Introduction

In this thesis we study the Robinson-Schensted correspondence and its variations by inserting two types of numbers, called regular and barred numbers.

In Chapter 1 the necessary combinatorial notions are introduced and the Robinson-Schensted algorithms of the first and second kind are discussed.

This is followed by a systematical approach to RS 1 and RS 2 insertion algorithms in Chapter 2, that proves to be very useful in understanding their basic combinatorial properties: A special type of tableaux, the so called (k, l)-semistandard tableaux, will be interpreted as representing chains of partitions, and the algorithms shall be studied by their "local" behaviour in the sense of these chains, i. e., by the effect that adding or removing an element to the chain has on the outcome of the algorithms, and a reformulation of the algorithms themselves as processes that compute doubly indexed families of partitions according to rules governing local configurations; these rules are derived from some recursion relations, and they only partially specify the global order of computation. Tabulating all the partitions in these families gives insightful pictorial representations of the computation, from which fundamental symmetry properties can be read off that are not at all obvious from their iterative definitions.

In Chapter 3 we first give the traditional definition of the Schützenberger algorithm, which is in terms of moving around entries through the squares of a diagram, and then consider what this means in terms of chains of partitions. This leads to recursion relations and a pictorial representation of the computation as for the RS 1 and RS 2 insertion algorithms.

In Chapter 4 we formulate and prove the central theorem that relates the two algorithms to each other, again using a family of partitions in the proof that helps to visualize the argument.

As a consequence of our study we generalize the classical Cauchy identities for Schur functions to the setting of (k, l)-hook Schur functions in Chapter 5. There we give a bijective proof of the (k, l)-hook Schur function Cauchy identity by generalizing the Knuth correspondence between matrices and pairs of column strict tableaux to the setting of (k, l)-hook Schur functions.

In chapter 6 we generalize the Knuth oscillating correspondence. This is a bijection between oscillating tableaux and symmetric matrices.

In Chapter 7 we show that the number of (k, l)-semistandard tableaux of shape λ does not depend on the ordering of the regular and barred numbers. For that reason we state a bijection between lattice paths and (k, l)-semistandard tableaux.

The main result of Chapter 8 is a generalization of the Robinson-Schensted correspondence to the class of dual graded graphs. This class extends the class of differential posets, for which a generalized Robinson-Schensted algorithm was constructed in Chapter 1. As particular cases of the general construction, we give examples including skew tableaux, shifted tableaux and binary trees.

In Chapter 9 we first discuss Fibonacci r-differential posets. Then we give an elementary approach to Fomin's insertion algorithm for Fibonacci tableaux.

Contents

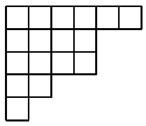
Chapter 1

Generalized Robinson-Schensted algorithm

1.1 Introduction

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$, the *Ferrers diagram* F_λ of shape λ, for short sh$F = \lambda$, is the set of left justified rows of cells or squares with λ_1 cells in the top row, λ_2 cells in the next row, etc. For instance, Figure 1.1 shows F_λ for $\lambda = (6, 4, 4, 2, 1)$.

Figure 1.1:



Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$, we write $\mu \leq \lambda$ if $l \leq k$ and $\mu_i \leq \lambda_i$ for $i = 1, \ldots, l$. If $\mu \leq \lambda$, the skew diagram of shape λ/μ consists of the squares of F_λ that remain after we remove the squares of F_μ.

Assume that we have an alphabet A consisting of two types of numbers, namely the numbers $1, 2, \ldots$, which we shall call *regular numbers* and the numbers $\bar{1}, \bar{2}, \ldots$, which we shall call *barred numbers* and an arbitrary ordering of them which might look like $1 < \bar{1} < \bar{2} < 2 < \bar{5} < 3$. Note that we have to maintain the natural ordering of the regular and barred numbers among themselves.

A (k,l)-*semistandard tableau* T of shape λ/μ is a filling of $F_{\lambda/\mu}$ with numbers from the set $\{1, 2, \ldots, k, \bar{1}, \bar{2}, \ldots, \bar{l}\}$ such that the numbers are increasing in each row and column according to the given order and the regular (resp. barred) numbers are weakly (resp. strictly) increasing from left to right in each row and strictly (resp. weakly)

Figure 1.2:

$$T = \begin{array}{cccc} 1 & 1 & 3 & \bar{4} \\ 2 & 2 & \bar{3} & \bar{4} \\ \bar{1} & \bar{2} & & \\ \bar{1} & & & \end{array}$$

increasing from top to bottom in each column. For example, in Figure 1.2 we display a (3,4)-semistandard tableau T with sh$T = (4, 4, 2, 1)$ and the ordering $1 < 2 < 3 < \bar{1} < \bar{2} < \bar{3} < \bar{4}$.

1.2 RS 1 and RS 2 insertion algorithms

The two generalizations of the Robinson-Schensted insertion algorithm which are most useful for constructing bijections that prove various symmetric function identities for the (k, l)-*hook Schur functions* (see Definition 5.0.7). The latter were first introduced by Berele and Regev [1] and occur naturally in the representation of Lie superalgebras. Therefore they are also called *super-Schur functions*.

Let us state the traditional *row insertion algorithms* [24]. To insert a number a into a given tableau T, we look in the top row of T for the least number which is greater than a, call it b. We replace ("bump") b with a in the top row, and view b as living temporarily in the crack between the two rows. Now proceed with b and the second row in exactly the same way, i. e., if c is the least number in the second row bigger than b, then b bumps c, and we view c as temporarily homeless between rows two and three. If a is greater than all the elements in the first row we place a at the end of this row and are finished. If a homeless number will be greater than all the others in the row below it, we place it at the end of this row, terminating the insertion procedure. To illustrate, insert 4 into the tableau T. We put elements that are inserted during the process in boldface type.

$$T = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 7 & \\ 6 & 9 & \\ 8 & & \end{array} \qquad \begin{array}{ccc} 1 & 3 & \mathbf{4} \\ 2 & 7 & \\ 6 & 9 & \\ 8 & & \end{array} \qquad \begin{array}{ccc} 1 & 3 & 4 \\ 2 & \mathbf{5} & \\ 6 & 9 & \\ 8 & & \end{array} \qquad \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & \mathbf{7} & \\ 8 & & \end{array} \qquad \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & 7 & \\ 8 & \mathbf{9} & \end{array}$$

For the *column insertion algorithm* we just replace the word 'row' by 'column' in the description of the row insertion algorithm.

The basic Robinson-Schensted correspondence is a bijection between permutations of n and pairs of standard tableaux (P, Q) which are fillings with numbers $1, 2, \ldots, n$ of the Ferrer's diagram of the same shape λ, where λ runs over all partitions of n.

If we start with a permutation π we proceed as follows. Initialize the tableaux P and Q to be empty. Successively insert the elements of the bottom line of π from left to right into the tableaux P. At the ith step, when the bumping process ceases, a new cell will have been added to P; add a cell to Q in the corresponding position, and put the value i inside it. Hence, at each step P and Q have the same shape, and Q records the order in which cells were added to P. An example follows forthwith.

Example 1.2.1 *If we apply the above to the permutation* $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$ *we get the following sequences of tableaux.*

$$\begin{array}{lccccc}
P_k: & 3 & \begin{array}{l}1\\3\end{array} & \begin{array}{ll}1&4\\3&\end{array} & \begin{array}{ll}1&2\\3&4\end{array} & \begin{array}{lll}1&2&5\\3&4&\end{array} \\[2em]
Q_k: & 1 & \begin{array}{l}1\\2\end{array} & \begin{array}{ll}1&3\\2&\end{array} & \begin{array}{ll}1&3\\2&4\end{array} & \begin{array}{lll}1&3&5\\2&4&\end{array}
\end{array}$$

It is easy to see from this definition of RS that we have a bijection. At each stage, we know by looking at the highest entry of Q which cell was added to P. We can then reverse the bumping procedure from the element, say c in that cell. Now c must have been bumped by the greatest element smaller than c in the row above it, say b. So we replace b with c, and view b as temporarily homeless between rows. Then we repeat this, allowing b to unbump the greatest element smaller than b in the row above. Eventually, we force an element out the top of the tableau, say a, which must have been the element we originally inserted into P. Meanwhile, by undoing each stage of the bumping process, we have restored P to its condition before a was inserted.

Let us now define two analogues of the Schensted bumping process for regular and barred numbers.

For the first generalization of the Schensted bumping algorithm, we bump the regular numbers from row to row as in the usual row insertion algorithm and we bump the barred numbers from column to column as in the usual column insertion algorithm. Therefore we have to distinguish two cases. To insert a regular number (resp. barred number) a into a given tableau T, we look in the top row (resp. leftmost column) of T for the least number x such that $x > a$. Say, x lies in the ith row and jth column counted from left to right and top to bottom. We replace x with a in the top row (resp. leftmost column). If x is a regular number then it bumps the least element $y > x$ in the $i + 1$th row. If x is a barred number then it bumps the least element $y > x$ in the $j + 1$th column. This process is repeated until for a regular number (resp. barred

number) z which has to be inserted holds $z \geq b$ for all b in some row (resp. column). In this case z is placed at the end of this row (resp. column).
Since in most of our examples we assume that the barred numbers are strictly larger than the regular numbers we call this order the natural one for short. In Figure 1.3 we give an example of inserting a number x into a (k,l)-semistandard tableau T via what we call RS 1 insertion. We assume the natural order. To illustrate, suppose $x = 1$ and

$$T = \begin{array}{llll} 1 & 1 & 2 & 2 \\ 3 & \bar{1} & & \\ \bar{1} & \bar{2} & & \\ \bar{2} & & & \end{array}$$

To follow the path of the insertion of x into T, we put elements that are inserted during the algorithm in boldface type.

Figure 1.3: RS 1 insertion

$$\begin{array}[t]{llll} 1 & 1 & 2 & 2 \\ 3 & \bar{1} & & \\ \bar{1} & \bar{2} & & \\ \bar{2} & & & \end{array} \quad \begin{array}[t]{llll} 1 & 1 & \mathbf{1} & 2 \\ 3 & \bar{1} & & \\ \bar{1} & \bar{2} & & \\ \bar{2} & & & \end{array} \quad \begin{array}[t]{llll} 1 & 1 & 1 & 2 \\ \mathbf{2} & \bar{1} & & \\ \bar{1} & \bar{2} & & \\ \bar{2} & & & \end{array} \quad \begin{array}[t]{llll} 1 & 1 & 1 & 2 \\ 2 & \bar{1} & & \\ \mathbf{3} & \bar{2} & & \\ \bar{2} & & & \end{array} \quad \begin{array}[t]{llll} 1 & 1 & 1 & 2 \\ 2 & \bar{1} & & \\ 3 & \bar{\mathbf{1}} & & \\ \bar{2} & & & \end{array} \quad \begin{array}[t]{llll} 1 & 1 & 1 & 2 \\ 2 & \bar{1} & \bar{\mathbf{2}} & \\ 3 & \bar{1} & & \\ \bar{2} & & & \end{array}$$

In our second generalization of the Schensted bumping algorithm, we bump the regular numbers and the barred numbers from row to row with the stipulation that we bump in such a way as to maintain the (k,l)-semistandard tableau conditions at each stage. This bumping process was first used by Berele and Remmel [22]. In Figure 1.4, we give an example of inserting a number x into a (k,l)-semistandard tableau T via this second bumping algorithm which we call RS 2 insertion. Once again we assume the natural order. We put elements that are inserted during the algorithm in boldface type.

Figure 1.4: RS 2 insertion

$$\begin{array}[t]{llll} 1 & 1 & 2 & 2 \\ 3 & \bar{1} & & \\ \bar{1} & \bar{2} & & \\ \bar{2} & & & \end{array} \quad \begin{array}[t]{llll} 1 & 1 & \mathbf{1} & 2 \\ 3 & \bar{1} & & \\ \bar{1} & \bar{2} & & \\ \bar{2} & & & \end{array} \quad \begin{array}[t]{llll} 1 & 1 & 1 & 2 \\ \mathbf{2} & \bar{1} & & \\ \bar{1} & \bar{2} & & \\ \bar{2} & & & \end{array} \quad \begin{array}[t]{llll} 1 & 1 & 1 & 2 \\ 2 & \bar{1} & & \\ \mathbf{3} & \bar{2} & & \\ \bar{2} & & & \end{array} \quad \begin{array}[t]{llll} 1 & 1 & 1 & 2 \\ 2 & \bar{1} & & \\ 3 & \bar{2} & & \\ \bar{\mathbf{1}} & & & \end{array} \quad \begin{array}[t]{llll} 1 & 1 & 1 & 2 \\ 2 & \bar{1} & & \\ 3 & \bar{2} & & \\ \bar{1} & & & \\ \bar{\mathbf{2}} & & & \end{array}$$

In the case of RS 2 insertion a barred number $\bar{x}$ bumps the least element a such that $\bar{x} \leq a$ when it is inserted into a row rather than bumping the least element a such that $x < a$ when a regular number is inserted into a row.

For both RS 1 and RS 2 insertion procedures it is routine to check that we always end up with a (k, l)-semistandard tableau after the insertion process and the insertion process can be reversed if we know the new cell that was created by the insertion process. Thus we can use either bumping procedure to get an analogue of the Robinson-Schensted correspondence between sequences $s = (s_0, s_1, \ldots, s_n)$ from $\{1, \ldots, k, \bar{1}, \ldots, \bar{l}\}$ and pairs (P_i, Q_i) of tableaux of the same shape where P_i is a (k, l)-semistandard tableau and Q_i is a standard tableau.

Example 1.2.2 (RS 2) *Assume the natural order. We give an illustration of the computation of (P, Q) with $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & \bar{2} & 3 & \bar{1} & 1 & \bar{2} & \bar{2} \end{matrix}$ via RS 2 insertion procedure. Assume that the barred numbers are strictly larger than the regular numbers.*

$$P_k: \quad \begin{matrix} 1 \end{matrix} \quad \begin{matrix} 1 & 1 \end{matrix} \quad \begin{matrix} 1 & 1 & \bar{2} \end{matrix} \quad \begin{matrix} 1 & 1 & 3 \\ \bar{2} & & \end{matrix} \quad \begin{matrix} 1 & 1 & 3 & \bar{1} \\ \bar{2} & & & \end{matrix} \quad \begin{matrix} 1 & 1 & 1 & \bar{1} \\ 3 & & & \\ \bar{2} & & & \end{matrix} \quad \begin{matrix} 1 & 1 & 1 & \bar{1} & \bar{2} \\ 3 & & & & \\ \bar{2} & & & & \end{matrix}$$

$$P = \begin{matrix} 1 & 1 & 1 & \bar{1} & \bar{2} \\ 3 & \bar{2} & & & \\ \bar{2} & & & & \end{matrix} \qquad Q = \begin{matrix} 1 & 2 & 3 & 5 & 7 \\ 4 & 8 & & & \\ 6 & & & & \end{matrix}.$$

Example 1.2.3 (RS 1) *Now we compute (P, Q) of the same sequence π but via RS 1 insertion.*

$$P_k: \quad \begin{matrix} 1 \end{matrix} \quad \begin{matrix} 1 & 1 \end{matrix} \quad \begin{matrix} 1 & 1 \\ \bar{2} & \end{matrix} \quad \begin{matrix} 1 & 1 & 3 \\ \bar{2} & & \end{matrix} \quad \begin{matrix} 1 & 1 & 3 \\ \bar{1} & \bar{2} & \end{matrix} \quad \begin{matrix} 1 & 1 & 1 \\ 3 & \bar{1} & \bar{2} \end{matrix} \quad \begin{matrix} 1 & 1 & 1 \\ 3 & \bar{1} & \bar{2} \\ \bar{2} & & \end{matrix}$$

$$P = \begin{matrix} 1 & 1 & 1 \\ 3 & \bar{1} & \bar{2} \\ \bar{2} & & \\ \bar{2} & & \end{matrix} \qquad Q = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & 6 \\ 7 & & \\ 8 & & \end{matrix}$$

Actually from most points of view, our second analogue of the Schensted correspondence is the most natural extension of the usual Robinson-Schensted correspondence via row insertion. Indeed we can reduce RS 2 to standard row insertion. That is, given an arbitrary ordering of the regular and barred numbers and a word $\omega = \omega_1 \ldots \omega_n$, we can associate a permutation $p(\omega) = p_1 \ldots p_n$ with ω by labelling the regular numbers

from left to right and the barred numbers from right to left in the following way: Label the smallest number a_1 with $1, \ldots, a_{11}$ where a_{11} denotes the number of times a_1 appears in ω. Then label the next smallest number a_2 by $a_{11}+1, \ldots, a_{11}+a_{22}$ where a_{22} denotes the number of times a_2 appears in ω, etc.
As an example assume the ordering $\bar{1} < 1 < \bar{2} < 2$ and $\omega = 2\,1\,\bar{1}\,\bar{2}\,1\,2\,\bar{1}$, then $p(\omega) = 6\,3\,2\,5\,4\,7\,1$. It is proven in [22] that if $p(\omega)$ is associated to (P, Q) via the standard Robinson-Schensted correspondence, then ω is associated to $(\bar{P}, Q)$ via RS 2 correspondence where $\bar{P}$ results from P by replacing each p_i by ω_i in P.

However, no such reduction is possible for our first analogue of the Robinson-Schensted correspondence. For example, assume that the barred number are strictly larger than the regular numbers and $\omega = \bar{3}\,\bar{2}\,1\,2\,3 \xrightarrow{RS1} (1\,2\,3\,\bar{2}\,\bar{3},\, 1\,2\,3\,4\,5)$ which would force the corresponding permutation $p(\omega)$ to be $p(\omega) = 1\,2\,3\,4\,5$. But then $p(\omega) \xrightarrow{RS1} (1\,2\,3\,4\,5,\, 1\,2\,3\,4\,5)$ and substituting p_i by ω_i in the left tableau results in $\bar{3}\,\bar{2}\,1\,2\,3$ not in $1\,2\,3\,\bar{2}\,\bar{3}$.
Now we want to define the key properties of the RS 1 and RS 2 correspondences. The situation is relatively straightforward for RS 2. We define a sequence $s = s_1, \ldots, s_n$ with each $s_i \in \{1, \ldots, k, \bar{1}, \ldots, \bar{l}\}$ for $i = 1, \ldots, n$ to be (k,l)*-increasing of type 2* if the regular numbers are weakly increasing and the barred numbers strictly increasing in s according to the given order of the numbers. For example, if we assume that the regular numbers are strictly smaller than the barred numbers, $1\,1\,2\,3\,\bar{1}\,\bar{2}\,\bar{4}$ is a (3,4)-increasing sequence of type 2. Similarly, we say s is a (k,l)*-decreasing sequence of type 2* if the barred numbers are weakly decreasing and the regular numbers strictly decreasing in s according to the given order of the numbers. For example, if we assume the order $\bar{1} < \bar{2} < \bar{3} < 1 < \bar{4} < 2$ then $\bar{4}\,\bar{4}\,2\,\bar{2}\,\bar{2}\,1\,\bar{1}$ (2,4)-decreasing sequence of type 2.

Definition 1.2.4 *For RS 2 insertion the biletters of a (k,l)-matrix word π with entries from the set $\{1, 2, \ldots, k, \bar{1}, \bar{2}, \ldots, \bar{l}\}$ are arranged so that the top halves are increasing; for those biletters of the form $\binom{i}{z}$ where i is fixed and a regular number, the bottom halves form a (k,l)-increasing sequence of type 2, and for those biletters of the form $\binom{\bar{i}}{z}$ where $\bar{i}$ is fixed and a barred number, the bottom halves form a (k,l)-decreasing sequence of type 2. We call the multiset of elements which appear in the top (resp.bottom) line of a matrix word the content of $\hat{\pi}$ (resp.$\check{\pi}$).*

Example 1.2.5 (RS 2) *Assume the natural order. An example for a (k,l)-matrix word for RS 2 insertion is*

$$\pi = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{2} \\ 1 & 2 & 3 & 1 & 1 & \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & 2 & \bar{2} & 3 & 2 \end{pmatrix}.$$

We even have a correspondence between (k,l)-matrix words π and pairs (P_i, Q_i) where both are (k,l)-semistandard tableaux of the same shape.

It is not immediately obvious that Q is a (k,l)-semistandard tableau. Since the top row $\hat{\pi}$ of the (k,l)-matrix word π is weakly increasing, we can prove that Q is a (k,l)-semsistandard tableau if we can establish the following two facts.

- For any regular number i, all the cells in Q which contain i lie in different columns of Q.
- For any barred number $\bar{i}$, all the cells in Q which contain $\bar{i}$ lie in different rows of Q.

These two facts will follow from our next lemma, stated in [22].

Lemma 1.2.6 (RS 2) *Suppose T is any (k,l)-semistandard tableau and T' is obtained from T by inserting $s_1,\ldots,s_n$ where $s_i \in \{1,\ldots,k,\bar{1},\ldots,\bar{l}\}$ for $i = 1,\ldots,n$. Let $c_1,\ldots,c_n$ the newly created cells in T' created by the insertion $s_1,\ldots,s_n$ respectively. Then*

(1) if $s_1,\ldots,s_n$ is a (k,l)-increasing sequence of type 2, then for all $i<j$, c_i lies in a column strictly to the left of that of c_j and in a row weakly below that of c_j, and

(2) if $s_1,\ldots,s_n$ is a (k,l)-decreasing sequence of type 2, then for all $i<j$, c_i lies in a column weakly to the right of that of c_j and c_i lies in a row which is strictly above that of c_j.

Proof: It is enough to prove the lemma for $n=2$. For (1), suppose s_1s_2 is a (k,l)-increasing sequence of type 2. Thus either s_1 and s_2 are regular numbers and $s_1 \le s_2$, s_1 is a regular number and s_2 is a barred number and $s_1 < s_2$ or $s_1 > s_2$ respectively, or s_1 and s_2 are barred numbers and $s_1 < s_2$.
Now there are three possible cases. That is,
(i) either both s_1 and s_2 do not bump any elements in which case they lie at the end of the first row or
(ii) one of the numbers s_1, s_2 bumps some entry b_i of T and the other one does not bump anything and it follows from the observation that the bumping path of b_i, i. e., the sequence of cells of T which have elements which are bumped in the process of inserting b_i in T, always moves weakly to the left in columns; or
(iii) both s_1 and s_2 bump elements b_i, b_j in the first row. The lemma holds by induction for inserting b_ib_j into T since b_ib_j is also a (k,l)- increasing sequence of type 2. Part (2) of the lemma follows from a similar analysis. □

Thus, for example, if $s = s_1,s_2,\ldots,s_n$ is a (k,l)-increasing sequence of type 2 and T' is obtained from T by inserting $s_1,\ldots,s_n$, then Lemma 1.2.6 tells us that the difference between the shapes of T' and T, $\lambda(T')/\lambda(T)$, is a skew row and the order in which the new cells were created in the insertion process is from left to right and bottom to top. Moreover, Lemma 1.2.6 tells us that if we reverse the RS 2 correspondence by removing a sequence of cells in a skew row from right to left and top to bottom, we will produce a (k,l)-decreasing sequence of type 2, then $\lambda(T')/\lambda(T)$ is a skew column, i. e., no two cells lie in the same row, and that the order in which the new cells were created is from right to left and from top to bottom.

Furthermore, Lemma 1.2.6 shows that we can recover from the Q tableau the order in which the cells of P were created in the insertion of $b_1, \dots, b_n$ into the empty tableau. That is, those cells in Q with smaller entries were created in P before those cells with larger entries.

Example 1.2.7 (RS 2) *Assume the following ordering* $1 < 2 < 3 < \bar{1} < \bar{2}$. *Let* $n = 5$ *and* $\pi = \begin{pmatrix} 1 & \bar{1} & \bar{1} & \bar{2} & \bar{2} \\ \bar{2} & \bar{1} & 3 & \bar{1} & 2 \end{pmatrix}$. *The tableaux P and Q constructed by the RS 2 algorithm are:*

$$\mathbf{P_k} : \emptyset \qquad \bar{2} \qquad \begin{array}{l} \bar{1} \\ \bar{2} \end{array} \qquad \begin{array}{l} 3 \\ \bar{1} \\ \bar{2} \end{array} \qquad \begin{array}{ll} 3 & \bar{1} \\ \bar{1} & \\ \bar{2} & \end{array} \qquad \begin{array}{ll} 2 & \bar{1} \\ 3 & \\ \bar{1} & \\ \bar{2} & \end{array}$$

$$\mathbf{Q_k} : \emptyset \qquad 1 \qquad \begin{array}{l} 1 \\ \bar{1} \end{array} \qquad \begin{array}{l} 1 \\ \bar{1} \\ \bar{1} \end{array} \qquad \begin{array}{ll} 1 & \bar{2} \\ \bar{1} & \\ \bar{1} & \end{array} \qquad \begin{array}{ll} 1 & \bar{2} \\ \bar{1} & \\ \bar{1} & \\ \bar{2} & \end{array}$$

The situation for the RS 1 correspondence is only a bit more complicated. Given a sequence s as defined before, we say s is a (k,l)*-decreasing sequence of type 1* if the regular numbers are strictly decreasing and the barred numbers are weakly increasing in s preserving the given order of the numbers. For example, if we assume that the regular numbers are strictly smaller than the barred numbers then $3\,2\,1\,\bar{1}\,\bar{1}\,\bar{2}\,\bar{4}$ is a (3,4)-decreasing sequence of type 1. Similarly, we say s is a (k,l)*-increasing sequence of type 1* if the barred numbers are strictly decreasing and the regular numbers are weakly increasing preserving the given order of the numbers. For example, let us assume the order $1 < \bar{1} < 2 < \bar{2} < \bar{3} < 3 < \bar{4}$. Then $\bar{4}\,2\,2\,\bar{3}\,3\,3\,3\,\bar{1}$ is a (3,4)-increasing sequence of type 1.

A (k,l)-matrix word associated with RS 1 insertion is defined as in Definition 1.2.4 replacing RS 2 by RS 1 insertion and (k,l)-increasing and decreasing sequences of type 2 by (k,l)-increasing and decreasing sequences of type 1.
Then the exact analogues of Lemma 1.2.6 hold with adequate substitutions.

Example 1.2.8 (RS 1) *Assume* $\bar{1} < 1 < 2 < \bar{2}$. *Let* $\pi = \begin{pmatrix} \bar{1} & \bar{1} & 1 & 1 & 2 \\ 2 & 1 & \bar{2} & 1 & \bar{1} \end{pmatrix}$.

$$
P_k: \quad \begin{array}{l} 2 \end{array} \qquad \begin{array}{l} 1 \\ 2 \end{array} \qquad \begin{array}{l} 1 \\ 2 \\ \bar{2} \end{array} \qquad \begin{array}{ll} 1 & 1 \\ 2 & \\ \bar{2} & \end{array} \qquad \begin{array}{ll} \bar{1} & 1 \\ 1 & \bar{2} \\ 2 & \end{array}
$$

$$
Q_k: \quad \begin{array}{l} \bar{1} \end{array} \qquad \begin{array}{l} \bar{1} \\ \bar{1} \end{array} \qquad \begin{array}{l} \bar{1} \\ \bar{1} \\ 1 \end{array} \qquad \begin{array}{ll} \bar{1} & 1 \\ \bar{1} & \end{array} \qquad \begin{array}{ll} \bar{1} & 1 \\ \bar{1} & 2 \\ \bar{1} & \end{array}
$$

Chapter 2

Equivalence of RS correspondence and Fomin's approach

We give here a proof of the equivalence of the Robinson-Schensted correspondence and Fomin's approach to it [5, 2, 23], which is in many ways more natural and general than the usual bumping procedure. It provides a pictorial way of looking at things that is often helpful in understanding certain properties, some of which are different to show from the bumping description. We follow closely van Leeuwen [16].

2.1 Definitions

For a partition λ in the set $\mathcal{P}_n$ of all partitions of n we introduce notations for the sets of its direct predecessors and successors in the lattice:

Definition 2.1.1 *Let $\lambda \in \mathcal{P}_n$. Then*

$$\lambda^- = \{\mu \in \mathcal{P}_{n-1} \mid \mu \subset \lambda\}$$

and

$$\lambda^+ = \{\mu \in \mathcal{P}_{n+1} \mid \mu \supset \lambda\}.$$

The relation "$\subset$" makes $\mathcal{P}$ itself into a poset, which is called the Young Lattice.

Clearly $\mu \in \lambda^-$ is equivalent to $\lambda \in \mu^+$; when it holds, the difference F_λ / F_μ consists of a single square s, which lies both at the end of a row and of a column of F_λ, while it lies one position beyond both the end of a row and of a column of F_μ. In this case we shall write $s = \lambda - \mu$ as well as $\lambda = \mu + s$ and $\mu = \lambda - s$, and call s a *corner* of λ, and a *cocorner* of μ. So the partition $\lambda = (6, 4, 4, 2, 1)$ whose diagram is displayed in Figure 1.1 has corners $(1, 6)$, $(3, 4)$, $(4, 2)$ and $(5, 1)$ and cocorners $(1, 7)$, $(2, 5)$, $(4, 3)$, $(5, 2)$ and $(6, 1)$.

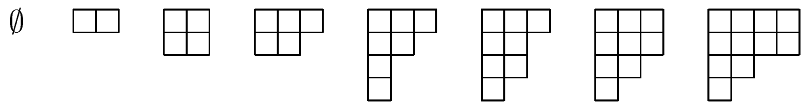
Definition 2.1.2 *We define the* type *of a (k, l)-semistandard tableau to be the sequence*

$$\tau(T) = (v_1, \ldots, v_k; \mu_1, \ldots, \mu_l)$$

where v_i (resp. μ_j) is the number of cells of λ/μ assigned to the integer i (resp. $\bar{j}$) in T. Equivalently, we may consider T to be a multichain in Young's lattice, i. e., a weakly increasing sequence of shapes

$$\mu = \lambda^0 \subseteq \cdots \subseteq \lambda^k = \lambda$$

by simply filling each horizontal (resp. vertical) skew shape λ^{i+1}/λ^i with the integer i (resp. $\bar{i}$). Two tableaux T, T' will be called similar *when $chT = chT'$. In this case T' can be obtained from T by renumbering the entries in an order preserving way. Therefore the tableau T of Figure 1.2 corresponds to the chain*



Definition 2.1.3 *The entries for the (k,l)-semistandard tableaux are arbitrary numbers from the set $\{1, \ldots, k, \bar{1}, \ldots, \bar{l}\}$. If the smallest number of a tableau appears more than once, then we define the smallest entry of the tableau as the leftmost element of the equal numbers if it is a regular number and it is the number placed in the row nearest to the top if it is a barred number. If the highest element of a tableau appears more than once, then the highest entry is the rightmost number (resp. the number in the row nearest to the bottom) of the equal numbers if it is a regular number (resp. barred number).*

2.2 RS insertion in terms of chains of partitions

(i) We denote the row (resp. column) insertion of m into a tableau T in the corner s of P by $P = T \leftarrow m$ (resp. $P = T \rightarrow m$).

(ii) Let s be a corner of $\lambda \in P$, and s' be the cocorner of λ in the row following that of s. We define $\rho^+(\lambda, \lambda - s) = \lambda + s'$ and $\rho^-(\lambda, \lambda + s') = \lambda - s$. We also define $\rho^+(\lambda) = \lambda + t$, where $t = (1, \lambda_1 + 1)$ is the cocorner of λ in row 1, so $\rho^+(\lambda)$ is the unique element of λ^+ that is not of the form $\rho^+(\lambda, \mu)$; for this case we define $\rho^-(\lambda, \lambda + t)$ to be an exceptional non-partition value. Define τ^+ and τ^- like ρ^+ and ρ^-, but replacing the word 'row' by 'column'.

(iii) For $m \in \mathbf{Z}$ and a tableau T define $m > T$ to mean that m exceeds all entries of T, $m \geq T$ that m is greater or equal than all entries of T and $m < T$ that all entries of T exceed m.

Now let us study the effect of the insertion procedure in terms of chains of partitions. The case $m \geq T$ for a regular number m (resp. $m > T$ for a barred number m) is special, since in that case the tableau has a different highest entry after insertion than before. It is also a very simple case, since the insertion involves only adding m to row 1 of T, so that $\mathrm{sh}(T \leftarrow m) = \rho^{+}(\mathrm{sh}T)$ and $(T \leftarrow m)^{-} = T$, where T^{-} denotes T with the largest element removed; in terms of chains of partitions we have

$$ch(T \leftarrow m) = \rho^{+}(shT) \,:\, chT \qquad \text{if} \begin{cases} m \geq T & \text{for a regular number } m \\ m > T & \text{for a barred number } m \end{cases} \tag{2.1}$$

where $\lambda : c$ denotes the chain in $\mathcal{P}$ formed by prepending λ to the chain c. Otherwise, the highest entry h of T (h a regular or barred number) will also be the highest entry of $T \leftarrow m$. Since the entries being moved during the insertion procedure form an increasing sequence, h either does not move at all, or is moved at the final step. Also the rule for finding the entry to replace is such that, in the case that h does in fact move, there is no other entry in that row that could have been replaced if h had been absent; therefore the presence of h does not affect the moves of any other entry. This means that insertion commutes with removal of the highest entry:

$$(T \leftarrow m)^{-} = T^{-} \leftarrow m \qquad \text{if} \begin{cases} m < T & \text{for a regular number } m \\ m \leq T & \text{for a barred number } m \end{cases} \tag{2.2}$$

So in order to determine $\mathrm{ch}(T \leftarrow m) = \mathrm{sh}(T \leftarrow m) \,:\, \mathrm{ch}(T \leftarrow m)^{-}$ it suffices to find $\mathrm{ch}(T^{-} \leftarrow m)$, which we may assume to be known by induction, and to determine $\mathrm{sh}(T \leftarrow m)$. Here we need to distinguish the cases that h does or does not move. In the latter case, since the final position of h lies outside $(T \leftarrow m)^{-}$, we have $\mathrm{sh}(T^{-} \leftarrow m) \neq \mathrm{sh}T$, and we necessarily have $\mathrm{sh}(T \leftarrow m) = \mathrm{sh}(T^{-} \leftarrow m) \cup \mathrm{sh}T$ (which has the right size since $\mathrm{sh}(T^{-} \leftarrow m) \cap \mathrm{sh}T = \mathrm{sh}T^{-}$). In the former case we have $\mathrm{sh}(T^{-} \leftarrow m)$ $= \mathrm{sh}T$, and the final position of h will be the first square outside T in the row below its initial position $\lceil T \rceil$; ($\lceil \quad \rceil$ denotes the cell of T with the highest entry) since $\lceil T \rceil$ is a corner of $\mathrm{sh}T$ this new square is indeed a cocorner of $\mathrm{sh}T$, and we conclude,

$$sh(T \leftarrow m) = \rho^{+}(shT, shT^{-}) \qquad \text{if} \begin{cases} m < T & \text{for a regular number } m \\ m \leq T & \text{for a barred number } m \end{cases} \tag{2.3}$$

$$\text{and } \mathrm{sh}(T^{-} \leftarrow m) = \mathrm{sh}T.$$

The facts collected so far, recorded in the equations (2.1)–(2.3), are sufficient to recursively compute $\mathrm{ch}(T \leftarrow m)$ in all cases, and since the set of entries of $T \leftarrow m$ is that of T with m added to it, they are sufficient to determine $T \leftarrow m$ completely. In passing we have shown that $\mathrm{ch}(T \leftarrow m)$ is indeed a proper chain of partitions, so that insertion preserves the tableau property; the crucial point is that when h moves down, it moves to a cocorner of $\mathrm{sh}T$ so that $\mathrm{sh}(T \leftarrow m) \in P$.

From these facts the analysis for the extraction procedure follows directly. Let P be a tableau with a corner s of sh$P$, then via the extraction procedure we obtain a tableau T with the entry m removed from row 1. The condition $m \geq T$ or $m > T$ for a regular or barred number respectively holds if and only if $s = \lceil P \rceil$ and $\rho^-(\mathrm{sh}P^-, \mathrm{sh}P) =$ an exceptional non-partition value; if so m is the entry of s and $T = P^-$. Otherwise we must have $P^- = T^- \leftarrow m$ by (2.2), and this will allow us to determine chT^-(and from that ch$T$) by induction, as soon as we know shT^-. Now $\mathrm{sh}T = \mathrm{sh}P - s$, and if this differs from shP^- we have $\mathrm{sh}T^- = (\mathrm{sh}P - s) \cap \mathrm{sh}P^- = \mathrm{sh}P^- - s$; in the remaining case $s = \lceil P \rceil$, we have $\mathrm{sh}T^- = \rho^-(\mathrm{sh}P^-, \mathrm{sh}P)$.
It is useful to give names to the configurations found: λ corresponds to sh$T$, κ to shT^-, μ to sh$(T^- \leftarrow m)$ and ν to sh$(T \leftarrow m)$.

Definition 2.2.1 (RS 2) *An arrangement of 4 partitions* $\begin{pmatrix} \lambda & \nu \\ \kappa & \mu \end{pmatrix}$ *with* $\lambda,\ \mu \in (\{\kappa\} \cup \kappa^+) \cap (\{\nu\} \cup \upsilon^-)$ *is called*

1. *a configuration of type 1-RS 2 if* $\kappa = \lambda = \mu$ *and* $\nu = \rho^+(\lambda)$
2. *a configuration of type 2-RS 2 if* $\kappa \neq \lambda = \mu$ *and* $\nu = \rho^+(\lambda, \kappa)$
3. *a configuration of type 3-RS 2 if* $\lambda, \mu \in \kappa^+, \kappa = \lambda \cap \mu$ *and* $\nu = \lambda \cup \mu$
4. *a configuration of type 0-RS 2 if* $\kappa = \lambda \wedge \mu = \upsilon$ *or* $\kappa = \mu \wedge \lambda = \nu$ *(possibly both)*

Now we return to RS 1 insertion. Since in the case of RS 1 insertion regular numbers are bumped from row to row and barred numbers from column to column equation (2.1) turns into

$$\mathrm{ch}(T \leftarrow m) = \rho^+(\mathrm{sh}T) : \mathrm{ch}T \quad \text{if}\, m \geq T \,\text{for a regular number}\, m \tag{2.4}$$

$$\mathrm{ch}(T \leftarrow m) = \tau^+(\mathrm{sh}T) : \mathrm{ch}T \quad \text{if}\, m \geq T \,\text{for a barred number}\, m \tag{2.5}$$

Equation (2.2) turns into

$$(T \leftarrow m)^- = T^- \leftarrow m \quad \text{if}\, m < T \,\text{for a regular or barred number}\, m \tag{2.6}$$

And the last equation (2.3) holds if the highest entry h of T is a regular number

$$\mathrm{sh}(T \leftarrow m) = \rho^+(\mathrm{sh}T, \mathrm{sh}T^-) \quad \text{if}\, m < T \,\text{and}\, \mathrm{sh}(T^- \leftarrow m) = \mathrm{sh}T \tag{2.7}$$

and for h a barred number we obtain

$$\mathrm{sh}(T \leftarrow m) = \tau^+(\mathrm{sh}T, \mathrm{sh}T^-) \quad \text{if}\, m < T \,\text{and}\, \mathrm{sh}(T^- \leftarrow m) = \mathrm{sh}T \tag{2.8}$$

It is useful to give names to the configurations found: λ corresponds to sh$T$, κ to shT^-, μ to sh$(T^- \leftarrow m)$ and ν to sh$(T \leftarrow m)$.

Definition 2.2.2 (RS 1) *An arrangement of 4 partitions* $\begin{pmatrix} \lambda & \nu \\ \kappa & \mu \end{pmatrix}$ *with* $\lambda,\, \mu \in (\{\kappa\} \cup \kappa^+) \cap (\{\nu\} \cup \upsilon^-)$ *is called*

1. a configuration of type 1-RS 1 if $\kappa = \lambda = \mu$ and $\nu = \rho^+(\lambda)$
2. a configuration of type $\bar{1}$-RS 1 if $\kappa = \lambda = \mu$ and $\nu = \tau^+(\lambda)$
3. a configuration of type 2-RS 1 if $\kappa \neq \lambda = \mu$ and $\nu = \rho^+(\lambda, \kappa)$
4. a configuration of type $\bar{2}$-RS 1 if $\kappa \neq \lambda = \mu$ and $\nu = \tau^+(\lambda, \kappa)$
5. a configuration of type 3-RS 1 if $\lambda,\, \mu \in \kappa^+$, $\kappa = \lambda \cap \mu$ and $\nu = \lambda \cup \mu$,
6. a configuration of type 0-RS 1 if $\kappa = \lambda \wedge \mu = \nu$ or $\kappa = \mu \wedge \lambda = \nu$ (possibly both).

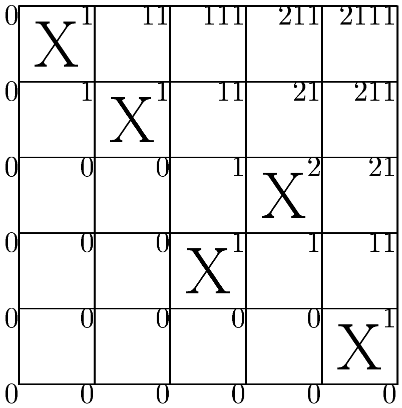
Although we have not yet encountered type 0-RS 1 respectively 0-RS 2, it will prove to be useful later. Observe that if we know that one of these configurations applies, then to determine ν uniquely it suffices to know the values of κ, λ, μ and whether type 1 applies; conversely, from λ, μ, ν we can always determine κ.

Theorem 2.2.3 (RS 2) *Applying the RS 2 insertion algorithm to the inverse of a permutation interchanges the tableaux:*

$$RS(\sigma) = (P, Q) \Longleftrightarrow RS(\sigma^-) = (Q, P) \qquad \textit{for all} \quad (k,l)-\textit{matrix words} \quad \sigma.$$

The proof given here comes about quite naturally if one views tableaux as chains of partitions, and is very similar to the proof given for Theorem 3.4.1.
Proof: Define a doubly indexed collection of tableaux $P^{[i,j]}$ for $0 \leq i, j \leq n$, by setting $P^{[i,j]}$ equal to the restriction of P_i to the set of its squares whose entry does not exceed j, where $P = \emptyset$ and $P_i = P_{i-1} \leftarrow \sigma_i$ for $i = 1, \ldots, n$ as in the definition of RS 2 (resp. RS 1) algorithm, and set $\lambda^{[i,j]} = \mathrm{sh}P^{[i,j]}$. Then it is obvious from the definitions that $\mathrm{ch}P = (\lambda^{[n,n]}, \lambda^{[n,n-1]}, \ldots, \lambda^{[n,0]})$ and $\mathrm{ch}Q = (\lambda^{[n,n]}, \lambda^{[n-1,n]}, \ldots, \lambda^{[0,n]})$. The sequences of partitions $(\lambda^{[i,n]}, \lambda^{[i,n-1]}, \ldots, \lambda^{[i,0]})$ are not in general chains in the Young lattice, because some partitions may be repeated (they are sometimes called multichains in analogy with multisets), but if we omit those partitions that are equal to their successor, then the remaining sequence equals $\mathrm{ch}P_i$. Any configuration $\left(\begin{smallmatrix} \lambda^{[i-1,j]} & \lambda^{[i-1,j-1]} \\ \lambda^{[i,j]} & \lambda^{[i,j-1]} \end{smallmatrix}\right)$ is of type 1-RS 2 if $j = \sigma_i$ (resp. type 1-RS 1 for σ_i a regular number or type $\bar{1}$-RS 1 for σ_i a barred number), of type 0 if $j < \sigma_i \vee i < \sigma_j^{-1}$, of type 3-RS 2 respectively 3-RS 1 if an entry j has not yet been inserted or if we take the restriction to entries smaller than the one currently being inserted of type 2-RS 2 (resp. type $\bar{2}$-RS 1 if the one inserted is a regular number and of type 2-RS 1 if it is a barred number). These configuration types are symmetric with respect to i and j, and it follows that if $\lambda^{[i,j]\,\prime}$ is the analogous collection of partitions for σ^{-1} in place of σ then $\lambda^{[i,j]\,\prime} = \lambda^{[j,i]}$, which immediately implies the theorem. $\square$

Example 2.2.4 (RS 2) *Assume the natural order. We give an illustration of the* $\lambda^{[i,j]}$, *displayed at the corners of the cells in the skew diagram for* $\sigma = \begin{pmatrix} 1 & \bar{1} & \bar{1} & \bar{2} & \bar{2} \\ \bar{2} & \bar{1} & 3 & \bar{1} & 2 \end{pmatrix}$. *The X's indicate the positions* $(i, p_i(\sigma))$ *in the skew diagram for* $p(\sigma) = (54231)$ *as defined in section 1.2.*



The right and the top edge can each be interpreted as chains of P and Q respectively. The tableaux P and Q constructed by the RS 2 algorithm are:

$$\mathbf{P_k} : \emptyset \quad \bar{2} \quad \begin{matrix} \bar{1} \\ \bar{2} \end{matrix} \quad \begin{matrix} 3 \\ \bar{1} \\ \bar{2} \end{matrix} \quad \begin{matrix} 3 & \bar{1} \\ \bar{1} & \\ \bar{2} & \end{matrix} \quad \begin{matrix} 2 & \bar{1} \\ 3 & \\ \bar{1} & \\ \bar{2} & \end{matrix}$$

$$\mathbf{Q_k} : \emptyset \quad 1 \quad \begin{matrix} 1 \\ \bar{1} \end{matrix} \quad \begin{matrix} 1 \\ \bar{1} \\ \bar{1} \end{matrix} \quad \begin{matrix} 1 & \bar{2} \\ \bar{1} & \\ \bar{1} & \end{matrix} \quad \begin{matrix} 1 & \bar{2} \\ \bar{1} & \\ \bar{1} & \\ \bar{2} & \end{matrix}$$

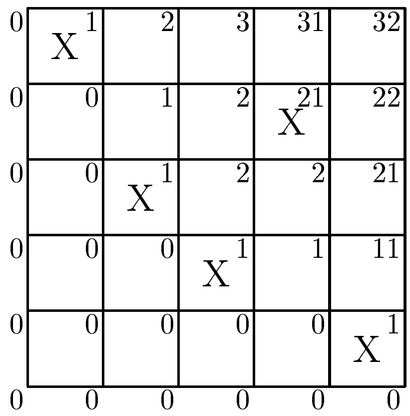
One may define the family $\lambda^{[i,j]}$ on an arbitrary region between two zig-zag paths from bottom right to top left with common end points, and allow other sequences of partitions than just empty ones along the bottom left path; knowledge of the partitions along this path and of the location of any configurations of type 1 within the region is then equivalent to knowledge of the partitions along the top right path.

Now let us illustrate the $\lambda^{[i,j]}$ written at the corners of cells in the skew diagram via RS 1 insertion.

Example 2.2.5 (RS 1) *Assume the natural order. Let* $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \bar{2} & \bar{1} & 3 & \bar{1} & 2 \end{pmatrix}$. *The X's indicate the positions* $(i, \sigma(i))$ *in the skew digram. The tableaux* P *and* Q *con-*

structed by the RS 1 algorithm are:

$$
\begin{array}{llllll}
P_k: & \bar{2} & \bar{1}\ \bar{2} & 3\ \bar{1}\ \bar{2} & \begin{array}{lll} 3 & \bar{1} & \bar{2} \\ \bar{1} & & \end{array} & \begin{array}{lll} 2 & \bar{1} & \bar{2} \\ 3 & \bar{1} & \end{array} \\[2ex]
Q_k: & 1 & 1\ 2 & 1\ 2\ 3 & \begin{array}{lll} 1 & 2 & 3 \\ 4 & & \end{array} & \begin{array}{lll} 1 & 2 & 3 \\ 4 & 5 & \end{array}
\end{array}
$$



Chapter 3

The Schützenberger algorithm

In this section we consider an algorithm due to Schützenberger that defines a non-trivial shape preserving transformation S of tableaux, under which the set of entries is replaced by the set of their *corresponding* elements (see Definition 3.3.4).

3.1 Involutory (k,l)-matrix words

Definition 3.1.1 *A (k,l)-involution is a (k,l)-matrix word σ with the following property. If $\binom{a}{b}$ for some regular or barred numbers a, b, $a \neq b$, appears k times in σ then $\binom{b}{a}$ also has to appear k times in σ.*
Let us denote the set of all (k,l)-semistandard tableaux of shape λ by $\mathcal{T}_{(k,l)}(\lambda)$.

We return to Theorem 2.2.3, and study the fixed points of the symmetry it expresses. Clearly, for any tableau P the (k,l)-matrix word $\sigma = RS^{-}(P,P)$ is an involution, and this defines a bijection from $\bigcup_{\lambda \in P_n} \mathcal{T}_{(k,l)}(\lambda)$ to the set of involutions in the set of all (k,l)-matrix words. Since we know that for this situation the family $\lambda^{[i,j]}$ satisfies $\lambda^{[i,j]} = \lambda^{[j,i]}$ for all i, j, we can find σ as follows with about half as much work as usual, by considering only positions with $i \geq j$. We successively compute the tableaux T_i corresponding to the sequences $(\lambda^{[i,i]}, \lambda^{[i,i-1]}, \ldots, \lambda^{[i,0]})$ for $i = n, \ldots, 0$ (so $T_n = P$); at each step we either find a fixed point of the involution σ, or a pair of points that are interchanged by σ, or nothing happens at all; these correspond to configurations on the main diagonal of type 1-RS 2, 2-RS 2 and 0-RS 2 respectively (type 3-RS 2 cannot occur). To start with the last possibility, if the entry i does not occur in the tableau T_i (because it has already been removed), then $\lambda^{[i,i]} = \lambda^{[i,i-1]} = \lambda^{[i-1,i]} = \lambda^{[i-1,i-1]}$, and $T_{i-1} = T_i$. Having decreased i in this manner until it occurs in T_i, we inspect its position $\lceil T_i \rceil$; if it lies in row 1, then we have found a fixed point $i = \sigma_i$, and $T_{i-1} = T_i^{-}$. Otherwise $\mathrm{sh}T_{i-1} = \rho^{-}(\mathrm{sh}T^{-}, \mathrm{sh}T)$, and we compute T_{i-1} by the extraction procedure of T_i^{-} for the appropriate square s; the numbers i and m that are removed in passing from T_i to T_{i-1} are exchanged by σ. When the empty tableau is reached, σ is completely determined. This computation has the following implication, which is due to Schützenberger [29].

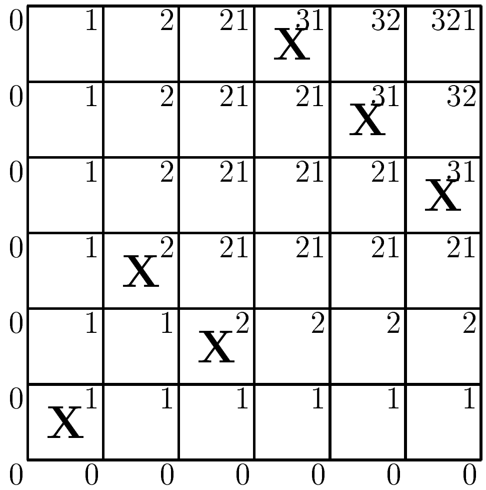
Proposition 3.1.2 *Let $\lambda \in \mathcal{P}_n$ and let $k = \sum_i (-1)^{i+1}\lambda_i$ be the number of columns of F_λ of odd length, then for each tableau $P \in \mathcal{T}_{(k,l)}(\lambda)$ the permutation $RS^{-1}(P,P)$ is an involution with k fixed points.*

Proof: Since for $i > 0$ the number of columns of F_λ of length i is $\lambda_i - \lambda_{i+1}$ it is clear that k is indeed the number of odd length columns. Whenever $\mu = \rho^-(\lambda, \nu)$, the partition μ is obtained from ν by decreasing two successive parts by 1, so $\sum_i (-1)^{i+1}\mu_i = \sum_i (-1)^{i+1}\nu_i$. Therefore, if $RS^{-1}(P,P)$ is computed as indicated above, then the value of the alternating sum for shT_i only changes when a fixed point $i = \sigma_i$ is found, and if so, it is one more for shT_i than for shT_{i-1}. It follows that the total number of fixed points will be k. $\square$

Example 3.1.3 *Assume the natural order. Let* $\sigma = \begin{pmatrix} 1 & 1 & 2 & \bar{1} & \bar{1} & \bar{2} \\ 1 & 2 & 1 & \bar{2} & \bar{1} & \bar{1} \end{pmatrix}$. *Then we have the following diagrams via the RS 2 algorithm:*

$$P_k: \quad \begin{matrix} 1 \end{matrix} \qquad \begin{matrix} 1 & 2 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 1 & \bar{2} \\ 2 & & \end{matrix} \qquad \begin{matrix} 1 & 1 & \bar{1} \\ 2 & \bar{2} & \end{matrix} \qquad \begin{matrix} 1 & 1 & \bar{1} \\ 2 & \bar{1} & \\ \bar{2} & & \end{matrix}$$

$$Q_k: \quad \begin{matrix} 1 \end{matrix} \qquad \begin{matrix} 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \qquad \begin{matrix} 1 & 1 & \bar{1} \\ 2 & & \end{matrix} \qquad \begin{matrix} 1 & 1 & \bar{1} \\ 2 & \bar{1} & \end{matrix} \qquad \begin{matrix} 1 & 1 & \bar{1} \\ 2 & \bar{1} & \\ \bar{2} & & \end{matrix}$$



3.2 Definition of the Schützenberger algorithm

The Schützenberger algorithm is based on the repeated application of a basic procedure that modifies a given tableau in a specific manner, and which we shall call the *deflation* procedure D, since it starts by emptying the square in the upper left-hand corner, and then proceeds to rearrange the remaining squares to form a proper semistandard tableau. The procedure can be reversed step by step, giving rise to an *inflation* procedure D^{-1}. More precisely, these procedures convert into each other the following sets of data: on the one hand a non-empty semistandard tableau P, and on the other hand a semistandard tableau T, a specified cocorner s of $\mathrm{sh}T$, and a number m that is smaller or equal than all entries of T; we write $(T, s, m) = D(P)$ and $P = D^{-1}(T, s, m)$. These procedures are such that we always have the following relations: the set of entries of P are those of T together with m, and $\mathrm{sh}P = \mathrm{sh}T + s$. Our description of these procedures is slightly informal; a more formal and elaborate description can be found in an excellent exposition by Knuth [14].

> *Deflation procedure.* Given a semistandard tableau P, the triple $(T, s, m) = D(P)$ is computed as follows. The first step is to put m equal to the smallest entry of P, and remove that entry, leaving an empty square at the origin. Then the following step is repeated until the empty square is a corner of the shape $\mathrm{sh}P$ of the original tableau. In the following we distinguish three cases according to the entries located directly to the right of and below the

empty square (if only one of these positions contains an entry, move that entry). When the position of the empty square finally is a corner of sh$P$, then s is defined to be this corner, and T is the tableau formed by the remaining non-empty squares.

Case 1: Both entries are barred numbers

$$\begin{array}{|c|c|}\hline & \bar{x} \\ \hline \bar{y} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline \bar{x} & \\ \hline \bar{y} \\ \cline{1-1}\end{array} \qquad \bar{x} \leq \bar{y}$$

$$\begin{array}{|c|c|}\hline & \bar{x} \\ \hline \bar{y} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline \bar{y} & \bar{x} \\ \hline \\ \cline{1-1}\end{array} \qquad \bar{x} > \bar{y}$$

Case 2: Both entries are regular numbers

$$\begin{array}{|c|c|}\hline & x \\ \hline y \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline x & \\ \hline y \\ \cline{1-1}\end{array} \qquad x < y$$

$$\begin{array}{|c|c|}\hline & x \\ \hline y \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline y & x \\ \hline \\ \cline{1-1}\end{array} \qquad x \geq y$$

Case 3: One of the entries is a regular and the other one is a barred number

$$\begin{array}{|c|c|}\hline & x \\ \hline \bar{y} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline x & \\ \hline \bar{y} \\ \cline{1-1}\end{array} \qquad x < \bar{y}$$

$$\begin{array}{|c|c|}\hline & x \\ \hline \bar{y} \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|}\hline \bar{y} & x \\ \hline \\ \cline{1-1}\end{array} \qquad \bar{y} < x$$

Example 3.2.1 *Let* $P = \begin{array}{cccc} 1 & 3 & 5 & 8 \\ 2 & 4 & 7 & 9 \\ 6 & 10 \end{array}$. *We let a dot indicate the position of the empty square as we perform the deflation procedure.*

$$\begin{array}{cccc} \bullet & 3 & 5 & 8 \\ 2 & 4 & 7 & 9 \\ 6 & 10 \end{array} \qquad \begin{array}{cccc} 2 & 3 & 5 & 8 \\ \bullet & 4 & 7 & 9 \\ 6 & 10 \end{array} \qquad \begin{array}{cccc} 2 & 3 & 5 & 8 \\ 4 & \bullet & 7 & 9 \\ 6 & 10 \end{array} \qquad \begin{array}{cccc} 2 & 3 & 5 & 8 \\ 4 & 7 & \bullet & 9 \\ 6 & 10 \end{array} \qquad \begin{array}{cccc} 2 & 3 & 5 & 8 \\ 4 & 7 & 9 & \bullet \\ 6 & 10 \end{array}$$

So in this case we have $m = 1$, $s = (2, 4)$ and $T = \begin{array}{cccc} 2 & 3 & 5 & 8 \\ 4 & 7 & 9 & \\ 6 & 10 \end{array}$.

Because the empty square moves either down or to the right in each step, termination is evidently guaranteed. That T is indeed a semistandard tableau can be seen by observing that at each stage of the process the regular (resp. barred) entries of the non-empty squares remain weakly increasing (resp. strictly increasing) along each row and strictly increasing (resp. weakly increasing) along each column. In fact, when there are entries both to the right and below the empty square, the definition of our cases is dictated by the semistandard tableau property. By the same consideration as stated above it also becomes clear that D is invertible, and that its inverse procedure D^{-1} can be defined as follows:

> *Inflation procedure.* Given a semistandard tableau T, a cocorner s of sh$T$ and a number m that is smaller or equal than the entries of T, the semistandard tableau $P = D^{-1}(T, s, m)$ is computed as follows. The first step is to attach an empty square to T at position s. Then the following step is repeated until the empty square is at the origin: Again we distinguish three cases according to the entries located directly to the left of and above it (if only one of these positions contains an entry, move that entry). When the empty square has arrived at the origin, it is filled with the number m to form the semistandard tableau P.

Case 1: Both entries are barred numbers

$$\begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \bar{x} \\ \hline \bar{y} & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \bar{x} \\ \hline & \bar{y} \\ \hline \end{array} \qquad \bar{x} \leq \bar{y}$$

$$\begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \bar{x} \\ \hline \bar{y} & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \\ \hline \bar{y} & \bar{x} \\ \hline \end{array} \qquad \bar{x} > \bar{y}$$

Case 2: Both entries are regular numbers

$$\begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \mathrm{x} \\ \hline \mathrm{y} & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \mathrm{x} \\ \hline & \mathrm{y} \\ \hline \end{array} \qquad x < y$$

$$\begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \mathrm{x} \\ \hline \mathrm{y} & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \\ \hline \mathrm{y} & \mathrm{x} \\ \hline \end{array} \qquad x \geq y$$

Case 3: One of the entries is a regular and the other one is a barred number

$$\begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \mathrm{x} \\ \hline \bar{y} & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \mathrm{x} \\ \hline & \bar{y} \\ \hline \end{array} \qquad x < \bar{y}$$

$$\begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \mathrm{x} \\ \hline \bar{y} & \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & \\ \hline \bar{y} & \mathrm{x} \\ \hline \end{array} \qquad \bar{y} < x$$

Example 3.2.2 *Let* $T = \begin{matrix} 2 & 3 & 5 & 8 \\ 4 & 7 & 9 \\ 6 & 10 \end{matrix}$ *,* $m = 1$*,* $s = (3,3)$*. Again a dot indicates the position of the empty square as we perform the inflation procedure.*

$$\begin{matrix} 2 & 3 & 5 & 8 \\ 4 & 7 & 9 \\ 6 & 10 & \bullet \end{matrix} \qquad \begin{matrix} 2 & 3 & 5 & 8 \\ 4 & 7 & 9 \\ 6 & \bullet & 10 \end{matrix} \qquad \begin{matrix} 2 & 3 & 5 & 8 \\ 4 & \bullet & 9 \\ 6 & 7 & 10 \end{matrix} \qquad \begin{matrix} 2 & 3 & 5 & 8 \\ \bullet & 4 & 9 \\ 6 & 7 & 10 \end{matrix} \qquad \begin{matrix} \bullet & 3 & 5 & 8 \\ 2 & 4 & 9 \\ 6 & 7 & 10 \end{matrix}$$

$$P = \begin{matrix} 1 & 3 & 5 & 8 \\ 2 & 4 & 9 \\ 6 & 7 & 10 \end{matrix}$$

One easily verifies that the procedure reverses D step by step, and also preserves the semistandard tableau property.

3.3 The effect of D in terms of chains of partitions

Definition 3.3.1 *(1) We define* $P^{\downarrow} = T$ *where the arrow is meant to suggest the lowest entry of* P *being squeezed out.*

(2) Let $x = (r, c)$ *and* y *be distinct squares. The relation* $x \| y$ *is defined to hold if either* $y = (r+1, c)$ *or* $y = (r, c+1)$*.*

This relation is not symmetric, because whenever we need it it will be clear that it can hold only in one direction.

Now let us state the effect of D in terms of partitions. In case P has only one square we obviously have $P^{\downarrow} = \emptyset$, so we assume that P has at least 2 squares. The highest entry h of P lies at some corner of $\mathrm{sh}P$, so it either does not move at all, or it moves in the final step into a square for which it is the only candidate; therefore its presence will not affect the movement of any other entry. This means that deflation commutes with removal of the highest entry:

$$P^{\downarrow -} = P^{-\downarrow}. \tag{3.1}$$

Consequently, if by induction we assume that we know $\mathrm{ch}P^{-\downarrow}$, then all that is needed to determine $\mathrm{ch}P^{\downarrow} = \mathrm{sh}P^{\downarrow} : \mathrm{ch}P^{\downarrow -}$ is to find $\mathrm{sh}P^{\downarrow}$. Here there are two cases to distinguish, namely whether h does or does not move. The former case applies when the final position $\mathrm{sh}P^{-}/\mathrm{sh}P^{-\downarrow}$ of the empty square in the computation of $P^{-\downarrow}$ is adjacent to the position $\lceil P \rceil$ of h ($\lceil\,\rceil$ denotes the square of P with the highest entry— remember our definition!), and if so, h moves into that square, making $\mathrm{sh}P^{\downarrow} = \mathrm{sh}P^{-}$. In the latter case the fact that h does not move can be expressed as $\lceil P^{\downarrow} \rceil = \lceil P \rceil$, and since

$\lceil P \rceil \notin \mathrm{sh}P^-$, we now obviously have $\mathrm{sh}P^\downarrow \neq \mathrm{sh}P^-$; since $\mathrm{sh}P^{\downarrow-}$ and $\mathrm{sh}P$ differ only by two squares, there are no more than two intermediate partitions, and the inequality determines $\mathrm{sh}P^\downarrow$ completely. Hence $\mathrm{ch}P^\downarrow$ is determined by (3.1) in combination with

$$shP^\downarrow = shP^- \Longleftrightarrow (shP^- - shP^{-\downarrow}) \| \lceil P \rceil.$$

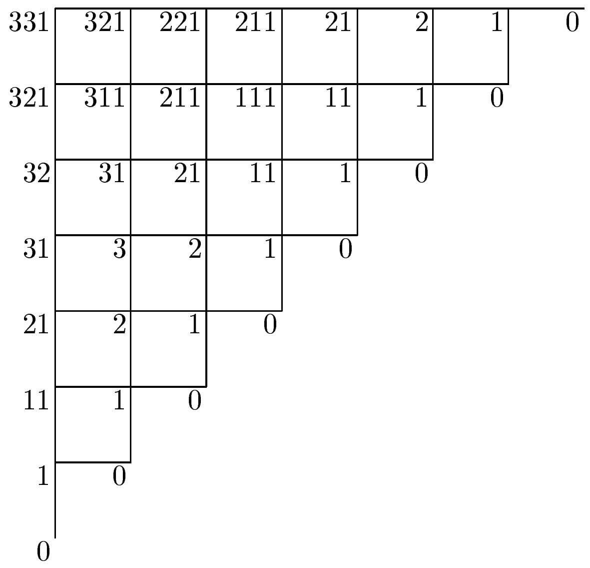
It is easy to see that the condition on the right is equivalent to the existence of only one intermediate partition between $\mathrm{sh}P^{\downarrow-}$ and $\mathrm{sh}P$, so the determination of $\mathrm{sh}P^\downarrow$ can be summarised by the condition $\mathrm{sh}P^{\downarrow-} \subset \mathrm{sh}P^\downarrow \subset \mathrm{sh}P$, and the rule that we have $\mathrm{sh}P^\downarrow \neq \mathrm{sh}P^-$ whenever that is possible. Although it may seem that we have used only a few aspects of the definition of D , we can in fact use the stated rule to recursively compute $\mathrm{ch}P^\downarrow$, and since the set of entries of $P^\downarrow$ is just that of P without the minimal entry, to compute $P^\downarrow$ itself. The situation for the inverse computation $P = D^{-1}(T, s, m)$ is quite similar, except that the basic step now precedes the recursive computation. We have $\mathrm{sh}P = \mathrm{sh}T + s$, and from (3.1) it follows that $T^- \subset \mathrm{sh}P^- \subset \mathrm{sh}P$; if this does not determine $\mathrm{sh}P^-$ completely, it is taken to be different from $\mathrm{sh}T$. Once $\mathrm{sh}P^-$ is determined, $\mathrm{ch}P^-$ is determined by recursive application of these rules. It is useful to attach names to the relations between several partitions that we have encountered here. So μ corresponds to $P^\downarrow$, λ to P^-, ν to $P^{\downarrow-}$ and κ to P.

Definition 3.3.2 *An arrangement of 4 partitions* $\left(\begin{smallmatrix}\kappa & \lambda \\ \mu & \nu\end{smallmatrix}\right)$ *with* $\lambda, \mu \in \kappa^- \cap \nu^+$ *is called*

1. *a configuration of type S1 if* $\lambda = \mu$ *and* $(\lambda - \nu) \| (\kappa - \lambda)$,
2. *a configuration of type S2 if* $\lambda \neq \mu$ *and* $\kappa = \lambda \cup \mu$, $\nu = \lambda \cap \mu$.

Example 3.3.3 *Assume the natural order. Let us give an illustration of this for*

$$P = \begin{array}{ccc} 1 & 1 & \bar{2} \\ 2 & 3 & \bar{2} \\ 3 & & \end{array}.$$



Next we define another ordering of the numbers from the set $\{1, \ldots, k, \bar{1}, \ldots, \bar{l}\}$ in the following way.

Definition 3.3.4 *Assume that we have an arbitrary ordering induced on the set* $\{1, \ldots, k, \bar{1}, \ldots, \bar{l}\}$. *We say that for the regular (resp. barred) elements the* $(1 + i)th$ *element corresponds to the* $(k - i)th$ *(resp.* $(l - i)th$*) element for* $i = 0, \ldots, k - 1$ *(resp.* $i = 0, \ldots, l-1$*). For example, if we have the following order* $\bar{1} < 2 < 3 < \bar{2} < 5 < \bar{4} < 7$ *then* $\bar{1} \leftrightarrow \bar{4}$, $2 \leftrightarrow 7$, $3 \leftrightarrow 5$, $\bar{2} \leftrightarrow \bar{2}$, $5 \leftrightarrow 3$, $\bar{4} \leftrightarrow \bar{1}$, $7 \leftrightarrow 2$. *Then we reorder those corresponding elements from right to left and obtain* $2 < \bar{1} < 3 < \bar{2} < 5 < 7 < \bar{4}$. *We denote corresponding elements by* $m \leftrightarrow m^c$.

The full Schützenberger algorithm S essentially consists of repetitions of the basic procedure. Repeating the application of the deflation procedure to P, we find a sequence of tableaux $P, P^{\downarrow}, P^{\downarrow\downarrow}, \ldots, \emptyset$, whose shapes form a saturated decreasing chain in the Young lattice starting in $\lambda = \mathrm{sh}P$; there is a unique tableau P^* for which this chain equals $\mathrm{ch}P^*$ and whose set of entries are the corresponding ones of those of P. The algorithm S has an inverse algorithm S^{-1}, which is just as easy to compute, but slightly more difficult to formulate. To compute $S^{-1}(P^*)$ one starts with an empty tableau, and successively computes tableaux whose shapes are the partitions occuring in $\mathrm{ch}P^*$, and each one is obtained from its predecessor by an appropriate application of D^{-1}; the final tableau so constructed is $S^{-1}(P^*)$. More precisely, one sets $P_0 = \emptyset$ and then successively $P_i = D^{-1}(P_{i-1}, s_i, m_i^c)$ for $i = 1, \ldots, n$, where $(m_1, \ldots, m_n)$ are the entries of P^* in weakly increasing order, and s_i is a cell with entry m_i in P^*; then $P = S^{-1}(P^*) = P_n$. It is obvious from the definition that S and S^{-1} commute with transposition: $S(P^t) = S(P)^t$ and $S^{-1}(P^t) = S^{-1}(P)^t$.

Example 3.3.5 *Let us assume the natural order. We display the successive stages* $P, P^{\downarrow}, P^{\downarrow\downarrow}, \ldots$*, the tableau* P^* *and* $S^{-1}(P^*)$.

$$P^{\downarrow\cdots\downarrow} \quad \begin{array}{llll} 1 & 2 & \bar1 & \bar2 \\ 2 & 3 & \bar1 & \\ \bar2 & \bar3 & & \\ \bar2 & & & \end{array} \qquad \begin{array}{llll} 2 & 2 & \bar1 & \bar2 \\ 3 & \bar1 & & \\ \bar2 & \bar3 & & \\ \bar2 & & & \end{array} \qquad \begin{array}{lll} 2 & \bar1 & \bar2 \\ 3 & \bar1 & \\ \bar2 & \bar3 & \\ \bar2 & & \end{array} \qquad \begin{array}{lll} 3 & \bar1 & \bar2 \\ \bar1 & \bar3 & \\ \bar2 & & \\ \bar2 & & \end{array} \qquad \begin{array}{ll} \bar1 & \bar2 \\ \bar1 & \bar3 \\ \bar2 & \\ \bar2 & \end{array}$$

$$\begin{array}{ll} \bar1 & \bar2 \\ \bar2 & \bar3 \\ \bar2 & \end{array} \qquad \begin{array}{ll} \bar2 & \bar3 \\ \bar2 & \\ \bar2 & \end{array} \qquad \begin{array}{ll} \bar2 & \bar3 \\ \bar2 & \end{array} \qquad \begin{array}{ll} \bar2 & \bar3 \end{array} \qquad \bar3 \qquad \emptyset$$

$$P^* = \begin{array}{|c|c|c|c|} \hline \bar1 & \bar2 & 1 & 2 \\ \hline \bar2 & \bar3 & 3 \\ \cline{1-3} \bar2 & 2 \\ \cline{1-2} \bar3 \\ \cline{1-1} \end{array}$$

$$S^{-}(P^*): \quad \emptyset \qquad \bar3 \qquad \begin{array}{ll} \bar2 & \bar3 \end{array} \qquad \begin{array}{ll} \bar2 & \bar3 \\ \bar2 & \end{array} \qquad \begin{array}{ll} \bar2 & \bar3 \\ \bar2 & \\ \bar2 & \end{array} \qquad \begin{array}{ll} \bar1 & \bar2 \\ \bar2 & \bar3 \\ \bar2 & \end{array} \qquad \begin{array}{ll} \bar1 & \bar2 \\ \bar1 & \bar3 \\ \bar2 & \\ \bar2 & \end{array}$$

$$\begin{array}{lll} 3 & \bar1 & \bar2 \\ \bar1 & \bar3 & \\ \bar2 & & \\ \bar2 & & \end{array} \qquad \begin{array}{lll} 2 & \bar1 & \bar2 \\ 3 & \bar1 & \\ \bar2 & \bar3 & \\ \bar2 & & \end{array} \qquad \begin{array}{llll} 2 & 2 & \bar1 & \bar2 \\ 3 & \bar1 & & \\ \bar2 & \bar3 & & \\ \bar2 & & & \end{array} \qquad \begin{array}{llll} 1 & 2 & \bar1 & \bar2 \\ 2 & 3 & \bar1 & \\ \bar2 & \bar3 & & \\ \bar2 & & & \end{array}$$

3.4 Involution property of S

The correspondence defined by Schützenberger algorithm is in fact an involution, although this is not obvious from the definition.

Theorem 3.4.1 *For all* $\lambda \in \mathcal{P}$ *the algorithm* S *defines an involution, i. e., for all tableaux* P *hold the equation*

$$S(P) = S^{-1}(P).$$

We display the successive stages of computing $S^{-1}(P)$ of our previous example.

$$\emptyset \qquad 3 \qquad \begin{array}{l} 2 \\ 3 \end{array} \qquad \begin{array}{ll} 2 & 2 \\ 3 & \end{array} \qquad \begin{array}{ll} 1 & 2 \\ 2 & 3 \end{array} \qquad \begin{array}{lll} \bar{3} & 1 & 2 \\ 2 & 3 & \end{array} \qquad \begin{array}{lll} \bar{3} & 1 & 2 \\ \bar{3} & 2 & 3 \end{array} \qquad \begin{array}{llll} \bar{2} & \bar{3} & 1 & 2 \\ \bar{3} & 2 & 3 & \end{array}$$

$$\begin{array}{llll} \bar{2} & \bar{3} & 1 & 2 \\ \bar{2} & 2 & 3 & \\ \bar{3} & & & \end{array} \qquad \begin{array}{llll} \bar{2} & \bar{3} & 1 & 2 \\ \bar{2} & 2 & 3 & \\ \bar{2} & & & \\ \bar{3} & & & \end{array} \qquad \begin{array}{llll} \bar{1} & \bar{2} & 1 & 2 \\ \bar{2} & \bar{3} & 3 & \\ \bar{2} & 2 & & \\ \bar{3} & & & \end{array}$$

Proof: Since the set of entries is clearly the same for $S(P)$ and $S^{-1}(P)$, it suffices to prove that $\mathrm{ch}S(P) = \mathrm{ch}S^{-1}(P)$. In view of (3.1), we may define a doubly indexed collection of tableaux $P^{[i,j]}$ for $i + j \leq n$ where $n = |shP|$, by setting for $P^{[0,0]} = P$, and for all applicable i, j : $P^{[i,j+1]} = (P^{[i,j]})^{-}$ and $P^{[i+1,j]} = (P^{[i,j]})^{\downarrow}$; furthermore we set $\lambda^{[i,j]} = \mathrm{sh}P^{[i,j]}$. Clearly we have $\mathrm{ch}P = (\lambda^{[0,0]}, \lambda^{[0,1]}, \ldots, \lambda^{[0,n]})$ and $\mathrm{ch}S(P) = (\lambda^{[0,0]}, \lambda^{[1,0]}, \ldots, \lambda^{[n,0]})$. Moreover, we have seen that any configuration $\left(\begin{smallmatrix} \lambda^{[i,j]} & \lambda^{[i,j+1]} \\ \lambda^{[i+1,j]} & \lambda^{[i+1,j+1]} \end{smallmatrix}\right)$ is one of type S1 or S2, and this determines $\lambda^{[i+1,j]}$, when the other three partitions are given. Since these configurations also occur in the description of the inflation procedure D^{-1} in terms of chains of partitions, it follows by an easy induction that for the intermediate tableaux P_i occurring in the construction of $S^{-1}(P)$ one has $\mathrm{ch}P_i = (\lambda^{[0,n-1]}, \lambda^{[1,n-1]}, \ldots, \lambda^{[i,n-1]})$; for $i = n$ this gives us $\mathrm{ch}S^{-1}(P) = \mathrm{ch}S(P)$. $\square$

We can generalize in a different way, since it is clear that by the same local application of the rules, the complete family of partitions $\lambda^{[i,j]}$ is not only determined by the values $\lambda^{[0,j]}$ along the top edge, or by the values $\lambda^{[i,0]}$ along the left edge, but also by any sequence of values starting with $\lambda^{[0,0]}$ and repeatedly going either one step down (increasing the first index by 1) or to the right (increasing the second index by 1) until the empty partition is reached, all values within the rectangle that encloses the path are still determined.

Chapter 4

Relating the two algorithms

In this section we give Knuth analogues for matters that involve both the RS 2 insertion procedure and the Schützenberger algorithm. We will show that for certain variations of a matrix word σ the corresponding tableaux are related to the (P, Q) tableaux corresponding to σ. The following theorem will be illustrated in Example 4.0.5.

Definition 4.0.2 *Let* $\sigma = \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$ *be a matrix word with (repeated) elements out of the set* $\{1, \ldots, n\}$. *Let us replace the largest element of* σ *by the smallest element of* σ*, the second largest element by the second smallest one and so on. Let us call those elements corresponding ones.*

Definition 4.0.3 *In order to compute the reverse matrix word* σ^r *we replace the top line* $\hat{\sigma}$ *by corresponding elements. Then we reorder the elements such that the top line is in weakly increasing order. For those biletters of the form* $\binom{i}{z}$ *with* i *a regular number, the bottom halves form a* (k, l)*-increasing sequence of type 2, and for those biletters of the form* $\binom{\bar{i}}{z}$ *with* $\bar{i}$ *a barred number, the bottom halves form a* (k, l)*-decreasing sequence of type 2 (for column insertion this turns out to be the right ordering).*
For another matrix word, say π*, that can be derived from* σ *we carry out the same steps as for the computation of* σ^r *but replacing the bottom line* $\check{\sigma}$ *of* σ *by corresponding elements.*
For π^r *we replace* $\hat{\sigma}$ *and* $\check{\sigma}$ *by corresponding elements and reorder them lexicographically. We also obtain* π^r *by replacing the top line of* π *by corresponding elements.*

Theorem 4.0.4 *For* $\lambda \in \mathcal{P}_n, P, Q \in \mathcal{T}_\lambda$*, and* σ *a matrix word, the following statements are equivalent:*

$$\begin{aligned} RS(\sigma) &= (P, Q) && (4.1)\\ RS^t(\sigma^r) &= (P, Q^*) && (4.2)\\ RS^t(\pi) &= (P^*, Q) && (4.3)\\ RS(\pi^r) &= (P^*, Q^*) && (4.4) \end{aligned}$$

where RS^t denotes the transpose Robinson-Schensted correspondence (column insertion).

σ^r is the reverse of σ, so if we look only at the left tableau P, the equivalence of (4.1) and (4.2) states that we obtain the same tableau P by applying the row insertion algorithm to σ and the column insertion algorithm to σ^r. From this fact and the commutation of S with transposition, one can immediately see that S is an involution, and using Theorem 2.2.3 one also obtains the equivalence of (4.3) and (4.4) with (4.1) and (4.2). The formulation of the theorem for regular numbers is due to Knuth [14]. He expresses the remarkable character of the theorem as follows (p.60):

> 'The reader is urged to try out these processes on some simple examples. The unusual nature of these coincidences might lead us to suspect that some sort of witchcraft is operating behind the scenes! No simple explanation for these phenomena is yet known; there seems to be no obvious way to prove even that π^r corresponds to tableaux having the same shape as P and Q.'

Example 4.0.5 *Let* $\sigma = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 & 1 \end{pmatrix}$. *Then we obtain according to the above description*

$$\sigma^r = \begin{pmatrix} 1 & 2 & 2 & 3 & 3 \\ 3 & 2 & 2 & 1 & 3 \end{pmatrix}, \pi = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 3 & 2 & 2 & 1 & 3 \end{pmatrix}, \pi^r = \begin{pmatrix} 1 & 2 & 2 & 3 & 3 \\ 3 & 1 & 2 & 2 & 3 \end{pmatrix}$$

$$RS^r(\sigma) = \left(\begin{matrix} 1 & 1 & 2 & 3 \\ 2 & & & \end{matrix}, \quad \begin{matrix} 1 & 1 & 2 & 2 \\ 3 & & & \end{matrix} \right) \tag{4.5}$$

$$RS^t(\sigma^r) = \left(\begin{matrix} 1 & 1 & 2 & 3 \\ 2 & & & \end{matrix}, \quad \begin{matrix} 1 & 2 & 3 & 3 \\ 2 & & & \end{matrix} \right) \tag{4.6}$$

$$RS^t(\pi) = \left(\begin{matrix} 1 & 2 & 2 & 3 \\ 3 & & & \end{matrix}, \quad \begin{matrix} 1 & 1 & 2 & 2 \\ 3 & & & \end{matrix} \right) \tag{4.7}$$

$$RS^r(\pi^r) = \left(\begin{matrix} 1 & 2 & 2 & 3 \\ 3 & & & \end{matrix}, \quad \begin{matrix} 1 & 2 & 3 & 3 \\ 2 & & & \end{matrix} \right) \tag{4.8}$$

The same theorem holds for (k,l)-matrix words. Recall Definitions 3.3.4 and 4.0.3 in order to compute the matrix words σ^r, π and π^r. We have a new ordering $\bar{1} < \bar{2} < \ldots < \bar{l} < 1 < 2 < \ldots < k$. When computing $RS^t(\sigma^r)$ and $RS(\pi^r)$ we have to take care of this new ordering.

Example 4.0.6 *Assume the natural order. Let* $\sigma = \begin{pmatrix} 1 & 2 & 2 & \bar{1} & \bar{2} & \bar{2} \\ \bar{2} & 1 & \bar{2} & 2 & \bar{1} & 2 \end{pmatrix}$. *Then we obtain the following (k,l)-matrix words according to the above description.*

$$\sigma^r = \begin{pmatrix} \bar{1} & \bar{1} & \bar{2} & 1 & 1 & 2 \\ 2 & \bar{1} & 2 & \bar{2} & 1 & \bar{2} \end{pmatrix} \quad \pi = \begin{pmatrix} 1 & 2 & 2 & \bar{1} & \bar{2} & \bar{2} \\ \bar{1} & 2 & \bar{1} & 1 & \bar{2} & 1 \end{pmatrix} \quad \pi^r = \begin{pmatrix} \bar{1} & \bar{1} & \bar{2} & 1 & 1 & 2 \\ 1 & \bar{2} & 1 & \bar{1} & 2 & \bar{1} \end{pmatrix}$$

$$RS^r(\sigma) = \left(\begin{array}{lll} 1 & 2 & 2 \\ \bar{1} & & \\ \bar{2} & & \\ \bar{2} & & \end{array}, \begin{array}{lll} 1 & 2 & \bar{2} \\ 2 & & \\ \bar{1} & & \\ \bar{2} & & \end{array}\right)$$

$$RS^t(\sigma^r) = \left(\begin{array}{lll} 1 & 2 & 2 \\ \bar{1} & & \\ \bar{2} & & \\ \bar{2} & & \end{array}, \begin{array}{lll} \bar{1} & \bar{2} & 1 \\ \bar{1} & & \\ 1 & & \\ 2 & & \end{array}\right)$$

$$RS^t(\pi) = \left(\begin{array}{lll} \bar{1} & 1 & 2 \\ \bar{1} & & \\ \bar{2} & & \\ 1 & & \end{array}, \begin{array}{lll} 1 & 2 & \bar{2} \\ 2 & & \\ \bar{1} & & \\ \bar{2} & & \end{array}\right)$$

$$RS^r(\pi^r) = \left(\begin{array}{lll} \bar{1} & 1 & 2 \\ \bar{1} & & \\ \bar{2} & & \\ 1 & & \end{array}, \begin{array}{lll} \bar{1} & \bar{2} & 1 \\ \bar{1} & & \\ 1 & & \\ 2 & & \end{array}\right)$$

Before we turn to the actual proof of the theorem, we need to make a trivial generalization of the construction describing the computation of S by means of a doubly indexed family of partitions. In that construction the sequences of partitions obtained by varying one index were chains, without repetitions. When combining with the Robinson-Schensted algorithm however, it will be useful to be able to deal with sequences with repetitions as well. To that end it suffices to allow, in addition to the configuration of type S1 and S2, a third type of configurations, that is like type 0-RS 2 except that the inclusions are reversed: define $\left(\begin{smallmatrix}\kappa & \lambda \\ \mu & \nu\end{smallmatrix}\right)$ to be of type S0 if and only if $\left(\begin{smallmatrix}\nu & \mu \\ \lambda & \kappa\end{smallmatrix}\right)$ is of type 0-RS 2. This simply allows a number of consecutive rows or columns to be replicated: when $\lambda^{[0,j]} = \lambda^{[0,j+1]} = \cdots = \lambda^{[0,k]}$ one will have $\lambda^{[i,j]} = \lambda^{[i,j+1]} = \cdots = \lambda^{[i,k]}$ for all i, as well as $\lambda^{[n-k,l]} = \lambda^{[n-k+1,l]} = \cdots = \lambda^{[n-j,l]}$ for all l; the reflection at the antidiagonal $i + j = n$ occurs because $\lambda^{[n-k,j]} = (0)$, although $(n-k, j)$ lies above the antidiagonal. Therefore the generalization is a very simple one: if for the sequence along the top edge one chooses a representing tableau P for which repetitions in the sequence correspond (as they did in the family of partitions used for the Robinson-Schensted algorithm) to entries that do not occur in P, then $S(P)$ similarly represents the sequence along the left edge. In particular, if we start out without any repetitions along one of these edges, then the new configurations will not occur anywhere, and we just get the original construction. Furthermore, define transposed configurations type 0^t-RS 2 type 3^t-RS 2 like type 0-RS 2-type 3-RS 2, but replacing ρ^+ by τ^+ (type 2^t-RS 2 and type 3^t-RS 2 are identical to type 2-RS 2 and type 3-RS 2, respectively).

Proof: Our proof will establish the equivalence of (4.1) and (4.2), which as we have seen is sufficient, in combination with results obtained earlier, to prove the remaining equivalences. The current proof will involve a family of partitions with three indices.

The idea is to define this family in such a way that by fixing one of the three indices we obtain doubly indexed families that correspond respectively to an application of RS 2 to a truncation of σ, an application of S (with repetitions allowed along the edges, as was just described), and an application of RS^t to a truncation of π. Fixing the index to the maximal value, n, will give the computations of respectively $RS(\sigma), S(Q)$, and $RS^t(\pi)$; fixing two indices of n will give respectively ch$P$, ch$Q$ and ch$S(Q)$. The existence of a family meeting these requirements will therefore prove the theorem, and the local laws for the doubly indexed families are sufficient to completely determine all values of the triply indexed family; in fact they overspecify this family, and the essential thing to prove is that the various ways in which the same partition can be determined are consistent with each other.

So we shall prove that there exists a family of partitions $\lambda^{[i,j,k]}$ for $0 \leq i,j,k \leq n$ and $i+k \geq n$ satisfying the following conditions.

(a) We have $\lambda^{[i,j,k]} = (0)$ whenever $j = 0$ or $i+k = n$.

(b) For $i+k > n$ and $j > 0$ the configuration $\left(\begin{smallmatrix} \lambda^{[i-1,j-1,k]} & \lambda^{[i-1,j,k]} \\ \lambda^{[i,j-1,k]} & \lambda^{[i,j,k]} \end{smallmatrix}\right)$ is of one of the types 0-RS 2 to 3-RS 2 and it is of type 1-RS 2 if and only if $j = \sigma_i$.

(c) For $i+k > n$ and $j > 0$ the configuration $\left(\begin{smallmatrix} \lambda^{[i,j-1,k-1]} & \lambda^{[i,j,k-1]} \\ \lambda^{[i,j-1,k]} & \lambda^{[i,j,k]} \end{smallmatrix}\right)$ is of one of the types 0^t-RS 2 to 3^t-RS 2, and it is of type 1^t-RS 2 if and only if $j = (\sigma^r_k)$ or equivalently $j = \sigma_{n+1-k}$.

(d) For $i+k > n+1$ the configuration $\left(\begin{smallmatrix} \lambda^{[i,j,k]} & \lambda^{[i-1,j,k]} \\ \lambda^{[i,j,k-1]} & \lambda^{[i-1,j,k-1]} \end{smallmatrix}\right)$ is of one of the types S0-S2.

Once the existence of the family of partitions $\lambda^{[i,j,k]}$ is established, the equivalence of (4.5) and (4.7) follows , since we have $chP = (\lambda^{[n,n,n]}, \lambda^{[n,n-1,n]}, \ldots, \lambda^{[n,0,n]})$, $chQ = (\lambda^{[n,n,n]}, \lambda^{[n-1,n,n]}, \ldots, \lambda^{[0,n,n]})$, and $chS(Q) = (\lambda^{[n,n,n]}, \lambda^{[n,n,n-1]}, \ldots, \lambda^{[n,n,0]})$. The existence proof is by induction on the triple (i,j,k): using the coordinatewise partial ordering on $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$, the induction hypothesis is that the partitions have been defined and the conditions established for all smaller triples. For $i+k = n+1$ we either have $j = \sigma_i = \sigma_{n+1-k}$, in which case $\lambda^{[i,j-1,k]} = (0)$ and $\lambda^{[i,j,k]} = (1)$, or $j \neq \sigma_i = \sigma_{n+1-k}$, in which case $\lambda^{[i,j-1,k]} = \lambda^{[i,j,k]}$; both cases satisfy (b) and (c).

In the remaining cases any subset of the indices i,j,k can be decreased by 1 , giving another valid index triple; to avoid unwieldy expressions we give names to the 8 partitions selected by such triples. Put $\kappa = \lambda^{[i-1,j-1,k-1]}, \lambda = \lambda^{[i,j-1,k-1]}, \mu = \lambda^{[i-1,j-1,k]}, \nu = \lambda^{[i,j-1,k]}$, and the same with primes when using j instead of $j-1$: $\kappa' = \lambda^{[i-1,j,k-1]}, \lambda' = \lambda^{[i,j,k-1]}, \mu' = \lambda^{[i-1,j,k]}, \nu' = \lambda^{[i,j,k]}$. These partitions can be arranged in an obvious way on the 8 vertices of a cube, and our conditions correspond to the 6 faces of that cube: the configurations $\left(\begin{smallmatrix} \nu & \mu \\ \lambda & \kappa \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \nu' & \mu' \\ \lambda' & \kappa' \end{smallmatrix}\right)$ should be of type S0-S2, the configurations $\left(\begin{smallmatrix} \kappa & \kappa' \\ \lambda & \lambda' \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mu & \mu' \\ \nu & \nu' \end{smallmatrix}\right)$ of type 0 to 3-RS 2, and configurations $\left(\begin{smallmatrix} \kappa & \kappa' \\ \mu & \mu' \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \lambda & \lambda' \\ \nu & \nu' \end{smallmatrix}\right)$ of type 0^t to 3^t-RS 2; of each of these pairs the first may be assumed to be of the required type by induction. The partitions κ, λ, ν and κ' together with knowledge of whether either of the conditions $j = \sigma_i$ and $j = (\sigma^r)_k$ holds, determine all the other partitions; in a case-by-case analysis we shall verify that they always do so consistently.

1. $j = \sigma_i$. Since $i + k > n + 1$, this excludes the possibility $j = (\sigma^r)_k$. We have $\kappa = \lambda = \kappa'$, and $\lambda' = \rho^+(\lambda)$; if $\nu = \lambda$, then the configuration $\left(\begin{smallmatrix}\mu & \mu' \\ \nu & \nu'\end{smallmatrix}\right)$ coincides with $\left(\begin{smallmatrix}\kappa & \kappa' \\ \lambda & \lambda'\end{smallmatrix}\right)$, and all conditions are satisfied. Otherwise $\nu \in \lambda^+$, $\mu = \nu = \mu'$ and $\nu' = \rho^+(\nu)$, and we must check that $\left(\begin{smallmatrix}\lambda & \lambda' \\ \nu & \nu'\end{smallmatrix}\right)$ is of type 2^t or 3^t and that $\left(\begin{smallmatrix}\nu' & \mu' \\ \lambda' & \kappa'\end{smallmatrix}\right) = \left(\begin{smallmatrix}\nu' & \nu \\ \lambda' & \lambda\end{smallmatrix}\right)$ is of type S1 or S2. From $\nu \in \lambda^+$ we get $\nu' = \lambda^+$ (the first part is increased in both cases), so that we have at least the proper inclusions. Now if $\nu' \neq \lambda'$ then types 3^t and S2 apply respectively; if on the other hand $\nu = \lambda'$, then $\nu' = \rho^+(\rho^+(\lambda))$, and we have types 2^t and S1, since for all partitions λ we have $\tau^+(\rho^+(\lambda), \lambda) = \rho^+(\rho^+(\lambda))$, and clearly the squares added by the successive applications of ρ^+ are adjacent.

2. $j = (\sigma^r)_k$. This case follows by symmetry from (1), namely by simultaneously interchanging the first and third index of the family and σ and σ^r, and transposing all partitions and squares.
   Having dealt with $j = \sigma_i$ and $j = (\sigma^r)_k$, the remaining cases where κ is equal to one of λ, μ and κ' are easy, since two opposite faces of the cube become equal and all conditions are satisfied. Hence we assume that this is not the case, and for convenience define the squares $p = \lambda - \kappa, q = \nu - \lambda$ and $r = \kappa' - \kappa$. The remaining cases are distinguished by the possibilities for the relative positions of p, q and r.

3. $r \notin \{p, q\}$. In this case all primed partitions are obtained from the unprimed ones by adding the square r, so $\left(\begin{smallmatrix}\nu' & \mu' \\ \lambda' & \kappa'\end{smallmatrix}\right)$ has the same type as $\left(\begin{smallmatrix}\nu & \mu \\ \lambda & \kappa\end{smallmatrix}\right)$, and the remaining faces have types 2 and s^t.

4. $r = p \not\parallel q$. Then we have $\mu = \kappa + q, \kappa' = \lambda = \kappa + p$ and $\mu' = \nu = \mu + p = \lambda + q$, so both $\left(\begin{smallmatrix}\kappa & \kappa' \\ \lambda & \lambda'\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\mu & \mu' \\ \nu & \nu'\end{smallmatrix}\right)$ have type 2: we have $\lambda' = \rho^+(\lambda, \kappa)$ and $\nu‘ = rho^+(\nu, \mu)$. Now we make a further distinction:

   (4a) $\lambda' = \nu$, which occurs if and only if q lies in the row following that of p. Now $\nu' = \nu + s$, where s is the cocorner of ν in the same row as q; since q is a corner of ν, we have a horizontally adjacent pair of squares $q \parallel s$. Furthermore, $\left(\begin{smallmatrix}\lambda & \lambda' \\ \nu & \nu'\end{smallmatrix}\right)$ is of type s^t, so we should have $\tau^+(\nu, \lambda) = \nu' = \nu + s$; indeed s is a cocorner of ν in the column following that of q. Also, $q \parallel s$ implies that $\left(\begin{smallmatrix}\nu' & \mu' \\ \lambda' & \kappa'\end{smallmatrix}\right) = \left(\begin{smallmatrix}\nu' & \nu \\ \nu & \lambda\end{smallmatrix}\right)$ is of type S1.

   (4b) $\lambda' \neq \nu$. Then since the cocorner t of λ in the row following that of p is distinct from 1, it is also a cocorner of ν, so $\lambda' = \lambda + t$ and $\nu' = \nu + t$, and $\left(\begin{smallmatrix}\lambda & \lambda' \\ \nu & \nu'\end{smallmatrix}\right)$ is of type 3^t. Furthermore, $\left(\begin{smallmatrix}\nu' & \mu' \\ \lambda' & \kappa'\end{smallmatrix}\right) = \left(\begin{smallmatrix}\nu' & \nu \\ \lambda' & \lambda\end{smallmatrix}\right)$ is of type S2.

5. $p \not\parallel q = r$. This case follows by symmetry from case (4).

6. $r = p \parallel q$, and p and q lie in the same row. Then $\lambda = \mu = \kappa' = \kappa + p$, and $\mu' = \tau^+(\lambda, \kappa) = \lambda + q = \nu$, so both $\binom{\kappa\ \kappa'}{\lambda\ \lambda'}$ and $\binom{\mu\ \mu'}{\nu\ \nu'}$ have type 2-RS 2: we have $\lambda' = \rho^+(\lambda, \kappa)$, and $\nu' = \rho^+(\nu, \mu)$. Now the cocorner t in the row following that of p is also the cocorner of ν in the row following that of q ,so $\lambda' = \lambda + t$ and $\nu' = \nu + t$, and $\binom{\lambda\ \lambda'}{\nu\ \nu'}$ is of type 3^t. Furthermore, $\binom{\nu'\ \mu'}{\lambda'\ \kappa'} = \binom{\nu'\ \nu}{\lambda'\ \lambda}$ is of type 2-RS 2.

7. $r = p \parallel q$, and p and q lie in the same column. This case follows by symmetry from case (6).

This completes all possible cases. $\square$

Chapter 5

(k, l)-hook Schur function identities

The reader who is not familiar with symmetric functions can consult [18]. Berele and Remmel [22] proved a number of symmetric function identities for the (k, l)-hook Schur function of shape λ, $HS_\lambda(x_1, \ldots, x_k; y_1, \ldots, y_l) = HS_\lambda(\mathbf{x}, \mathbf{y})$ (see Definition 5.0.7 below), by partly algebraic and combinatorial means.
Let us have an alphabet $A = \{1, \ldots, k, \bar{1}, \ldots, \bar{l}\}$ with k different regular numbers and l different barred numbers and an arbitrary ordering of those numbers. We let $\mathcal{T}_{(k,l)}(\lambda/\mu)$ denote the set of all (k, l)-semistandard tableaux of shape λ/μ. Thus $\mathcal{T}_{(k,0)}(\lambda/\mu)$ is the set of all column strict tableau of shape λ/μ filled with regular numbers from $\{1, \ldots, k\}$. Note that, $\mathcal{T}_{(k,0)}(\lambda/\mu) \subset \mathcal{T}_{(k,l)}(\lambda/\mu)$. Given $T \in \mathcal{T}_{(k,l)}(\lambda)$ or for that matter any filling T of F_λ with the numbers from $\{1, \ldots, k, \bar{1}, \ldots, \bar{l}\}$, we define the type of T as $\tau(T) = (\upsilon_1, \ldots, \upsilon_k; \mu_1, \ldots, \mu_l)$ where υ_i = number of times i occurs in T and μ_j = number of times $\bar{j}$ occurs in T. We define the weight of such a T by $\omega(T) = x_1^{\upsilon_1} \ldots x_k^{\upsilon_k} y_1^{\mu_1} \ldots y_l^{\mu_l}$. For example, for T in Figure 5.1, let us assume the natural order. Then $\tau(T) = (2, 2, 1; 2, 1, 1, 2)$ and $\omega(T) = x_1^2 x_2^2 x_3 y_1^2 y_2 y_3 y_4^2$.

Figure 5.1:

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & \bar{4} \\ \hline 2 & 2 & \bar{3} & \bar{4} \\ \hline \bar{1} & \bar{2} \\ \cline{1-2} \bar{1} \\ \cline{1-1} \end{array}$$

Definition 5.0.7 *We can then define the (k, l)-hook Schur function $HS_\lambda(\mathbf{x}, \mathbf{y})$ by*

$$HS_\lambda(x_1, \ldots, x_k; y_1, \ldots, y_l) = \sum_{T \in \mathcal{T}_{(k,l)}(\lambda)} \omega(T)$$

and the (k,l)-hook Schur function $HS_{\lambda/\mu}(x_1,\ldots,x_k;y_1,\ldots,y_l)$ by

$$HS_{\lambda/\mu}(x_1,\ldots,x_k;y_1,\ldots,y_l)=\sum_{T\in\mathcal{T}_{(k,l)}(\lambda/\mu)}\omega(T).$$

The notions of (k,l)-semistandard skew tableaux and (k,l)-hook Schur functions are the obvious extensions of their definitions to skew shapes λ/μ. The reason for calling the HS_λ's (k,l)-hook Schur functions is due to the fact that $HS_\lambda(x_1,\ldots,x_k;y_1,\ldots,y_l)=0$ unless the shape λ fits inside the (k,l)-hook consisting of all cells in the first k rows and l columns. We should also note that $(k,0)$-hook Schur functions (resp. skew Schur functions) are simply the usual Schur functions $S_\lambda(x_1,\ldots,x_k)$ (resp. skew Schur functions $S_{\lambda/\mu}(x_1\ldots,x_k)$). Moreover it is clear that $HS_{\lambda/\mu}(\mathbf{x},\mathbf{y})$ can be expressed in terms of skew Schur functions in the $x's$ and the $y's$ by

$$HS_{\lambda/\mu}(\mathbf{x},\mathbf{y})=\sum_{\mu\leq\beta\leq\lambda}S_{\beta/\mu}(\mathbf{x})S_{\lambda'/\beta'}(\mathbf{y})$$

where for a partition λ, λ' denotes the conjugate partition corresponding to λ, i. e., the partition induced by the columns of F_λ. The analogue of the Cauchy identity looks like

$$\sum_\lambda HS_\lambda(\mathbf{x},\mathbf{s})HS_\lambda(\mathbf{y},\mathbf{t})=\prod_{i,j}^{k_1,k_2}\frac{1}{1-x_iy_j}\prod_{i,j}^{l_1,l_2}\frac{1}{a-s_it_j}\prod_{i,j}^{k_1,l_2}(1+x_it_j)\prod_{i,j}^{k_2,l_1}(1+y_is_j)\quad(5.1)$$

To prove the analogue of the Cauchy identity, Berele and Remmel [22] generalized the Knuth correspondence [13] via the RS 2 correspondence.

Let $\mathcal{M}^{r,c}(k_1,l_1,k_2,l_2)$ denote the set of all $(k_1+l_1)\times(k_2+l_2)$ matrices $M=|M_{i,j}|$ where M has row sums $r=(r_1,\ldots,r_{k_2};r_{k_2+1},\ldots,r_{k_2+l_2})$ and column sums $c=(c_1,\ldots,c_{k_1};c_{k_1+1},\ldots,c_{k_1+l_1})$ such that

(i) $M_{i,j}$ is a nonnegative integer if either $i\leq k_1$ and $j\leq k_2$ or $k_1<i\leq k_1+l_1$ and $k_2<j\leq k_2+l_2$ and

(ii) $M_{i,j}\in\{0,1\}$, otherwise.

Given a matrix $M\in\mathcal{M}(k_1,l_1,k_2,l_2)$, we associate a word $\sigma(M)$ of biletters with M as follows. First we label the columns of M from left ot right with $1,\ldots,k_1,\bar{1},\ldots,\bar{l}_1$ and the rows of M from top to bottom with $1,\ldots,k_2,\bar{1},\ldots,\bar{l}_2$. Then for each entry $M_{i,j}$ of M, $\sigma(M)$ will contain $M_{i,j}$ biletters of the form $\binom{p}{q}$, where p is the label of the ith row and q is the label of the jth column of M. Finally the biletters of $\sigma(M)$ are arranged so that the top halves increase; for those biletters of the form $\binom{i}{z}$ with i a regular number, the bottom halves form a (k_1,l_1)-increasing sequence of type 2, and for those biletters of the form $\binom{\bar{i}}{z}$ with $\bar{i}$ a barred number, the bottom halves form a (k_1,l_1)-decreasing sequence of type 2. It is easy to check that the conditions on M ensure that the biletters of $\sigma(M)$ can always be arranged in such a manner.

Example 5.0.8 *Assume the natural order. We compute $\sigma(M)$ for $M \in \mathcal{M}^{r,c}(3,2,2,2)$, where $r = (3,4,4,3)$ and $c = (3,3,2,2,4)$.*

$$
\begin{array}{cccccc}
 & 1 & 2 & 3 & \bar{1} & \bar{2} \\
 & & & & & \\
M = & 1 & 1 & 1 & 0 & 0 & 1 \\
 & 2 & 0 & 0 & 1 & 1 & 2 \\
 & 0 & 1 & 0 & 1 & 2 & \bar{1} \\
 & 0 & 1 & 1 & 0 & 1 & \bar{2}
\end{array}
$$

$\sigma(M) = \binom{1}{1}\binom{1}{2}\binom{1}{3}\binom{2}{1}\binom{2}{1}\binom{2}{\bar{1}}\binom{2}{\bar{2}}\binom{\bar{1}}{\bar{2}}\binom{\bar{1}}{\bar{2}}\binom{\bar{1}}{\bar{1}}\binom{\bar{1}}{2}\binom{\bar{2}}{\bar{2}}\binom{\bar{2}}{3}\binom{\bar{2}}{2}$. We define the weight of M by $\omega(M) = \prod_{i=1}^{n} \omega\binom{a_i}{b_i}$ where $\omega\binom{a}{b} = x_b y_a$ if both a and b are regular, $\omega\binom{a}{b} = s_b t_a$ if both a and b are barred, $\omega\binom{a}{b} = x_b t_a$ if b is regular and a is barred, and finally $\omega\binom{a}{b} = s_b y_a$ if b is barred and a is regular. For instance, the weight of the matrix M in our example is $\omega(M) = x_1^3 x_2^3 x_3^2 s_1^2 s_2^4 y_1^3 y_2^4 t_1^4 t_2^3$. Then it is not difficult to see that the generating function of matrices $\mathcal{M}(k_1, l_1, k_2, l_2)$ by weight is just

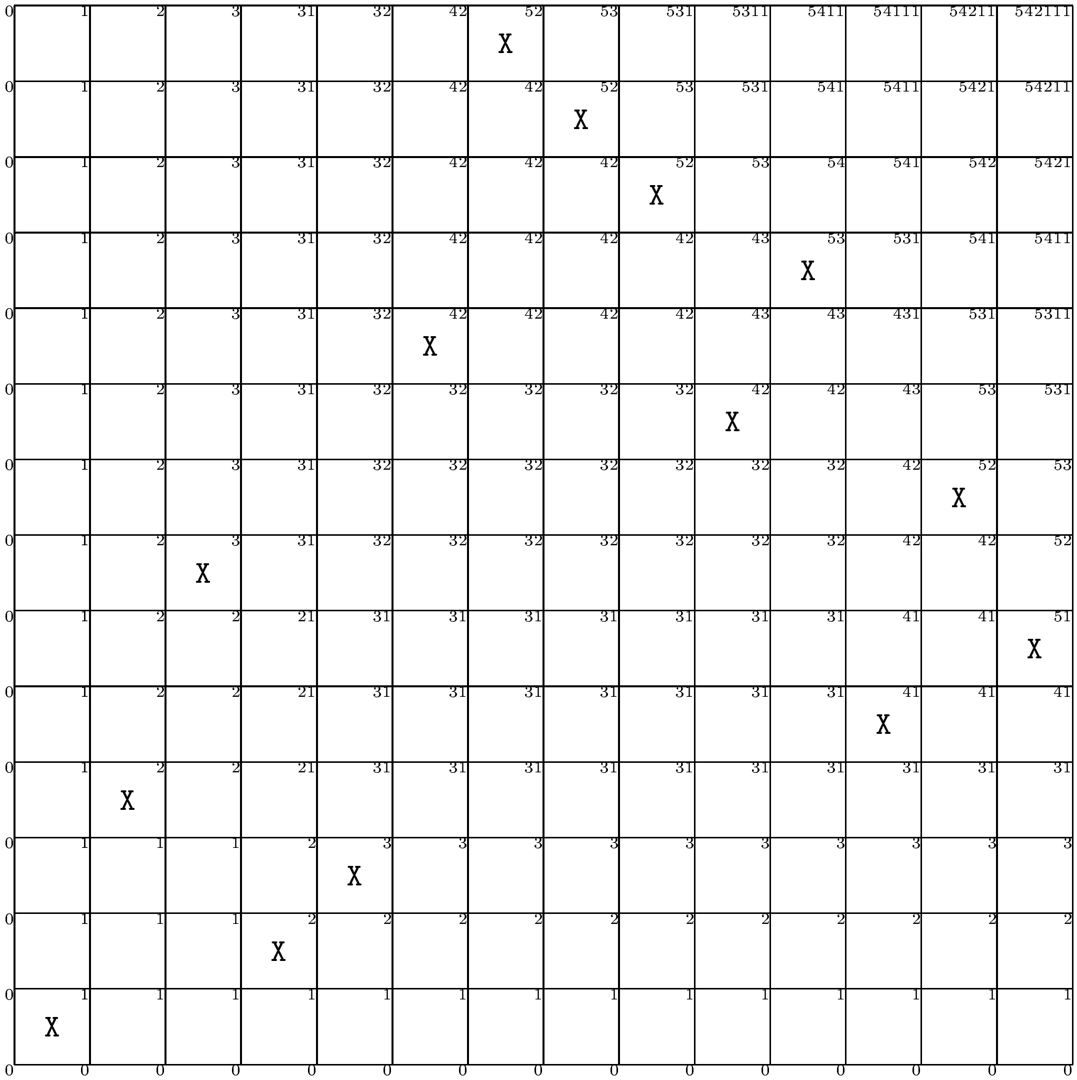
$$\sum_{M \in \mathcal{M}(k_1,l_1,k_2,l_2)} \omega(M) = \prod_{i,j=1}^{k_1,k_2} \frac{1}{1 - x_1 y_j} \prod_{i,j=1}^{l_1,l_2} \frac{1}{1 - s_i t_j} \prod_{i,j=1}^{k_1,l_2} (1 + x_i t_j) \prod_{i,j=1}^{l_1,k_2} (1 + s_i y_i) \qquad (5.2)$$

Since (5.2) gives a combinatorial interpretation to the right hand side of (5.1), it immediately follows that a bijective proof of (5.1) is given by the following theorem.

Theorem 5.0.9 *For any k_1, k_2, l_1, l_2, r, c, there is a weight preserving bijection ϕ between $\mathcal{M}^{r,c}(k_1, k_2, l_1, l_2)$ and (k_1, l_1)-semistandard tableaux P and $\tau(P) = r$ and (k_2, l_2)-semistandard tableaux Q and $\tau(Q) = c$. The bijection ϕ is an immediate application of the RS 2 correspondence.*

Let us rewrite $\sigma(M)$ to $\pi = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{2} & \bar{2} & \bar{2} \\ 1 & 2 & 3 & 1 & 1 & \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{1} & 2 & \bar{2} & 3 & 2 \end{pmatrix}$. Then we obtain via RS 2 insertion the following tableaux (P, Q) and via Fomin's approach we have the following diagram.

$$
P = \begin{array}{ccccc}
1 & 1 & 1 & 2 & 2 \\
2 & 3 & 3 & \bar{2} & \\
\bar{1} & \bar{2} & & & \\
\bar{1} & & & & \\
\bar{2} & & & & \\
\bar{2} & & & &
\end{array}
\qquad
Q = \begin{array}{ccccc}
1 & 1 & 1 & 2 & 2 \\
2 & 2 & \bar{1} & \bar{2} & \\
\bar{1} & \bar{2} & & & \\
\bar{1} & & & & \\
\bar{1} & & & & \\
\bar{2} & & & &
\end{array}
$$



The fact that P has the desired properties is immediate from the basic properties of RS 2 insertion and the facts that Q satisfies the (k_2, l_2)- semistandard conditions and that the steps of the bijection may be reversed follows from Lemma 1.2.6.

Chapter 6

A Knuth oscillating correspondence

Definition 6.0.10 *The skew shape λ/μ is called a* horizontal strip *if it has at most one cell in a column, and a* vertical strip *if it has at most one cell in a row.*

Definition 6.0.11 *An oscillating (k,l)-semistandard tableau of length $2k$ is a sequence of shapes*

$${}_{\lambda}S_{\mu}^{2k} := (\lambda = \mu^0 \subseteq \mu^1 \supseteq \mu^2 \subseteq \ldots \subseteq \mu^{2k-1} \supseteq \mu^k = \mu)$$

where the shapes alternately increase and decrease and any two adjacent shapes differ either by a horizontal or vertical strip. We define ${}_{\lambda}G_{\mu}(U_1, D_1, U_2, D_2, \ldots, U_k, D_k)$ to be the set of all oscillating (k,l)-semistandard tableaux of length $2k$ from shape λ to shape μ which either go up by horizontal strips of size μ_i and down by horizontal strips of size d_i or go up by vertical strips of size μ_i and down by vertical strips of size d_i and ${}_{\lambda}g_{\mu}(U_1, D_1, U_2, D_2, \ldots, U_k, D_k) := |{}_{\lambda}G_{\mu}(U_1, D_1, U_2, D_2, \ldots, U_k, D_k)|$.

Definition 6.0.12 *Let M be a symmetric $k \times k$ matrix. We define modified row- and column- sum vectors, which stop summing at the main diagonal, as follows. Set*

$$r_i = \sum_{j=1}^{i} a_{ij}, \qquad c_j = \sum_{i=j}^{k} a_{ij}.$$

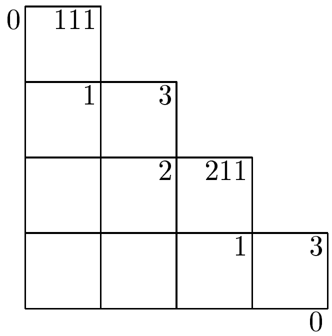
Now set $r = (r_1, r_2, \ldots, r_k)$ and $c = (c_1, c_2, \ldots, c_k)$.

Theorem 6.0.13 *Fix a positive integer k. There is a bijection between tableaux $R \in_{\emptyset} G_{\emptyset}(U_1, D_1, U_2, D_2, \ldots, U_k, D_k)$ and symmetric matrices M with nonnegative integer entries where $r = (D_1, \ldots, D_k)$ and $c = (U_1, \ldots, U_k)$.*

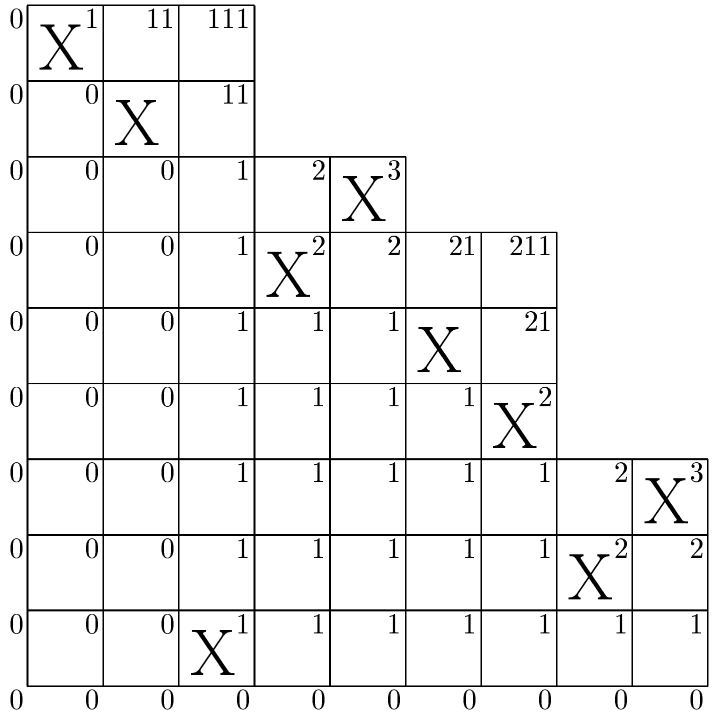
Example 6.0.14 *For an interesting example we start with*

$$R = (\emptyset, 111, 1, 3, 2, 211, 1, 3, \emptyset),$$

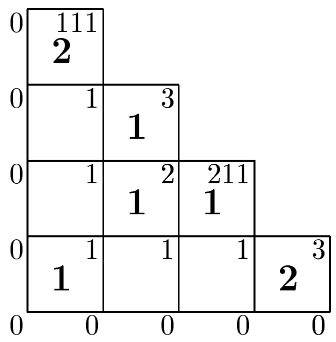
which we initially represent as



We refine it as usual and compute the interior to get



which we rewrite more compactly as



Reading off the entries of our last diagram we obtain the symmetric matrix

$$M = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

Note that there is a bijective correspondence between matrix words σ and matrices $M = (m_{ij})$ with nonnegative integral entries m_{ij} which is the number of times the pair $\binom{i}{j}$ appears in σ. If we put the barred numbers $\bar{1}, \bar{2}$ for vertical strips and the regular numbers $1, 2$ for horizontal strips and label the columns of M from left to right with $\bar{1}, 1, \bar{2}, 2$ and the rows of M from top to bottom with $\bar{1}, 1, \bar{2}, 2$ according to the oscillating (k, l)-semistandard tableaux R, M can also be thought of as

$$\sigma = \begin{pmatrix} \bar{1} & \bar{1} & \bar{1} & 1 & 1 & \bar{2} & \bar{2} & \bar{2} & 2 & 2 & 2 \\ \bar{1} & \bar{1} & 2 & 1 & \bar{2} & 1 & \bar{2} & \bar{2} & \bar{1} & 2 & 2 \end{pmatrix}.$$

If we start with M or σ we can construct R by drawing the appropriate diagram and reading off the oscillating (k, l)-semistandard tableau R along the diagonal. chains

Chapter 7

Bijection of (k,l)-semistandard tableaux and lattice paths

Next we want to show that the (k,l)-hook Schur functions $HS_\lambda(\mathbf{x},\mathbf{y})$ are independent of the ordering of the alphabet. In order to do this we give a combinatorial proof of the classical Jacobi-Trudi identity extended to the setting of (k,l)-hook Schur functions:

$$HS_{\lambda/\mu}(\mathbf{x},\mathbf{y}) = det\left(HS_{\hat{\lambda}_i/\hat{\mu}_i}(\mathbf{x},\mathbf{y})\right) \tag{7.1}$$

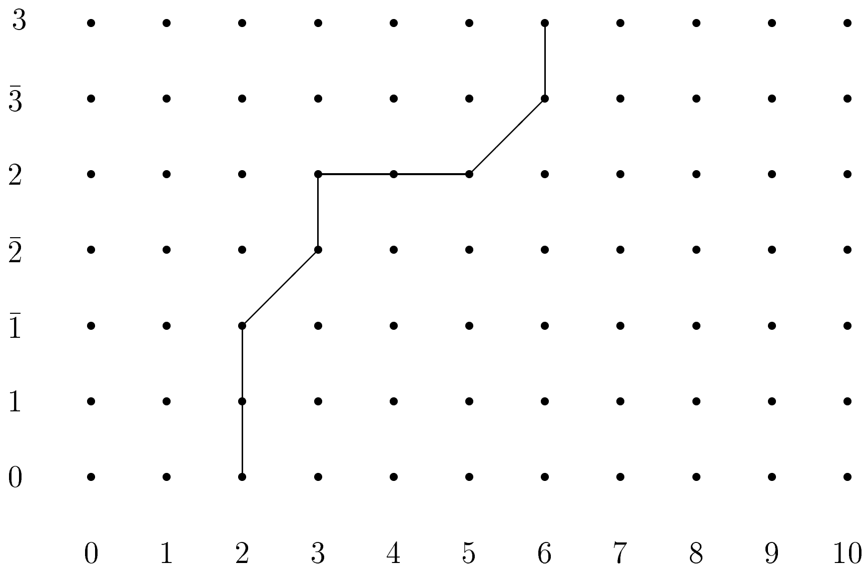
In this equation $\hat{\lambda}$ (resp. $\hat{\mu}$) denotes the partition obtained from $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ (resp. $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$) by adding $\delta = (n-1, n-2, \ldots, 0)$. So $\hat{\lambda} = \lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n + 0)$. For example, if $\lambda = (7,5,5,3)$ then $\hat{\lambda} = (10,7,6,3)$.

In the following we show that there is a bijection between the set of all (k,l)-semistandard tableaux of shape λ/μ, $\mathcal{T}_{(k,l)}, (\lambda/\mu)$ and nonintersecting paths in the grid. We consider lattice paths in the basic grid pictured in Figure 7.1. The rows are labeled according to the given ordering of the numbers. In our special case, we assume the ordering $1 < \bar{1} < \bar{2} < 2 < \bar{3} < 3$ on our alphabet $A = \{1,2,3,\bar{1},\bar{2},\bar{3}\}$. We let (s,t) denote the point in the sth row and tth column of the grid. We let P_i^j denote the collection of lattice paths connecting $(0,i)$ to (l,j) where l denotes the highest element in the given ordering. A lattice path consists of directed segments which are either vertical $\uparrow$ or horizontal $\rightarrow$ if the end points of the segments are regular numbers and which are either vertical $\uparrow$ or diagonal $\nearrow$ if their end points are barred numbers. Note that this implies P_i^j is empty unless $j \geq i$. Given a path p in P_i^j, we define the weight $\omega(p)$ of p to be the product of the weights of the individual segment of p where the weight of any vertical segment is 1, the weight of a horizontal segment ending at (s,t) is x_s, and the weight of a diagonal segment ending at $(\bar{s},t)$ is $y_{\bar{s}}$. For example, the path p pictured in Figure 7.1 is an element in P_2^6 and has weight $x_2^2 y_2 y_3$.

It is then clear from our definitions that

$$HS_{j-i}(\mathbf{x},\mathbf{y}) = \sum_{p \in P_i^j} \omega(p) \tag{7.2}$$

Figure 7.1:



where $HS_{j-i}(\mathbf{x},\mathbf{y}) = 0$ if $(j-i)$ is negative and $HS_0(\mathbf{x},\mathbf{y}) = 1$. We let

$$P_{i_1\cdots i_n}^{j_1\cdots j_n} = P_{i_1}^{j_1} \times \cdots \times P_{i_n}^{j_n}$$

and define the weight of $(p_1,\ldots,p_n)$ by $\omega(p_1,\ldots,p_n) = \prod_{k=1}^n \omega(p_k)$. By S_n we denote the set of all permutations of n. With these conventions, it is also easy to see that

Proposition 7.0.15

$$\det\left(HS_{\hat{\lambda}_i/\hat{\mu}_i}(\mathbf{x},\mathbf{y})\right) = \sum_{\sigma\in S_n} sgn(\sigma) \sum_{(p_1,\ldots,p_n)\in P_{\hat{\mu}_1\cdots\hat{\mu}_n}^{\hat{\lambda}_{\sigma(1)}\cdots\hat{\lambda}_{\sigma(n)}}} \omega(p_1,\ldots,p_n)$$

Next we let

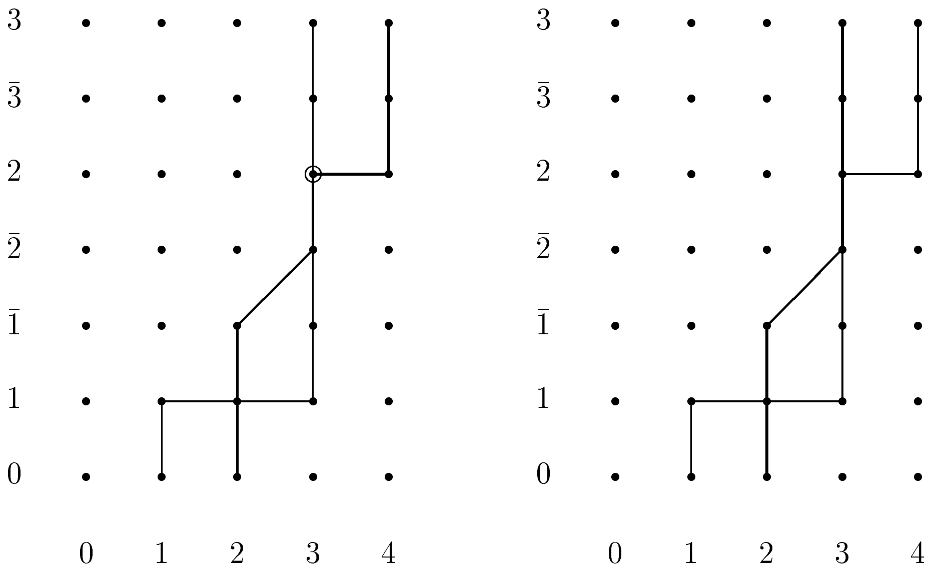
$$P_\mu^\lambda = \cup_{\sigma\in S_n} P_{\hat{\mu}_1\cdots\hat{\mu}_n}^{\hat{\lambda}_{\sigma(1)}\cdots\hat{\lambda}_{\sigma(n)}}.$$

We say an n-tuple of paths $(p_1,\ldots,p_n)$ in P_μ^λ is intersecting if two of the paths go through the same point and is nonintersecting otherwise. Let NP_μ^λ denote the set of all nonintersecting paths in P_μ^λ. The proof of (7.1) now comes down to proving two lemmas. The first will state that the right hand side of Proposition 7.0.15 reduces to the sum of the weights of the nonintersecting paths and the second will state that there is a weight preserving bijection F between $\mathcal{T}_{(k,l)}(\lambda/\mu)$ and NP_μ^λ.

Lemma 7.0.16

$$\sum_{(p_1,p_2,\ldots,p_n)\in NP_\mu^\lambda} \omega(p_1,\ldots,p_n) = \sum_{\sigma\in S_n} sgn(\sigma) \sum_{(p_1,\ldots,p_n)\in P_{\hat{\mu}_1\cdots\hat{\mu}_n}^{\hat{\lambda}_{\sigma(1)}\cdots\hat{\lambda}_{\sigma(n)}}} \omega(p_1,\ldots,p_n)$$

Figure 7.2:



Proof: We give an involution I on P_μ^λ whose fixed points are NP_μ^λ and which pairs off any intersecting n-tuple of paths $p_1, \ldots, p_n)$ with another intersecting n-tuple of paths $I(p_1, \ldots, p_n)$ with the same weight but opposite sign. The involution I is very simple. For any given intersecting n-tuple of paths $(p_1, \ldots, p_n)$, we locate the point (s,t) in the grid which is farthest to the right and the highest one which has at least two paths which go through (s,t). Then we locate the pair of paths $p - i$ and p_j with the smallest indices which go through (s,t) and swith the tails of p_i and p_j. That is $I(p_1, \ldots, p_n) = (p'_1, \ldots, p'_n)$ where $p'_k = p_k$ if $k \notin \{i,j\}$, p'_i is the path which follows p_i to (s,t) and then follows p_j and p'_j is the path that follows p_j to (s,t) and then follows p_i. Figure 7.2 gives an example of the map I.

It is easy to see that I^2 equals the identity and that I is weight preserving. Moreover, if $(p_1, \ldots, p_n) \in P_{\hat{\mu}_1 \cdots \hat{\mu}_n}^{\hat{\lambda}_{\sigma(1)} \cdots \hat{\lambda}_{\sigma(n)}}$ then $I(p_1, \ldots, p_n) \in P_{\hat{\mu}_1 \cdots \hat{\mu}_n}^{\hat{\lambda}_{\tau(1)} \cdots \hat{\lambda}_{\tau(n)}}$ where σ and τ simply differ by a transposition $(i,j) \in S_n$. Thus $(p_1, \ldots, p_n)$ have opposite signs and hence I proves the lemma. $\square$

Finally to complete the proof of (7.1), we need only prove the following.

Lemma 7.0.17

$$\sum_{T \in \mathcal{T}_{(k,l)}(\lambda/\mu)} \omega(T) = \sum_{(p_1, \ldots, p_n) \in NP_\mu^\lambda} \omega(p_1, \ldots, p_n)$$

Proof: Suppose $\lambda \geq \mu$. It is easy to see that if $(p_1, \ldots, p_n)$ is nonintersecting, then $(p_1, \ldots, p_n) \in P_{\hat{\mu}_1 \cdots \hat{\mu}_n}^{\hat{\lambda}_1 \cdots \hat{\lambda}_n}$. Now if $T \in \mathcal{T}_{(k,l)}(\lambda/\mu)$, we shall code each row of T with a path.

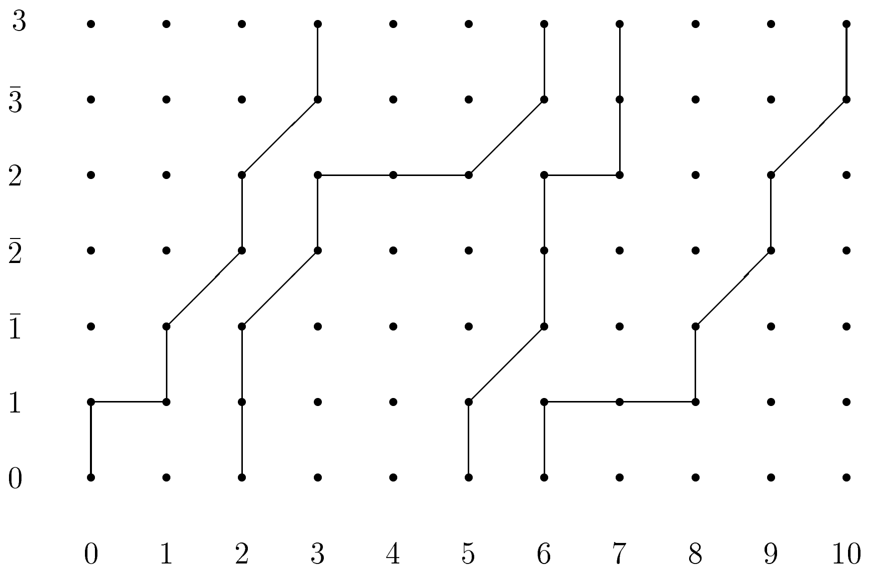
That is, the kth row of T from the top will have length $\lambda_k - \mu_k$ and we will code it by a path p_k in $P_{\hat{\mu}_k}^{\hat{\lambda}_k}$. We code the kth row of T reading from left to right by the path p_k whose horizontal segments from bottom to top respectively end at $(s_i, \hat{\mu}_k + l)$ where s_i is a regular number in the lth column of the kth row in T and whose diagonal segments from bottom to top respectively end at $(t_j, \hat{\mu}_k + l)$ where t_j is a barred number in the lth column of the kth row of T. The path p_k is then completed by adding vertical segments which start at $(0, \hat{\mu}_k)$ and connect all horizontal segments and diagonal segments above so that the path ends at $(\bar{l}, \hat{\lambda}_k)$. Then $F(T) = (p_1, \ldots, p_n)$. Figure 7.0.18 gives an example of the bijection F. It is easy to see that F is weight preserving and that we can reverse the process and start with any $(p_1, \ldots, p_n) \in NP_{\mu}^{\lambda}$ and produce a filling T of F_λ such that the kth row of T is coded by p_k so that F^{-1} is also defined. Our definitions ensure that if $T = F^{-1}(p_1, \ldots, p_n)$ then within each row of T, the entries are increasing from left to right according to the given order and there are no two equal barred numbers. Thus, the essential fact we need to check in order to prove that F is a bijection is that the nonintersecting property of $(p_1, \ldots, p_n)$ ensures that the columns of T satisfy the conditions for a (k, l)-semistandard tableau and vice versa. Let r denote the number of rows of T. Fix a column j of T, let x and y denote the elements of T in the jth column which lie in the ith row and $(i + 1)$th row of T reading from top to bottom. Thus x sits on top of y in T.
x is coded by a horizontal segment which starts at $(x, j - 1 + r - i)$ and ends at $(x, j + r - i)$ if x is a regular number. x is coded by a diagonal segment which starts at $(z, j - 1 + r - i)$ and ends at $(x, j + r - i)$ if x is a barred number. z denotes the greatest number strictly smaller than x in the given order. y is coded by a horizontal segment which starts at $(y, j - 1 + r - i - 1)$ and ends at $(y, j + r - i - 1)$ if y is a regular number. y is coded by a diagonal segment which starts at $(z, j - 1 + r - i - 1)$ and ends at $(y, j + r - i - 1)$ if y is a barred number. z is defined as above.

By considering the picture, one can see that $(p_1, \ldots, p_r)$ is nonintersecting if and only if the segment coded by y lies strictly above the segment coded by x for all such pairs x and y. But then it is a routine case by case verification for the following cases x and y are both regular, x is regular and y is barred, x is barred and y is regular or x and y are barred to see that the segment coded by y lies strictly above the segment coded by x if and only if x and y satisfy the appropriate conditions required for a (k, l)-semistandard tableau. $\square$

The definition of (k, l)-hook Schur functions (see Definition 5.0.7) $HS_{\lambda/\mu}(\mathbf{x}, \mathbf{y}) = \sum_T \omega(T)$, together with the above lemmas show that the number of (k, l)-semistandard tableaux of shape λ/μ does not depend on the ordering of the regular and barred numbers.

Example 7.0.18 *Let* $\lambda = (7, 5, 5, 3)$, $\mu = (3, 3, 1)$. *Then we obtain* $\hat{\lambda} = (10, 7, 6, 3)$ *and*

$$\hat{\mu} = (6,5,2,0). \ T = \begin{array}{ccccccc} & & & 1 & 1 & \bar{2} & \bar{3} \\ & & & \bar{1} & 2 & & \\ & \bar{2} & 2 & 2 & \bar{3} & & \\ 1 & \bar{2} & \bar{3} & & & & \end{array}$$



Chapter 8

RS-Algorithms for Dual Graded Graphs

The main result of this chapter is a generalization of the Robinson-Schensted correspondence to the class of dual graded graphs. This class extends the class of differential posets, for which a generalized Robinson-Schensted algorithms was constructed earlier in chapter 1.
As particular cases of the general construction, we give examples including the RS algorithm for skew tableaux, the Sagan-Worley-Haiman algorithm for the graph of shifted shapes and the binary tree. First of all let us develop the general theory about dual graded graphs [3, 4, 6].

8.1 Graded graphs

The terminology and notation used follows [30, 31, 27].

Definition 8.1.1 *A graded graph is a triple $G = (P, \rho, E)$ where*

- *P is a discrete set of vertices,*
- *$\rho : P \to \mathbf{Z}$ is a rank function,*
- *E is a multiset of arcs (x, y), where $\rho(y) = \rho(x) + 1$.*

The sets $P_n = \{x : \rho(x) = n \in \mathcal{Z}\}$ are called levels of G. Finiteness of levels is always assumed and some of them may be empty.
A graded graph is completely determined by the adjacency matrices of the bipartite graphs by its consecutive levels. Thus graded graphs can be studied as linear-algebraic objects.

Definition 8.1.2 *Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be a pair of graded graphs with a common set of vertices and common rank function. Define an oriented graded graph $G = (P, \rho, E_1, E_2)$ by directing the G_1-edges up (in the direction of increasing rank) and the G_2-edges down (in the direction of decreasing rank). Let K be a field of characteristic zero. The formal linear combinations of vertices over K form the vector space KP. Now it is natural to introduce the up and down operators U, D associated with the graph $G = (G_1, G_2)$ by*

$$Ux = \sum_y a_1(x, y)y$$

and

$$Dy = \sum_x a_2(x, y)x$$

where $a_i(x, y)$ is the multiplicity of the edge (x, y) in G_i.

Definition 8.1.3 *The restrictions of U and D onto subspaces KP_n are denoted U_n and D_n. Graded graphs G_1 and G_2 with a common set of vertices and common rank function are said to be dual if the following linear commutation relations hold:*

$$D_{n+1}U_n = U_{n-1}D_n + I_n$$

where the I-path is a degenerate path in the case $x = y$.

These relations mean combinatorially that

1. if x and y are different vertices of rank n then the number of DU– paths from x to y is the number of UD–paths joining x and y.

2. if x is a vertex of rank n then the number of DU–loops from x to x equals the number of UD–loops plus 1.

8.2 RS Algorithms for Skew Tableaux

In the next section we generalize the skew RS algorithm [25] to tableaux and words with two types of numbers; it contains the most general formulae from which all previous identities can be derived. Then we give an example which we solve via Fomin's approach. Finally we give a formula including (k, l)-hook Schur functions.

The Fundamental Algorithm

For skew tableaux there are two types of row insertion: external and internal.
External insertion is very much like Schensted's original procedure. Initially one has a (k,l)-semistandard tableau P of shape λ/μ and an element m to be added to P. To start, a regular (resp. barred) number m replaces the smallest p_{1j} such that $p_{1j} > m$ (resp. $p_{1j} \geq m$); in the case where m is a regular (resp. barred) number and $m \geq p_{1j}$ (resp. $m > p_{1j}$) for all j, it is placed at the row's right end. If a p_{1j} was displaced from the first row then it is inserted into the second using the same rules as above. This process continues until some element comes to rest at the end of a row. The only need for caution in the skew case is when something is to be inserted into row i which is empty . If this happens, we must have $\lambda_i = \mu_i$ and we put the element in cell $(i, \lambda_i + 1)$. If external row insertion of m into P yields P we will write $R_m(P) = P$.
With *internal row insertion*, no new element is added to the tableau P. Instead we start with an inner corner of the shape λ/μ of P which is a cell (i,j) with $(i,j) \in \lambda/\mu$ but $(i-1,j), (i,j-1) \notin \lambda/\mu$. To start the process p_{ij} is removed from row i and inserted, using the usual rules, into row $i+1$. The insertion then continues in a normal fashion, ending with an element that occupies a cell at the end of some row. The internal row insertion operator will be denoted R_{ij}.

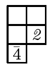
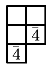
Since the objects we will be dealing with are now multisets (sets with repetition) A, B, etc., we will define $A \cup B$ to be the multiset where the number of repetitions of an element k is the sum of the number of copies of k in A and B.

Theorem 8.2.1 *Let α be a fixed partition. Then the map $(\pi, T, U) \leftrightarrow (P, Q)$ is a bijection between (k,l)-matrix words π with T, U (k,l)- semistandard tableaux of shape α/μ, on the one hand, and P, Q (k,l)-semistandard tableaux of shape λ/α, on the other, such that $\check{\pi} \cup T = P$ and $\hat{\pi} \cup U = Q$.*

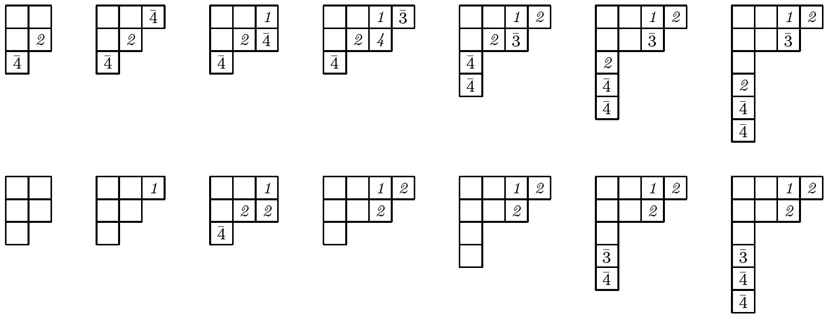
Proof: First, given (π, T, U) we construct (P, Q) as follows. Let $(P_0, Q_0) = (T, \emptyset)$, where $\emptyset$ is the unique empty tableau of shape λ/λ, i. e., Q_0 is just the Ferrers diagram for λ. Let the largest element of $\hat{\pi} \cup U$ be m. The pairs (P_k, Q_k), $k = 0, \ldots, m$ are constructed iteratively as follows. To obtain P_k internally insert all the elements of P_{k-1} corresponding to k's in Q_{k-1} followed by all the elements of $\check{\pi}$ paired with k's in $\hat{\pi}$. In both cases the insertions proceed from left to right. Placing k's in the appropriate squares of Q_{k-1} results in Q_k.

Note that when inserting an element in a row it still displaces the first entry strictly greater than itself and that each insertion during step k begins in a row at least as high as the one just previous (if any). Thus in constructing P_k each insertion comes to rest to the right of the preceeding one so that Q_k remains a (k,l)-semistandard tableaux. This fact also makes it easy to define the inverse, since we know we must begin deletion with the square whose entry is a barred number k which is nearest to the bottom or with the square of the rightmost k, k a regular number in Q_k and proceed to the left. $\square$

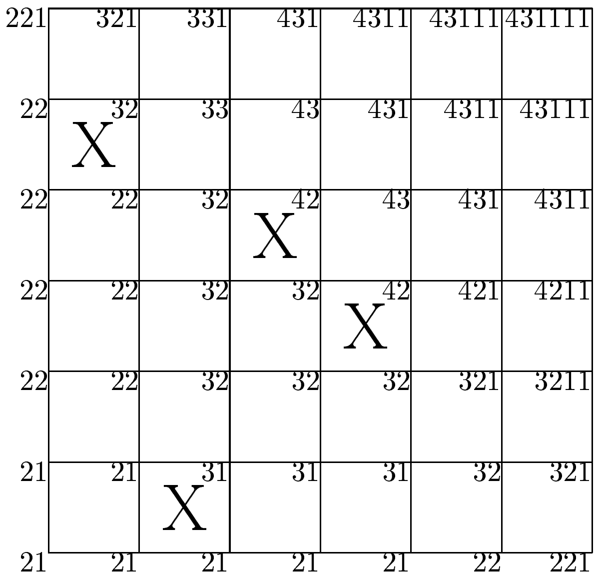
Example 8.2.2 *Let us assume the natural order. As an example of this construction let* $\pi = \begin{pmatrix} 1 & 2 & 2 & \bar{3} \\ \bar{4} & 1 & \bar{3} & 2 \end{pmatrix}$

$T =$  $\qquad U =$ 

The following list of pairs (P_k, Q_k) *is in order of increasing* k *from left to right:*



Example 8.2.3 *Now we reinterpret Example 8.2.2 using Fomin's approach. In the picture below,* P *is the right edge,* Q *the upper edge,* T *the left edge, and* U *the lower edge. The* (k, l)*-matrix word is represented by* X*'s, as usual.*



Proof: All the hard work has already been done in chapter 2. The skew tableaux P and Q represent the chains on the upper boundary, while T and U represent the lower boundary and π is represented by X's in the diagram. $\square$
The enumerative corollary of Theorem 8.2.1 is

Corollary 8.2.4 *Since (k,l)-hook Schur functions HS_λ count (k,l)-hook semistandard tableaux we have the following generalization of the famous Cauchy identity [18]. Fix a partition α. Then*

$$\sum_\lambda HS_{\lambda/\alpha}(\mathbf{x},\mathbf{s})HS_{\lambda/\alpha}(\mathbf{y},\mathbf{t}) =$$

$$\prod_{i,j}\frac{1}{1-x_iy_j}\prod_{i,j}\frac{1}{1-s_it_j}\prod_{i,j}(1+x_it_j)\prod_{i,j}(1+y_is_j)\sum_\mu HS_{\alpha/\mu}(\mathbf{x},\mathbf{s})HS_{\alpha/\mu}(\mathbf{y},\mathbf{t}).$$

As mentioned before, Knuth [13] was able to extend the Robinson-Schensted algorithm to tableaux with repetitions. If there are no barred numbers at all we have an analog of theorem 8.2.1 for matrix words and generalized tableaux.

Corollary 8.2.5 *Fix a partition α. Then*

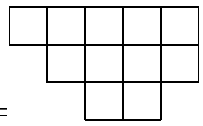
$$\sum_\lambda s_{\lambda/\alpha}(\mathbf{x})s_{\lambda/\alpha}(\mathbf{y}) = \sum_\mu s_{\alpha/\mu}(\mathbf{x})s_{\alpha/\mu}(\mathbf{y})\prod_{i,j}(1-x_iy_j)^{-1}.$$

8.3 Shifted Tableaux

There is another family of arrays, the shifted tableaux, that exhibit many of the nice properties of their "left-justified" cousins. It is therefore not surprising that skew shifted tableaux are relatively well behaved. Below we will outline the necessary definitions and notation for this case. Consult [24] or [34].

Definition 8.3.1 *A partition $\lambda = (\lambda_1,\ldots,\lambda_k)$ is strict if $\lambda_1 > \lambda_2 > \cdots > \lambda_k$. Given a strict partition the corresponding shifted Ferrers diagram λ^* has row i shifted $i-1$ spaces to the right.*

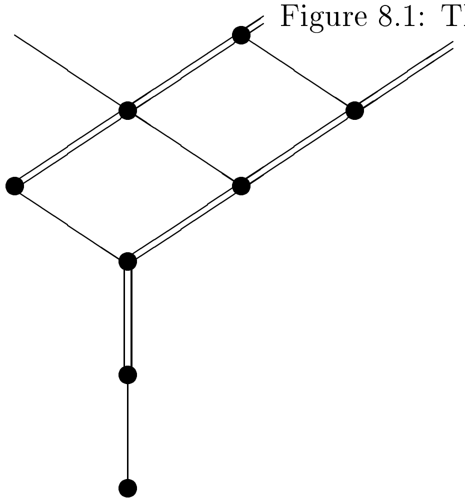
For example, if we have $\lambda = (5,4,2)$ then the shifted tableau

$P^* =$ 

The cells $(i,i) \in \lambda^*$ are called the diagonal; all other cells are off-diagonal. The number of diagonal cells is called the length of λ and denoted $l(\lambda)$. In the example above $l(\lambda) = 3$ corresponding to the diagonal squares $(1,1)$, $(2,2)$ and $(3,3)$.

Definition 8.3.2 *The vertices of the graph of shifted shapes $\mathcal{SY}$ are the Young diagrams with strictly decreasing row lengths. The dual graph for $\mathcal{SY}$ is the same graph but with the edges which correspond to adding non-diagonal cells doubled; see figure 8.1.*

Figure 8.1: The dual graph with multiple edges



We give an informal description of the shifted Schensted algorithm introduced by Sagan-Worley and Haiman.
It's necessary to be able to distinguish certain elements of a shifted tableau P^*. In previous papers [24]or [34] this was done by circling these elements. Since this is not the most convenient convention typographically, we will underline numbers which are distinguished, e. g., $\underline{k}$. The entries of our tableau are taken from the set $\{\underline{1}, 1, \underline{2}, 2, \underline{3}, 3, \ldots\}$ with the total order $\underline{1} < 1\underline{2} < 2\underline{3} < 3 < \cdots$ and the restriction that $\underline{m}$ and m cannot both occur in the tableau. If we do not wish to commit ourselves as to whether an integer is distinguished or not, we will write $\underline{k}$. Of particular interest will be the tableau, where only off-diagonal entries can be underlined.
We must now modify the definition of the insertion operators R_m and $R_{i,j}$ from section 8.2 to apply to a shifted tableau P^*. We distinguish between two insertion algorithms. If a diagonal element is never displaced during insertion then we have the usual Schensted insertion. For example

$$R_4 \left(\begin{array}{cccc} 1 & 2 & 3 & 6 \\ & 5 & 7 & \end{array} \right) = \left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ & 5 & 6 & \\ & & 7 & \end{array} \right).$$

If, at some point, p^*_{ii} is displaced then it is inserted into the $i+1$th column (replacing the smallest element larger than itself if one exists). The process continues column by column until some entry comes to rest at the lower end of a column. Since entries are moving to the right and up during column insertion, the diagonal will never be intersected again. An insertion of this type is called non-Schensted. For example, a non- Schensted internal insertion might look like

$$R_{1,4}\left(\ \right) = \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & 1 & 3 \\ \hline & \ & \ & 2 & 6 \\ \hline & & 4 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 3 & 6 \\ \hline & \ & \ & 1 & 5 \\ \hline & & 2 & 4 \\ \hline \end{array}$$

$$R_3\begin{pmatrix} 1 & 2 & 4 & 7 \\ & 5 & 6 & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 6 & 7 \\ & 4 & 5 & & \end{pmatrix}.$$

Finally, we will wish to consider internal insertions that only displace entries by columns in P^*. Let (i,j) be an inner corner of the shape $\lambda^* mu^*$ of P^*. $C_{i,j}(P^*)$ will be the tableau obtained by removing p^*_{ij} from column j and inserting it in column $j+1$, etc., until an element comes to rest at the end of some column. Such an operation will always be considered non-Schensted. By way of illustration,

$$C_{1,4}\left(\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & 1 & 3 \\ \hline & \ & \ & 2 & 6 \\ \hline & & 4 & 5 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & 1 & 3 \\ \hline & \ & \ & 2 & 6 \\ \hline & & 4 & 5 \\ \hline \end{array}$$

Theorem 8.3.3 (shifted Schensted correspondence) *There is a bijectin between permutations π of n and pairs (P,Q) of standard shifted tableaux such that sh $P =$ sh Q and Q has a subset of its off-diagonal elements underlined.*

Proof: If $\pi = x_1 x_2 \cdots x_n$ is one-line notation then we can find the corresponding pair (P,Q) using the following procedure.

Starting with P and Q both empty, we insert the elements $x_1, x_2, \ldots, x_n$ into P. After inserting x_k, a new cell (i,j) will become filled at the end of a row and column of P. We then set

$$q_{ij} = \begin{cases} k & \text{if the insertion of} x_k \text{was a Schensted move} \\ \underline{k} & \text{if the insertion of } x_k \text{was a non-Schensted move} \end{cases}$$

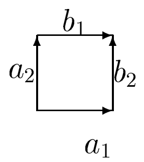
Because π is a permutation of n we are assured that P and Q are standard with n cells. Also our choice of the cell (i,j) above ensures that at every stage sh $P =$ sh Q. Finally note that a non-Schensted move must involve the bumping of a diagonal element into the next column so q_{ij} will never be underlined for any i.

Fomin's approach for the graph of shifted shapes and an R-correspondence coincide with the above described algorithm of Sagan-Worley [34] and Haiman [11].

Let $G_1 = \mathcal{SY}$. Since the dual graph G_2 does have multiple edges, the"vertex version" of the generalized Schensted algorithm does not apply and the "edge version" will only be given.

First of all, to describe an R-correspondence, color the edges of $\mathcal{SY}$ in black, blue and red as follows. Each edge a adds a cell B to a certain shifted shape λ. If B lies on the main diagonal then a is colored black. Otherwise color one of the edges which add B to λ blue and another one red (this corresponds to two kinds of entries in the usual shifted tableau as described above). Now we can define an R-correspondence Φ for the dual pair of graphs $(G_1, G_2) = G$.

Lemma 8.3.4 *Let α be a generalized permutation of the cells of the diagram, The edges a_1, $b_1 \in E_1$ are oriented up and a_2, $b_2 \in E_2$ are oriented down. We have the following*



picture: *We define the functions Φ and $\Psi = \Phi^{-1}$ as follows: For Φ we have the following four cases:*

1. *b_1 and b_2 add different cells $\Rightarrow$*
   a_1 and a_2 add the same cells as b_1 and b_2, respectively;
   a_2 has the same color as b_2; $\alpha := 0$;

2. *b_1 and b_2 add a cell into the kth row, $k \geq 2$, b_2 is blue or black $\Rightarrow$*
   a_1 and a_2 add a cell into the $(k-1)$st row; a_2 is blue; $\alpha := 0$;

3. *b_1 and b_2 add a cell into the first row, b_2 is blue or black $\Rightarrow$*
   a_1 and a_2 are degenerate; $\alpha := 1$;

4. *b_1 and b_2 add a cell into the kth column, b_2 is red $\Rightarrow$*
   a_1 and a_2 add a cell into the $(k-1)$st column; a_2 is red or black; $\alpha := 0$;

For Ψ we have:

1. *a_1 and a_2 add different cells $\Rightarrow$*
   b_1 and b_2 add the same cells as a_1 and a_2, respectively;
   b_2 has the same color as a_2;

2. *a_1 and a_2 add the same cell, a_2 is blue $\Rightarrow$*
   b_1 and b_2 add a cell into the next row; b_2 is blue or black;

3. *a_1 and a_2 add the same cell, a_2 is red or black $\Rightarrow$*
   b_1 and b_2 add a cell into the next column; b_2 is red;

4. *a_1 and a_2 are degenerate, $\alpha = 1 \Rightarrow$*
   b_1 and b_2 add a cell into the first row; b_2 is blue or black;

Proof: It is straightforward to verify that these procedures are well-defined and that Φ and Ψ are inverse bijections between appropriate sets.

Example 8.3.5 *Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$. We have the following diagram:*

0	1	2	21	X 31	32
0	1	X 2	21	21	31
0	1	1	2	2	X 3
0	1	1	X 2	2	2
0	X 1	1	1	1	1
0	0	0	0	0	0

Now we apply the algorithm of Sagan-Worley as described above. For the P and Q tableau we get:

$$P_k: \quad \begin{array}{c} 1 \end{array} \qquad \begin{array}{cc} 1 & 4 \end{array} \qquad \begin{array}{cc} 1 & 2 \\ 4 & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 5 \\ 4 & & \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & \end{array} \tag{8.1}$$

$$Q: \quad \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \end{array} \tag{8.2}$$

When we apply the algorithm to the inverse permutation we obtain the P and Q tableaux exchanged as for the classical Robinson-Schensted correspondence. Now we give an example when a diagonal element is moved.

Example 8.3.6 *Let* $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & \underline{5} \\ 1 & 4 & 2 & 5 & \underline{3} \end{pmatrix}$. *Then we have the following diagram:*

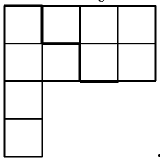
0	1	2	21	X 31	41
0	1	X 2	21	21	31
0	1	1	2	2	X 3
0	1	1	X 2	2	2
0	X 1	1	1	1	1
0	0	0	0	0	0

Now we apply the non-Schensted insertion for the underlined number as described above. For the P and Q tableaux we get:

$$P_k: \quad 1 \qquad 1\ \ 4 \qquad \begin{array}{ll} 1 & 2 \\ & 4 \end{array} \qquad \begin{array}{lll} 1 & 2 & 5 \\ & 4 & \end{array} \qquad \begin{array}{llll} 1 & 2 & \underline{3} & 5 \\ & 4 & & \end{array} \tag{8.3}$$

$$Q = \begin{array}{llll} 1 & 2 & 4 & \underline{5} \\ & 3 & & \end{array} \tag{8.4}$$

Again we obtain the P and Q tableaux exchanged by applying the algorithm to the inverse permutation.

A Greene analogue $v_\Phi(\sigma)$ for the shifted case was found by Worley [34] and Sagan [24]. It can be expressed in terms of the ordinary Greene invariant. Namely, for $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, compute the Young diagram $\lambda(\sigma_n\cdots\sigma_2\sigma_1\sigma_1\sigma_2\cdots\sigma_n)$ and partition it in halves along the main diagonal (the diagonal itself is contained in the lower part). These halves always have the same shifted shape, and this is $v_\Phi(\sigma)$. For our example we obtain  .

One last bit of terminology before stating the main result of this section. If A, B are sets of integers, some of which have been underlined, we write $A \equiv B$ to mean that A and B are the same up to underlining, e. i., $\{1, \underline{3}, \underline{4}, 5, 7\} \equiv \{\underline{1}, 3, \underline{4}, 5, \underline{7}\}$. We do not permit a set A to have both $k \in A$ and $\underline{k} \in A$.

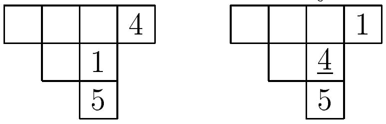
Theorem 8.3.7 *Let n be a fixed positive integer and α a fixed strict partition. Then the map*

$$(\pi, T^*, U^*) \leftrightarrow (P^*, Q^*)$$

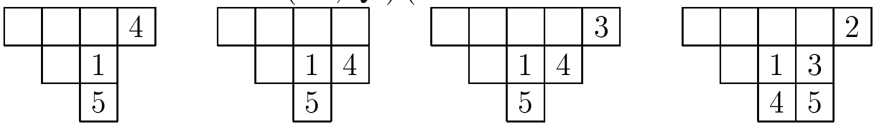
defined below is a bijection between a partial permutation π, a partial tableau T^* of shape α^*/μ^*, a partial tableau U^* of the same shape as T^* where only off-diagonal entries can be underlined such that $\breve{\pi} \cup T^* = \{1, \ldots, n\} \equiv \hat{\pi} \cup U^*$, on the one hand, and a partial tableau P^* of shape λ^*/α^*, a partial tableau Q^* of the same shape as P^* where only off-diagonal entries can be underlined.

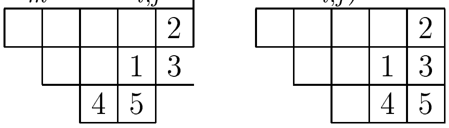
Proof: We will construct $(P_0^*, Q_0^*) = (T^*, \emptyset_\alpha^*)$, $(P_1^*, Q_1^*), \ldots, (P_n^*, Q_n^*) = (P^*, Q^*)$ as follows. At the kth step we determine whether $k \in \hat{\pi}$ or $\underset{\sim}{k} \in U^*$.
In the former case we have $\genfrac{}{}{0pt}{}{k}{m}$ in π and set $P_k^* = R_m(P_{k-1}^*)$. If the insertion is Schensted

then we place k in the corresponding position of Q^*_{k-1} to form Q^*_k. However, if it was non-Schensted then $\underline{k}$ is put in that position.
Now suppose $\tilde{k} \in U^*$ appearing, say, in cell (i, j). If $u^*_{i,} = k$ then $P^*_k = R_{i,j}(R^*_{k-1})$ with Q^*_k formed as before. If instead $u^*_{ij} = \underline{k}$ then $P^*_k = C_{i,j}(P^*_{k-1})$ with $\underline{k}$ being added to form Q^*_k.
Note that we will have $\underline{k} \in Q^*$ if and only if the kth insertion was non-Schensted, i. e., step k ended in column insertion. This guarantees that Q^* has no underlined elements on the diagonal. It also makes construction of the inverse map possible by flagging which deletions are to be started by columns rather than by rows.
Finally, we want to give an example where the lower boundary $\partial^-(S)$ of the skew diagram is not trivial. Let be $\pi = \binom{2\ 3}{3\ 2}$.



We will illustrate all five possible insertions for (P^*, Q^*) (Schensted and non-Schensted R_m and $R_{i,j}$ as well as $C_{i,j}$).





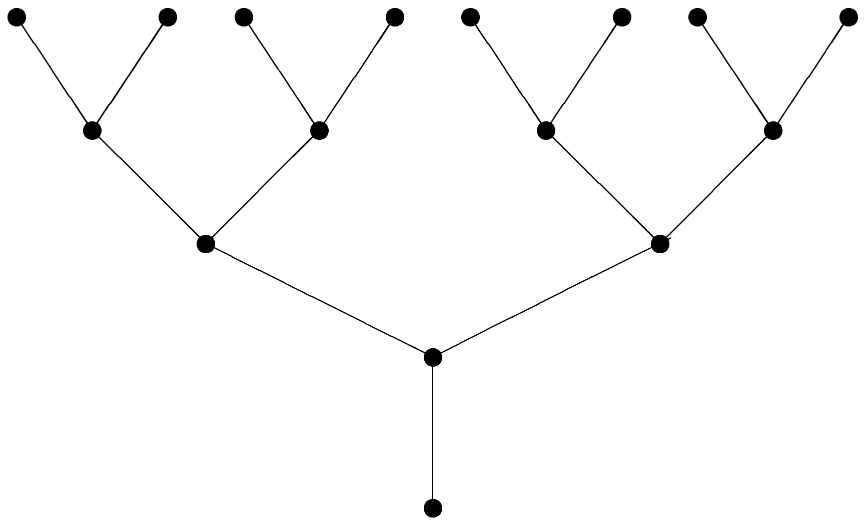
We get the following diagram:

421	431	531	532	542	543
42	43	53	531	532	542
32	44	52	53	531	541
32	42	42	52	521	531
32	42	42	42	421	431
31	41	41	41	42	421

8.4 Lifted Binary tree and subword order

Definition 8.4.1 *A binary tree is a tree with either 2 or no edges at each vertex. A lifted binary tree has an additional element at the bottom connected with one edge of*

Figure 8.2: The lifted binary tree



the binary tree. The vertices of the lifted binary tree can be naturally labeled by the bit strings for the nonnegative integers: $0, 1, 10, 11, 100, 101, 110, \ldots$ *so that 0 is the root; 1 is the only vertex of rank 1; 10 and 11 are the vertices of rank 2, etc. (see figure 8.4). Generally, the rank is the length of the string, except for the case* $\rho(0) = 0$.

Definition 8.4.2 *The graph BinWord has the same set of vertices as the lifted binary tree, the same rank function, and the following covering relation:*

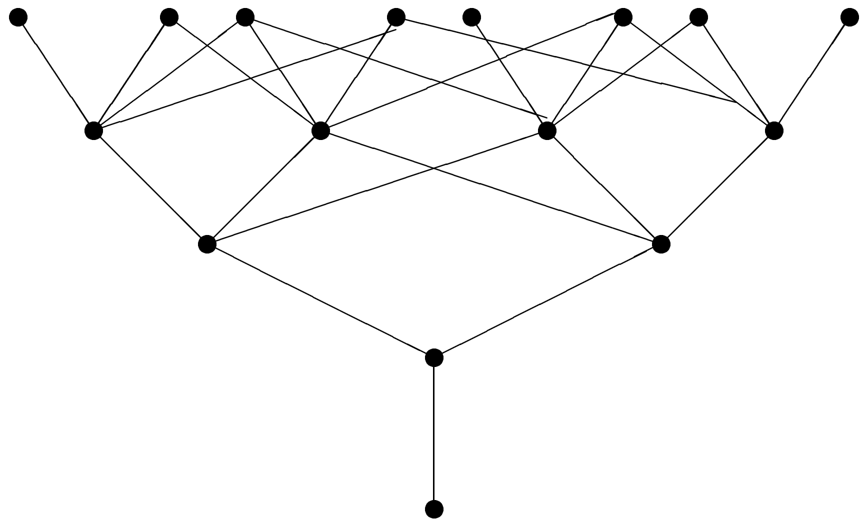
$$x \prec y \Leftrightarrow x \text{ can be obtained by deleting a single symbol from } y.$$

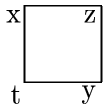
In addition, $0 \prec 1$. *For example,* 101001 *covers* 11001, 10001, 10101 *and* 10100. *We emphasize that BinWord is a graph without multiple edges (see figure 8.4).*

There are many simple ways to define an R-correspondence for this dual pair. The following one seems to be the most natural.

Lemma 8.4.3 *Let* G_1 *be the lifted binary tree and* G_2 *the dual graph BinWord. Define the function* Φ, Ψ *as follows (Remark that we only list the cases where* x *and* y *have the same rank since the other ones are trivial).* x, y, t *and* z *are the apices of a cell in a skew diagram.*

Figure 8.3: The graph BinWord





Φ:

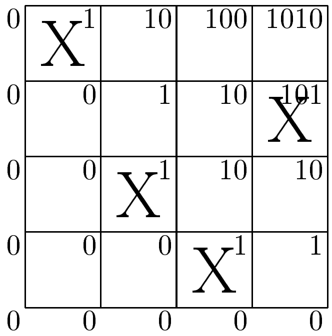
1. $x \neq y$ **or** $z = \omega 0$ for some $\omega \Rightarrow t := x$ with the last symbol removed; $\alpha := 0$;
2. $x = y$ **and** $z = \omega 1$ for some $\omega \Rightarrow t := x$; $\alpha := 1$;

Ψ:

1. $x \neq y \Rightarrow z := y\omega$; (here ω is the last symbol of x; so $x = t\omega$).
2. $x = y \neq t \Rightarrow z := y0$;
3. $x = y = t$, $\alpha = 1 \Rightarrow z := y1$ (if $y = 0$ then $y1$ means 1)

The corresponding maps Φ and Ψ are inverse to each other, and Φ is an R-correspondence in G. We obtain a bijection between permutations and paths in BinWord. Let us give an example to illustrate the algorithm.

Example 8.4.4 *Let $\pi = (4213)$. Then we have the following diagram:*



The corresponding map v_Φ assigns binary words (vertices of BinWord) to permutations. This map can be directly defined as follows.

Proposition 8.4.5 *Let σ be a permutation of n. Then the binary word $v_\Phi(\sigma)$ is a descent word of σ. Formally, if $\sigma = \sigma_1 \cdots \sigma_n$, then $v_\Phi(\sigma) = \omega_1 \cdots \omega_n$ where*

$$\omega_1 = 1; \qquad \omega_k = \begin{cases} 0, & \sigma_{k-1} > \sigma_k \\ 1, & \sigma_{k-1} < \sigma_k \end{cases} \qquad k = 2, \ldots, n;$$

in addition, $v_\Phi(\sigma) = 0$ when $n = 0$.

Proof: Induction on the size of σ based on the above algorithm. In our example we read σ starting at the right top end down to the bottom and from right to left and obtain $\sigma = (4132)$ and $v_\Phi = (1010)$.

Chapter 9

The Fibonacci Lattice

A further example of differential posets is the Fibonacci lattice. We will derive Fomin's insertion algorithm for Fibonacci tableaux by reinterpreting a Fibonacci tableau as a chain in the differential poset. The interested reader may consult [32] for deeper understanding.

9.1 Definitions

Definition 9.1.1 *Let x and y be elements of a poset P. If y covers x, i. e., $x \leq y$ and $\rho(y) = \rho(x) + 1$ we will write $x \prec y$.*

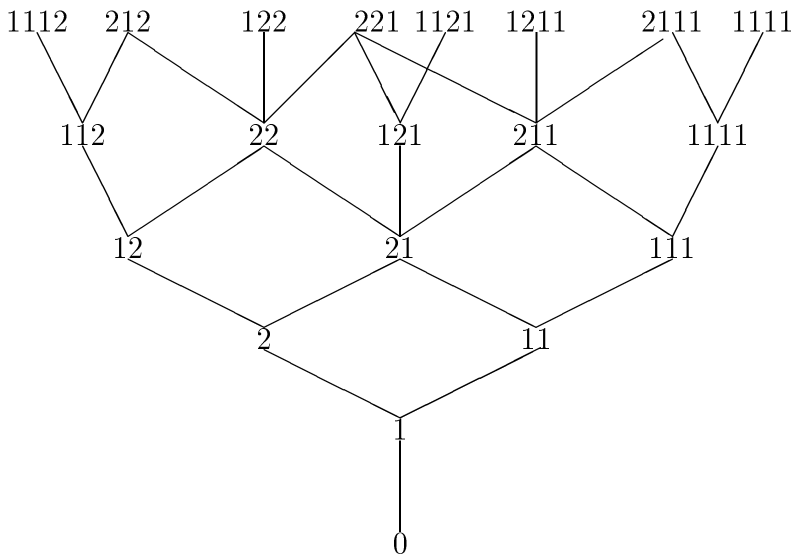
Definition 9.1.2 *Let r be a positive integer. A poset P is called* r-differential *if it satisfies $DU = UD + rI$. Combinatorially this means that*

1. *P is locally finite, graded and has a $\hat{\emptyset}$ element.*
2. *If $x \neq y$ in P and there are exactly k elements of P which are covered by both x and y, then there are exactly k elements of P which cover both x and y.*
3. *If $x \in P$ and x covers exactly k elements of P, then x is covered by exactly $k + r$ elements of P.*

Let r be a positive integer and A_r be an alphabet of $r + 1$ letters. We will regard the letters as consisting of the number 1 with r different colors, denoted $1_1, 1_2, \ldots, 1_r$ together with the number 2. Thus $A_r = \{1_1, 1_2, \ldots, 1_r, 2\}$. Let A_r^* denote the free monoid generated by A_r, i. e., A_r^* consists of all finite words $a_1, a_2, \ldots, a_k$ (including the empty word $\emptyset$) of elements of A_r.

Definition 9.1.3 *We define the* Fibonacci r-differential poset *$Z(r)$ as follows: As a set, $Z(r)$ coincides with A_r^*. If $\omega \in Z(r)$ define ω' to be covered by ω, written as $\omega' \prec \omega$, if either*

Figure 9.1: The Fibonacci Lattice $Z(1)$



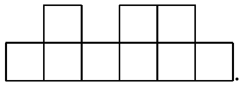
- *ω' is obtained from ω by changing a 2 to some 1_i provided that the only letters to the left of this 2 are also 2's or*

- *ω' is obtained from ω by deleting the first letter of the form 1_i.*

The rank of a word ω is the sum of the entries in ω. The process of reflection-extension, due to D. Wagner, applies to any r-differential poset. Geometrically, one can think of $Z(r)$ as built up inductively in the following manner: Start with a single element which will be the $\hat{0}$ of the poset P. Add the elements $1_1, 1_2, \ldots, 1_r$ above it, to yield the rank 1 elements. Now reflect P about the rank 1 elements, which in this case places exactly one element (the reflection of $\hat{0}$) at level two, and label that element by prepending the number 2. Then add elements $1_i\, 1_j$ by prepending each of $1_1, 1_2, \ldots, 1_r$ to each of the rank 1 elements. In general, once the poset from rank 0 to rank i $P_{[0,i]}$ has been constructed, we construct $P_{[i,i+1]}$ as follows: First reflect $P_{[i-1,i]}$ about the rank i elements and prepend the number 2 to each element at level $i+1$; this insures that condition 2 of definition 9.1.2 will continue to be satisfied. Then add elements above each of the elements of P_i by prepending in turn $1_1, 1_2, \ldots, 1_r$; this ensures that condition 3 of Definition 9.1.2 will continue to be satisfied. Figure 9.1 shows $Z(1)$.

9.2 The Fibonacci tableau

In the following we assume that $r = 1$. A word in the alphabet A_1 is called a snake and can be pictured as a shape. This bijection does not hold for $r > 1$. For example $121221 \in A_1$ becomes



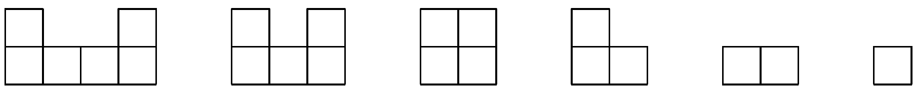
The chain of a snake of rank(n) in $Z(1)$ can be represented as a Fibonacci tableau, i.e., a filling of such a shape by integers $1, \ldots, n$ indicating when respective columns get added. At the moment n, a new box filled with n can be added to a current tableau. This new box is either placed into the leftmost available position in the top row (in the first column of length 1) or inserted into the bottom row at any place to the left of this position, thus dividing the tail of the snake. As an example, the growth process

$$\emptyset \to 1 \to 11 \to 21 \to 22 \to 212 \to 2112$$

corresponds to the tableau

$$T = \begin{array}{cccc} 3 & & & 4 \\ 2 & 6 & 5 & 1 \end{array}.$$

Note that this construction has an essential distinction from the classical Young's case: when a new box is inserted into a snake, the old boxes move. Thus the position of a box can change in the course of the growth. This could not be otherwise since the lattice $Z(1)$ is not distributive.
Let us put this construction in other words. Define T^- to be the restriction of T with its highest entry removed. Now define recursively $\mathrm{ch}T = \mathrm{sh}T : \mathrm{ch}T^-$ where the right side denotes a chain in $Z(1)$ formed by prepending $\mathrm{sh}T$ to the chain $\mathrm{ch}T^-$; terminating with the empty tableau $\emptyset$. We will view any Fibonacci tableau T as representing $\mathrm{ch}T$; clearly any saturated chain in $Z(1)$ can be represented as $\mathrm{ch}T$ for some Fibonacci tableau T, and T is completely determined by $\mathrm{ch}T$ together with its set of entries.
For $T = \begin{array}{cccc} 3 & & & 4 \\ 2 & 6 & 5 & 1 \end{array}$, we have

chT=  $\emptyset$

An arrangement of positive integers within a snakeshape ought to obey certain conditions in order to represent a valid Fibonacci tableau.

Lemma 9.2.1 *A filling of a snakeshape with positive integers is a row strict tableau* $\Longleftrightarrow$

1. *for any pair* $\boxed{\substack{B\\A}}$ *the inequality* $A \leq B$ *holds*

2. *to the right of any* $\boxed{\substack{B\\A}}$ *there are no numbers from the interval* (A, B) *in any row*

3. *if the position above* $\boxed{A}$ *is not occupied yet then to the right of* $\boxed{A}$ *there are no numbers greater than* A *in any row.*

Now we consider $Z(1)$ as defined in 9.1.3.

Definition 9.2.2 *Let* λ *be a snake of rank* n*. We denote by* λ^+ *(resp.* λ^-*) all snakeshapes of rank* $n+1$ *(resp.* $n-1$*) obtained from* λ *by adding (resp. removing) a square. Let* $\Lambda^- = \{\lambda^-\}$ *and* $\Lambda^+ = \{\lambda^+\}$*. We observe that if* λ *has* m *predecessors, then it has* $m+1$ *successors. According to the definition of cover relations for* $Z(1)$ *an element of* Λ^+ *is either obtained from* λ *by adding a 1 in the leftmost position or by replacing a 1 by 2 provided there is no other 1 left to that one. We call the* $m+1$ *positions at which a cell can be added to* λ *the addible cells of* λ *and indicate those with* $\boxed{+}$ *in the snake of* λ*. An element of* Λ^- *is obtained from* λ *by deleting the first 1 or replacing a 2 by 1 provided there is no 1 to the left of that 2. The* m *positions from which a cell can be removed are called removable cells of* λ *and indicated with* $\boxed{-}$ *in the snake of* λ*.* λ^{+-} *is a snake of rank* n *obtained by first enlarging* λ *by an addible cell and then deleting a different removable cell of the resulting snake. We denote* $\Lambda^{+-} = \{\lambda^{+-}\}$*. So it follows that* $\lambda \notin \Lambda^{+-}$*.*

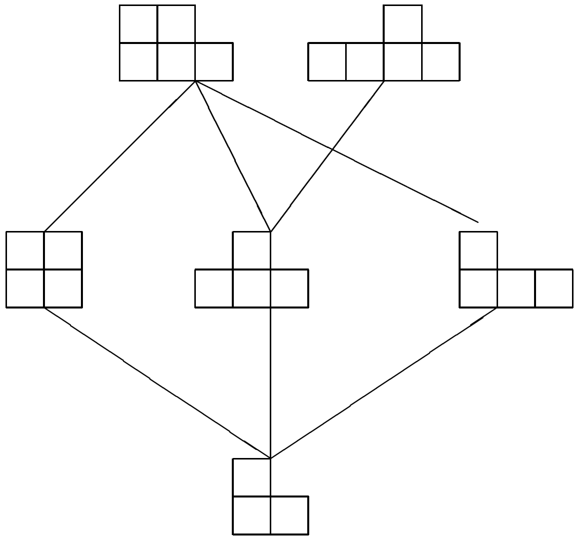
Note that we always have a $\boxed{-}$ in the bottom row in a column with only one cell provided there is no column with two cells to the left of it. There is always a $\boxed{+}$ in the cell above a $\boxed{-}$ in the bottom row. We always have a $\boxed{-}$ in the top row in the first column with two cells. Note that we always have a $\boxed{+}$ to the left of the first column in the bottom row (corresponds to prepending a 1). For each $\boxed{-}$ in the top row one has a $\boxed{+}$ in the bottom row immediately after the column with the two cells. It follows again $|\Lambda^+| = |\Lambda^-| + 1$ for all $\lambda \in Z(1)$.

9.3 Fomin's insertion for the Fibonacci tableau

Assume T is a Fibonacci tableau. Then T satisfies the conditions of Lemma 9.2.1. Assume the integer a_0 is not an entry of T. Attach a new box $\boxed{a_0}$ just to the left of T in the bottom row. Then find all entries of T which are greater than a_0 and sort them :

$$a_0 < a_1 < \cdots < a_k.$$

Figure 9.2: The shape $\lambda = (121)$, the shapes λ^+, λ^- and λ^{+-}



λ^+

λ^{+-}, λ

λ^-

Now put a box just above a_0 and move the a_i chainwise according to the following rule:

$$\emptyset \leftarrow a_1 \leftarrow \cdots \leftarrow a_k$$

that is, a_1 moves into the new box, a_2 moves to the old place of a_1, etc. The box that was occupied by a_k disappears; if it was situated in the bottom row the left and right parts of the snake are concatenated .

Example 9.3.1 *Let us insert* $a_0 = 2$ *into the Fibonacci tableau* $\begin{array}{cccc} & 5 & & 3 \\ 7 & 4 & 6 & 1 \end{array}$. *After Fomin's insertion of* a_0 *we obtain* $\begin{array}{cccc} 3 & 6 & & 4 \\ 2 & 5 & 7 & 1 \end{array}$.

In the following we give an elementary explanation of Fomin's insertion for the Fibonacci tableau.
The numbers f_λ of standard Fibonacci tableaux of snakeshape λ with rank n satisfy 2 fundamental recursions:

$$f_\lambda = \sum_{\lambda^-} f_{\lambda^-} \qquad \text{and} \qquad (n+1) f_\lambda = \sum_{\lambda^+} f_{\lambda^+}. \tag{9.1}$$

These recursions together imply algebraically the formula

$$n! = \sum f_\lambda^2 \tag{9.2}$$

The reason why these well-known formulas for standard Young tableaux also holds for $Z(1)$ is simply because the Young lattice and the Fibonacci lattice $Z(1)$ are 1-differential, i.e., fulfilling the recurrent commutation rule $DU = UD + I$. Following closely an article by A. Garsia and T. J. McLarnan [7] we show that a combinatorial analysis of this algebraic argument produces the Fomin insertion for Fibonacci tableaux.

9.4 Proof of the enumerative identities

The downward recursion $f_\lambda = \sum_{\lambda^-} f_{\lambda^-}$ follows from the bijection between Fibonacci tableaux and the set of saturated decreasing chains in $Z(1)$ starting in λ.

The proof of the upward recursion $(n+1)f_\lambda = \sum_{\lambda^+} f_{\lambda^+}$ is an easy induction on $n = \text{rank}(\lambda)$. If $\text{rank}(\lambda) = 0$, $\lambda = \emptyset$ and $f_\emptyset = f_{\square}$.
Suppose that the upward recursion is true for all snakes of rank smaller than n. The downward recursion from λ says that

$$\sum_{\lambda^-} f_{\lambda^-} = f_\lambda. \tag{9.3}$$

Summing the downward recursion from the λ^+ yields

$$(m+1)f_\lambda + \sum_{\lambda^{+-}} f_{\lambda^{+-}} = \sum_{\lambda^+} f_{\lambda^+}. \tag{9.4}$$

This is because each λ^+ has λ as a predecessor, and each λ^{+-} has a unique λ^+ as a successor. Similarly, the sum of the upward recursions from the snakes λ^- is

$$f_\lambda + \sum_{\lambda^{+-}} f_{\lambda^{+-}} = n \sum_{\lambda^-} f_{\lambda^-}. \tag{9.5}$$

Combining formulas (9.3), (9.4) and (9.5) then gives

$$(n+1)f_\lambda \overset{9.3}{=} f_\lambda + n \sum_{\lambda^-} f_{\lambda^-} \overset{9.4}{=} (m+1)f_\lambda + \sum_{\lambda^{+-}} f_{\lambda^{+-}} \overset{9.5}{=} \sum_{\lambda^+} f_{\lambda^+}. \tag{9.6}$$

Formula (9.2) follows at once from (9.1):

$$(n+1)\sum_{\lambda\vdash n} f_\lambda^2 = \sum_{\lambda\vdash n}\sum_{\lambda^+} f_{\lambda^+} f_\lambda = \sum_{\lambda\vdash(n+1)}\sum_{\lambda^-} f_\lambda f_{\lambda^-} = \sum_{\lambda\vdash(n+1)} f_\lambda^2. \tag{9.7}$$

Since clearly $1! = \sum_{\lambda\vdash 1} f_\lambda^2$, this proves (9.2).

9.5 Combinatorial proof of the recursions

To prove (9.2) combinatorially, we need a bijection $\Phi : (S, i) \leftrightarrow T$, where T varies over all standard Fibonacci tableau whose shape is in Λ^+ and $1 \leq i \leq n+1$. S is a partial Fibonacci tableau of rank(n) with integers $1, 2, \ldots, i-1, i+1, \ldots, n+1$. The three equalities in (9.6) suggests the following rough recipe for computing Φ.

1. Given (S, i) with rank $S = n$ and $1 \leq i \leq n+1$. Remove the largest element from S to produce S^-.
2. Let $T^- = \Phi(S^-, i)$. That is, use the upward recursion Φ to associate T^- with the pair (S^-, i). Since sh$T^- \in \Lambda^{+-}$ is a successor of sh$S^- \in \Lambda^-$ in $Z(1)$, we have either sh$T^- \in \Lambda^{+-}$ of sh$T^- = shS$.
3. As long as sh$T^- \in \Lambda^{+-}$ we put $n+1$ in an addible cell of shT^-. Since every λ^{+-} is contained in exactly one λ^-, this can be done in only one way.

In order to get an exact algorithm for Φ , we have to settle two details.

1. What happens if $i = n+1$, in which case $\Phi(S^-, i)$ makes no sense?
2. What happens if sh$T = \lambda$?

Define a bumping scheme β as an injection $\beta : \Lambda^- \to \Lambda^+$. The snakeshape of S will have a unique addible cell not in the image of β. This cell is called the lone cell of sh$S$.

The bumping scheme from which follows Fomin's insertion for Fibonacci tableaux maps any removable cell in the bottom row to the addible cell above it, any removable cell in the top row to the following addible cell in between two columns. The leftmost addible cell is always the lone cell.

More explicitly, the tableau $T = \Phi(S, i)$ is constructed as follows.

Case 1: If $i = n+1$, T is obtained by adding $n+1$ to S in the lone square of λ.

Case 2: Now $i \neq n+1$, $T^- = \Phi(S^-, i)$. If sh$T^- \neq \lambda$, then sh$T^- = \lambda^{+-}$ is the predecessor of a unique shape $\lambda^+ \in \Lambda^+$. The shape λ^+ is obtained by adding to λ^{+-} the unique square of λ which is not in λ^{+-}. To produce T, add $n+1$ to T^- in this square.

Case 3: Suppose sh$T^- = \lambda$. Let $\lambda^+ = \beta(shS^-)$. Produce T from T^- by adding $n+1$ to the unique square in λ^+/λ. In other words, place $n+1$ in the addible square of λ associated by the bumping scheme with the removable square which contained $n+1$ in S.

Proposition 9.5.1 $\Phi(S, i)$ *can be computed as follows:*

(1) Remove from the Fibonacci tableau S all entries $j > i$, let us call the tableau $S_{\leq i}$; place i in the lone cell of the resulting tableau and switch it left-right. Call the result T^i. Now construct a sequence $T^{i+1}, \ldots, T^{n+1}$; T is obtained by switching left-right T^{n+1} using the rules:

(2) If the cell containing j in S is vacant in T^{j-1}, produce T^j from T^{j-1} by adding j to this cell.

(3) If on the other hand, the cell containing j in S is occupied in T^{j-1} then it will be a removable cell of T^{j-1}. Produce T^j by adding j to T^{j-1} in the addible cell associated with this removable cell by the bumping scheme.

(4) If the corresponding addible cell is already occoupied in T^j then place j in the addible cell corresponding to the one which was occupied.

Example 9.5.2 *Let Φ be the insertion scheme induced by the Fibonacci insertion bumping scheme. Let $i = 2$ and $S = \begin{array}{cccc} & 5 & & 3 \\ 7 & 4 & 6 & 1 \end{array}$. The index of T_l indicates rule (l), $l = 1, 2, 3, 4$.*

$$T^2_{(1)} = 2\,1 \quad T^3_{(2)} = \begin{array}{cc} & 3 \\ 1 & 2 \end{array} \quad T^4_{(3)} = \begin{array}{cc} 4 & 3 \\ 1 & 2 \end{array} \quad T^5_{(3)} = \begin{array}{ccc} 4 & & 3 \\ 1 & 5 & 2 \end{array} \quad T^6_{(3)} = \begin{array}{ccc} 4 & 6 & 3 \\ 1 & 5 & 2 \end{array} \quad T^7_{(4)} = \begin{array}{cccc} 4 & & 6 & 3 \\ 1 & 7 & 5 & 2 \end{array}$$

$$T = \begin{array}{cccc} 3 & 6 & & 4 \\ 2 & 5 & 7 & 1 \end{array}$$

Proof: An induction establishes that if $S_{\leq j}$ is the Fibonacci tableau obtained from S by deleting all entries $> j$, then either $shT^j = shS_{\leq j}$ or $shT^j = sh(S_{\leq j})^{+-}$. This guarantees that the sequence in (2) and (3) is well defined. It is easy to see that the algorithm described here satisfies the recursive description given above for Φ. In particular rule (1) corresponds to case 1 in the definition of Φ, rule (2) to case 2 and rule (3) to case 3.

We now turn our attention to the proof of $n! = \sum_\lambda f_\lambda^2$. In order to prove this formula combinatorially, we need a bijection

$$\rho : \sigma \longleftrightarrow (P, Q)$$

between permutations and pairs of fibonacci tableaux of the same shape. The algebra in (9.7) tells us exactly how to compute ρ recursively. The left hand expression counts pairs (P, Q) of tableaux together with an integer i. The first equality says to insert i into P using Φ, i. e., to employ the upward recursion (9.1). The second equality is just a relabeling. The last equality coming from the downward recursion (9.1) says to add a square containing $n+1$ to Q so as to produce a tableau of the same shape as the new

P tableau.
In short, given a permutation σ one proceeds as in the Fibonacci insertion algorithm to insert the letters of σ one at a time (using the insertion scheme Φ) to produce the P tableau. The Q tableau records the positions at which new squares are added to the boundary of the P tableau at each stage of this algorithm. Since Φ is a bijection, each stage of this algorithm is invertible; so ρ is a bijection as well.

It's useful to attach names to the relations between several snakeshapes that we have encountered here.

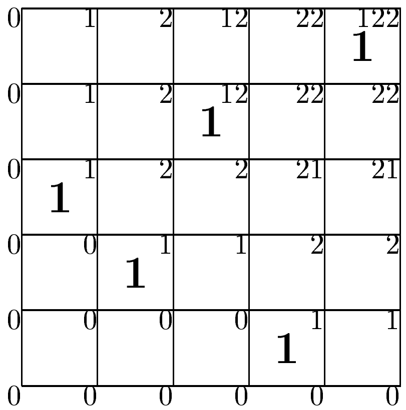
Definition 9.5.3 *An arrangement of 4 partitions* $\binom{\lambda\ \nu}{\kappa\ \mu}$ *with* $\lambda,\ \mu \in (\{\kappa\}\cup\kappa^{+})\cap(\{\nu\}\cup\nu^{-})$ *is called*

1. *a configuration of type F1 if* $\kappa = \lambda = \mu$ *and* $\nu = 1\kappa$,
2. *a configuration of type F2 if* $rank(\kappa) \neq rank(\lambda) = rank(\mu)$ *and* $\nu = 2\kappa$,
3. *a configuration of type F3 if* $\lambda,\ \mu \in \kappa^{+}$, $\kappa = \mu \cap \lambda$ *and* $\nu = \lambda \cap \mu$.

Example 9.5.4 *Let* $\sigma = 32415$. *Then we obtain the following Fibonacci tableau via Fomin's insertion:*

$$\begin{array}{c} \\ 3 \end{array} \qquad \begin{array}{c} 3 \\ 2 \end{array} \qquad \begin{array}{cc} & 3 \\ 4 & 2 \end{array} \qquad \begin{array}{cc} 2 & 4 \\ 1 & 3 \end{array} \qquad \begin{array}{ccc} & 2 & 4 \\ 5 & 1 & 3 \end{array}$$

We have the following diagram where the right side denotes the chain corresponding to P *and the top side the chain for* Q.



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