

and the $(2m+1-i)$-th row of A with

$$\sum_{s=0}^{m-i}(\text{row } 2m+1-i-2s \text{ of A})$$

for $i = 1, ..., m-1$. Perform analogous operations on the columns of the resulting matrix. Note that these operations do not change the value of $\mathrm{Pf}(A)$ and so $\mathrm{Pf}(A) = \mathrm{Pf}(A')$. It follows that for $1 \le i, j \le m$,

$$\begin{aligned} A'_{i,j} &= \sum_{s=0}^{m-j}\sum_{r=0}^{m-i} x_{j-i+2s-2r} \\ &= \sum_{t=-m}^{m-i-j}(\min\{t+m+1, m-i+1\} - \max\{0, t+j\})x_{j+i+2t}. \end{aligned}$$

By assumption X is skew-symmetric, so $x_r = -x_{-r}$ for all $1 \le r \le 2m$. One may also verify that

$$\begin{aligned} &\min\{t+m+1, m-i+1\} \\ &\qquad - \max\{0, t+j\} = \min\{(-t-i-j)+m+1, m-i+1\} \\ &\qquad\qquad\qquad\qquad - \max\{0, (-t-i-j)+j\}, \end{aligned}$$

whence $A'_{i,j}$ vanishes for $1 \le i, j \le m$. It is not hard to convince oneself that replacing i and j with $2m-i+1$ and $2m-j+1$ respectively in the above summation gives the (i,j)-entry of A' for $m+1 \le i, j \le 2m$, and so these entries also vanish.

For $1 \le i \le m$ and $m+1 \le j \le 2m$,

$$\begin{aligned} A'_{i,j} &= \sum_{s=0}^{j-m-1}\sum_{r=0}^{m-i} x_{j-i-2r-2s} \\ &= \sum_{t=1}^{j-i}(\min\{t, m-i+1\} - \max\{0, t-j+m\})x_{i-j+2t}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} &\min\{(j-i-t), m-i+1\} - \max\{0, (j-i-t)-j+m\} \\ &\qquad\qquad = \min\{t, m-i+1\} - \max\{0, t-j+m\} + 1, \end{aligned}$$

so again by the skew-symmetry of X,

$$A'_{i,j} = \bar{x}_{i,j},$$

where

$$\bar{x}_{i,j} = x_{j-i} + x_{j-i-2} + \cdots + x_{|2m-i-j+1|+1} \tag{4.5}$$

for $1 \le i \le m$ and $m+1 \le j \le 2m$.

Having performed exactly the same operations on both the rows and columns of A it follows that A' is also skew-symmetric, thus for $m+1 \le i \le 2m$ and $1 \le j \le m$, $A'_{i,j} = -A'_{j,i}$.

For the remaining (i,j)-entries of A' it suffices to consider only those entries for which $1 \le i \le 2m$ and $2m+1 \le j \le 2m+l$. These columns are affected by the row operations alone so for $1 \le i \le m$ the (i,j)-entry of A' is

$$\sum_{s=0}^{m-i} y_{i+2s,j-2m},$$

while an analogous argument shows that for $m+1 \leq i \leq 2m$, $a'_{i,j}$ is equal to

$$\sum_{s=0}^{i-m-1} y_{i-2s,j-2m}.$$

By the symmetry of Y it is clear that for $j \in \{2m+l+1, ..., 2m+2l\}$ and $1 \leq i \leq m$,

$$A'_{i,j} = \sum_{s=0}^{m-i} y_{2m+1-i-2s,j-2m-l},$$

while for $m+1 \leq i \leq 2m$ and $j \in \{2m+l+1, ..., 2m+2l\}$ the (i,j)-entry of A' is

$$\sum_{s=0}^{i-m-1} y_{2m+1-i+2s,j-2m-l}.$$

Note that A' has now been completely determined since these row and column operations leave Z unchanged.

Construct a new matrix $A'' = (A''_{i,j})_{1 \leq i,j \leq 2m+2l}$ by perfoming the following operations on A':

(i) Add column $(2m+l+j)$ to column $(2m+j)$ for $j = 1, ..., l$ (resp. rows);
(ii) Subtract column $(m+j)$ from column $(m+1-j)$ for $j = 1, ..., m$ (resp. rows).

Consider the effect of the first set of row and column operations stated above. For $1 \leq i,j \leq 2l$,

$$A''_{i+2m,j+2m} = \begin{cases} 0, & 1 \leq i,j \leq l, \\ z_{i,j} + z_{i+l,j}, & 1 \leq i \leq l, l+1 \leq j \leq 2l, \\ z_{i,j} + z_{i,j+l}, & l+1 \leq i \leq 2l, 1 \leq j \leq l, \\ z_{i,j}, & l+1 \leq i,j \leq 2l, \end{cases}$$

by the second property in the statement of the lemma.

The (i,j)-entry of A'' for $1 \leq i \leq m$ and $2m+1 \leq j \leq 2m+l$ becomes

$$A''_{i,j} = \sum_{s=0}^{m-i} y_{i+2s,j-2m} + \sum_{s=0}^{m-i} y_{2m+1-i-2s,j-2m},$$

while for $m+1 \leq i \leq 2m$,

$$A''_{i,j} = \sum_{s=0}^{i-m-1} y_{i-2s,j-2m} + \sum_{s=0}^{i-m-1} y_{2m+1-i+2s,j-2m},$$

and crucially (by the third property of the statement of the lemma), $A''_{i,j} = A''_{2m+1-i,j}$ for $1 \leq i \leq m$ and $2m+1 \leq j \leq 2m+l$.

Now consider the effect of the second set of operations applied to A'. Again by the skew-symmetry of A' this leaves all entries unchanged except those for which $1 \leq i \leq m$ and $2m+1 \leq j \leq 2m+2l$. For $2m+1 \leq j \leq 2m+l$ this entry vanishes, hence the resulting matrix has the form

$$A'' = \begin{pmatrix} 0 & \bar{X} & 0 & Y_1 \\ -\bar{X}^t & 0 & Y_2 & Y_3 \\ 0 & -Y_2^t & 0 & Z_1 \\ -Y_1^t & -Y_3^t & -Z_1^t & Z_2 \end{pmatrix},$$

where for $1 \leq i,j \leq m$,

$$(\bar{X})_{i,j} = \bar{x}_{i,j+m}$$

is given by (4.5) and

$$
\begin{aligned}
(Y_1)_{i,j} &= \sum_{s=0}^{m-i}(y_{2m+1-i-2s,j} - y_{i+2s,j}) && 1 \le i \le m, 1 \le j \le l,\\
(Y_2)_{i,j} &= \sum_{s=0}^{i-1}(y_{i+m-2s,j} + y_{m+1-i+2s,j}) && 1 \le i \le m, 1 \le j \le l,\\
(Y_3)_{i,j} &= \sum_{s=0}^{i-1} y_{m+1-i+2s,j} && 1 \le i \le m, 1 \le j \le l,\\
(Z_1)_{i,j} &= z_{i,j+l} + z_{i+l,j+l} && 1 \le i, j \le l,\\
(Z_2)_{i,j} &= z_{i+l,j+l} && 1 \le i, j \le l.
\end{aligned}
$$

By rearranging rows and columns in exactly the same way A'' may be brought into the form

$$
\begin{pmatrix} 0 & 0 & \bar{X} & Y_1 \\ 0 & 0 & -Y_2^t & Z_1 \\ -\bar{X}^t & Y_2 & 0 & Y_3 \\ -Y_1^t & -Z_1^t & -Y_3^t & Z_2 \end{pmatrix}. \tag{4.6}
$$

Since the same operations have been performed on both rows and columns this leaves the Pfaffian of A'' unchanged. By the well-known identity

$$
\mathrm{Pf}\begin{pmatrix} 0 & P \\ -P^t & Q \end{pmatrix} = (-1)^{\binom{n}{2}} \det(P),
$$

where P is an arbitrary $n \times n$ matrix, it follows that

$$
\mathrm{Pf}(A) = \mathrm{Pf}(A'') = (-1)^{\binom{m+l}{2}} \det \begin{pmatrix} \bar{X} & Y_1 \\ -Y_2^t & Z_1 \end{pmatrix}.
$$

Reversing the order of the rows 1 to m and multiplying the last l rows and columns by -1 gives

$$
\begin{aligned}
\mathrm{Pf}(A) &= (-1)^{\left(\binom{m+l}{2}+\binom{m}{2}\right)} \det \begin{pmatrix} \widehat{X} & \widehat{Y}_1 \\ \widehat{Y}_2 & \widehat{Z} \end{pmatrix},\\
&= (-1)^{\binom{l}{2}} \det \begin{pmatrix} \widehat{X} & \widehat{Y}_1 \\ \widehat{Y}_2 & \widehat{Z} \end{pmatrix},
\end{aligned}
$$

where $\widehat{X}, \widehat{Y}_1, \widehat{Y}_2$, and $\widehat{Z}$ are exactly those blocks asserted in the lemma. □

Remark 4.6.2

By letting $l = 0$ in the previous lemma we recover Gordon's original result from [**43**].

Applying Lemma 4.6.1 directly to the matrix V yields

$$
\mathrm{Pf}(V) = (-1)^{\binom{k}{2}} \det(\bar{V}),
$$

where $\bar{V} = (\bar{V}_{i,j})_{1 \le i,j \le b+k}$ is the matrix with (i, j)-entries given by

$$
\bar{V}_{i,j} = \begin{cases} V_{i+j-1} + V_{i+j-3} + \cdots + V_{|i-j|+1}, & 1 \le i, j \le b, \\ \sum_{s=0}^{i-1}(V_{b+1-i+2s,b+j} - V_{b+i-2s,b+j}), & 1 \le i \le b, b+1 \le j \le b+k, \\ \sum_{s=0}^{j-1}(V_{j+b-2s,b+i} + V_{b+1-j+2s,b+i}), & b+1 \le i \le b+k, 1 \le j \le b, \\ 0, & \textit{otherwise}. \end{cases}
$$

We can construct one final matrix, $\widehat{V} = (\widehat{V}_{i,j})_{1 \le i,j \le b+k}$, by performing one last set of row and column operations on $\bar{V}$: for $i = 1, 2, \ldots, b-1$ subtract the $(b-i)$-th row from row $(b+1-i)$, and perform analogous operations on the columns.

For $1 \leq i, j \leq b$ we have

$$\begin{aligned}\widehat{V}_{i,j} &= \bar{V}_{i,j} - \bar{V}_{i-1,j} - (\bar{V}_{i,j-1} - \bar{V}_{i-1,j-1}) \\ &= V_{i+j-1} - V_{i-j} - V_{i+j-2} + V_{i-j+1} \\ &= \binom{2a}{a+j-i} + \binom{2a}{a+j+i-1},\end{aligned}$$

while if $b+1 \leq i \leq b+k$ and $1 \leq j \leq b$ we have

$$\begin{aligned}\widehat{V}_{i,j} &= \bar{V}_{i,j} - \bar{V}_{i,j-1} \\ &= V_{b+j,b+i} + V_{b+1-j,b+i} \\ &= \binom{a+r_{i-b}}{a/2+r_{i-b}/2+j} + \binom{a+r_{i-b}}{a/2+r_{i-b}/2+1-j} \\ &= \binom{a-l_{i-b}+1}{a/2-l_{i-b}/2+j},\end{aligned}$$

where we suppose $l_i = -r_{k+1-i}$.

Finally consider $\widehat{V}$ for $1 \leq i \leq b$ and $b+1 \leq j \leq b+k$. We have

$$\begin{aligned}\widehat{V}_{i,j} &= \bar{V}_{i,j} - \bar{V}_{i-1,j} \\ &= V_{b+1-i,b+j} - V_{b+i,b+j} + 2\sum_{s=1}^{i-1}(V_{b+i+1-2s,b+j} - V_{b-i+2s,b+j}) \\ &= \binom{a+r_{j-b}}{a/2+r_{j-b}/2+1-i} - \binom{a+r_{j-b}}{a/2+r_{j-b}/2+i} + 2\sum_{s=2-i}^{i-1}\binom{a+r_{j-b}}{a/2+r_{j-b}/2+s} \\ &= \frac{2(a-4(i-1)i+r_{j-b})}{(a+r_{j-b})(a+2i+r_{j-b})}\binom{a+r_{j-b}}{a/2+r_{j-b}/2-i+1}.\end{aligned}$$

We have thus proved the following proposition.

Proposition 4.6.3

The number of vertically symmetric tilings of $H \setminus (L,R)$, where $r < 0$ for every $r \in R$, $|R| = k$, and $L = \{-r : r \in R\}$ is given by

$$\pm\det(\widehat{V}),$$

where $\widehat{V} := (\widehat{V}_{i,j})_{1\leq i,j\leq b+k}$ has entries given by

$$\widehat{V}_{i,j} = \begin{cases} \binom{2a}{a+j-i} + \binom{2a}{a+j+i-1} & 1 \leq i,j \leq b, \\ \frac{2(a-4(i-1)i+r_{j-b})}{(a+r_{j-b})(a+2i+r_{j-b})}\binom{a+r_{j-b}}{a/2+r_{j-b}/2-i+1} & 1 \leq i \leq b, b+1 \leq j \leq b+k, \\ \binom{a-l_{i-b}+1}{a/2-l_{i-b}/2+j} & b+1 \leq i \leq b+k, 1 \leq j \leq b, \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 4.6.4

Compare Proposition 4.6.3 with the lattice path matrix $P_{\widehat{S},\widehat{E}}$ from Proposition 4.3.1 for the same set of holes – these two matrices agree everywhere except in the entries where $1 \leq i \leq b$ and $b+1 \leq j \leq b+k$.

It is clear from the preceding remark and proposition that an analogue of Ciucu and Krattenthaler's factorisation theorem where the 2-holes are oriented in the opposite direction may not be written in the simple form

$$M(H \setminus (L,R)) = M_{-}(H \setminus (L,R)) \cdot M_{|}(H \setminus (L,R)).$$

By evaluating the matrix $\widehat{V}$, however, we may obtain an analogous, more general result to the one given in [**20**].

THEOREM 4.6.5 – TG [**40**]

The matrix $\widehat{V}$ described above has LU-decomposition

$$\widehat{V} = L' \cdot U'$$

where $L' = (L'_{i,j})_{1\le i,j\le b+k}$ is the lower triangular matrix with entries given by

$$L'_{i,j} = \begin{cases} A'(a,i,j), & 1 \le j \le i \le b, \\ B'(a,l_{i-b},j), & b+1 \le i \le b+k, 1 \le j \le b, \\ E'(a,i,j), & b+1 \le j \le i \le b+k, \\ 0, & \textit{otherwise}, \end{cases}$$

and $U' = (U'_{i,j})_{1\le i,j\le b+k}$ is the upper triangular matrix with entries given by

$$U'_{i,j} = \begin{cases} C'(a,i,j), & 1 \le i \le j \le b, \\ D'(a,i,r_{j-b}), & 1 \le i \le b, b+1 \le j \le b+k, \\ F'(a,i,j), & b+1 \le i < j \le b+k, \\ 0, & \textit{otherwise}, \end{cases}$$

where

$$\begin{aligned} A'(a,i,j) &= \frac{\Gamma(a+1)\Gamma(i+j-1)\Gamma(2j+a)}{\Gamma(2j-1)\Gamma(i-j+1)\Gamma(j-i+a+1)\Gamma(i+j+a)}, \\ B'(a,l,j) &= \frac{(-1)^{j+1}\Gamma(j+a)\Gamma(2j+a)\Gamma(a-l+2)\Gamma(j+\frac{l}{2}+\frac{a}{2}-1)}{\Gamma(j)\Gamma(2j+2a)\Gamma(\frac{a}{2}-\frac{l}{2}+1)\Gamma(\frac{l}{2}+\frac{a}{2})\Gamma(j-\frac{l}{2}+\frac{a}{2}+1)}, \\ C'(a,i,j) &= \frac{\Gamma(a+1)\Gamma(i+j-1)\Gamma(2i+2a)}{\Gamma(j-i+1)\Gamma(2i+a-1)\Gamma(i-j+a+1)\Gamma(i+j+a)}, \\ D'(a,i,r) &= \frac{(-1)^{i+1}\Gamma(2i-1)\Gamma(i+a)\Gamma(a+r+1)\Gamma(\frac{a}{2}+i-\frac{r}{2}-1)}{\Gamma(i)\Gamma(2i+a-1)\Gamma(\frac{a}{2}+\frac{r}{2}+1)\Gamma(\frac{a}{2}-\frac{r}{2})\Gamma(i+\frac{a}{2}+\frac{r}{2}+1)} \\ &\quad + \frac{(-1)^{i+1}\Gamma(2i-1)\Gamma(i+a+1)\Gamma(a+r+1)\Gamma(i+\frac{a}{2}-\frac{r}{2}-1)}{\Gamma(i-1)\Gamma(2i+a-1)\Gamma(\frac{a}{2}+\frac{r}{2}+1)\Gamma(\frac{a}{2}-\frac{r}{2}+1)\Gamma(i+\frac{a}{2}+\frac{r}{2}+1)}, \end{aligned}$$

and $E'(a,i,j)$ and $F'(a,i,j)$ are entries satisfying

$$\sum_{s=1}^{\min(i,j)} E'(a,i,s)F'(a,s,j) = -\sum_{t=1}^{b} B'(a,l_{i-b},t)D'(a,t,r_{j-b})$$

for $b+1 \le i,j \le b+k$.

PROOF. The proof of this theorem is straightforward since we need only check that

$$\sum_{s=1}^{i} A'(a,i,s)D'(a,s,r_{j-b}) = \widehat{V}_{i,j} \tag{4.7}$$

for $1 \le i \le b$ and $b+1 \le j \le b+k$ (the other identities arising from the LU-decomposition have already been established in the proof of Proposition 4.3.1).

Implementing Zeilberger's algorithm [**66**, **76**, **77**] as in the proof of Theorem 4.4.1 (and also Theorem 4.4.5) shows that the sum on the left hand side satisfies the following recurrence in i

$$(a+2i+r+2)(a-2i(i-1)+r)\sum_{s=1}^{i+1} A'(a,i+1,s)D'(a,s,r)$$
$$- (a-2i+r+2)(a-2i(i+1)+r)\sum_{s=1}^{i} A'(a,i,s)D'(a,s,r) = 0,$$

as does the expression

$$\frac{2(a-4(i-1)i+r)}{(a+r)(a+2i+r)}\binom{a+r}{a/2+r/2-i+1}.$$

Once (4.7) has been verified for $i=1$ and 2 the proof is complete. □

This leads to the following corollary, the proof of which is much the same as the proof of Corollary 4.4.4 and Corollary 4.4.6.

Corollary 4.6.6 – TG [**40**]

The number of vertically symmetric tilings of $H \setminus (L,R)$ where $r<0$ for all $r \in R$ is

$$M_{|}(H\setminus(L,R)) = \prod_{i=1}^{a}\frac{2i+2b-1}{2i-1}\prod_{1\le i<j\le a}\frac{i+j+2b-1}{i+j-1}\cdot|\det(Q'_{L,R})|,$$

where $Q'_{L,R} = (Q'_{l,r})_{l\in L, r\in R}$ is the matrix with entries given by

$$Q'_{l,r} = -\sum_{s=1}^{b} B'(a,l,s)D'(a,s,r).$$

If $|R| = 1$ in the above theorem then we obtain an analogous result to Ciucu and Krattenthaler [**20**] where in our case the 2-hole has been flipped and points *towards* the free boundary. For $|R| \geq 1$ this result is analogous to Lemma 4.4.9.

Open Problem 4.6.7

Corollary 4.6.6 and Lemma 4.4.9 together count vertically symmetric tilings of $H \setminus (L,R)$ where either $r>0$ for each $r \in R$, or instead $r<0$ for all r (and of course symmetry dictates that in both cases $L = \{-r : r \in R\}$). These two formulas cover sets of horizontally collinear holes in one half of the hexagon H that are either all left pointing or all right pointing, but what happens if some of the sets of contiguous holes consist of both left and right pointing collinear 2-holes?

We have now established a number of enumerative formulas for different classes of tilings of hexagons that contain horizontally collinear holes induced by sets of 2-holes. The following chapter is devoted to the asymptotic analysis of the entries of $\widehat{Q}_{L,R}$, $\check{Q}_{L,R}$, and $Q'_{L,R}$.

Remark 4.6.8

It should be noted that while the formula presented in Corollary 4.6.6 corresponds to vertically symmetric tilings of $H_{a,2b,a} \setminus (L,R)$, a similar result for half hexagons where the length of the vertical edges is odd can be obtained using the results found in [**40**] and [**42**].

CHAPTER
FIVE

THE EMERGENCE OF PHYSICAL PRINCIPLES

In this chapter we examine the formulas from Chapter 4 as the lengths of the boundaries of the tileable regions $H \setminus (L, R)$ and $V \setminus R$ are sent to infinity and the distances between fixed sets of horizontally collinear 2-holes is large. To this end we shall suppose throughout that H is a hexagon with sides of length $an, 2bn, an, an, 2bn, an$ (going clockwise from the south-west side), and that V is the corresponding half hexagon. We shall also assume that any sets of contiguous 2-holes are homogeneous (that is, sets of contiguous holes are either all left pointing or all right pointing) and thus induce left or right pointing equilateral triangular holes of even side length.

Remark 5.0.1

The contition on the holes described above implies that for any $r \in R$ and $l \in L$, either $r \geq l+2$ or $r \leq l-6$.

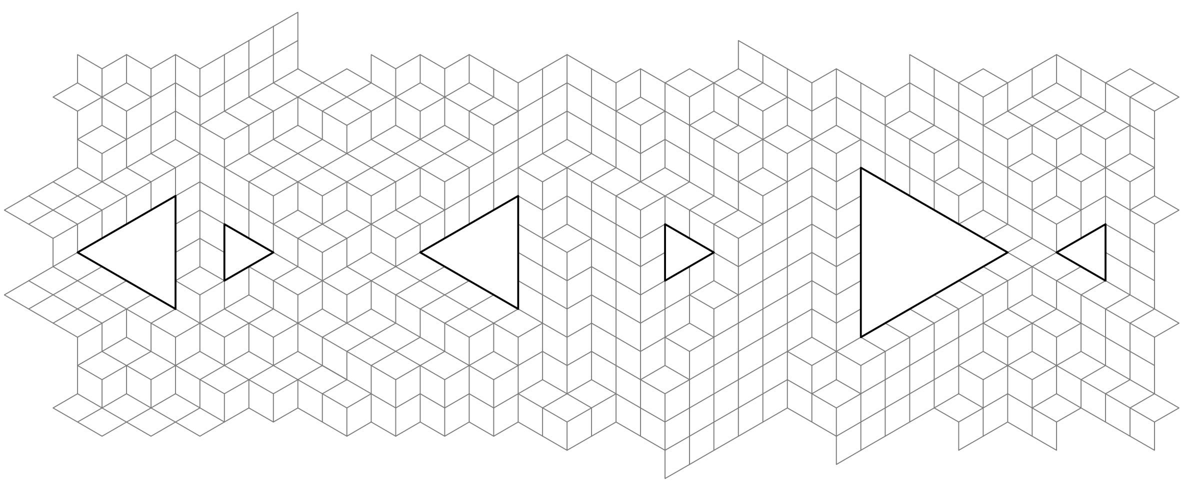


Figure 28. A set of horizontally collinear holes within a section of a tiling of the plane.

5.1. Collinear interactions

For fixed sets of 2-holes L and R contained in H consider the correlation function (see Section 2.8) as the lengths of the outer edges of $H \setminus (L, R)$ are sent to infinity

$$\omega_H(L, R) = \lim_{n\to\infty} \frac{M(H \setminus (L, R))}{M(H)}. \tag{5.1}$$

In this limit the outer boundary of H becomes the entire plane, thus $\omega_H(L, R)$ may be interpreted as a measure of how easy or difficult it is to tile the plane that contains 2-hole punctures indexed by L

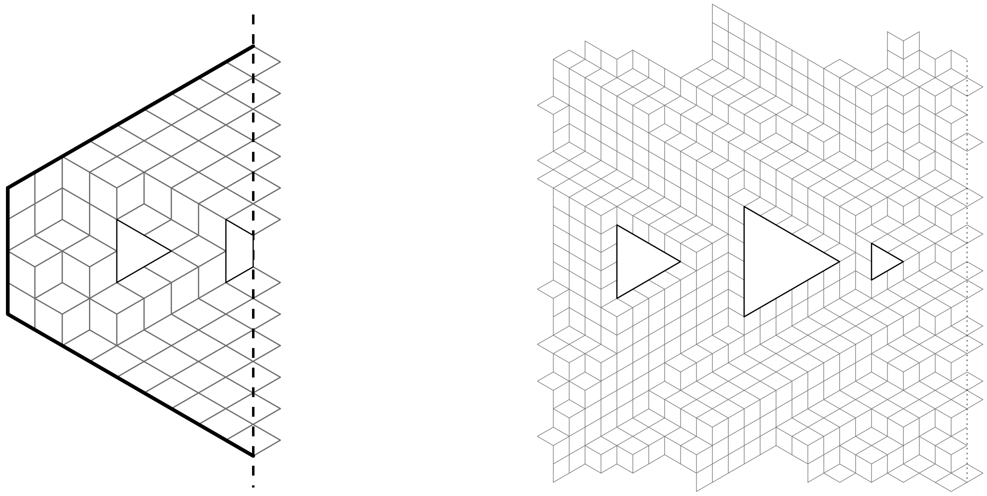


FIGURE 29. A tiling of the finite region $V_{9,4,9} \setminus \{-5,-1\}$ and a set of holes in the sea but near the shore.

and R, compared to the ways we can tile the plane without any holes. We say that the correlation function in this case is a measure of the interactions between holes within a *sea of unit rhombi* (see Figure 28).

Substituting our expression from Theorem 4.4.7 into the right hand side of equation (5.1) we see that

$$\omega_H(L,R) = \lim_{n\to\infty} \left(|\det(\check{Q}_{L,R}) \cdot \det(\widehat{Q}_{L,R})| \right).$$

Since the size of each matrix $\check{Q}_{L,R}$ and $\widehat{Q}_{L,R}$ is fixed and independent of n the problem reduces to studying the asymptotics of the individual entries:

$$\check{Q}_{l,r} = \frac{1}{r-l+1}\binom{r-l+1}{r/2-l/2} - \sum_{s=1}^{bn} \check{B}(an,l,s)\check{D}(an,s,r) \tag{5.2}$$

and

$$\widehat{Q}_{l,r} = \binom{r-l+1}{r/2-l/2+1} - \sum_{s=1}^{bn} \widehat{B}(an,l,s)\widehat{D}(an,s,r). \tag{5.3}$$

REMARK 5.1.1

There are many different ways in which one may obtain the plane as a limit of finite shapes. We shall soon see that the rate at which the sides of H approach infinity plays an important role in determining the asymptotics of $\omega_H(L,R)$.

Consider the correlation function for the region $V \setminus R$,

$$\omega_V(R) := \lim_{n\to\infty} \frac{M(V \setminus R)}{M(V)}.$$

The limit shape as the sides of this half hexagon are sent to infinity corresponds to the left half of the plane, constrained on the right by a vertical free boundary. A fixed set of right pointing holes R induces a set of collinear holes that lie within a sea of unit rhombi, but instead of interacting out at sea they are located in the vicinity of some sort of straight shoreline (a vertical free boundary, see Figure 29). Thanks to Corollary 4.6.6 we have

$$\omega_V(R) = \lim_{n\to\infty} \left(|\det(Q'_{L,R})| \right).$$

The following sections are devoted to uncovering closed asymptotic expressions for $\omega_H(L,R)$ and $\omega_V(R)$ when the holes are separated by large distances.

5.2. Interactions near the shore

We begin with the correlation function $\omega_V(R)$. As the size of $Q'_{L,R}$ is fixed and independent of n, in order to study the asymptotics of $\omega_V(R)$ we need only consider the entries of $Q'_{L,R}$ as $n\to\infty$, that is,

$$\lim_{n\to\infty}\left(-\sum_{s=1}^{bn} B'(an,l,s)D'(an,s,r)\right)$$

where

$$\begin{aligned} B'(an,l,s) &= \frac{(-1)^{s+1}\Gamma(s+an)\Gamma(2s+an)\Gamma(an-l+2)\Gamma(j+\frac{l}{2}+\frac{an}{2}-1)}{\Gamma(s)\Gamma(2s+2an)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{l}{2}+\frac{an}{2})\Gamma(s-\frac{l}{2}+\frac{an}{2}+1)},\\ D'(an,s,r) &= \frac{(-1)^{s+1}\Gamma(2s-1)\Gamma(s+an)\Gamma(an+r+1)\Gamma(\frac{an}{2}+s-\frac{r}{2}-1)}{\Gamma(s)\Gamma(2s+an-1)\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}-\frac{r}{2})\Gamma(s+\frac{an}{2}+\frac{r}{2}+1)}\\ &\quad+\frac{(-1)^{s+1}\Gamma(2s-1)\Gamma(s+an+1)\Gamma(an+r+1)\Gamma(s+\frac{an}{2}-\frac{r}{2}-1)}{\Gamma(s-1)\Gamma(2s+an-1)\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}-\frac{r}{2}+1)\Gamma(s+\frac{an}{2}+\frac{r}{2}+1)}. \end{aligned}$$

Proposition 5.2.1 – TG **[41]**

Suppose $2b/a\sim\mu$ as $n\to\infty$. The entries of $Q'_{L,R}$ in the limit are given by

$$\left(\frac{2}{\mu+1}\right)^{r-l+2}\frac{(\mu(\mu+2))^{1/2}}{\pi(r-l+1)}\cdot {}_2F_1\left[\begin{matrix}1,\ -\frac{1}{2}\\ \frac{r}{2}-\frac{l}{2}+\frac{3}{2}\end{matrix};-(\mu(\mu+2))^{-1}\right].$$

Proof. Each entry of $Q'_{L,R}$ may be written as

$$\begin{aligned} -\Bigg(&\frac{\Gamma(an+1)\Gamma(an+2)\Gamma(an-l+2)\Gamma(an+r+1)}{\Gamma(2an+2)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}-\frac{l}{2}+2)\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+2)}\\ &\qquad\times\sum_{s=0}^{bn-1}\frac{(an+1)_s(\frac{an}{2}+\frac{3}{2})_s(\frac{an}{2}+\frac{l}{2})_s(\frac{an}{2}-\frac{r}{2})_s(\frac{1}{2})_s}{(\frac{an}{2}+\frac{1}{2})_s(\frac{an}{2}-\frac{l}{2}+2)_s(\frac{an}{2}+\frac{r}{2}+2)_s(an+\frac{3}{2})_s\Gamma(s+1)}\\ &+\frac{2\Gamma(an+2)\Gamma(an+4)\Gamma(\frac{an}{2}+\frac{l}{2}+1)\Gamma(an-l+2)\Gamma(an+r+1)}{\Gamma(2an+4)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}-\frac{l}{2}+3)\Gamma(\frac{an}{2}+\frac{l}{2})\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+3)}\\ &\qquad\times\sum_{s=0}^{bn-2}\frac{(an+3)_s(\frac{an}{2}+\frac{5}{2})_s(\frac{an}{2}+\frac{l}{2}+1)_s(\frac{an}{2}-\frac{r}{2}+1)_s(\frac{3}{2})_s}{(\frac{an}{2}+\frac{3}{2})_s(\frac{an}{2}-\frac{l}{2}+3)_s(\frac{an}{2}+\frac{r}{2}+3)_s(an+\frac{5}{2})_s\Gamma(s+1)}\Bigg). \end{aligned}\tag{5.4}$$

A finite series

$$\sum_{s=0}^{bn-1}\mathrm{Expr}(s)$$

may be expressed as the limit of an infinite series

$$\lim_{\epsilon\to 0}\left(\sum_{s=0}^{\infty}\frac{(1-bn)_s(an+bn+1)_s}{(an+bn+1+\epsilon)_s(1-bn+\epsilon)_s}\cdot\mathrm{Expr}(s)\right),$$

therefore we may translate (5.4) into a sum of two *very well-poised hypergeometric series*[42]

$$
\begin{aligned}
-\lim_{\epsilon\to 0}\Bigg(&\frac{\Gamma(an+1)\Gamma(an+2)\Gamma(an-l+2)\Gamma(an+r+1)}{\Gamma(2an+2)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}-\frac{l}{2}+2)\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+2)}\\
&\times {}_7F_6\left[\begin{matrix} an+1+\epsilon,\ \frac{an}{2}+\frac{3}{2}+\frac{\epsilon}{2},\ \frac{an}{2}+\frac{l}{2}+\frac{\epsilon}{2},\ \frac{an}{2}-\frac{r}{2}+\frac{\epsilon}{2},\ \frac{1}{2}+\frac{\epsilon}{2},\ an+bn+1,\ 1-bn \\ \frac{an}{2}+\frac{1}{2}+\frac{\epsilon}{2},\ \frac{an}{2}-\frac{l}{2}+2+\frac{\epsilon}{2},\ \frac{an}{2}+\frac{r}{2}+2+\frac{\epsilon}{2},\ an+\frac{3}{2}+\frac{\epsilon}{2},\ 1-bn+\epsilon,\ an+bn+1+\epsilon\end{matrix};1\right]\\
&+\frac{2\Gamma(an+2)\Gamma(an+4)\Gamma(\frac{an}{2}+\frac{l}{2}+1)\Gamma(an-l+2)\Gamma(an+r+1)}{\Gamma(2an+4)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}-\frac{l}{2}+3)\Gamma(\frac{an}{2}+\frac{l}{2})\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+3)}\\
&\times {}_7F_6\left[\begin{matrix} an+3+\epsilon,\ \frac{an}{2}+\frac{5}{2}+\frac{\epsilon}{2},\ \frac{an}{2}+\frac{l}{2}+1+\frac{\epsilon}{2},\ \frac{an}{2}-\frac{r}{2}+1+\frac{\epsilon}{2},\ \frac{3}{2}+\frac{\epsilon}{2},\ an+bn+2,\ 2-bn \\ \frac{an}{2}+\frac{3}{2}+\frac{\epsilon}{2},\ \frac{an}{2}-\frac{l}{2}+3+\frac{\epsilon}{2},\ \frac{an}{2}+\frac{r}{2}+3+\frac{\epsilon}{2},\ an+\frac{5}{2}+\frac{\epsilon}{2},\ 2-bn+\epsilon,\ an+bn+2+\epsilon\end{matrix};1\right]\Bigg).
\end{aligned}
\tag{5.5}
$$

> **Remark 5.2.2**
>
> Dealing with hypergeometric series such as those stated above can at first appear a little daunting, however Christian Krattenthaler's excellent package `HYP`[a] is an incredibly useful tool for manipulating and transforming them into more appealing expressions.
>
> [a] Available at `http://www.mat.univie.ac.at/~kratt/hyp_hypq/hyp.html`

To each of the series above we may apply the following transformation formula (see [**70**, (2.4.1.1), reversed])

$$
\begin{aligned}
&{}_7F_6\left[\begin{matrix} a,\ a/2+1,\ b,\ c,\ d,\ e,\ -n \\ a/2,\ a-b+1,\ a-c+1,\ a-d+1,\ a-e+1,\ a+n+1\end{matrix};1\right]\\
&\qquad=\frac{(a+1)_n(a-d-e+1)_n}{(a-d+1)_n(a-e+1)_n}\,{}_4F_3\left[\begin{matrix} a-b-c+1,\ d,\ e,\ -n \\ a-b+1,\ a-c+1,\ -a+d+e-n\end{matrix};1\right],
\end{aligned}
\tag{5.6}
$$

thereby obtaining

$$
\frac{(an+2+\epsilon)_{bn-1}((1+\epsilon)/2-bn)_{bn-1}}{(an+(3+\epsilon)/2)_{bn-1}(1-bn+\epsilon)_{bn-1}}\,{}_4F_3\left[\begin{matrix} 2-\frac{l}{2}+\frac{r}{2},\ \frac{1}{2}+\frac{\epsilon}{2},\ bn+an+1,\ 1-bn \\ \frac{an}{2}-\frac{l}{2}+2+\frac{\epsilon}{2},\ \frac{an}{2}+\frac{r}{2}+2+\frac{\epsilon}{2},\ \frac{3}{2}-\frac{\epsilon}{2}\end{matrix};1\right]
\tag{5.7}
$$

for the upper hypergeometric series in (5.5) and

$$
\frac{(an+4+\epsilon)_{bn-2}((1+\epsilon)/2-bn)_{bn-2}}{(an+(5+\epsilon)/2)_{bn-2}(2-bn+\epsilon)_{bn-2}}\,{}_4F_3\left[\begin{matrix} 2-\frac{l}{2}+\frac{r}{2},\ \frac{3}{2}+\frac{\epsilon}{2},\ bn+an+2,\ 2-bn \\ \frac{an}{2}-\frac{l}{2}+3+\frac{\epsilon}{2},\ \frac{an}{2}+\frac{r}{2}+3+\frac{\epsilon}{2},\ \frac{5}{2}-\frac{\epsilon}{2}\end{matrix};1\right]
\tag{5.8}
$$

for the lower. As ϵ vanishes the pre-factor in (5.7) may be written as

$$
\frac{\Gamma(2an+2)\Gamma(2bn)\Gamma(an+bn)\Gamma(an+bn+1)}{\Gamma(an+1)\Gamma(an+2)\Gamma(bn)^2\Gamma(2an+2bn)},
$$

so combining this with the corresponding pre-factor in (5.5) yields

$$
\begin{aligned}
&\frac{\Gamma(2bn)\Gamma(an+bn)\Gamma(an-l+2)\Gamma(an+bn+1)\Gamma(an-l+2)\Gamma(an+r+1)}{\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}-\frac{l}{2}+2)\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+2)\Gamma(2an+2bn)\Gamma(bn)^2}\\
&\qquad\qquad\times {}_4F_3\left[\begin{matrix} 2-\frac{l}{2}+\frac{r}{2},\ \frac{1}{2},\ bn+an+1,\ 1-bn \\ \frac{an}{2}-\frac{l}{2}+2,\ \frac{an}{2}+\frac{r}{2}+2,\ \frac{3}{2}\end{matrix};1\right]
\end{aligned}
\tag{5.9}
$$

[42] The ${}_pF_q$ hypergeometric series, denoted ${}_pF_q\left[\begin{matrix}a_1,\ \dots,\ a_p\\ b_1,\ \dots,\ b_q\end{matrix};z\right]$, is defined to be $\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}$, where $(\alpha)_\beta$ is the Pochhammer symbol, that is, $(\alpha)_\beta := \alpha\cdot(\alpha+1)\cdots(\alpha+\beta-1)$ for $\beta > 0$, while $(\alpha)_0 := 1$. A hypergeometric series ${}_pF_q\left[\begin{matrix}a_1,\ a_2,\ \dots,\ a_p\\ b_1,\ b_2,\ \dots,\ b_q\end{matrix};z\right]$ is *well-poised* if $a_1+1 = b_1+a_2 = \cdots = b_q+a_p$. If in addition we have $a_2 = a_1/2+1$ then the series is *very well-poised*.

for the upper summand. We may easily apply Stirling's approximation[43] to the pre-factor of (5.9) as n grows, thereby obtaining

$$O(n^{-1}) \cdot {}_4F_3\left[\begin{matrix} 2-\frac{l}{2}+\frac{r}{2},\ \frac{3}{2},\ bn+an+2,\ 2-bn \\ \frac{an}{2}-\frac{l}{2}+3,\ \frac{an}{2}+\frac{r}{2}+3+,\ \frac{5}{2} \end{matrix};1\right].$$

REMARK 5.2.3

We use big-O notation in the above, wherein $f(x) = O(g(x))$ if and only if there exists some positive real number M and a real number x_0 such that

$$|f(x)| \leq M|g(x)|$$

for all $x \geq x_0$.

Since in the limit (5.9) vanishes we are left with the lower hypergeometric series in (5.5) to consider. The pre-factor in (5.8) may be re-written as

$$\frac{\Gamma(2an+4)\Gamma(2bn)\Gamma(an+bn)\Gamma(an+bn+2)}{6\Gamma(an+2)\Gamma(an+4)\Gamma(bn-1)\Gamma(bn)\Gamma(2an+2bn)}$$

as $\epsilon \to 0$, thus when we replace the power series in (5.5) with the expression in (5.8) we obtain for the lower summand

$$\frac{(-1)(\frac{an}{2}+\frac{l}{2})\Gamma(2bn)\Gamma(an+bn)\Gamma(an+bn+2)\Gamma(an-l+2)\Gamma(an+r+1)}{3\Gamma(bn-1)\Gamma(bn)\Gamma(2an+2bn)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}-\frac{l}{2}+3)\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+3)}$$
$$\times\, {}_4F_3\left[\begin{matrix} 2-\frac{l}{2}+\frac{r}{2},\ \frac{3}{2},\ bn+an+2,\ 2-bn \\ \frac{an}{2}-\frac{l}{2}+3,\ \frac{an}{2}+\frac{r}{2}+3,\ \frac{5}{2} \end{matrix};1\right].$$

Since in these symmetric tilings we have $r \leq l-6$ for all $l \in L$, $r \in R$ the above hypergeometric series is finite and terminates for all $s \geq \frac{l}{2}-\frac{r}{2}-1$, thus we may happily interchange the sum and limit in the expression above as $n \to \infty$. Liberally applying Stirling's approximation and supposing that as n grows $2b/a \to \mu$ for some real $\mu > 0$ we obtain

$$-\frac{2^{r-l+2}(\mu(\mu+2))^{3/2}}{3\pi}\,{}_2F_1\left[\begin{matrix} \frac{3}{2},\ \frac{r}{2}-\frac{l}{2}+2 \\ \frac{5}{2} \end{matrix};-\mu(\mu+2)\right]. \tag{5.10}$$

We can transform the above expression using the following identity (see **[70**, (1.8.10)] with the sum reversed on the right hand side)

$${}_2F_1\left[\begin{matrix} a,\ -n \\ c \end{matrix};z\right] = \frac{(1-z)^n(a)_n}{(c)_n}\,{}_2F_1\left[\begin{matrix} -n,\ c-a \\ 1-a-n \end{matrix};(1-z)^{-1}\right] \tag{5.11}$$

thereby obtaining

$$\left(\frac{2^{r-l+2}(\mu(\mu+2))^{3/2}}{3\pi}\right)\cdot\frac{3(\mu+1)^{l-r-4}}{(r-l+1)}\,{}_2F_1\left[\begin{matrix} 2-\frac{l}{2}+\frac{r}{2},\ 1 \\ \frac{r}{2}-\frac{l}{2}+\frac{3}{2} \end{matrix};\frac{1}{(\mu+1)^2}\right]. \tag{5.12}$$

Permuting the arguments in the top row of the ${}_2F_1$ series and applying the following transformation formula which may be found in **[70**, (1.7.1.3)]

$${}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix};z\right] = (1-z)^{-a}\,{}_2F_1\left[\begin{matrix} a,\ c-b \\ c \end{matrix};-\frac{z}{1-z}\right] \tag{5.13}$$

reduces (5.12) to

$$\left(\frac{2}{\mu+1}\right)^{r-l+2}\frac{(\mu(\mu+2))^{1/2}}{\pi(r-l+1)}\cdot{}_2F_1\left[\begin{matrix} 1,\ -\frac{1}{2} \\ \frac{r}{2}-\frac{l}{2}+\frac{3}{2} \end{matrix};-(\mu(\mu+2))^{-1}\right]$$

as the limit of $Q'_{l,r}$ as $n \to \infty$. □

[43]That is, $n! \sim \sqrt{2\pi n}\left(\frac{n}{\exp(1)}\right)^n$ for large n, where $n! = \Gamma(n+1)$.

We now have an asymptotic expression for the entries of $Q'_{L,R}$ as the boundary of the half hexagon grows infinitely large. The limit shape obtained is thus the left half plane constrained on the right by a free boundary, containing some set of horizontally collinear right pointing 2-holes. Supposing the Euclidean distances between the holes are proportional to some real variable $\tau > 0$ (as in the statement of Conjecture 2.8.4): our goal now is to examine what happens as τ grows large. It is clear that unless $\mu = 1$ the entries of $Q'_{L,R}$ either blow up or shrink exponentially, thus we now focus our attention on the case where $2b/a \to 1$, that is, the sides of $V \setminus R$ are (in the limit) the same size and approach infinity at the same rate.

The set R indexes a set of right pointing 2-holes in the left half plane, some of which may lie contiguously and thus induce a larger equilateral triangular hole of even side length (remember that k-many contiguous 2-holes is equivalent to an equilateral triangle with sides of length $2k$). Suppose we have

$$r_1 > r_2 > \cdots > r_p,$$

for $R = \{r_1, r_2, \ldots, r_p\}$ and similarly for $L = \{l_1, l_2, \ldots, l_p\}$,

$$l_1 < l_2 < \cdots < l_p.$$

The sets R and L can both be written as unions of subsets corresponding to induced holes, that is, $R = \bigcup_{i=1} R_i$ and $L = \bigcup_{i=1} L_i$ for $1 \le i \le u$ where u is the number of induced holes in $V \setminus R$.[44] Each subset of R is thus

$$R_\beta = \{r - 2p + 2 : 1 \le p \le |R_\beta|, r = \max(R_\beta)\},$$

and similarly

$$L_\alpha = \{l + 2q - 2 : 1 \le q \le |L_\alpha|, l = \min(L_\alpha)\}.$$

According to Proposition 5.2.1 the sub-matrix with rows and columns of $Q'_{L,R}$ indexed by a set of contiguous left pointing holes $L_\alpha \subseteq L$ and right pointing holes $R_\beta \subseteq R$ has entries given by

$$\frac{\sqrt{3}}{2\pi}\sum_{s=0}^{\infty}\frac{(-1/2)_s(-1/3)^s}{(\frac{r}{2}-\frac{l}{2}-j-i+1/2)_{s+1}},$$

where $r = \max(R_\beta)$ and $l = \min(L_\alpha)$.

Let $Q'|_r^l$ denote this $|L_\alpha| \times |R_\beta|$ sub-matrix and suppose that, proceeding from row $|L_\alpha| - 1$ to the first row of $Q'|_r^l$, we subtract row i from row $i+1$. The (i,j)-entries of $Q'|_r^l$ are then

$$\frac{\sqrt{3}}{2\pi}\sum_{s=0}^{\infty}\frac{(-1/2)_s(-1/3)^s}{(\frac{r}{2}-\frac{l}{2}-j-i+3/2)_s}\left(\frac{1}{(\frac{r}{2}-\frac{l}{2}-j-i+1/2)}-\frac{1}{(\frac{r}{2}-\frac{l}{2}-j-i+3/2+s)}\right)$$
$$=\frac{\sqrt{3}}{2\pi}\sum_{s=0}^{\infty}\frac{(s+1)(-1/2)_s(-1/3)^s}{(\frac{r}{2}-\frac{l}{2}-j-i+1/2)_{s+1}}.$$

Suppose we successively perform the operation described above to $Q'|_r^l$ a total of $|L_\alpha| - 1$ times, proceeding on the k-th iteration from row $|L_\alpha| - 1$ to row k. The entries of $Q'|_r^l$ are thus

$$\frac{\sqrt{3}}{2\pi}\sum_{s=0}^{\infty}\frac{(s+1)_{i-1}(-1/2)_s(-1/3)^s}{(\frac{r}{2}-\frac{l}{2}-j-i+1/2)_{s+i}}.$$

If we now perform a similar set of operations on the *columns* of $Q'|_r^l$ we see that the (i,j)-entries of this sub-matrix are given by

$$\frac{\sqrt{3}}{2\pi}\sum_{s=0}^{\infty}\frac{(s+1)_{i+j-2}(-1/2)_s(-1/3)^s}{(\frac{r}{2}-\frac{l}{2}-j-i+1/2)_{s+i+j-1}}.$$

[44] Since we are dealing with the vertically symmetric case the number of left and right pointing induced holes is the same.

REMARK 5.2.4

Performing the set of row operations described above is equivalent to constructing a matrix $\tilde{Q}'|_r^l$ in which row i is given by

$$\sum_{t=0}^{i-1}(-1)^t\binom{i-1}{t}\cdot(\text{row } i-t \text{ of } Q'|_r^l).$$

Performing the set of column operations on $\tilde{Q}'|_r^l$ is thus equivalent to constructing a matrix $\mathbf{Q}'|_r^l$ in which the j-th column is given by

$$\sum_{t=0}^{j-1}(-1)^t\binom{j-1}{t}\cdot(\text{row } j-t \text{ of } \tilde{Q}'|_r^l).$$

If $R=\bigcup_{i=1}^u R_i$ then the matrix $Q'_{L,R}$ may be written in block form as

$$\begin{pmatrix} Q'|_{r_1}^{l_1} & Q'|_{r_2}^{l_1} & \dots & Q'|_{r_u}^{l_1} \\ Q'|_{r_1}^{l_2} & Q'|_{r_2}^{l_2} & \dots & Q'|_{r_u}^{l_2} \\ \vdots & \vdots & \ddots & \vdots \\ Q'|_{r_1}^{l_u} & Q'|_{r_2}^{l_u} & \dots & Q'|_{r_u}^{l_u} \end{pmatrix}$$

where each block $Q'|_{r_\beta}^{l_\alpha}$ has (i,j)-entries given by

$$\frac{\sqrt{3}}{2\pi}\sum_{s=0}^{\infty}\frac{(-1/2)_s(-1/3)^s}{(\frac{r_\beta}{2}-\frac{l_\alpha}{2}-j-i+1/2)_{s+1}},$$

for $l_\alpha=\min(L_\alpha)$, $r_\beta=\max(R_\beta)$ where $L_\alpha\subseteq L$ and $R_\beta\subseteq R$, and each block is of size $|L_\alpha|\times|R_\beta|$. We may easily apply sets of row and column operations as described above to this larger matrix $Q'_{L,R}$ so that the (i,j)-entry of each block is given by

$$\frac{\sqrt{3}}{2\pi}\cdot\sum_{s=0}^{\infty}\frac{(s+1)_{i+j-2}(-1/2)_s(-1/3)^s}{(\frac{r_\beta}{2}-\frac{l_\alpha}{2}-j-i+1/2)_{s+i+j-1}}.$$

Note that performing these operations does not change the value of the determinant of $Q'_{L,R}$.

As τ grows large the entries in each block are asymptotically

$$\frac{\sqrt{3}}{2\pi}\cdot\frac{\Gamma(i+j-1)}{(\frac{r_\beta}{2}-\frac{l_\alpha}{2})^{i+j-1}}+O\left((\tfrac{r_\beta}{2}-\tfrac{l_\alpha}{2})^{-(i+j)}\right).$$

By slightly re-writing the above expression as

$$\left(\frac{(\sqrt{3})^{i+j}\Gamma(i)\Gamma(j)}{2\pi}\right)\frac{\binom{i+j-2}{i-1}}{(\sqrt{3}(r_\beta/2-l_\alpha/2))^{i+j-1}}$$

it is easy to see that

$$\det(Q'_{L,R})\sim\left(\prod_{L_\alpha\in L}\left(\prod_{i=1}^{|L_\alpha|}\frac{3^{i/2}\Gamma(i)}{\sqrt{2\pi}}\right)\right)\cdot\left(\prod_{R_\beta\in R}\left(\prod_{j=1}^{|R_\beta|}\frac{3^{j/2}\Gamma(j)}{\sqrt{2\pi}}\right)\right)\cdot\det(\mathbf{Q}_{L,R})$$

where $\mathbf{Q}_{L,R}$ is the matrix given by

$$\begin{pmatrix}
\frac{\binom{0}{0}}{(y_1-x_1)} & \frac{\binom{1}{0}}{(y_1-x_1)^2} & \cdots & \frac{\binom{|R_1|-1}{0}}{(y_1-x_1)^{|R_1|}} & & \frac{\binom{0}{0}}{(y_u-x_1)} & \frac{\binom{1}{0}}{(y_u-x_1)^2} & \cdots & \frac{\binom{|R_u|-1}{0}}{(y_u-x_1)^{|R_u|}} \\
\frac{\binom{1}{1}}{(y_1-x_1)^2} & \frac{\binom{2}{1}}{(y_1-x_1)^3} & \cdots & \frac{\binom{|R_1|-1}{1}}{(y_1-x_1)^{|R_1|+1}} & & \frac{\binom{1}{1}}{(y_u-x_1)^2} & \frac{\binom{2}{1}}{(y_u-x_1)^3} & \cdots & \frac{\binom{|R_u|}{1}}{(y_u-x_1)^{|R_u|+1}} \\
\vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\
\frac{\binom{|L_1|-1}{|L_1|-1}}{(y_1-x_1)^{|L_1|}} & \frac{\binom{|L_1|}{|L_1|-1}}{(y_1-x_1)^{|L_1|+1}} & \cdots & \frac{\binom{|R_1|+|L_1|-2}{|L_1|-1}}{(y_1-x_1)^{|R_1|+|L_1|-1}} & & \frac{\binom{|L_1|-1}{|L_1|-1}}{(y_u-x_1)^{|L_1|}} & \frac{\binom{|L_1|}{|L_1|-1}}{(y_u-x_1)^{|L_1|+1}} & \cdots & \frac{\binom{|R_u|+|L_1|-2}{|L_1|-1}}{(y_u-x_1)^{|R_u|+|L_1|-1}} \\
 & & \vdots & & & & & \vdots & \\
\frac{\binom{0}{0}}{(y_1-x_u)} & \frac{\binom{1}{0}}{(y_1-x_u)^2} & \cdots & \frac{\binom{|R_1|-1}{0}}{(y_1-x_u)^{|R_1|}} & & \frac{\binom{0}{0}}{(y_u-x_u)} & \frac{\binom{1}{0}}{(y_u-x_u)^2} & \cdots & \frac{\binom{|R_u|-1}{0}}{(y_u-x_u)^{|R_u|}} \\
\frac{\binom{1}{1}}{(y_1-x_u)^2} & \frac{\binom{2}{1}}{(y_1-x_u)^3} & \cdots & \frac{\binom{|R_1|-1}{1}}{(y_1-x_u)^{|R_1|+1}} & & \frac{\binom{1}{1}}{(y_u-x_u)^2} & \frac{\binom{2}{1}}{(y_u-x_u)^3} & \cdots & \frac{\binom{|R_u|}{1}}{(y_u-x_u)^{|R_u|+1}} \\
\vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\
\frac{\binom{|L_u|-1}{|L_u|-1}}{(y_1-x_u)^{|L_u|}} & \frac{\binom{|L_u|}{|L_u|-1}}{(y_1-x_u)^{|L_u|+1}} & \cdots & \frac{\binom{|R_1|+|L_u|-2}{|L_u|-1}}{(y_1-x_u)^{|R_1|+|L_u|-1}} & & \frac{\binom{|L_u|-1}{|L_u|-1}}{(y_u-x_u)^{|L_u|}} & \frac{\binom{|L_u|}{|L_u|-1}}{(y_u-x_u)^{|L_u|+1}} & \cdots & \frac{\binom{|R_u|+|L_u|-2}{|L_u|-1}}{(y_u-x_u)^{|R_u|+|L_u|-1}}
\end{pmatrix}$$

in which $x_\alpha = \sqrt{3}\cdot l_\alpha/2$ and $y_\beta = \sqrt{3}\cdot r_\beta/2$.

A determinant such as this has a closed form evaluation – it is a special case of the following result of Ciucu (see [**12**, Theorem 8.1]) which may also be found in [**53**, Theorem 25].

Theorem 5.2.5 – Ciucu [12]

Let $s_1, s_2, \ldots, s_m \geq 1$ and $t_1, t_2, \ldots, t_n \geq 1$ be integers. Write $S = \sum_{i=1}^m s_i$, $T = \sum_{i=1}^n t_i$, and assume that $S \geq T$. Let $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ be indeterminates. Define N to be the $S \times S$ matrix

$$\begin{bmatrix} A & B \end{bmatrix}$$

whose blocks are given by

$$A = \begin{pmatrix} A|_{y_1}^{x_1} & A|_{y_2}^{x_1} & \ldots & A|_{y_n}^{x_1} \\ A|_{y_1}^{x_2} & A|_{y_2}^{x_2} & \ldots & A|_{y_n}^{x_2} \\ \vdots & \vdots & \ddots & \vdots \\ A|_{y_1}^{x_m} & A|_{y_2}^{x_m} & \ldots & A|_{y_n}^{x_m} \end{pmatrix}, \quad B = \begin{pmatrix} B|_{x_1} \\ B|_{x_2} \\ \vdots \\ B|_{x_m} \end{pmatrix},$$

where

$$A|_{y_j}^{x_i} = \begin{pmatrix} \frac{\binom{0}{0}}{(y_j-x_i)} & \frac{\binom{1}{0}}{(y_j-x_i)^2} & \cdots & \frac{\binom{t_j-1}{0}}{(y_j-x_i)^{t_j}} \\ \frac{\binom{1}{1}}{(y_j-x_i)^2} & \frac{\binom{2}{1}}{(y_j-x_i)^3} & \cdots & \frac{\binom{t_j-1}{1}}{(y_j-x_i)^{t_j+1}} \\ \vdots & \vdots & & \vdots \\ \frac{\binom{s_i-1}{s_i-1}}{(y_j-x_i)^{s_i}} & \frac{\binom{s_i}{s_i-1}}{(y_j-x_i)^{s_i+1}} & \cdots & \frac{\binom{t_j+s_i-2}{s_i-1}}{(y_j-x_i)^{t_j+s_i-1}} \end{pmatrix}$$

and

$$B|_{x_i} = \begin{pmatrix} \binom{0}{0}(x_i)^0 & \binom{1}{0}(x_i)^1 & \dots & \binom{S-T-1}{0}(x_i)^{S-T-1} \\ \binom{0}{1}(x_i)^{-1} & \binom{1}{1}(x_i)^0 & \dots & \binom{S-T-1}{1}(x_i)^{S-T-2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \binom{0}{s_i-1}(x_i)^{1-s_i} & \binom{1}{s_i-1}(x_i)^{2-s_i} & \dots & \binom{S-T-1}{s_i-1}(x_i)^{S-T-s_i} \end{pmatrix}.$$

Then we have

$$\det(N) = \frac{\prod_{1\le i<j\le m}(x_j - x_i)^{s_i s_j} \prod_{1\le i<j\le n}(y_i - y_j)^{t_i t_j}}{\prod_{i=1}^m \prod_{j=1}^n (y_j - x_i)^{s_i t_j}}.$$

Substituting $\mathbf{Q}_{L,R}$ for A and supposing B is the empty matrix in the above theorem yields

$$\det(Q'_{L,R}) \sim \left(\prod_{L_\alpha\in L}\left(\prod_{i=1}^{|L_\alpha|}\frac{3^{i/2}\Gamma(i)}{\sqrt{2\pi}}\right)\right)\cdot\left(\prod_{R_\beta\in R}\left(\prod_{j=1}^{|R_\beta|}\frac{3^{j/2}\Gamma(j)}{\sqrt{2\pi}}\right)\right) \\ \times \frac{\prod_{1\le i<j\le u}(x_j - x_i)^{|L_i|\cdot|L_j|}\prod_{1\le i<j\le u}(y_i - y_j)^{|R_i|\cdot|R_j|}}{\prod_{i=1}^u\prod_{j=1}^u (y_j - x_i)^{|L_i|\cdot|R_j|}}.$$

Remember that in Section 2.8 we defined a statistic, $q(t)$, on a hole t given by the number of right pointing unit triangles minus the number of left pointing ones that comprise it. It follows that if L_α indexes a set of contiguous left pointing 2-holes then $q(L_\alpha) = -2|L_\alpha|$, since each left pointing 2-hole has charge -2. Similarly for a right pointing hole R_β we have $q(R_\beta) = 2|R_\beta|$. It is also clear that the expression in the denominator of the entries of $\mathbf{Q'}_{L,R}$,

$$\sqrt{3}(r_\beta/2 - l_\alpha/2),$$

is equal (up to sign) to the Euclidean distance between the mid-points of the vertical sides of the holes indexed by l_α and r_β. We have thus proved the following theorem.

Theorem 5.2.6 – TG [**40**]

Consider tilings of $V \setminus R$. As $n \to \infty$ suppose $2b/a \to 1$ and let $\mathcal{H}$ index the fixed set of holes induced by R, $\mathcal{H}'$ the reflection of those holes in the vertical free boundary. The interaction between holes indexed by $\mathcal{H}$ that are separated by large distances in a sea of unit rhombi and the vertical free boundary is asymptotically

$$\omega_V(R) \sim \prod_{h\in\mathcal{H}\cup\mathcal{H}'} C_h \prod_{1\le i<j\le|\mathcal{H}\cup\mathcal{H}'|} \mathrm{d}(h_i, h_j)^{\frac{1}{4}q(h_i)q(h_j)}$$

where

$$C_h = \left(\prod_{i=1}^{|q(h)|/2}\frac{3^{i/2}\Gamma(i)}{\sqrt{2\pi}}\right).$$

5.3. The method of image charges

Once more we briefly dip into the world of physical systems. Imagine we have a set of point charges $Q = \{Q_1, Q_2, \dots, Q_n\}$ in the left half of the two dimensional plane that induce an electric field in the presence of a straight-line equipotential conductor that borders the half plane to the right (see Figure 30). This means that each point charge feels a force from the electric field induced by the other charges not only directly but also via the conducting surface. At first sight it would seem that

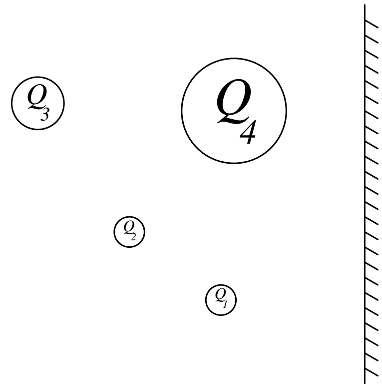


FIGURE 30. A set of point charges Q near a straight-line semi-conductor.

the electrostatic potential energy of such a system is quite complicated to calculate,[45] however this task is made far simpler by applying another physical principle known as *the method of image charges*.

According to Feynman [**31**, Chapter 6, Section 7], if we let $Q' = \{Q'_1, Q'_2, \dots, Q'_n\}$ be the set of point charges obtained by reflecting Q about the conductor and setting $\mathrm{ch}(Q'_i) = -\mathrm{ch}(Q_i)$ for all $1 \leq i \leq n$ then the total electrostatic potential energy of the system of charges and the equipotential conductor $U_{Q\cup|}$ satisfies

$$U_{Q\cup|} = \frac{1}{2} U_{Q\cup Q'},$$

that is, it is half the total electrostatic potential energy of the system obtained by replacing the conductor with the *images* of the charges within it, where each *imaginary charge* Q'_i has the opposite electrical charge to that of Q_i (see Figure 31). It follows that the total electrostatic potential energy of this system is

$$U_{Q\cup|} = \frac{1}{2}\left(\frac{k_c}{2}\sum_{1\leq i<j\leq|Q\cup Q'|} \mathrm{ch}(Q_i)\,\mathrm{ch}(Q_j)(-\ln(\mathrm{d}(Q_i, Q_j)))\right),$$

where $Q_i, Q_j \in Q \cup Q'$. It follows that if we have an electric field induced by a set of point charges Q in the presence of a straight-line equipotential conductor then the probability of finding them arranged at a set of mutual distances $\{\mathrm{d}(Q_i, Q_j) : Q_i, Q_j \in Q\}$ is proportional to

$$\exp(k_c/\kappa\mathbb{T}) \prod_{1\leq i<j\leq|Q\cup Q'|} \mathrm{d}(Q_i, Q_j)^{\frac{1}{4}\mathrm{ch}(Q_i)\,\mathrm{ch}(Q_j)}.$$

Compare this with Theorem 5.2.6 – if we suppose the right pointing collinear holes $h_1, \dots, h_{|\mathcal{H}|}$ correspond to a set of point charges $Q_1, \dots, Q_{|\mathcal{H}|}$ and set $\mathrm{ch}(Q_i) = q(h_i)$ then they agree up to a multiplicative constant. Once again we have somewhat unexpectedly observed yet another well-known physical principle emerging independently from our mathematical analysis. Theorem 5.2.6 therefore provides yet more evidence for the apparent link between rhombus tilings and two dimensional physical systems conjectured by Ciucu in [**11**].

5.4. Interactions out at sea

We will now uncover an asymptotic expression for $\omega_H(L, R)$, which according to Ciucu's Factorisation Theorem [**6**] is equal to

$$\omega_{\widehat{H}}(L, R) \cdot \omega_{\check{H}}(L, R),$$

where

$$\omega_{\widehat{H}}(L, R) = \lim_{n\to\infty} |\det(\widehat{Q}_{L,R})|$$

[45] It is certainly more complicated than if there were no semi-conductor present.

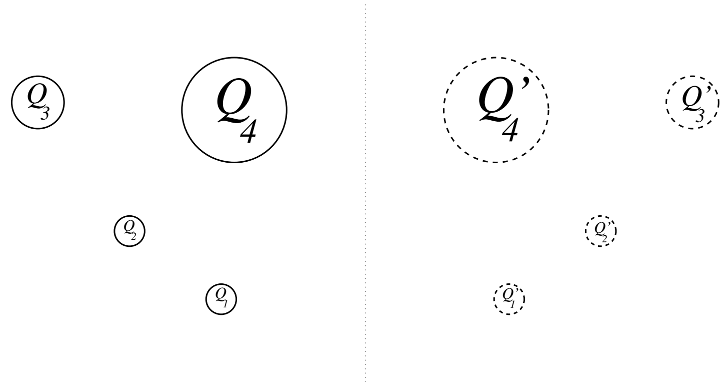


FIGURE 31. A set of point charges Q together with their corresponding imaginary charges Q'.

and

$$\omega_{\breve{H}}(L,R) = \lim_{n\to\infty} |\det(\breve{Q}_{L,R})|.$$

We begin with the latter correlation function, which gives the interaction between unit triangular dents that lie in the nooks of the fixed horizontal boundary that constrains rhombus tilings of the lower half plane (see Figure 32).

THEOREM 5.4.1 – TG [**40**]

Consider tilings of the region $\breve{H} \setminus (L,R)$. Suppose that as $n \to \infty$, $2b/a \to 1$, and let $\mathcal{H}$ index the holes induced by $L \cup R$. The interaction between holes separated by large distances is asymptotically

$$\omega_{\breve{H}}(L,R) \sim \prod_{h\in\mathcal{H}} C_h \prod_{1\le i<j\le|\mathcal{H}|} \mathrm{d}(h_i,h_j)^{\frac{1}{4}q(h_i)q(h_j)}$$

in which

$$C_h = \left(\prod_{i=1}^{|q(h)|/2} \frac{3^{i/2}\Gamma(i)}{\sqrt{2\pi}}\right).$$

PROOF. We begin by extracting the asymptotic behaviour of $\breve{Q}_{l,r}$ as $n \to \infty$. Suppose first that $r \ge l+2$. We may express the matrix entry $\breve{Q}_{l,r}$ as

$$\frac{1}{r-l+1}\binom{r-l+1}{r/2-l/2} - \sum_{s=0}^{\infty} \breve{B}(an,l,s+1)\breve{D}(an,s+1,r) + \sum_{t=bn}^{\infty} \breve{B}(an,l,t+1)\breve{D}(an,t+1,l), \quad (5.14)$$

where

$$\breve{B}(a,l,j) = \frac{(-1)^{j+1}\Gamma(j+a-1)\Gamma(2j+a)\Gamma(a-l+1)\Gamma(j+\frac{l}{2}+\frac{a}{2}-1)}{2\,\Gamma(j)\Gamma(2j+2a-2)\Gamma(\frac{a}{2}-\frac{l}{2}+1)\Gamma(\frac{l}{2}+\frac{a}{2})\Gamma(j-\frac{l}{2}+\frac{a}{2}+1)},$$

$$\breve{D}(a,i,r) = \frac{(-1)^{i+1}\Gamma(2i+1)\Gamma(i+a)\Gamma(a+r+1)\Gamma(i+\frac{a}{2}-\frac{r}{2}-1)}{2\,\Gamma(2i+a-1)\Gamma(i+1)\Gamma(\frac{a}{2}-\frac{r}{2})\Gamma(\frac{a}{2}+\frac{r}{2}+1)\Gamma(i+\frac{a}{2}+\frac{r}{2}+1)}.$$

The sum over s in (5.14) may be written as a hypergeometric series:

$$\frac{\Gamma(an)\Gamma(an+2)\Gamma(an-l+1)\Gamma(an+r+1)}{2\Gamma(2an)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}-\frac{l}{2}+2)\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+2)}$$
$$\times\, {}_5F_4\left[\begin{matrix} an+1,\ \frac{an}{2}+\frac{3}{2},\ \frac{3}{2},\ \frac{l}{2}+\frac{an}{2},\ \frac{an}{2}-\frac{r}{2} \\ \frac{an}{2}+\frac{1}{2},\ an+\frac{1}{2},\ \frac{an}{2}-\frac{l}{2}+2,\ \frac{an}{2}+\frac{r}{2}+2 \end{matrix};1\right]. \quad (5.15)$$

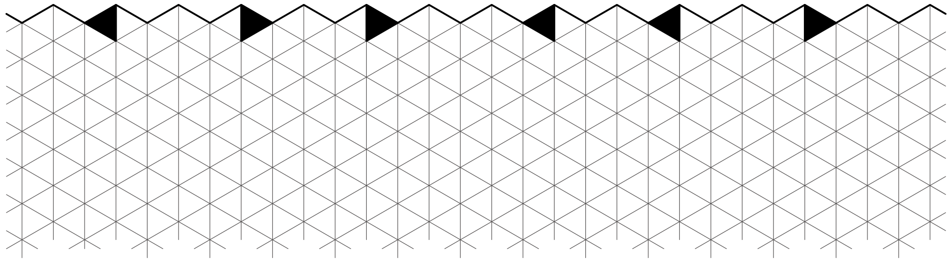


FIGURE 32. A set of triangular dents on the fixed horizontal boundary of the lower half plane obtained as the limit shape of $\widehat{H} \setminus (L, R)$.

According to Slater [**70**, Appendix III.12] this hypergeometric series satisfies the following summation formula

$$
{}_5F_4\left[\begin{matrix} a,\ \frac{a}{2}+1,\ b,\ c,\ d \\ \frac{a}{2},\ a-b+1,\ a-c+1,\ a-d+1 \end{matrix};1\right] = \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-b-c-d+1)}{\Gamma(a+1)\Gamma(a-b-c+1)\Gamma(a-b-c+1)\Gamma(a-c-d+1)}. \tag{5.16}
$$

Applying this formula to (5.15) yields

$$
\frac{\Gamma(an)\Gamma(an-\frac{1}{2}+1)\Gamma(an-l+1)\Gamma(\frac{r}{2}-\frac{l}{2}+\frac{1}{2})\Gamma(an+r+1)}{2\Gamma(2an)\Gamma(\frac{an}{2}-\frac{l}{2}+\frac{1}{2})\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{r}{2}-\frac{l}{2}+2)\Gamma(\frac{an}{2}+\frac{r}{2}+\frac{1}{2})\Gamma(\frac{an}{2}+\frac{r}{2}+1)},
$$

which can be shown to equal

$$
\frac{1}{r-l+1}\binom{r-l+1}{r/2-l/2},
$$

hence (5.14) reduces to

$$
\sum_{s=0}^{\infty} \breve{B}(an, l, s+1+bn)\breve{D}(an, s+1+bn, l).
$$

This sum may again be expressed as a hypergeometric series

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix} bn+\frac{an}{2}+\frac{3}{2},\ bn+\frac{3}{2},\ \frac{l}{2}+bn+\frac{an}{2},\ bn+an+1,\ bn+\frac{an}{2}-\frac{r}{2},\ 1 \\ bn+\frac{an}{2}+\frac{1}{2},\ bn+an+\frac{1}{2},\ bn-\frac{l}{2}+\frac{an}{2}+2,\ bn+1,\ bn+\frac{an}{2}+\frac{r}{2}+2 \end{matrix};1\right] \\
&\quad\times \frac{\Gamma(2bn+3)\Gamma(an-l+1)\Gamma(bn+an)\Gamma(bn+an+1)\Gamma(2bn+an+2)}{4\,\Gamma(bn+1)\Gamma(bn+2)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{l}{2}+\frac{an}{2})\Gamma(2bn+an+1)\Gamma(2bn+2an)} \\
&\quad\times \frac{\Gamma(an+r+1)\Gamma(\frac{l}{2}+bn+\frac{an}{2})\Gamma(bn+\frac{an}{2}-\frac{r}{2})}{\Gamma(\frac{an}{2}-\frac{r}{2})\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(bn-\frac{l}{2}+\frac{an}{2}+2)\Gamma(bn+\frac{an}{2}+\frac{r}{2}+2)},
\end{aligned} \tag{5.17}
$$

which in turn can be expressed as the limit of a ${}_7F_6$ series

$$
\lim_{\epsilon\to 0}\ {}_7F_6\left[\begin{matrix} 2bn+an+1+\epsilon,\ V \\ W \end{matrix};1\right], \tag{5.18}
$$

where V and W are the lists

$$
(bn+\tfrac{an}{2}+\tfrac{3}{2}+\tfrac{\epsilon}{2}, bn+\tfrac{3}{2}+\tfrac{\epsilon}{2}, \tfrac{l}{2}+bn+\tfrac{an}{2}+\tfrac{\epsilon}{2}, 1+\tfrac{\epsilon}{2}, bn+an+1+\tfrac{\epsilon}{2}, bn+\tfrac{an}{2}-\tfrac{r}{2}+\tfrac{\epsilon}{2})
$$

and

$$
(bn+\tfrac{an}{2}+\tfrac{1}{2}+\tfrac{\epsilon}{2}, bn+an+\tfrac{1}{2}+\tfrac{\epsilon}{2}, bn-\tfrac{l}{2}+\tfrac{an}{2}+2+\tfrac{\epsilon}{2}, 2bn+an+1+\tfrac{\epsilon}{2}, bn+1+\tfrac{\epsilon}{2}, bn+\tfrac{an}{2}+\tfrac{r}{2}+2+\tfrac{\epsilon}{2})
$$

respectively. Such a series satisfies the following transformation formula

$$
\begin{aligned}
{}_7F_6&\left[\begin{matrix} a,\ \frac{a}{2}+1,\ b,\ c,\ d,\ e,\ a-e+n+1 \\ \frac{a}{2},\ a-b+1,\ a-c+1,\ a-d+1,\ a-e+1,\ e-n \end{matrix};1\right] \\
&= \frac{\Gamma(a-d+1)\Gamma(a-c+1)\Gamma(a-b+1)\Gamma(a-b-c-d+1)}{\Gamma(a-c-d+1)\Gamma(a-b-d+1)\Gamma(a-b-c+1)\Gamma(a+1)} \\
&\qquad \times {}_4F_3\left[\begin{matrix} b,\ c,\ d,\ -n \\ a-e+1,\ -a+b+c+d,\ e-n \end{matrix};1\right], \qquad (5.19)
\end{aligned}
$$

which can also be found in Slater's book [**70**, (4.3.6.4) reversed]. Applying this transformation to (5.18), permuting the elements in the hypergeometric series and letting ϵ tend to zero we obtain

$$
\frac{(2bn+2an-1)(2bn-l+an+2)}{2(an-l-1)(2bn+an+1)}\,{}_4F_3\left[\begin{matrix} 1,\ \frac{l}{2}+bn+\frac{an}{2},\ \frac{r}{2}-\frac{an}{2}+1,\ bn+\frac{3}{2} \\ bn+1,\ \frac{l}{2}-\frac{an}{2}+\frac{3}{2},\ bn+\frac{an}{2}+\frac{r}{2}+2 \end{matrix};1\right]. \qquad (5.20)
$$

One final transformation formula (see Slater [**70**, (4.3.5.1)]),

$$
\begin{aligned}
{}_4F_3&\left[\begin{matrix} a,\ b,\ c,\ -n \\ e,\ f,\ a+b+c-e-f-n+1 \end{matrix};1\right] \\
&= {}_4F_3\left[\begin{matrix} -n,\ a,\ a+c-e-f-n+1,\ a+b-e-f-n+1 \\ a+b+c-e-f-n+1,\ a-e-n+1,\ a-f-n+1 \end{matrix};1\right] \\
&\qquad \times \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}, \qquad (5.21)
\end{aligned}
$$

applied to the ${}_4F_3$ series in (5.20) gives

$$
\frac{bn(2bn+2an-1)(2bn+an-l+2)}{(r-l+1)(2bn+an+1)(2bn+an-r-2)}\,{}_4F_3\left[\begin{matrix} \frac{r}{2}-\frac{an}{2}+1,\ 1,\ \frac{r}{2}-\frac{l}{2}+2,\ \frac{an}{2}+\frac{r}{2}+\frac{1}{2} \\ bn+\frac{an}{2}+\frac{r}{2}+2,\ \frac{r}{2}-bn-\frac{an}{2}+2,\ \frac{r}{2}-\frac{l}{2}+\frac{3}{2} \end{matrix};1\right],
$$

or equivalently

$$
\frac{bn(2bn+2an-1)(2bn+an-l+2)}{(bn+an+1)(2bn+an-r-2)}\sum_{s=0}^{\infty}\frac{(\frac{r}{2}-\frac{an}{2}+1)_s(\frac{r}{2}-\frac{l}{2}+2)_s(\frac{an}{2}+\frac{r}{2}+\frac{1}{2})_s}{2(bn+\frac{an}{2}+\frac{r}{2}+2)_s(\frac{r}{2}-bn-\frac{an}{2}+2)_s(\frac{r}{2}-\frac{l}{2}+\frac{1}{2})_{s+1}}. \qquad (5.22)
$$

The sum on the right is bounded:

$$
\sum_{s=0}^{\infty}\frac{(\frac{r}{2}-\frac{an}{2}+1)_s(\frac{r}{2}-\frac{l}{2}+2)_s(\frac{an}{2}+\frac{r}{2}+\frac{1}{2})_s}{2(bn+\frac{an}{2}+\frac{r}{2}+2)_s(\frac{r}{2}-bn-\frac{an}{2}+2)_s(\frac{r}{2}-\frac{l}{2}+\frac{1}{2})_{s+1}} \le \sum_{s=0}^{\infty}\frac{(\frac{r}{2}-\frac{l}{2}+2)_s}{2(\frac{r}{2}-\frac{l}{2}+\frac{1}{2})_{s+1}}\left(\frac{a}{(a+2b)}\right)^{2s}
$$

and since $r \ge l+2$ it follows that

$$
\frac{(\frac{r}{2}-\frac{l}{2}+2)_s}{2(\frac{r}{2}-\frac{l}{2}+\frac{1}{2})_{s+1}} \le 1
$$

for all $s \ge 0$, thus we can safely interchange the sum and limit signs when we let n tend to infinity in (5.22).

Replacing the ${}_6F_5$ series in (5.17) with (5.20) where the ${}_4F_3$ hypergeometric series has been replaced with (5.22) we obtain

$$
\frac{16b^{3/2}(a+b)^{3/2}a^{-l+r+1}\left(\frac{a}{2}+b\right)^{l-r}}{\pi(a+2b)^4}
$$

in the limit for the pre-factor (via Stirling's approximation) while the hypergeometric series reduces to

$$
{}_2F_1\left[\begin{matrix} 1,\ \frac{r}{2}-\frac{l}{2}+2 \\ \frac{r}{2}-\frac{l}{2}+\frac{3}{2} \end{matrix};\frac{a^2}{(a+2b)^2}\right].
$$

If we suppose also that $2b/a \to \mu$ for some real $\mu > 0$, once we perform one final transformation formula (transformation (5.13) from Section 5.2) we see that for $r \geq l+2$

$$\lim_{n\to\infty} \check{Q}_{l,r} = \left(\frac{2}{\mu+1}\right)^{r-l+2} \frac{(\mu(\mu+2))^{1/2}}{\pi(r-l+1)} {}_2F_1\left[\begin{matrix} 1,\ -\frac{1}{2} \\ \frac{r}{2}-\frac{l}{2}+\frac{3}{2} \end{matrix}; -(\mu(\mu+2))^{-1}\right],$$

which is precisely the expression obtained in Proposition 5.2.1.

Suppose now that $r \leq l-6$. Then $\check{Q}_{l,r}$ is given by

$$-\sum_{s=0}^{bn-1} \check{B}(an,l,s+1)\check{D}(an,s+1,r),$$

which may be expressed as the following limit of a hypergeometric series

$$\frac{\Gamma(an)\Gamma(an+2)\Gamma(an-l+1)\Gamma(an+r+1)}{2\,\Gamma(2an)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}-\frac{l}{2}+2)\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+2)} \lim_{\epsilon\to 0}\left({}_7F_6\left[\begin{matrix} V \\ W \end{matrix}; 1\right]\right), \tag{5.23}$$

where

$$V = (\epsilon+an+1, \tfrac{\epsilon}{2}+\tfrac{an}{2}+\tfrac{3}{2}, \tfrac{\epsilon}{2}+\tfrac{l}{2}+\tfrac{an}{2}, \tfrac{\epsilon}{2}+\tfrac{an}{2}-\tfrac{r}{2}, \tfrac{\epsilon}{2}+\tfrac{3}{2}, bn+an+1, 1-bn)$$

and

$$W = (\tfrac{\epsilon}{2}+\tfrac{an}{2}+\tfrac{1}{2}, \tfrac{\epsilon}{2}-\tfrac{l}{2}+\tfrac{an}{2}+2, \tfrac{\epsilon}{2}+\tfrac{an}{2}+\tfrac{r}{2}+2, \tfrac{\epsilon}{2}+an+\tfrac{1}{2}, \epsilon-bn+1, \epsilon+bn+an+1).$$

By applying the following transformation formula,

$$\begin{aligned} &{}_7F_6\left[\begin{matrix} a,\ \frac{a}{2}+1,\ b,\ c,\ d,\ e,\ -n \\ \frac{a}{2},\ a-b+1,\ a-c+1,\ a-d+1,\ a-e+1,\ a+n+1 \end{matrix}; 1\right] \\ &\qquad = \frac{(a+1)_n(a-d-e+1)_n}{(a-d+1)_n(a-e+1)_n} {}_4F_3\left[\begin{matrix} a-b-c+1,\ d,\ e,\ -n \\ a-b+1,\ a-c+1,\ -a+d+e-n \end{matrix}; 1\right] \end{aligned} \tag{5.24}$$

(see [**70**, (2.4.1.1) reversed]) to (5.23) and letting ϵ tend to zero we obtain (after performing some cancellations)

$$\begin{aligned} &-\frac{2^{r-l+2}\Gamma(bn+\frac{3}{2})\Gamma(\frac{an}{2}-\frac{l}{2}+\frac{1}{2})\Gamma(bn+an+1)\Gamma(\frac{an}{2}+\frac{r}{2}+\frac{1}{2})}{3\pi\,\Gamma(bn)\Gamma(\frac{an}{2}-\frac{l}{2}+2)\Gamma(bn+an-\frac{1}{2})\Gamma(\frac{an}{2}+\frac{r}{2}+2)} \\ &\qquad\qquad \times {}_4F_3\left[\begin{matrix} 2-\frac{l}{2}+\frac{r}{2},\ \frac{3}{2},\ bn+an+1,\ 1-bn \\ \frac{an}{2}+2-\frac{l}{2},\ \frac{an}{2}+\frac{r}{2}+2,\ \frac{5}{2} \end{matrix}; 1\right]. \end{aligned} \tag{5.25}$$

Since $r \leq l-6$ this expression is quite clearly a terminating hypergeometric series, thus as n tends to infinity (5.25) reduces to

$$-\frac{2^{r-l+2}(\mu(\mu+1))^{3/2}}{3\pi} {}_2F_1\left[\begin{matrix} 2-\frac{l}{2}+\frac{r}{2},\ \frac{3}{2} \\ \frac{5}{2} \end{matrix}; -\mu(\mu+2)\right].$$

Compare this with (5.10) in the previous section – they agree entirely! It follows that the matrix manipulations that give rise to the closed form determinant evaluation in the previous section may also be applied to $\check{Q}_{L,R}$, completing the proof. □

REMARK 5.4.2

This result is somewhat surprising, since if $L = \{-r : r \in R\}$ and all elements in R are negative then $\omega_{\check{H}}(L,R) = \omega_V(R)$, that is, the interaction between a set of right pointing holes and a free boundary is the same as the interaction between sets of unit holes that lie along a fixed zig-zag horizontal boundary of a sea of unit rhombi.

We now turn our attention to $\omega_{\widehat{H}}(L,R)$. As $n \to \infty$ we obtain the half plane constrained by a fixed horizontal zig-zag boundary that contains a set of trapezoid holes as the limit shape (see Figure 33).

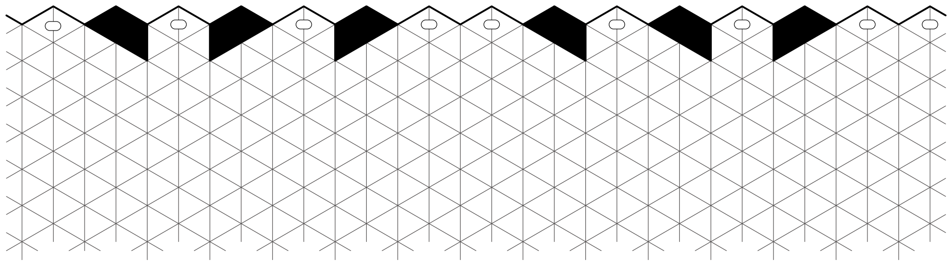


FIGURE 33. A set of trapezoid dents on the fixed horizontal boundary of the lower half plane obtained as the limit of $\widehat{H} \setminus (L, R)$.

PROPOSITION 5.4.3 – TG [**40**]

Suppose that $2b/a \to \mu$ as $n \to \infty$ for some real $\mu > 0$. Then the entries of $\widehat{Q}_{L,R}$ are given by

$$\left(\frac{2}{\mu+1}\right)^{r-l+2} \frac{(\mu(\mu+2))^{-1/2}}{\pi(r-l+3)} {}_2F_1\left[\begin{matrix}1, \frac{1}{2} \\ \frac{r}{2}-\frac{l}{2}+\frac{5}{2}\end{matrix}; -(\mu(\mu+2))^{-1}\right]$$

for $l \in L$ and $r \in R$.

PROOF. In order to extract the limit

$$\lim_{n\to\infty} \widehat{Q}_{l,r}$$

we can apply precisely the same set of manipulations as we did in the previous proof to uncover

$$\lim_{n\to\infty} \check{Q}_{l,r}.$$

The full proof can be found in Appendix B. □

Note the similarity between Proposition 5.4.3 and Proposition 5.2.1. Again the same boundary effect can be observed – if $\mu \neq 1$ then for holes that are separated by large distances the entries of $\widehat{Q}_{L,R}$ either blow up or shrink exponentially, thus we focus once more on the situation when $\mu = 1$. Clearly in this case we may use the same set of matrix manipulations as in the previous section in order to obtain a closed form evaluation of $\widehat{Q}_{L,R}$.

COROLLARY 5.4.4 – TG [**40**]

Consider the weighted tilings of the region $\widehat{H} \setminus (L, R)$. Suppose as $n \to \infty$, $2b/a \to 1$ and let $\mathcal{H}$ index the set of holes that induced by $L \cup R$. For holes that are separated by large distances their interaction is asymptotically

$$\omega_{\widehat{H}}(L,R) \sim \prod_{h\in\mathcal{H}} C'_h \prod_{1\leq i<j\leq|\mathcal{H}|} \mathrm{d}(h_i,h_j)^{\frac{1}{4}q(h_i)q(h_j)}$$

where

$$C'_h = \left(\prod_{i=1}^{|q(h)|/2} \frac{3^{(i-1)/2}\Gamma(i)}{\sqrt{2\pi}}\right).$$

The result above pertains to the interactions between trapezoids that lie on a fixed horizontal zig-zag boundary that borders a sea of rhombi, where the tilings are weighted in a strange fashion. However by placing some mild assumptions on L and R we may easily obtain the following lemma (which follows from Lemma 4.4.9), thus generalising the asymptotic result of [**20**] and also confirming the analogy between tilings and the method of images for left pointing triangular holes.

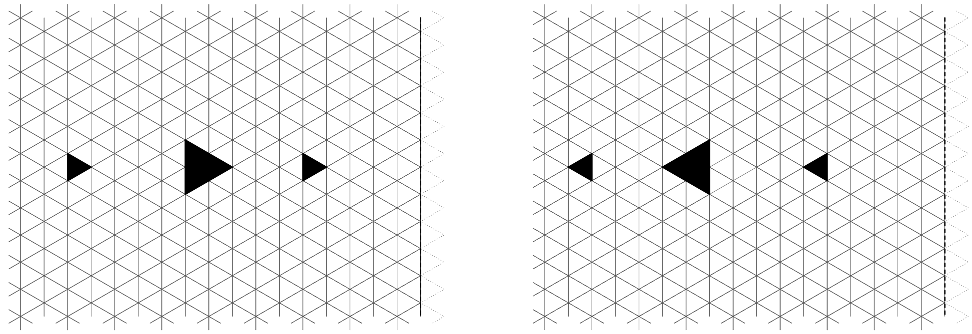


FIGURE 34. Two configurations of holes that interact slightly differently in a sea of rhombi.

LEMMA 5.4.5 – TG [**40**]

Suppose $l < 0$ for all $l \in L$ and consider tilings of $V \setminus L$ as $n \to \infty$, where $2b/a \to 1$. Let $\mathcal{H}$ index the set of holes induced by L, $\mathcal{H}'$ the reflection of those holes in the vertical free boundary. If the holes in $\mathcal{H}$ are separated by large distances then their interaction with this boundary is asymptotically

$$\omega_V(L) \sim \prod_{h \in \mathcal{H} \cup \mathcal{H}'} C'_h \prod_{1 \leq i < j \leq |\mathcal{H} \cup \mathcal{H}'|} \mathrm{d}(h_i, h_j)^{\frac{1}{4} q(h_i) q(h_j)}$$

where

$$C'_h = \left(\prod_{i=1}^{|q(h)|/2} \frac{3^{(i-1)/2} \Gamma(i)}{\sqrt{2\pi}} \right).$$

REMARK 5.4.6

A curious, counter-intuitive phenomenon becomes apparent when we compare Lemma 5.4.5 with Theorem 5.2.6. Suppose $L = R$ and consider the correlation functions $\omega_V(L)$ and $\omega_V(R)$. Clearly we have

$$\omega_V(R)/\omega_V(L) \sim \prod_{h \in \mathcal{H}} 3^{|q(h)|/2},$$

thus the interaction between right pointing holes differs from the interaction obtained when we reflect each hole about its vertical edge. Within the system of tilings this means that the orientation of the holes seems to still make its presence felt on the rhombi, even when the holes are separated by large distances from each other and the free boundary (see Figure 34).

OPEN PROBLEM 5.4.7

This orientation-interaction phenomenon is really quite odd, after all one would expect that at large distances the orientation of the holes has very little impact on the ways in which the plane can be tiled. This poses a further question: is this an artefact of this specific system, or does it allude to a more general principle underlying two dimensional tiling problems?

The final result of this chapter is the following Corollary, which arises from combining Corollary 5.4.4 with Theorem 5.4.1.

COROLLARY 5.4.8 – TG [**40**]

Consider tilings of $H \setminus (L, R)$ and suppose $2b/a \to 1$ as $n \to \infty$. Let $\mathcal{H}$ index the set of fixed holes induced by L and R. If the holes in $\mathcal{H}$ are separated by large distances then their interaction in a sea *of unit rhombi is asymptotically*

$$\omega_H(L,R) \sim \prod_{h\in\mathcal{H}} C_h^* \prod_{1\le i<j\le|\mathcal{H}|} \mathrm{d}(h_i,h_j)^{\frac{1}{2}q(h_i)q(h_j)},$$

where $\mathcal{H} = L \cup R$ *and*

$$C_h^* = \frac{1}{3^{|q(h)|/4}} \prod_{i=1}^{|q(h)/2|} \frac{3^i\Gamma(i)^2}{2\pi}.$$

Open Problem 5.4.9

The above corollary proves Ciucu's conjecture for collinear holes of even side length, but what if we consider sets of holes induced by sets of contiguous collinear 2-holes that are both right and left pointing? This would require taking a closer look at the determinant evaluations in the previous sections for such sets of holes.

CHAPTER
SIX

DIMERS ARE FOREVER

Until now we have studied rhombus tilings of holey hexagons via their representation as families of non-intersecting lattice paths between points on the (half) integer lattice. The set of ways in which one may skin a cat, however, is not a singleton, and in this chapter we outline a method for counting *dimer coverings*[46] of bipartite planar graphs that will allow us to count rhombus tilings in a completely different way.

6.1. Kasteleyn's method

Let $G = (V, E)$ be a *bipartite planar graph*[47] consisting of a set of equinumerous black and white vertices $V = \{b_1, \ldots, b_n, w_1, \ldots, w_n\}$ and a set of edges E between them, and suppose that G is embedded on a sphere (such a graph is sometimes referred to as a *bipartite planar combinatorial map*- from now on, simply a *map*). Within this context a *matching* of G is a subset of its edges, say $E' \subseteq E$, together with the vertices to which they are incident, say $V' \subseteq V$, such that every vertex in V' is incident with precisely one edge in E'. A matching is *perfect* if $V' = V$ (see Figure 35, where the matchings are indicated by solid lines in the centre and right hand diagrams).

Suppose we label the black and white vertices of G from $b_1, b_2, \ldots, b_n$ and $w_1, w_2, \ldots, w_n$ respectively and attach to its edges weights taken from some commutative ring, thereby obtaining a *weighted map* G_w where the weight of an edge that connects two adjacent vertices $b_i, w_j \in V$ is denoted $w(b_i, w_j)$.[48]

For some $\sigma \in \mathfrak{S}_n$ let

$$P_m(B, W_\sigma) := \prod_{i=1}^{n} w(b_i, w_{\sigma(i)})$$

denote the *weighted perfect matching* in which each black vertex b_i in $B := \{b_1, b_2, \ldots, b_n\}$ is matched with the white vertex $w_{\sigma(i)}$ in $W := \{w_1, w_2, \ldots, w_n\}$. The sum over all weighted perfect matchings of G is thus

$$\sum_{\sigma \in \mathfrak{S}_n} P_m(B, W_\sigma).$$

Remark 6.1.1

Observe that the function $P_m(B, W_\sigma)$ vanishes unless σ matches each black vertex to an adjacent white one. In other words, $P_m(B, W_\sigma) = 0$ unless σ gives rise to a perfect matching between black and white vertices.

[46] Otherwise known as *perfect matchings*.

[47] That is, a graph that can be embedded in the plane with vertices that are either black or white, where any two adjacent vertices differ in colour.

[48] Note that if b_i, w_j are non-adjacent then we set $w(b_i, w_j) = 0$.

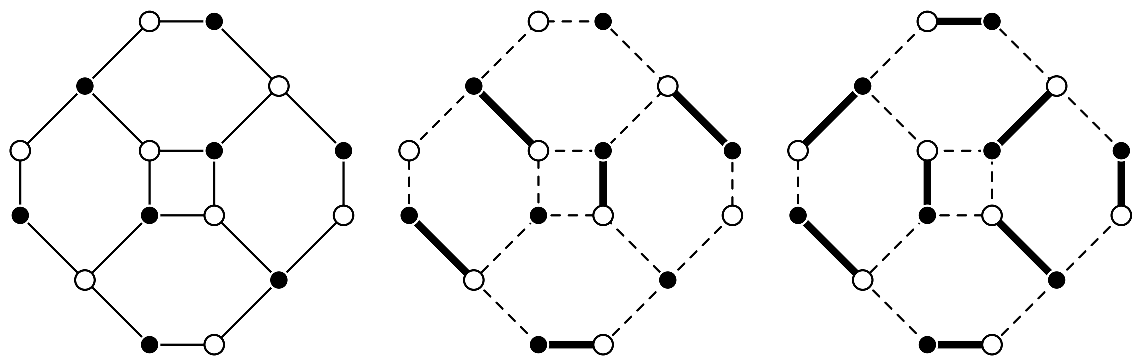


FIGURE 35. A bipartite planar graph, a matching, and a perfect matching.

We define the *weighted bi-adjacency matrix* of G_w to be the $n \times n$ matrix A_{G_w} with i-th row and j-th column indexed by the vertices b_i and w_j respectively, where each (i, j)-entry is given by $w(b_i, w_j)$. The previous expression may thus be re-written as

$$\sum_{\sigma \in \mathfrak{S}_n} \prod_{i}^{n} (A_{G_w})_{b_i, w_{\sigma(i)}}, \tag{6.1}$$

which is otherwise known as the *permanent* of A_{G_w} (denoted $\mathrm{per}(A_{G_w})$). By setting the weight of all edges between adjacent vertices to 1 the above expression yields the number of perfect matchings of G.

If our goal is to find a closed form evaluation for the expression in (6.1) then at first sight one may be forgiven for thinking that we have reached a dead end. The permanent of a matrix is, after all, a somewhat enigmatic function whose properties are not particularly well-understood; indeed, Valiant's algebraic variations of the *P vs. NP* problem **[74]** may be phrased in terms of the complexity of its computation. A great deal more is known, however, about the determinant of a matrix – the much-loved distant relative of the permanent which, as we have already seen, has a comparative abundance of useful, well-understood properties.

How, then, may we relate the permanent of a matrix to its determinant? We know that the determinant is simply the permanent where each summand is multiplied by the signature of the corresponding permutation, but can we compute the permanent of a matrix in terms of its determinant? To date it seems that there exists no general method that allows us to express one in terms of the other, however Kasteleyn **[50]** showed that for matrices arising in the context of counting perfect matchings of maps this is indeed possible.

6.2. Admissible orientations

Suppose we endow the surface of the sphere on which G_w is embedded with an orientation in the clockwise direction. Let us orient the edges of G_w so that each edge is directed from a black vertex to a white one, thereby obtaining an *oriented weighted map* (see Figure 36, left). Kasteleyn showed that within such maps it is *always* possible to change the direction of a finite (possibly empty) set of edges so that around each face of G_w an odd number of edges agree with the orientation of the surface of the sphere (when the edges are viewed from the centre of each face).[49] Such an orientation is called *admissible* and we will denote by G_w^+ the weighted map G_w whose edges have been re-directed so that its orientation is admissible (see Figure 36, right). We encode the construction of an admissible orientation within the weighting of G_w by multiplying by -1 the weights of those edges that are

[49] Since G_w is embedded on a sphere its outer boundary is also a face.

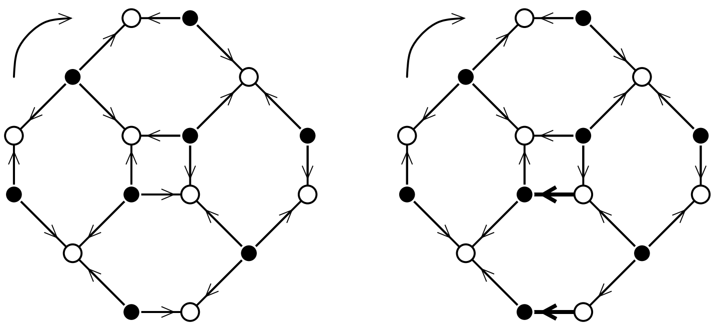


FIGURE 36. An orientation that is not admissible (left) and an admissible one obtained by changing the direction of the two edges in bold (right).

directed from white vertices to black.[50] The weighted bi-adjacency matrix of G_w^+, denoted $A_{G_w^+}$, is referred to as the *Kasteleyn matrix* corresponding to G_w.

THEOREM 6.2.1 – KASTELEYN [50]

The number of (weighted) perfect matchings of G_w is equal to

$$\mathrm{per}(A_{G_w}) = |\det(A_{G_w^+})|.$$

REMARK 6.2.2

Kasteleyn's method is in fact more general than it appears here and one may use a similar approach to count weighted perfect matchings of any planar graph. In this case one considers the Pfaffian, rather than determinant, of a weighted adjacency matrix whose rows (and columns) are indexed by *all* the vertices of the corresponding map. For bipartite planar graphs, however, a straightforward argument shows that such a computation reduces to the situation described above.

6.3. Poking holes in Kasteleyn's approach

Kasteleyn's method is certainly very general, and although when it was first described Kasteleyn seems to have had little interest in counting dimer coverings of graphs that contain gaps, it turns out that with a sprinkling of linear algebra we can extend these ideas in order to count dimer coverings of maps that contain gaps or holes in their interior.

Let G_w^+ be an admissibly oriented weighted map and suppose now that V is a set of vertices contained within it. Consider the map $G_w^+ \setminus V$ obtained by removing the set of vertices V (and all edges adjacent to them) from G_w^+. If V does not consist of equinumerous sets of white and black vertices then it is clear that no dimer covering of $G_w^+ \setminus V$ exists, so we shall assume that V consists of k-many black and k-many white vertices. The orientation of $G_w^+ \setminus V$ is either admissible or not, and in order to distinguish between these two cases we make the following definition.

DEFINITION 6.3.1

A set of vertices V contained in an admissibly oriented weighted map G_w^+ is called *admissibility preserving* if the orientation of the weighted map obtained by removing V from G_w^+ (that is, $G_w^+ \setminus V$) remains admissible.

[50]This is sometimes referred to as a *Kasteleyn twist*.

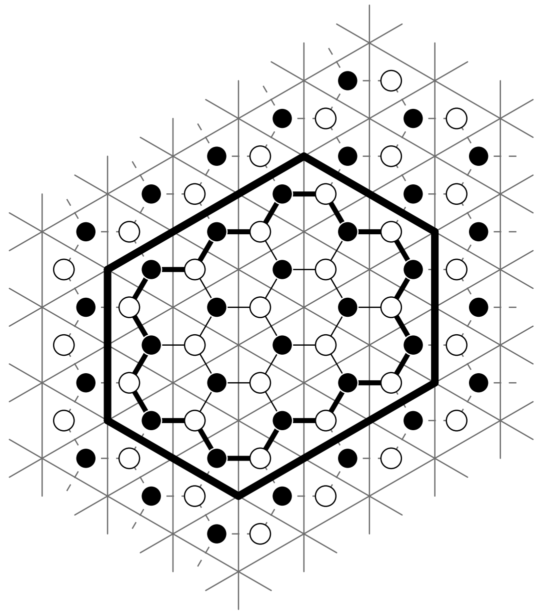


FIGURE 37. The hexagon $H_{2,2,3} \subset \mathscr{T}$ and the corresponding sub-graph $G_{2,2,3} \subset \mathscr{H}$.

REMARK 6.3.2

It may well be the case that upon removing V from G_w^+ there remain a number of vertices in $G_w^+ \setminus V$ that have degree 1. The edges connected to these vertices will always be included within every dimer covering of $G_w^+ \setminus V$ (we say that they are *forced dimers*), thus we may safely remove them together with their incident vertices. In this way V *induces* a larger set of gaps upon its removal from G_w^+, and removing forced dimers does not change the admissibility of its orientation.

With respect to counting dimer coverings of $G_w^+ \setminus V$, it follows from the arguments above that as long as V is an admissibility preserving set of vertices then this number is given by

$$|\det(A_{G_w^+} \setminus V)|, \tag{6.2}$$

where $A_{G_w^+} \setminus V$ denotes the sub-matrix obtained by deleting from $A_{G_w^+}$ those rows and columns that are indexed by the vertices belonging to V.

By way of a lemma often attributed to Jacobi[51] we can calculate the determinant in 6.2 in an alternative way, since

$$\det(A_{G_w^+} \setminus V) = \det(A_{G_w^+}) \cdot \det(((A_{G_w^+})^{-1})_V), \tag{6.3}$$

where $((A_{G_w^+})^{-1})_V$ denotes the sub-matrix obtained deleting from the inverse of $A_{G_w^+}$ all but those rows and columns indexed by V.

REMARK 6.3.3

The expression on the right hand side of (6.3) is similar in flavour to the formulas established in the previous chapters in the sense that it reduces calculating the number of dimer coverings of a map that contains defects to a product involving the number of dimer coverings of the original map (that is, $\pm\det(A_{G_w^+})$) together with a determinant evaluation of a matrix whose size is dependent on the vertices that have been removed (that is, $\det(((A_{G_w^+})^{-1})_V)$).

[51] This is a standard result from linear algebra that says that a minor of a matrix A is equal to the determinant of A times the complementary minor of A^{-1}, see for example [**1**, p. 98].

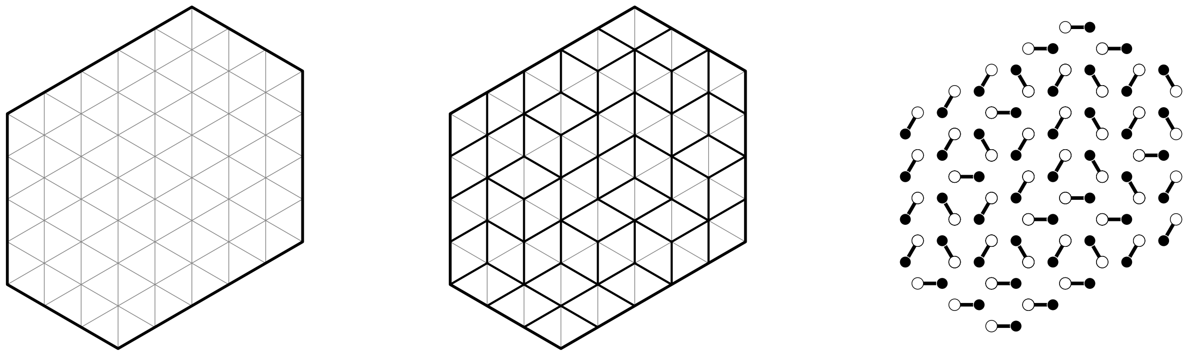


FIGURE 38. A rhombus tiling of $H_{3,3,5}$ and its corresponding perfect matching of $G_{3,3,5}$.

Kasteleyn's method clearly provides us with a powerful set of tools with which we can enumerate dimer coverings of maps that may contain gaps in their interior. In the following section we shall describe how rhombus tilings of hexagons on the triangular lattice can be represented as dimer coverings of sub-graphs of the hexagonal lattice, thus enabling us to attack the problem of counting tilings of holey hexagons from this new perspective.

6.4. From rhombus tilings to perfect matchings

Let us return to the triangular lattice $\mathscr{T}$ that was first described in Section 2.7 of Chapter 2. As previously discussed $\mathscr{T}$ consists of unit equilateral triangles that are either left or right pointing, however we may form from $\mathscr{T}$ a completely different lattice by replacing all left pointing unit triangles with black vertices, all right pointing triangles with white vertices, and then connecting with an edge pairs of vertices that correspond to unit triangles that share precisely one edge on $\mathscr{T}$. This new lattice has a honeycomb structure and is referred to as the *hexagonal lattice* (we shall denote it $\mathscr{H}$, see Figure 37).[52]

Under this construction the hexagon[53] H that lives on $\mathscr{T}$ corresponds to an *hexagonal sub-graph* of $\mathscr{H}$ that we denote G. More formally, G is defined to be the sub-graph of $\mathscr{H}$ whose outer boundary is determined by beginning at the centre of an hexagonal face and traversing faces that share a common edge via $(a-1)$ north-west edges, then $(b-1)$ north edges, then $(c-1)$ north-east edges, then $(a-1)$ south-east edges, $(b-1)$ south edges, and finally $(c-1)$ south-west edges (again see Figure 37).

A rhombus tiling of H is simply a partition of all unit triangles contained within it into pairs, where each pair consists of a left and right pointing unit triangle that share an edge. It is quite clear that with respect to $\mathscr{H}$ each rhombus tiling of H corresponds to a distinct perfect matching of G (see Figure 38). The number of perfect matchings of G is therefore

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$$

(this is MacMahon's box formula from Theorem 2.2.4).

6.5. Orienting hexagonal sub-graphs of the hexagonal lattice

Clearly in order to utilise Kasteleyn's method we must first ensure we have an admissible orientation of G. We begin by endowing the plane with a sense of rotation in the clockwise direction and directing the edges of G from black vertices to white. It is obvious that a single hexagonal face contained in G with edges directed in this way is already comprised of an odd number of edges

[52] The hexagonal lattice is sometimes called the "dual" of the triangular lattice.

[53] From now on assume that for positive integers a, b, c, H denotes the semi-regular hexagon with sides of length a, b, c, a, b, c going from the south-west side as in Chapter 2.

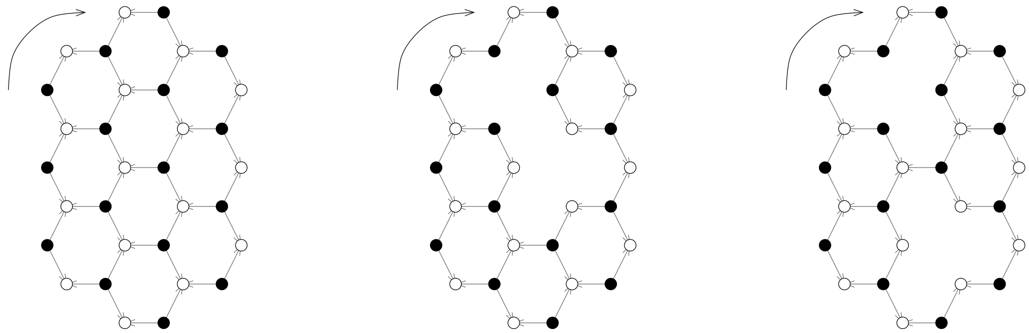


FIGURE 39. The admissibly oriented hexagonal sub-graph $G_{2,3,2}$ with two different pairs of vertices removed from its interior.

whose direction agrees with the orientation of the plane. Once we have convinced ourselves that this holds for the outer boundary (which is also considered a face), we see that this orientation of G is already admissible (see Figure 39).

It follows that if

$$B := \{b_1, b_2, \ldots, b_{ab+bc+ca}\}$$

denotes the labelled black vertices in G and

$$W := \{w_1, w_2, \ldots, w_{ab+bc+ca}\}$$

the labelled white ones then the Kasteleyn matrix corresponding to G,

$$A_G := ((A_G)_{b_i,w_j})_{b_i,w_j \in G}$$

is simply the ordinary bi-adjacency matrix of G, that is, its entries are given by

$$(A_G)_{b_i,w_j} = \begin{cases} 1 & b_i, w_j \textit{ adjacent}, \\ 0 & \textit{otherwise}. \end{cases}$$

According to Theorem 6.7.3 we already know that

$$|\det(A_G)| = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2},$$

however in order to really make use of the theory outlined in Section 6.3 (in particular, equation (6.3)) we first need to understand what it means for a set of vertices in G to be admissibility preserving, and also to explicitly establish the entries of the inverse of A_G. We shall address the former in the remainder of this section while the latter shall be postponed until the next.

According to Definition 6.3.1, a set of vertices contained in an admissibly oriented graph is deemed to be admissibility preserving if the orientation of the sub-graph obtained upon their removal is again admissible. With respect to our hexagonal sub-graph of the hexagonal lattice, G, a pair of vertices $V = \{v, v'\}$ contained in the interior of G preserve admissibility if they are located on the same face within G. We thus call such a pair of vertices *face connected* (otherwise we say that v and v' are *unconnected*). Figure 39 shows $G_{2,3,2}$ with different pairs of vertices removed, one pair that preserves admissibility (centre) and another that does not (right).

Suppose now that $V = \bigcup_i V_i$ is a set of vertices contained in G that consists of equinumerous black and white vertices (say, k-many in each colour class), and let us assume that each V_i is a *connected* set of vertices (by which we mean that V_i consists either of a single vertex, or instead for any $v \in V_i$ there exists at least one other $v' \in V_i, v \neq v'$ such that v and v' are face connected). Further suppose that for any $v \in V_i$ and $v' \in V_j, i \neq j$, the vertices v and v' are unconnected. The set V is thus an unconnected union of sets of connected vertices, and after a moment's thought it is easy to convince ourselves that V preserves admissibility as long as the number of white and black

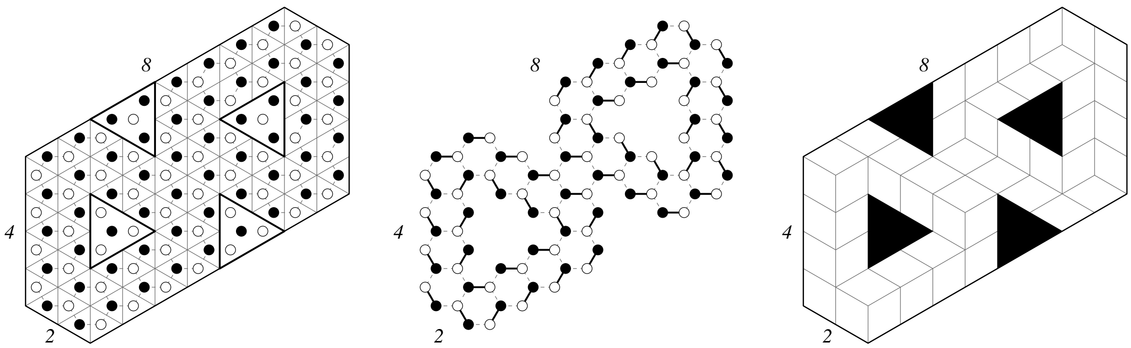


FIGURE 40. The sub-graph $G_{2,4,8}$ with holes created by removing an unconnected set of connected vertices.

vertices in each connected set V_i have the same parity. Figure 40 shows the graph $G_{2,4,8}$ with such a set of vertices removed.

6.6. Inverting the Kasteleyn matrix

Our goal now is to establish the entries of the inverse Kasteleyn matrix corresponding to G. We begin by observing that according to Cramer's rule each (i, j)-entry of A_G^{-1} is equal to

$$(-1)^{i+j}\frac{\det(A_G \setminus \{b_j, w_i\})}{\det(A_G)}. \tag{6.4}$$

As we already have a formula for the denominator we turn our attention to the numerator

$$\det(A_G \setminus \{b_j, w_i\}). \tag{6.5}$$

Let us take a closer at the matrix $A_G \setminus \{b_j, w_i\}$. It is certainly the bi-adjacency matrix of the graph $G \setminus \{b_j, w_i\}$ where all directed edges between adjacent vertices have a weight of 1, thus if $PM(G \setminus \{b_j, w_i\})$ denotes the set of perfect matchings of $G \setminus \{b_j, w_i\}$ we have

$$|PM(G \setminus \{b_j, w_i\})| = \sum_{\sigma\in\mathfrak{S}_{ab+bc+ca-1}} \prod_{i=1}^{ab+bc+ca-1} (A_G \setminus \{b_j, w_i\})_{i,\sigma(i)}, \tag{6.6}$$

that is, $|PM(G \setminus \{b_j, w_i\})|$ is equal to the permanent of the matrix $A_G \setminus \{b_j, w_i\}$. Since each summand in the right hand side of the above expression is 1 if b_i and $w_{\sigma(i)}$ are adjacent for $1 \le i \le ab + bc + ca - 1$, and zero otherwise, each non-zero term in the sum corresponds to a perfect matching of $G \setminus \{b_j, w_i\}$.

By taking instead the determinant of $A_G \setminus \{b_j, w_i\}$ we introduce into (6.6) a sign in front of each summand

$$\det(A_G \setminus \{b_j, w_i\}) = \sum_{\sigma\in\mathfrak{S}_{ab+bc+ca-1}} \operatorname{sgn}(\sigma) \prod_{i=1}^{ab+bc+ca-1} (A_G \setminus \{b_j, w_i\})_{i,\sigma(i)}, \tag{6.7}$$

so we may crudely express this determinant as

$$|PM^+(G \setminus \{b_j, w_i\})| - |PM^-(G \setminus \{b_j, w_i\})|,$$

where $PM^+(G \setminus \{b_j, w_i\})$ denotes the set of perfect matchings that are counted with a positive sign in (6.7), and $PM^-(G \setminus \{b_j, w_i\})$ those that are counted with a negative one. We therefore say that $\det(A_G \setminus \{b_j, w_i\})$ counts the number of *signed perfect matchings* of $G \setminus \{b_j, w_i\}$.

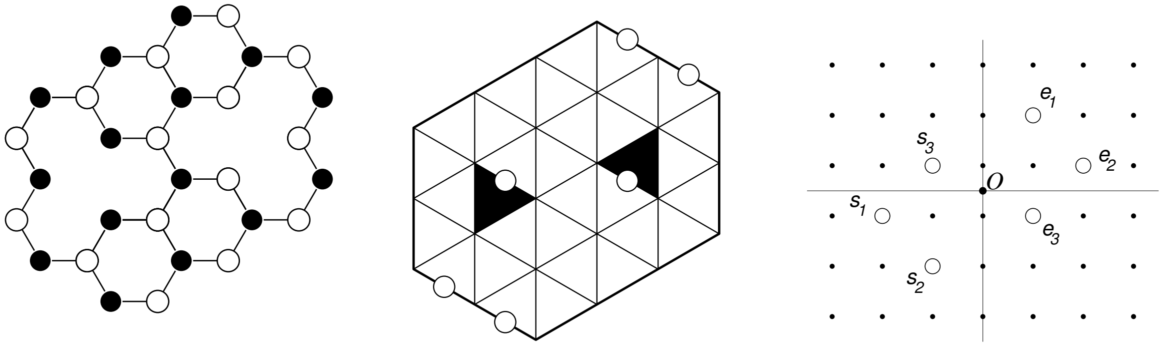


FIGURE 41. The hexagonal sub-graph $G_{2,2,3} \setminus \{b, w\}$, the corresponding holey hexagon $H \setminus \{l, r\}$, and the corresponding set of points in the (half) integer lattice.

REMARK 6.6.1

Observe that Kasteleyn's method provides us with a way to weight the edges of $G \setminus \{b_j, w_i\}$ to ensure that the weights of the perfect matchings of $G \setminus \{b_j, w_i\}$ are such that in the determinant of $A_{G\setminus\{b_j,w_i\}^+}$ (the Kasteleyn matrix corresponding to $G \setminus \{b_j, w_i\}$), all perfect matchings have the same sign. It follows that if b_j and w_i are admissibility preserving then

$$\det(A_{G\setminus\{b_j,w_i\}^+}) = \pm \det(A_G \setminus \{b_j, w_i\}).$$

Consider the hexagon corresponding to $G \setminus \{b_j, w_i\}$ on $\mathscr{T}$. This is a semi-regular hexagon H from which a pair of unit triangles have been removed – one left pointing, l (corresponding to b_j), and one right, r (corresponding to w_i), that is, $G \setminus \{b_j, w_i\}$ corresponds to $H \setminus T$ where $T = \{l, r\}$.

According to the method outlined in Chapter 3 tilings of $H \setminus \{l, r\}$ (equivalently, perfect matchings of $G \setminus \{b_j, w_i\}$) correspond to families of non-intersecting lattice paths that begin at the set of points[54]

$$S := \{s_1, s_2, \ldots, s_{a+1}\},$$

where

$$s_i = \begin{cases} (i - \frac{1+a+c}{2}, \frac{a-b+1}{2} - i), & 1 \le i \le a, \\ (x(r), y(r)), & i = a+1, \end{cases}$$

and end at

$$E := \{e_1, e_2, \ldots, e_{a+1}\},$$

where

$$e_j := \begin{cases} (j - \frac{1+a-c}{2}, \frac{a+b+1}{2} - j), & 1 \le j \le a, \\ (x(l), y(l)), & j = a+1. \end{cases}$$

REMARK 6.6.2

The hexagon $H \setminus T$ gives rise to a set of points in the (half) integer lattice that can be obtained via the approach outlined in Chapter 3. In particular the left pointing unit triangle l corresponds to the point $(x(l), y(l)) \in \mathbb{Z}_{a,c} \times \mathbb{Z}_{a,b}$, and similarly r corresponds to $(x(r), y(r))$ (see Figure 41).

[54] That is, points in $\mathbb{Z}_{a,c} \times \mathbb{Z}_{a,b}$.

We may thus form the lattice path matrix $P_{S,E} := (P_{i,j})_{1\leq i,j\leq a+1}$, where

$$P_{i,j} := \begin{cases} \binom{b+c}{c+j-i} & 1 \leq i,j \leq a, \\ \binom{(b+c)/2-x(r)-y(r)}{j-x(r)-(a-c+1)/2} & i = a+1, 1 \leq j \leq a \\ \binom{x(l)+y(l)+(b+c)/2}{x(l)-i+(a+c+1)/2} & 1 \leq i \leq a, j = a+1, \\ \binom{x(l)+y(l)-x(r)-y(r)}{x(l)-x(r)} & i = j = a+1. \end{cases}$$

Recalling Theorem 3.4.2 (that of Lindström, and Gessel and Viennot) we have

$$\pm \det(P_{S,E}) = \sum_{\sigma\in\mathfrak{S}_{a+1}} \operatorname{sgn}(\sigma) N(S, E_\sigma), \tag{6.8}$$

where $N(S, E_\sigma)$ denotes the number of families of non-intersecting lattice paths in which each vertex s_i of S is connected to vertex $e_{\sigma(i)}$ of E. We may thus crudely re-write the determinant in (6.8) as

$$|LP^+(S, E)| - |LP^-(S, E)|,$$

where $LP^+(S, E)$ denotes the set of families of lattice paths that are counted with a positive sign in (6.8), while $LP^-(S, E)$ denotes those counted with a negative one.

Remark 6.6.3

As discussed in Remark 3.4.4, if the start and end points S and E lie in such a way that every non-zero contribution to the sum in (6.8) has the same sign then the number of signed families of non-intersecting lattice paths agrees with $\pm$ the number of tilings of $H \setminus \{l, r\}$. In particular this holds when l and r correspond to a pair of admissibility preserving vertices (see Section 6.8).

6.7. Holey matrimony

We are already well-versed in the relationship between perfect matchings and families of non-intersecting lattice paths, however it turns out that the bijection between these sets of objects can be refined even further.

Theorem 6.7.1 – Cook and Nagel **[24]**

The perfect matchings that belong to $PM^+(G \setminus \{b_j, w_i\})$ are in bijection either with those families of lattice paths in $LP^+(G \setminus \{b_j, w_i\})$, or instead with those in $LP^-(G \setminus \{b_j, w_i\})$.

An immediate consequence of Theorem 6.7.1 is the following

$$\det(A_G \setminus \{b_j, w_i\}) = \pm \det(P_{S,E}). \tag{6.9}$$

This is an incredibly useful result for it allows us to marry together these two distinct approaches to counting (signed) tilings. Removing vertices from the interior of G results in having to consider a possible sign change that is *locally* dependent on exactly how the remaining vertices are matched in the region surrounding the gap, however Cook and Nagel's result shows that this potential sign change is reflected on a *global* scale within the lattice path set-up. This simplifies our task greatly, since in order to determine the entries of A_G^{-1} we simply need to evaluate the determinant of $P_{S,E}$. In order to do this we shall adopt our favoured approach which has proved so fruitful in previous chapters.

Remark 6.7.2

The regions and tilings considered in **[24]** are in fact far more general than those stated in the above theorem – Cook and Nagel's result holds for any equilateral triangular region on $\mathscr{T}$ containing any number of holes that admits a tiling by rhombi. Of course we can specialise these regions to the hexagons with which we are concerned.

Theorem 6.7.3 – TG [41]

The lattice path matrix $P_{S,E}$ corresponding to $G \setminus \{b_j, w_i\}$ has LU-decomposition

$$P_{S,E} = L \cdot U$$

where $L = (L_{i,j})_{1 \le i,j \le a+1}$ has entries given by

$$L_{i,j} := \begin{cases} A(b,c,i,j) & 1 \le j \le i \le a, \\ B(a,b,c,x(r),y(r),j) & i = a+1, 1 \le j \le a, \\ E(a,b,c,x(r),y(r)) & i = j = a+1, \\ 0 & \textit{otherwise} \end{cases}$$

and $U = (U_{i,j})_{1 \le i,j \le a+1}$ is given by

$$U_{i,j} := \begin{cases} C(b,c,i,j) & 1 \le i \le j \le a, \\ D(a,b,c,x(l),y(l),i) & 1 \le i \le a, j = a+1, \\ F(a,b,c,x(l),y(l)) & i = j = a+1, \\ 0 & \textit{otherwise,} \end{cases}$$

where

$$\begin{aligned} A(b,c,i,j) :=& \frac{\Gamma(c+1)\Gamma(i)\Gamma(b+j)}{\Gamma(j)\Gamma(b+i)\Gamma(i-j+1)\Gamma(c-i+j+1)}, \\ B(a,b,c,x(r),y(r),j) :=& \sum_{s=1}^{j} \frac{(-1)^{j-s}\Gamma(b+j)\Gamma(c+s)\Gamma(b+j-s)}{\Gamma(b)\Gamma(s)\Gamma(j-s+1)\Gamma(b+c+j)} \binom{\frac{b}{2}+\frac{c}{2}-x(r)-y(r)}{s-\frac{1}{2}(a-c+1)-x(r)}, \\ C(b,c,i,j) :=& \frac{\Gamma(b+1)\Gamma(j)\Gamma(b+c+i)}{\Gamma(b+i)\Gamma(c+j)\Gamma(j-i+1)\Gamma(b+i-j+1)}, \\ D(a,b,c,x(l),y(l),i) :=& \sum_{s=1}^{i} \frac{(-1)^{i-s}\Gamma(i)\Gamma(b+s)\Gamma(c+i-s)}{\Gamma(c)\Gamma(s)\Gamma(b+i)\Gamma(i-s+1)} \binom{\frac{b}{2}+\frac{c}{2}+x(l)+y(l)}{\frac{1}{2}(a+c+1)+x(l)-s}, \end{aligned}$$

and $E(a,b,c,x(r),y(r)), F(a,b,c,x(l),y(l))$ satisfy

$$\begin{aligned} E(a,b,c,x(r),y(r))F(a,b,c,x(l),y(l)) + \sum_{s=1}^{a} B(a,b,c,x(r),y(r),s)D(a,b,c,x(l),y(l),s) \\ = \binom{x(l)+y(l)-x(r)-y(r)}{x(l)-x(r)}. \end{aligned}$$

Proof. The equality involving the entries $E(a,b,c,x(r),y(r))$ and $F(a,b,c,x(l),y(l))$ follows immediately from expressing $P_{S,E}$ in terms of its LU-decomposition, thus in order to complete the proof we must show the following:

(i) $\sum_{s=1}^{\min\{i,j\}} A(b,c,i,s)C(b,c,s,j) = \binom{b+c}{c+j-i}$;

(ii) $\sum_{s=1}^{i} A(b,c,i,s)D(a,b,c,x(l),y(l),s) = \binom{x(l)+y(l)+(b+c)/2}{x(l)-i+(a+c+1)/2}$;

(iii) $\sum_{s=1}^{j} B(a,b,c,x(r),y(r),s)C(b,c,s,j) = \binom{(b+c)/2-x(r)-y(r)}{j-x(r)-(a-c+1)/2}$.

By once more employing Zeilberger's algorithm [66, 76, 77] we see that sum on the left side of the first equality satisfies the following two recurrences:

$$(b+i-j+1)\sum_{s=1}^{i+1} A(b,c,i+1,s)C(b,c,s,j) + (i-j-c)\sum_{s=1}^{i} A(b,c,i,s)C(b,c,s,j) = 0; \tag{6.10}$$

and

$$(c-i+j+1)\sum_{s=1}^{j+1} A(b,c,i,s)C(b,c,s,j+1) + (j-i-b)\sum_{s=1}^{j} A(b,c,i,s)C(b,c,s,j) = 0. \tag{6.11}$$

It is straightforward to check that

$$(b+i-j+1)\binom{b+c}{c+j-i-1}+(i-j-c)\binom{b+c}{b+j-i}=0 \tag{6.12}$$

and

$$(c-i+j+1)\binom{b+c}{c+j-i+1}+(j-i-b)\binom{b+c}{c+j-i}=0, \tag{6.13}$$

so the equality holds in general once we have verified that (6.10) and (6.12) hold for $i=1$ and $i=2$, and similarly that (6.11) and (6.13) hold for $j=1$ and $j=2$.

For the second equality note that by interchanging the summations we obtain

$$\sum_{s=0}^{i-1}\sum_{t=s}^{i-1}\frac{(-1)^{t-s}c\Gamma(i)\Gamma(b+s+1)\Gamma(c+t-s)}{\Gamma(s+1)\Gamma(b+i)\Gamma(i-t)\Gamma(t-s+1)\Gamma(c-i+t+2)}\binom{\frac{b}{2}+\frac{c}{2}+x(l)+y(l)}{\frac{a}{2}+\frac{c}{2}+\frac{1}{2}+x(l)-s},$$

the inner sum of which may be expressed as a ${}_2F_1$ hypergeometric series

$$\sum_{s=0}^{i-1}\frac{\Gamma(c+1)\Gamma(i)\Gamma(b+s+1)}{\Gamma(s+1)\Gamma(b+i)\Gamma(i-s)\Gamma(c-i+s+2)}\binom{\frac{b}{2}+\frac{c}{2}+x(l)+y(l)}{\frac{a}{2}+\frac{c}{2}+\frac{1}{2}+x(l)-s}{}_2F_1\left[\begin{matrix}c,\ s-i+1\\ c-i+s+2\end{matrix};1\right].$$

When faced with an expression such as this there are myriad transformation and summation identities that one may turn to in order to try to simplify things. In our case it turns out that a straightforward application of the Chu-Vandermonde identity,

$${}_2F_1\left[\begin{matrix}a,\ -n\\ c\end{matrix};1\right]=\frac{(c-a)_n}{(c)_n},$$

(which may be found in [**70**, 1.7.7; Appendix III.4]) yields

$$\frac{(s-i+2)_{i-s-1}}{(c-i+s+2)_{i-s-1}},$$

where $(\alpha)_\beta$ is the Pochhammer symbol (see Footnote 42). For $s<i-1$ the above term vanishes, thus proving (ii).

Precisely the same method can be used to prove the third equality (that is, interchanging the sums and applying the Chu-Vandermonde identity), thus once this equality has been verified under initial conditions the proof is complete. □

COROLLARY 6.7.4 – TG [**41**]

The determinant of the lattice path matrix $P_{S,E}$ corresponding to $G\setminus\{b_j,w_i\}$ is given by

$$M(H)\cdot\left(\binom{x(l)+y(l)-x(r)-y(r)}{x(l)-x(r)}-\sum_{s=1}^{a}B(a,b,c,x(r),y(r),s)D(a,b,c,x(l),y(l),s)\right).$$

This follows from the fact that $L_{i,i}=1$ for $1\le i\le a$, thus the determinant of $P_{S,E}$ is the product of the diagonal entries of U and entry $L_{a+1,a+1}$,

$$\left(\prod_{i=1}^{a}A(b,c,i,i)\cdot C(b,c,i,i)\right)\cdot(E(a,b,c,x(r),y(r))F(a,b,c,x(l),y(l)))$$

(it turns out that the product on the left is another re-packaging of MacMahon's box formula). We immediately obtain the following enumerative result.

Corollary 6.7.5 – TG [**41**]

Suppose b_j and w_i are face connected vertices in G. Then the number of perfect matchings of $G \setminus \{b_j, w_i\}$ is

$$M(H) \cdot \left| \left(\binom{x(l)+y(l)-x(r)-y(r)}{x(l)-x(r)} - \sum_{s=1}^{a} B(a,b,c,x(r),y(r),s) D(a,b,c,x(l),y(l),s) \right) \right| .$$

There is one final observation to make before stating the main theorem of this chapter. The sign on the right in (6.9) can be easily controlled by labelling the vertices in $G \setminus \{b_j, w_i\}$ in a consistent manner. Consider the sets of labelled vertices contained in G from Section 6.5:

$$B := \{b_1, b_2, \ldots, b_{ab+bc+ca}\}$$

and

$$W := \{w_1, w_2, \ldots, w_{ab+bc+ca}\}.$$

Suppose we remove vertex b_j from B and w_i from W, letting $B' := B \setminus \{b_j\}$ and $W' := W \setminus \{w_i\}$. We may then re-label each vertex $b'_k \in B'$ according to the following convention

$$b'_k := \begin{cases} b_k & 1 \le k < j, \\ b_{k-1} & \textit{otherwise}, \end{cases}$$

and similarly for the vertices in W'. It follows that either

$$\det(A_G \setminus \{b_j, w_i\}) = \det(P_{S,E})$$

for any pair of vertices $b_j \in B$, $w_i \in W$, or instead

$$\det(A_G \setminus \{b_j, w_i\}) = -\det(P_{S,E})$$

for all such pairs. Everything is now in place for us to state our main result.

Theorem 6.7.6 – TG [**41**]

The inverse Kasteleyn matrix corresponding to the sub-graph G of the hexagonal lattice $\mathscr{H}$ consisting of black and white vertices ($\{b_1, b_2, \ldots, b_{ab+bc+ca}\}$ and $\{w_1, w_2, \ldots, w_{ab+bc+ca}\}$ respectively) is equal to $(\pm 1) \cdot K$, where $K = (K_{w_i,b_j})_{w_i,b_j \in G}$ is the matrix with entries given by

$$(-1)^{i+j} \cdot \left(\binom{x(l)+y(l)-x(r)-y(r)}{x(l)-x(r)} - \sum_{s=1}^{a} \frac{g(a,b,c,x(l),y(l),s) g(a,c,b,-y(r),-x(r),s)}{\binom{b+c+s-1}{b+s-1}\binom{b+s-1}{s-1}} \right),$$

in which

$$g(u,v,w,x,y,z) := \sum_{t=1}^{z} (-1)^{z-t} \binom{v+t-1}{t-1} \binom{w+z-t-1}{w-1} \binom{\frac{v}{2}+\frac{w}{2}+x+y}{x-t+\frac{u}{2}+\frac{w}{2}+\frac{1}{2}},$$

and the points $(x(r), y(r)), (x(l), y(l)) \in \mathbb{Z}_{a,c} \times \mathbb{Z}_{a,b}$ are determined by the locations of w_i and b_j (respectively) inside G.

Proof. Replace the numerator in (6.4) with the result from Corollary 6.7.4, taking heed of the preceding observation regarding the signs of the entries. This completes the proof. ☐

By combining the above result with the arguments from Section 6.3 we can easily deduce the following corollary.

Corollary 6.7.7 – TG [41]

Suppose $V = \{b_1, b_2, \dots, b_k, w_1, w_2, \dots, w_k\}$ is a set of admissibility preserving vertices contained in G. Then the number of perfect matchings of $G \setminus V$ is

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}\cdot|\det((K)_V)|,$$

where $(K)_V$ denotes the complementary minor of the matrix K.

6.8. Applications of the main result

Corollary 6.7.7 is useful for a number of reasons. The condition on the vertices that we remove from G (that they are admissibility preserving) is not particularly strong, thus (under the translation of these graphs into hexagons with holes on the triangular lattice) it gives equivalent formulas for a swathe of earlier results, bringing together a large number of previously disparate formulas for different classes of holes under one roof. Further to this, when we come to consider the interaction of a fixed set of holes the problem can easily be reduced, by virtue of Corollary 6.7.7 and the definition of the correlation function in Section 2.8, to extracting the asymptotic behaviour of the entries of a determinant whose size is dependent only on the sizes of the holes.

Recall that a pair of vertices $V = \{v, v'\}$ contained in G are admissibility preserving if they are face connected. On the triangular lattice the sub-graph $G \setminus V$ corresponds to a holey hexagon $H \setminus T$, where $T = \{t_1, t_2\}$ is a pair of unit triangles. If v and v' are face connected in G then the unit triangles t_1 and t_2 must either touch at a point (we say they are *point connected*, see Figure 42 left), share precisely one edge (that is, *edge connected*, see Figure 42 centre), or instead each shares a point or an edge with the outer boundary of H (see Figure 42 right).

If T consists of two triangles that point in the same direction then the number of tilings of $H \setminus T$ is clearly zero[55] so suppose for the time being that $t_1 = l$ is a left pointing unit triangle, and $t_2 = r$ a right pointing one. In such a case the determinant of $(K)_{v,v'}$ is the determinant of a matrix consisting of a single entry, thus Corollary 6.7.7 yields an enumerative formula that involves a triple sum.

Suppose l and r touch at a point so that they form a little bow-tie consisting of unit triangles, as in Figure 42 (left). Tilings of such holey hexagons are considered by Eisenkölbl in [**28**], where the centre of the bow-tie is located at a lattice distance k along the horizontal symmetry axis from the left side of the hexagon with sides of length $n, 2m, n, n, 2m, n$.

Since l and r correspond to admissibility preserving vertices in G, the regions containing a unit triangular bow-tie whose tilings are counted by the formula in Corollary 6.7.7 are clearly far more general than those considered in [**28**]. Indeed if we let $a = 2m$, $b = n$ and $c = n$ and suppose that $x(l) = y(l) = \lfloor \frac{n}{2} \rfloor + k$ and $(x(r), y(r)) = (x(l)+1, x(l)+1)$ for some non-negative integer k then we recover an equivalent expression to the one given by Eisenkölbl.

Open Problem 6.8.1

Eisenkölbl's formula is decidedly more concise than the one given in Corollary 6.7.7, thus it would be interesting to see exactly how the triple sum should be manipulated in order to obtain the corresponding product formula.

Suppose now that l and r share an edge, as in Figure 42 (centre). In this case Corollary 6.7.7 provides us with an exact formula for the number of tilings of H that contain the fixed rhombus created by joining l to r. There exist already at least two different formulas that enumerate such tilings – one due to Fischer [**33**], the other Johansson [**46**, Equation 4.37] (both are triple sums that enumerate the number of tilings containing a fixed horizontal rhombus at a specified location). If

[55] In this case the left and right pointing unit triangles that remain are no longer equinumerous, thus $H \setminus T$ is not tileable.

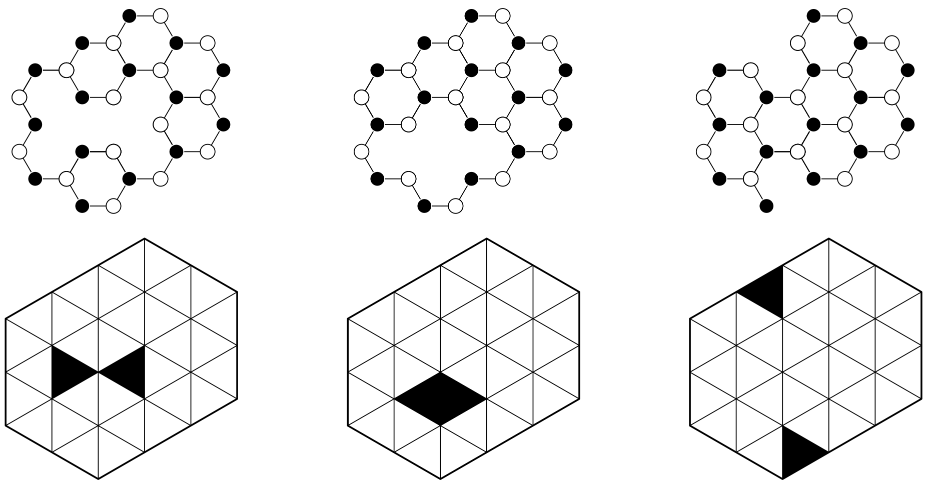


FIGURE 42. Holey hexagons with pairs of triangles removed that correspond to face connected pairs of vertices.

we suppose that for $(x(l), y(l)) \in \mathbb{Z}_{a,c} \times \mathbb{Z}_{a,b}$ we have

$$(x(r), y(r)) = (x(l) + 1, y(l))$$

then our formula from Corollary 6.7.7 gives an alternative expression to those found in [**33**] and [**46**].

OPEN PROBLEM 6.8.2

Given that there are now three separate formulas that count tilings with a central fixed rhombus, one involving a single sum and the others differing triple sums, it would be interesting to understand exactly how to obtain one from the other. Further to this, Krattenthaler [**52**] has conjectured that such a triple sum for a horizontal fixed rhombus may be simplified further, thus Corollary 6.7.7 could perhaps lead to a proof of this apparent phenomenon.

Consider the case where b_j and w_i lie somewhere along the outer boundary of G. They are clearly face connected, thus when we translate $G \setminus \{b_j, w_i\}$ into a region of $\mathscr{T}$ we see that Corollary 6.7.7 gives equivalent expressions for two recent results of Ciucu and Fischer (namely Proposition 3 and Proposition 4 from [**17**]). These two results, which enumerate tilings of the region H containing a pair of dents along its boundary, was established via the method of Kuo condensation [**13**, **58**].

We can extend this idea further – suppose $V \subset G$ is a set $\{w_1, w_2, \ldots, w_k, b_1, b_2, \ldots, b_k\}$ of vertices that lie on the boundary face of G. On the triangular lattice V corresponds to a set of unit triangles $T = \{l_1, l_2, \ldots, l_k, r_1, r_2, \ldots, r_k\}$ along the boundary of H, where each pair of unit triangles in T corresponds to a pair of admissibility preserving vertices in V (see Figure 43, left). According to Corollary 6.7.7 the number of tilings of $H \setminus T$ is

$$M(H) \cdot |\det((K)_V)| = M(H)^{1-k} \cdot |\det((A_G \setminus \{w_i, b_j\})_{w_i, b_j \in V})|.$$

Since each pair of vertices w_i, b_j is admissibility preserving it follows that

$$M(H \setminus T) = M(H)^{1-k} \cdot |\det(\pm M(H \setminus \{r_i, l_j\}))|.$$

It turns out that the sign of the entries in the right hand side above can be completely controlled. This comes from carefully studying the signatures of the permutations that give rise to families of non-intersecting paths from S to E for different pairs of unit triangles located along different edges of $H \setminus \{l, r\}$: if l and r lie on adjacent edges of H then the sign of $\det(A_G \setminus \{b_j, w_i\})$ is the same as that of $\det(A_G)$, otherwise it changes. If we specialise a set of holes $l_1, l_2, \ldots, l_k$ and $r_k, r_{k-1}, \ldots, r_1$ to appear in that cyclic order around the boundary of $H \setminus \{l_1, l_2, \ldots, l_k, r_1, r_2, \ldots, r_k\}$ then we recover Ciucu's generalised Kuo condensation for tilings of hexagons [**13**, Corollary 2.4].

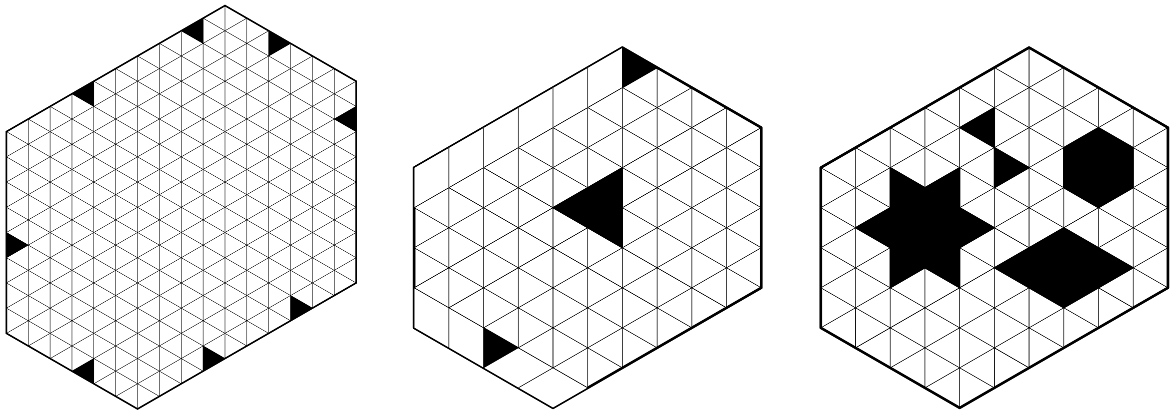


FIGURE 43. A hexagon with arbitrary dents, an unbalanced holey hexagon, a hexagon containing a set of arbitrary holes.

Further to this, we can also use Corollary 6.7.7 to derive an expression for enumerating tilings of so-called unbalanced hexagons that have dents on their boundary as discussed in [**17**, Theorem 2]. If the set T corresponds to a set of unit triangles that when removed induce (by forcing certain rhombi) an unbalanced hexagon with dents on its boundary (see Figure 43, centre) then we obtain a determinant version of Theorem 2 from Ciucu and Fischer's paper [**17**], where the entries of the matrix are given explicitly by Theorem 6.7.6.

Corollary 6.7.7 has yet more applications – suppose now that $V = \bigcup_i V_i$ is an unconnected union of sets of connected vertices V_i in G, where $V = \{b_1, b_2, \ldots, b_k, w_1, w_2, \ldots, w_k\}$. On the triangular lattice such a set of vertices corresponds to a union of distinct regions $T = \bigcup_i T_i$ where either T_i consists of a single vertex, otherwise for each $t \in T_i$ there exists at least one other $t' \in T_i, t \neq t'$ such that t and t' are either edge or point connected. Further to this, the conditions on V ensure that no $t \in T_i$ is edge or point connected to any $t' \in T_j$ for $i \neq j$, thus T is an unconnected union of sets of connected unit triangles. By removing T from H we obtain a hexagon containing a set of holes where each hole (indexed by T_i) is induced by removing the set of unit triangles in T_i. If the parity of left and right pointing unit triangles in each T_i is the same then again by Corollary 6.7.7 we have

$$M(H \setminus T) = M(H) \cdot |\det((K_{w_i,b_j})_{w_i,b_j \in V})|.$$

This means that we can calculate the number of tilings of hexagons that contain a wide variety of holes as the determinant of a matrix whose size is dependent only on the regions we have *removed*. This is of course useful from an enumerative perspective, but perhaps the most important potential application of Corollary 6.7.7 arises when we zoom out and look at the effect of the holes on tilings of the entire plane. If we fix a set of holes T inside H that correspond to a set of admissibility preserving vertices V in G and let the sides of the hexagon tend to infinity then the interaction between the holes is given by

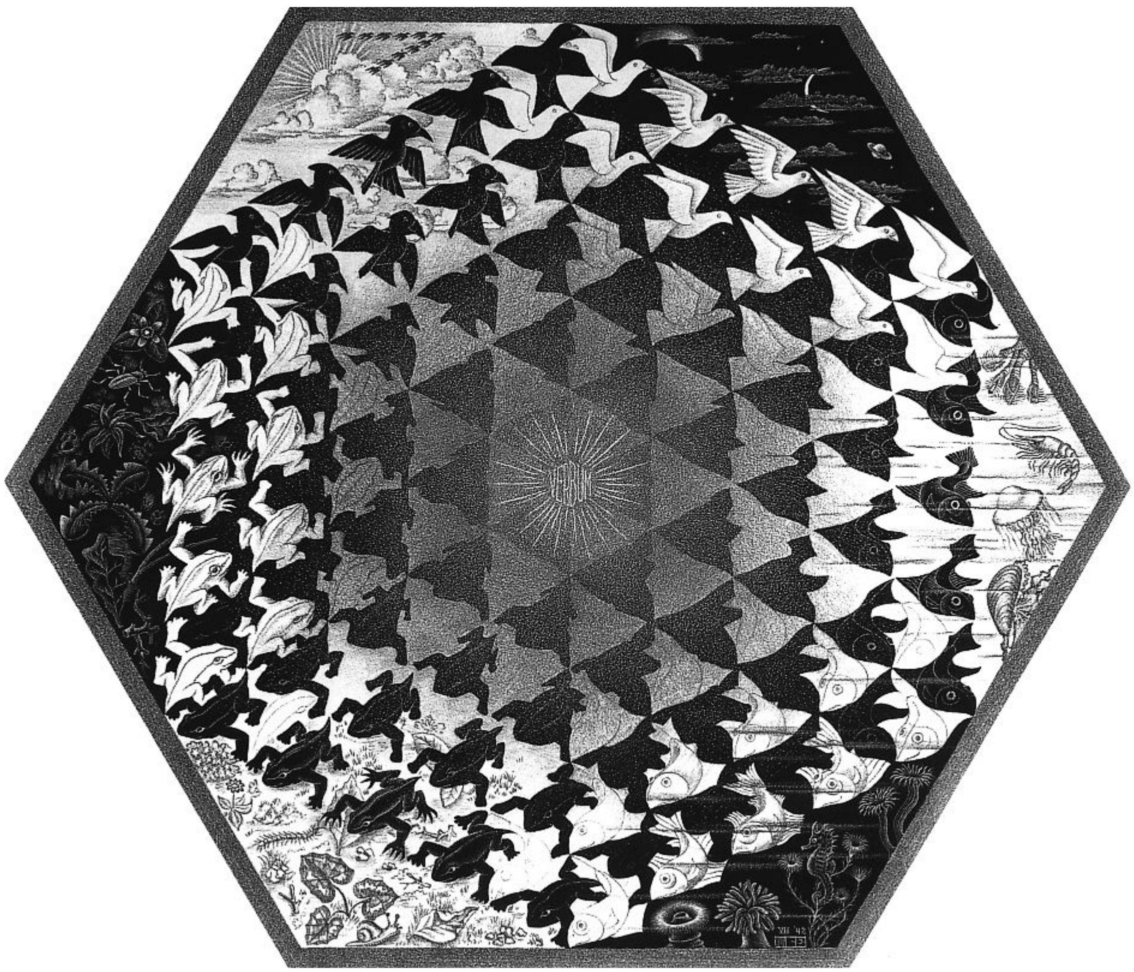
$$\omega(T) = \lim_{n\to\infty} |\det((K_{w_i,b_j})_{w_i,b_j \in V})|,$$

that is, for a fixed set of holes, $\omega(T)$ is equal to the determinant of a matrix with entries given by

$$\lim_{n\to\infty} K_{w_i,b_j}.$$

OPEN PROBLEM 6.8.3

If the asymptotics of these entries could be obtained we would have a proof of Ciucu's conjecture for an incredibly general class of holes, perhaps even the most general class of holes to date. It seems that this is not entirely impossible to do and may well come from applying the saddle-point method to a contour integral corresponding to K_{w_i,b_j}.



Verbum by M. C. Escher (1942).

APPENDIX A

Proof of Theorem 4.4.5 and Corollary 4.4.6. The expression involving the matrix entries $\widehat{E}(a,i,j)$ and $\widehat{F}(a,i,j)$ follows immediately from taking the LU-decomposition of $P_{\widehat{S},\widehat{E}}$, thus in order to prove Theorem 4.4.5 we need to show that the following equalities hold:

(i) $\sum_{s=1}^{\min(i,j)} \widehat{A}(a,i,s)\widehat{C}(a,s,j) = \binom{2a}{a+j-i} + \binom{2a}{a+j+i-1}$;

(ii) $\sum_{s=1}^{j} \widehat{B}(a,l,s)\widehat{C}(a,s,j) = \binom{a-l+1}{a/2-l/2+j}$;

(iii) $\sum_{s=1}^{i} \widehat{A}(a,i,s)\widehat{D}(a,r,s) = \binom{a+r+1}{a/2+r/2+j}$.

For the first equality, note that since

$$\widehat{A}(a,i,s)\widehat{C}(a,s,j) = \widehat{A}(a,j,s)\widehat{C}(a,s,i)$$

we may assume without loss of generality that $\min(i,j) = i$. Zeilberger's algorithm [**66**,**76**,**77**] shows that the sum on the left hand side of (i) satisfies the following recurrence:

$$\begin{aligned}(a-i+j)(i+j-a-1)\sum_{s=1}^{i}\widehat{A}(a,i,s)\widehat{C}(a,s,j)&\\ +2(a(a+1)-i(i+1)+j(j-1))\sum_{s=1}^{i+1}\widehat{A}(a,i+1,s)\widehat{C}(a,s,j)&\\ -(a+i+j-2)(a+i+j+1)\sum_{s=1}^{i+2}\widehat{A}(a,i+2,s)\widehat{C}(a,s,j)&=0.\end{aligned}$$

It can be easily checked that the right hand side of (i) satisfies the very same recurrence in i, thus the equality holds once it has been verified under initial conditions.

For the second and third equalities note that

$$\widehat{B}(a,\gamma,s)\widehat{C}(a,s,\delta) = \widehat{A}(a,\delta,s)\widehat{D}(a,-\gamma,s),$$

thus it suffices to show

$$\sum_{s=1}^{\delta}\widehat{B}(a,\gamma,s)\widehat{C}(a,s,\delta) = \binom{a+\gamma+1}{a/2+\gamma/2+\delta}. \tag{A.1}$$

Implementing Zeilberger's algorithm for the sum on the left hand side above satisfies the following recurrence

$$(2\delta-\gamma-a-2)\sum_{s=1}^{\delta}\widehat{B}(a,\gamma,s)\widehat{C}(a,s,\delta) + (a+2\delta+\gamma+2)\sum_{s=1}^{\delta+1}\widehat{B}(a,\gamma,s)\widehat{C}(a,s,\delta+1) = 0,$$

as does the right hand side of A.1 thus the proof of Theorem 4.4.5 is complete once we have verified these two equalities under initial conditions.

Since $\widehat{L}_{i,i} = 1$ for $1 \leq i \leq b$ the determinant of the matrix $P_{\widehat{S},\widehat{E}}$ is equal to

$$\left(\prod_{s=1}^{b} U_{s,s}\right) \det(\widehat{L}^* \cdot \widehat{U}^*),$$

where $\widehat{L}^* = (\widehat{L}_{b+i,b+j})_{1\leq i,j \leq k}$ and $\widehat{U}^* = (\widehat{U}_{b+i,b+j})_{1\leq i,j\leq k}$. It turns out that the left-most product (over s) is equal to the formula that counts vertically symmetric tilings of $H_{a,2b,a}$ (see Theorem 2.6.2), whereas the product on the right hand side is the determinant of a matrix with (i, j)-entries given by

$$\widehat{P}_{b+i,b+j} - \sum_{s=1}^{b} \widehat{B}(a, l_i, s)\widehat{D}(a, s, r_j).$$

This completes the proof of Corollary 4.4.6.

$\square$

APPENDIX B

Proof of Proposition 5.4.3. Suppose first that $r \geq l+2$ (see Remark 5.0.1). Each entry $\widehat{Q}_{l,r}$ is given by

$$\binom{r-l+1}{r/2-l/2+1} - \sum_{s=1}^{\infty} \widehat{B}(an,l,s)\widehat{D}(an,s,r) + \sum_{t=bn+1}^{\infty} \widehat{B}(an,l,t)\widehat{D}(an,t,r).$$

The sum over s above may be written in hypergeometric notation as

$$\frac{\Gamma(an+1)\Gamma(an+2)\Gamma(an-l+2)\Gamma(an+r+2)}{\Gamma(2an+2)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}-\frac{l}{2}+2)\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+2)} \\ \times {}_5F_4\left[\begin{matrix} an+1,\ \frac{an}{2}+\frac{3}{2},\ \frac{1}{2},\ \frac{l}{2}+\frac{an}{2},\ \frac{an}{2}-\frac{r}{2} \\ \frac{an}{2}+\frac{1}{2},\ an+\frac{3}{2},\ \frac{an}{2}-\frac{l}{2}+2,\ \frac{an}{2}+\frac{r}{2}+2 \end{matrix};1\right], \tag{B.1}$$

to which we can apply the summation formula (5.16) from Section 5.4 (this may be found in Slater [**70**, Appendix III.12]), yielding

$$\frac{\Gamma(an+\frac{3}{2})\Gamma(\frac{r}{2}-\frac{l}{2}+\frac{3}{2})\Gamma(an+1)\Gamma(an-l+2)\Gamma(an+r+2)}{\Gamma(\frac{r}{2}-\frac{l}{2}+2)\Gamma(\frac{an}{2}-\frac{l}{2}+\frac{3}{2})\Gamma(\frac{an}{2}+\frac{r}{2}+\frac{3}{2})\Gamma(2an+2)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+1)}$$

which can easily be shown to equal

$$\binom{r-l+1}{r/2-l/2+1}.$$

The sum over t is, hypergeometrically speaking, equal to

$${}_6F_5\left[\begin{matrix} bn+\frac{an}{2}+\frac{3}{2},\ bn+\frac{1}{2},\ \frac{l}{2}+bn+\frac{an}{2},\ bn+an+1,\ bn+\frac{an}{2}-\frac{r}{2},\ 1 \\ bn+\frac{an}{2}+\frac{1}{2},\ bn+an+\frac{3}{2},\ bn-\frac{l}{2}+\frac{an}{2}+2,\ bn+1,\ bn+\frac{an}{2}+\frac{r}{2}+2 \end{matrix};1\right] \\ \times \frac{\Gamma(an-l+2)\Gamma(an+r+2)\Gamma(2bn+1)\Gamma(an+bn+1)^2\Gamma(an+2bn+2)}{\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}+\frac{l}{2})\Gamma(\frac{an}{2}-\frac{r}{2})\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(bn+1)^2\Gamma(an+2bn+1)\Gamma(2an+2bn+2)} \\ \times \frac{\Gamma(\frac{an}{2}+\frac{l}{2}+bn)\Gamma(\frac{an}{2}+bn-\frac{r}{2})}{\Gamma(\frac{an}{2}-\frac{l}{2}+bn+2)\Gamma(\frac{an}{2}+bn+\frac{r}{2}+2)} \tag{B.2}$$

which may instead be expressed as the following limit of a ${}_7F_6$ series

$$\lim_{\epsilon\to 0}\ {}_7F_6\left[\begin{matrix} 2bn+an+1+\epsilon,\ V \\ W \end{matrix};1\right], \tag{B.3}$$

where V and W are the lists

$$(bn+\tfrac{an}{2}+\tfrac{3}{2}+\tfrac{\epsilon}{2}, bn+\tfrac{1}{2}+\tfrac{\epsilon}{2}, \tfrac{l}{2}+bn+\tfrac{an}{2}+\tfrac{\epsilon}{2}, 1+\tfrac{\epsilon}{2}, bn+an+1+\tfrac{\epsilon}{2}, bn+\tfrac{an}{2}-\tfrac{r}{2}+\tfrac{\epsilon}{2})$$

and

$$(bn+\tfrac{an}{2}+\tfrac{1}{2}+\tfrac{\epsilon}{2}, bn+an+\tfrac{3}{2}+\tfrac{\epsilon}{2}, bn-\tfrac{l}{2}+\tfrac{an}{2}+2+\tfrac{\epsilon}{2}, 2bn+an+1+\tfrac{\epsilon}{2}, \\ bn+1+\tfrac{\epsilon}{2}, bn+\tfrac{an}{2}+\tfrac{r}{2}+2+\tfrac{\epsilon}{2})$$

respectively. We may again apply (5.19) from Section 5.4 (see [**70**, (4.3.6.4) reversed]) to the preceding hypergeometric series, thus as $\epsilon \to 0$ we obtain

$$\frac{(an+bn+\frac{1}{2})(\frac{an}{2}+bn-\frac{l}{2}+1)}{(an+2bn+1)(\frac{an}{2}-\frac{l}{2}+\frac{1}{2})}\,{}_4F_3\left[\begin{matrix}1,\ \frac{l}{2}+bn+\frac{an}{2},\ \frac{r}{2}-\frac{an}{2}+1,\ bn+\frac{1}{2}\\ bn+1,\ \frac{l}{2}-\frac{an}{2}+\frac{1}{2},\ bn+\frac{an}{2}+\frac{r}{2}+2\end{matrix};1\right]. \tag{B.4}$$

Applying transformation (5.21) from Section 5.4 (see Slater [**70**, (4.3.5.1)]) to the ${}_4F_3$ series above and replacing the ${}_6F_5$ series in (B.2) with the resulting expression we obtain

$$\frac{\Gamma(an-l+2)\Gamma(an+r+2)\Gamma(2bn+1)\Gamma(an+bn+1)^2\Gamma(an+2bn+2)}{\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}+\frac{l}{2})\Gamma(\frac{an}{2}-\frac{r}{2})\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(bn+1)^2\Gamma(an+2bn+1)\Gamma(2an+2bn+2)}$$
$$\times\frac{\Gamma(\frac{an}{2}+\frac{l}{2}+bn)\Gamma(\frac{an}{2}+bn-\frac{r}{2})}{\Gamma(\frac{an}{2}-\frac{l}{2}+bn+2)\Gamma(\frac{an}{2}+bn+\frac{r}{2}+2)}\times\frac{(an+bn+\frac{1}{2})(\frac{an}{2}+bn-\frac{l}{2}+1)(bn)(\frac{l}{2}-\frac{an}{2}-\frac{1}{2})}{(an+2bn+1)(\frac{an}{2}-\frac{l}{2}+\frac{1}{2})(bn+\frac{an}{2}-\frac{r}{2}-1)(\frac{l}{2}-\frac{r}{2}-\frac{3}{2})}$$
$$\times\,{}_4F_3\left[\begin{matrix}\frac{r}{2}-\frac{an}{2}+1,\ 1,\ \frac{r}{2}-\frac{l}{2}+2,\ \frac{an}{2}+\frac{r}{2}+\frac{3}{2}\\ \frac{an}{2}+bn+\frac{r}{2}+2,\ 2-\frac{an}{2}-bn+\frac{r}{2},\ \frac{r}{2}-\frac{l}{2}+\frac{5}{2}\end{matrix};1\right].$$

Just as in the proof of Theorem 5.4.1 the series above is dominated by a convergent series, thus supposing $2b/a \to \mu$ as $n \to \infty$ (where $\mu > 0$ is real valued) the above expression reduces in the limit to

$$\frac{(\mu(\mu+2))^{1/2}2^{r-l+1}(\mu+1)^{l-r-4}}{\pi(\frac{r}{2}-\frac{l}{2}+\frac{3}{2})}\,{}_2F_1\left[\begin{matrix}1,\ \frac{r}{2}-\frac{l}{2}+2\\ \frac{r}{2}-\frac{l}{2}+\frac{5}{2}\end{matrix};(\mu+1)^{-2}\right]$$

to which we can apply (5.13) from Section 5.2 (which may also be found in [**70**, (1.7.1.3)]) thereby obtaining for $r \geq l+2$

$$\lim_{n\to\infty}\widehat{Q}_{l,r}=\left(\frac{2}{\mu+1}\right)^{r-l+2}\frac{(\mu(\mu+2))^{-1/2}}{\pi(r-l+3)}\,{}_2F_1\left[\begin{matrix}1,\ \frac{1}{2}\\ \frac{r}{2}-\frac{l}{2}+\frac{5}{2}\end{matrix};-(\mu(\mu+2))^{-1}\right].$$

Suppose now that $r \leq l-6$. Then $\widehat{Q}_{l,r}$ is given by

$$-\sum_{s=0}^{bn-1}\widehat{B}(an,l,s+1)\widehat{D}(an,r,s+1)$$

which may be expressed as the following limit of a hypergeometric series

$$\frac{\Gamma(an+1)\Gamma(an+2)\Gamma(an-l+2)\Gamma(an+r+2)}{\Gamma(2an+2)\Gamma(\frac{an}{2}-\frac{l}{2}+1)\Gamma(\frac{an}{2}-\frac{l}{2}+2)\Gamma(\frac{an}{2}+\frac{r}{2}+1)\Gamma(\frac{an}{2}+\frac{r}{2}+2)}\lim_{\epsilon\to 0}\left({}_7F_6\left[\begin{matrix}V\\W\end{matrix};1\right]\right),$$

where

$$V=(\epsilon+an+1,\tfrac{\epsilon}{2}+\tfrac{an}{2}+\tfrac{3}{2},\tfrac{\epsilon}{2}+\tfrac{l}{2}+\tfrac{an}{2},\tfrac{\epsilon}{2}+\tfrac{an}{2}-\tfrac{r}{2},\tfrac{\epsilon}{2}+\tfrac{1}{2},bn+an+1,1-bn)$$

and

$$W=(\tfrac{\epsilon}{2}+\tfrac{an}{2}+\tfrac{1}{2},\tfrac{\epsilon}{2}-\tfrac{l}{2}+\tfrac{an}{2}+2,\tfrac{\epsilon}{2}+\tfrac{an}{2}+\tfrac{r}{2}+2,\tfrac{\epsilon}{2}+an+\tfrac{3}{2},\epsilon-bn+1,\epsilon+bn+an+1).$$

By applying transformation formula (5.24) from Section 5.4 and letting ϵ tend to zero we obtain (after performing some cancellations)

$$-\frac{2^{-l+r+2}\Gamma\left(bn+\frac{1}{2}\right)\Gamma(an+bn+1)\Gamma\left(\frac{1}{2}(-l+an+3)\right)\Gamma\left(\frac{1}{2}(an+r+3)\right)}{\pi\Gamma(bn)\Gamma\left(an+bn+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(-l+an+4)\right)\Gamma\left(\frac{1}{2}(an+r+4)\right)}$$
$$\times\,{}_4F_3\left[\begin{matrix}2-\frac{l}{2}+\frac{r}{2},\ \frac{1}{2},\ bn+an+1,\ 1-bn\\ \frac{an}{2}+2-\frac{l}{2},\ \frac{an}{2}+\frac{r}{2}+2,\ \frac{3}{2}\end{matrix};1\right].$$

Once again we have a series that terminates for $s \geq \frac{l}{2} - \frac{r}{2} - 1$, thus as n tends to infinity this expression reduces to

$$-\frac{2^{r-l+2}(\mu(\mu+2))^{1/2}}{3\pi} {}_2F_1\left[\begin{matrix}\frac{1}{2},\ 2-\frac{l}{2}+\frac{r}{2} \\ \frac{3}{2}\end{matrix};-\mu(\mu+2)\right].$$

Applying the transformation formula (5.11) from Section 5.2 reduces the above expression to

$$\frac{(\mu(\mu+2))^{1/2}2^{r-l+1}(\mu+1)^{l-r-4}}{\pi(\frac{r}{2}-\frac{l}{2}+\frac{3}{2})} {}_2F_1\left[\begin{matrix}1,\ \frac{r}{2}-\frac{l}{2}+2 \\ \frac{r}{2}-\frac{l}{2}+\frac{5}{2}\end{matrix};(\mu+1)^{-2}\right]$$

which agrees entirely with $\widehat{Q}_{l,r}$ for $r > l$ above.

$\square$

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