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Hypergeometric closed forms

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Finally, for all the love, happiness and encouragement that she provided to me, I dedicate this master thesis to my girlfriend, Daniela.

PROLOGUE

My interest in the search of patterns in integer sequences led me to start bachelor studies in mathematics years ago. These patterns, in many cases, were expressed through "closed forms".

During the bachelor, my attempts to understand how to formalize these and other notions made me curious to know about the foundations of mathematics, where I learnt the important concept of recursive function.

Later, my interests in logic brought me to Vienna, where I began master studies specialized in this area. During this time, Professor Krattenthaler showed me a class of sequences that I had not known about until then: the ones having hypergeometric closed form.

This thesis reflects partially what I have been studying since then, and its exposition is organized in five chapters:

- the first one provides the formal definition of sequence having hypergeometric closed form, and shows how to solve linear difference equations with polynomial coefficients, with the help of Petkovšek's complete Hyper algorithm,
- the second and third one formalize and show how to solve the problems of indefinite and definite hypergeometric summation respectively, by explaining Gosper's algorithm and Zeilberger's creative telescoping algorithm,
- the fourth one could be seen as a little summary of famous hypergeometric identities,
- the fifth one is an introduction to the theory of $R\Pi\Sigma^*$- extensions, which generalizes the algorithmic machinery presented in the previous chapters.

VORWORT

Mein Interesse an der Suche von Bildungsvorschriften ganzzahliger Folgen war der Grund, weshalb ich mich für ein Bachelorstudium der Mathematik entschied. Diese Bildungsvorschriften wurden in vielen Fällen in geschlossener Form dargestellt.

Meine Neugier an den Grundlagen der Mathematik entsprang meinem Bestreben während dem Bachelorstudium, besagte und andere Ausdrücke zu formulieren, wodurch ich das bedeutende Konzept rekursiver Funktionen kennenlernte.

Später führte mich mein Interesse an der Logik nach Wien, wo ich mit dem Masterstudium in diesem Bereich begann. Während dieser Zeit zeigte mir Professor Krattenthaler eine Klasse von Folgen, von der ich bis zu diesem Zeitpunkt nichts wusste: Folgen, die eine geschlossene hypergeometrische Form besitzen.

Diese Masterarbeit spiegelt zum Teil wieder, womit ich mich seitdem beschäftigt habe und ist in fünf Kapitel gegliedert:

- Im ersten Kapitel wird die formale Definition von Folgen mit geschlossener hypergeometrischer Form vorgestellt. In diesem Kapitel wird ausserdem gezeigt, wie lineare Differenzengleichungen mit Polynomkoeffizienten mithilfe des sogenannten kompletten Petkovšek-Hyper-Algorithmus gelöst werden können.
- Wie Probleme indefiniter und definiter hypergeometrischer Summation mit dem Algorithmus von Gosper bzw. Zeilbergers kreativem Teleskop-Algorithmus gelöst werden können, wird im zweiten und dritten Kapitel gezeigt.
- Das vierte Kapitel bietet eine kleine Übersicht berühmter hypergeometrischer Identitäten.
- Das fünfte ist eine Einführung in die Theorie von $R\Pi\Sigma^*$- extensions, die eine Verallgemeinerung der Algorithmen der vorherigen Kapitel darstellt.

CONTENTS

1 LINEAR DIFFERENCE EQUATIONS WITH POLYNOMIAL COEFFICIENTS

The main objective of this first chapter is to explain how to decide constructively if a given linear difference equation with polynomial coefficients has hypergeometric closed form, by using the so-called Petkovšek's complete Hyper algorithm. All these concepts will be formalized within the text.

During this chapter, let K be a field of characteristic zero, and recall that, for all $\alpha \in K$, $a, b \in K^{\mathbb{N}}$ and $n \in \mathbb{N}$, $(a+b)(n) := a(n)+b(n)$, $(a \cdot b)(n) := a(n) \cdot b(n)$ and $(\alpha *_K a)(n) := \alpha \cdot a(n)$ ($*_K$ will be denoted simply by $*$, if the field is clear from the context).

In addition, recall that, by convention, given $f \in K^{\mathbb{N}}$, $j \in \mathbb{N}$ and $i \in \mathbb{Z}$ such that $i < j$, $\sum_{k=j}^{i}(f(k)) = 0$ and $\prod_{k=j}^{i}(f(k)) = 1$.

1.1 THE SHIFT OPERATOR

The first main concept for working with difference equations is the so-called shift operator. It will be introduced in this section, together with some related notions.

Proposition 1.1.1. Sequences over a field of characteristic zero form an algebra
$(K^{\mathbb{N}}, +, \cdot, *)$ is a K-algebra (cf. Section 8.2 of [Petkovšek et al.]).

Recall that:

- K-**alg** (resp. **CRing**, **Field**, **Grp**, **Vect**$_K$, $\mathbb{Z}$-**Mod**) denotes the category of K-algebras (resp. commutative unital rings, fields, groups, K-vector spaces, $\mathbb{Z}$-modules),
- given two objects A, B from K-**alg** (resp. **CRing**, **Field**, **Grp**, **Vect**$_K$, $\mathbb{Z}$-**Mod**),
  - $A \leqslant_{K\text{-}\mathbf{alg}} B$ (resp. $A \leqslant_{\mathbf{CRing}} B$, $A \leqslant_{\mathbf{Field}} B$, $A \leqslant_{\mathbf{Grp}} B$, $A \leqslant_{\mathbf{Vect}_K} B$, $A \leqslant_{\mathbb{Z}\text{-}\mathbf{Mod}} B$) denotes that A is a K-subalgebra (resp. subring, subfield, subgroup, K-vector subspace, $\mathbb{Z}$-submodule) of B,
  - $A \lesssim_{K\text{-}\mathbf{alg}} B$ (resp. $A \lesssim_{\mathbf{CRing}} B$, $A \lesssim_{\mathbf{Field}} B$, $A \lesssim_{\mathbf{Grp}} B$, $A \lesssim_{\mathbf{Vect}_K} B$, $A \lesssim_{\mathbb{Z}\text{-}\mathbf{Mod}} B$) denotes that A is isomorphic to a K-subalgebra (resp. subring, subfield, subgroup, K-vector subspace, $\mathbb{Z}$-submodule) of B,
  - $A \trianglelefteq_{K\text{-}\mathbf{alg}} B$ (resp. $A \trianglelefteq_{\mathbf{CRing}} B$, $A \trianglelefteq_{\mathbf{Grp}} B$) denotes that A is a K-ideal (resp. an ideal, a normal subgroup) of B,
  - $\mathrm{End}_{K\text{-}\mathbf{alg}}(A)$ (resp. $\mathrm{End}_{\mathbf{CRing}}(A)$, $\mathrm{End}_{\mathbf{Field}}(A)$, $\mathrm{End}_{\mathbf{Grp}}(A)$, $\mathrm{End}_{\mathbf{Vect}_K}(A)$, $\mathrm{End}_{\mathbb{Z}\text{-}\mathbf{Mod}}(A)$) denotes the set of K-algebra (resp. unital ring, field, group, K-linear, $\mathbb{Z}$-module) endomorphisms over A,
  - $\mathrm{Aut}_{K\text{-}\mathbf{alg}}(A)$ (resp. $\mathrm{Aut}_{\mathbf{CRing}}(A)$, $\mathrm{Aut}_{\mathbf{Field}}(A)$, $\mathrm{Aut}_{\mathbf{Grp}}(A)$, $\mathrm{Aut}_{\mathbf{Vect}_K}(A)$, $\mathrm{Aut}_{\mathbb{Z}\text{-}\mathbf{Mod}}(A)$) denotes the set of K-algebra (resp. unital ring, field, group, K-linear, $\mathbb{Z}$-module) automorphisms over A.

Proposition 1.1.2. A field is embeddable into its algebra of sequences
$K \lesssim_{\mathbf{CRing}} K^{\mathbb{N}}$.

Proof Let $\phi : K \longrightarrow K^{\mathbb{N}}$ such that $\phi(\alpha)(n) = \alpha$, for all $\alpha \in K$ and $n \in \mathbb{N}$. ϕ is clearly a **CRing**-homomorphism, so $K \lesssim_{\mathbf{CRing}} K^{\mathbb{N}}$. $\square$

Proposition 1.1.3. The algebra of sequences can not be embedded into another field
There exist no field F such that $K^{\mathbb{N}} \lesssim_{\mathbf{CRing}} F$.

Proof Let $a, b \in K^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a(n) = \frac{1+(-1)^n}{2}$ and $b(n) = \frac{1-(-1)^n}{2}$. $a(n) \cdot b(n) = 0$, for all $n \in \mathbb{N}$, so a and b are zero divisors of $K^{\mathbb{N}}$ and thus there exists no field F such that $K^{\mathbb{N}} \lesssim_{\mathbf{CRing}} F$. □

Definition 1.1.4. Let f be a function. Then f is said to be a K-**operator** if $f : K^{\mathbb{N}} \longrightarrow K^{\mathbb{N}}$.

For example, given $f : K^{\mathbb{N}} \longrightarrow K^{\mathbb{N}}$ such that $f(a)(n) = ((-1) * a)(n) = (-1) \cdot a(n) = -a(n)$, for all $a \in K^{\mathbb{N}}$ and $n \in \mathbb{N}$, f is a K-operator.

Note that $\{f \mid f \text{ is a linear } K\text{-operator}\} = \mathrm{End}_{\mathbf{Vect}_K}(K^{\mathbb{N}})$.

Definition 1.1.5. Let f be a K-operator. Then f is said to be the **shift** K-**operator** if $f(a)(n) = a(n+1)$, for all $a \in K^{\mathbb{N}}$ and $n \in \mathbb{N}$, fact which is denoted by $f = \mathcal{N}_K$.

For example, given $a \in K^{\mathbb{N}}$ such that $a(n) = 2 \cdot n$, for all $n \in \mathbb{N}$, $\mathcal{N}_K(a)(n) = a(n+1) = 2 \cdot n + 2$, for all $n \in \mathbb{N}$.

Since the field is usually clear from the context, $\mathcal{N}_K$ will be frequently denoted by simply $\mathcal{N}$.

Proposition 1.1.6. The shift operator is linear
$\mathcal{N} \in \mathrm{End}_{\mathbf{Vect}_K}(K^{\mathbb{N}})$.

Proof Let $a, b \in K^{\mathbb{N}}$, $\alpha, \beta \in K$ and $n \in \mathbb{N}$. $\mathcal{N}(\alpha * a + \beta * b)(n) = (\alpha * a + \beta * b)(n+1) = (\alpha * a)(n+1) + (\beta * b)(n+1) = \alpha \cdot a(n+1) + \beta \cdot b(n+1) = \alpha \cdot \mathcal{N}(a)(n) + \beta \cdot \mathcal{N}(b)(n) = (\alpha * \mathcal{N}(a))(n) + (\beta * \mathcal{N}(b))(n) = (\alpha * \mathcal{N}(a) + \beta * \mathcal{N}(b))(n)$. □

Proposition 1.1.7. Multiplication by a fixed sequence is linear
Let $u \in K^{\mathbb{N}}$ and f the K-operator such that $f(a) = u \cdot a$, for all $a \in K^{\mathbb{N}}$. Then $f \in \mathrm{End}_{\mathbf{Vect}_K}(K^{\mathbb{N}})$.

Proof Let $a, b \in K^{\mathbb{N}}$ and $\alpha, \beta \in K$. $f(\alpha * a + \beta * b) = u \cdot (\alpha * a + \beta * b) = \alpha * (u \cdot a) + \beta * (u \cdot b) = \alpha * f(a) + \beta * f(b)$. □

Proposition 1.1.8. Linear operators form a noncommutative algebra
$(\mathrm{End}_{\mathbf{Vect}_K}(K^{\mathbb{N}}), +, \circ, *_K)$ is a noncommutative algebra (cf. Section 8.2 of [Petkovšek et al.]).

Definition 1.1.9. Let f be a K-operator. Then f is said to be the **antidifference** K-**operator** if $f = \mathcal{N} - 1$, fact which is denoted by $f = \Delta_K$.

For example, given $a \in K^{\mathbb{N}}$ such that $a(n) = 2 \cdot n + 1$, for all $n \in \mathbb{N}$, $\Delta_K(a)(n) = (\mathcal{N} - 1)(a)(n) = (\mathcal{N}(a) - a)(n) = \mathcal{N}(a)(n) - a(n) = a(n+1) - a(n) = 2 \cdot (n+1) + 1 - (2 \cdot n + 1) = 2$, for all $n \in \mathbb{N}$.

Since the field is usually clear from the context, it will be frequent to denote Δ_K by simply Δ.

1.2 LINEAR RECURRENCE OPERATORS

Having defined the shift operator, the next step is to introduce the so-called linear recurrence operators, which will form a fundamental tool in order to develop a theory about resolution of linear difference equations.

From now on, if there is no confusion, sequences in presence of K-operators will be often identified with their formulas, i.e. given $a \in K^{\mathbb{N}}$ and a K-operator f, $f(a)$ will be often denoted by $f(a(n))$, but keep in mind that a is not being evaluated in any concrete value of n (evaluation of $f(a)$ in a value n will be denoted by $f(a)(n)$, as usual). Eg. if $a(n) = \frac{1+(-1)^n}{2}$, for all $n \in \mathbb{N}$, then $f(a)$ will be denoted by $f\left(\frac{1+(-1)^n}{2}\right)$.

In addition, from now on, $F : K^{\mathbb{N}} \times \{f \mid f \text{ is a K-operator}\} \longrightarrow \{f \mid f \text{ is a K-operator}\}$ such that $(u\ F\ f)(a) := F(u, f)(a) = u \cdot f(a)$, for all $a, u \in K^{\mathbb{N}}$ and K-operator f, will be denoted by $\bullet_K$ (or even simply by $\bullet$, if the field is clear from the context).

For example, given $a \in K^{\mathbb{N}}$, $((1 \bullet \mathcal{N}^0 + e^n \bullet \mathcal{N}^1) \circ (1 \bullet \mathcal{N}^0 + n \bullet \mathcal{N}^1))(a(n)) = (1 \bullet \mathcal{N}^0 + e^n \bullet \mathcal{N}^1)(a(n) + n \cdot a(n+1)) = a(n) + (n + e^n) \cdot a(n+1) + e^n \cdot (n+1) \cdot a(n+2)$.

Definition 1.2.1. Let $L \in \mathrm{End}_{\mathbf{Vect}_K}(K^{\mathbb{N}})$. Then L is said to be a **linear recurrence K-operator** if $p(\mathcal{N}) = L$, for some $p(t) \in \mathrm{End}_{\mathbf{Vect}_K}(K^{\mathbb{N}})[t]$.

For example, given $f : K^{\mathbb{N}} \longrightarrow K^{\mathbb{N}}$ such that $f(a)(n) = 2 \cdot n^3 \cdot a(n+2) + 5 \cdot a(n)$, for all $a \in K^{\mathbb{N}}$ and $n \in \mathbb{N}$, f is a linear recurrence K-operator, since $f = (2 \cdot n^3) \bullet \mathcal{N}^2 + 0 \bullet \mathcal{N}^1 + 5 \bullet \mathcal{N}^0$.

Definition 1.2.2. Let L be a linear recurrence K-operator and $r \in \mathbb{N}$. Then r is said to be the **order** of L if there exist $a_0, ..., a_r \in K^{\mathbb{N}}$ such that $L = \sum_{i=0}^{r} (a_i \bullet \mathcal{N}^i)$ and $a_0 \cdot a_r \neq 0$, fact which is denoted by $r = \mathrm{order}(L)$.

For example, $\mathrm{order}((2 \cdot n^3) \bullet \mathcal{N}^2 + 0 \bullet \mathcal{N}^1 + 5 \bullet \mathcal{N}^0) = 2$.

Definition 1.2.3. Let S be a set. Then S is said to be the **solution space of a linear difference K-equation** if $S = \{x \in K^{\mathbb{N}} \mid L(x) = b\}$, for some linear recurrence K-operator L and $b \in K^{\mathbb{N}}$.

Definition 1.2.4. Let S be a set. Then S is said to be the **solution space of an homogeneous linear difference K-equation** if $S = \{x \in K^{\mathbb{N}} \mid L(x) = 0\}$ (i.e. if $S = \mathrm{Ker}(L)$), for some linear recurrence K-operator L.

For example, given $S = \{x \in \mathbb{R}^{\mathbb{N}} \mid \exists\, \alpha, \beta \in \mathbb{R} \text{ such that } x(n) = 2^{\frac{n}{2}} \cdot (\alpha \cdot (-1)^n + \beta),\ \forall\, n \in \mathbb{N}\}$, S is the solution of an homogeneous linear difference K-equation, since $S = \{x \in \mathbb{R}^{\mathbb{N}} \mid x(n+2) - 2 \cdot x(n),\ \forall\, n \in \mathbb{N}\} = \{x \in \mathbb{R}^{\mathbb{N}} \mid (1 \bullet \mathcal{N}^2 + 0 \bullet \mathcal{N}^1 + (-2) \bullet \mathcal{N}^0)(x) = 0\}$.

Proposition 1.2.5. The solution space of a linear difference equation is an affine space
Let S be the solution space of a linear difference K-equation. Then S is an affine K-subspace of $K^{\mathbb{N}}$ and, if S is the solution of a homogeneous linear difference K-equation, then $S \leqslant_{\mathbf{Vect}_K} K^{\mathbb{N}}$ (cf. Section 8.2 of [Petkovšek et al.]).

The dimension of a solution space does not always coincide with order of the linear recurrence operator, eg.:

- $\dim_K\left(\left\{x \in K^{\mathbb{N}} \mid \frac{1+(-1)^n}{2} \cdot \mathcal{N}^1(x(n)) + \frac{1-(-1)^n}{2} \cdot \mathcal{N}^0(x(n)) = 0\right\}\right) =$
$\dim_K\left(\left\{x \in K^{\mathbb{N}} \mid \forall\, n \in \mathbb{N},\ \frac{1+(-1)^n}{2} \cdot x(n+1) + \frac{1-(-1)^n}{2} \cdot x(n) = 0\right\}\right) =$
$\dim_K\left(\{x \in K^{\mathbb{N}} \mid \forall\, n \in \mathbb{N},\ [[n \text{ is even} \Rightarrow x(n+1) = 0] \wedge [n \text{ is odd} \Rightarrow x(n) = 0]]\}\right) =$
$\dim_K\left(\{x \in K^{\mathbb{N}} \mid \forall\, n \in \mathbb{N},\ x(2 \cdot n + 1) = 0\}\right) = \infty$,
- $\dim_K\left(\left\{x \in K^{\mathbb{N}} \mid \frac{1+(-1)^n}{2} \cdot \mathcal{N}^1(x(n)) + (-1) \cdot \mathcal{N}^0(x(n)) = 0\right\}\right) =$
$\dim_K\left(\left\{x \in K^{\mathbb{N}} \mid \forall\, n \in \mathbb{N},\ \frac{1+(-1)^n}{2} \cdot x(n+1) = x(n)\right\}\right) =$
$\dim_K(\{x \in K^{\mathbb{N}} \mid \forall\, n \in \mathbb{N},\ [[n \text{ is even} \Rightarrow x(n+1) = x(n)] \wedge [n \text{ is odd} \Rightarrow x(n) = 0]]\}) =$
$\dim_K(\{x \in K^{\mathbb{N}} \mid \forall\, n \in \mathbb{N},\ x(n) = 0\}) = \dim_K(\{0\}) = 0$,

- $\dim_K(\{x \in K^{\mathbb{N}} \mid (n-1)\cdot(n-4)\cdot(n-7)\cdot \mathcal{N}^1(x(n)) + n\cdot(n-3)\cdot(n-6)\cdot\mathcal{N}^0(x(n)) = 0\}) =$
  $\dim_K(\{x \in K^{\mathbb{N}} \mid \forall\, n \in \mathbb{N},\ (n-1)\cdot(n-4)\cdot(n-7)\cdot x(n+1) + n\cdot(n-3)\cdot(n-6)\cdot x(n) = 0\}) =$
  $\dim_K\left(\left\{x \in K^{\mathbb{N}} \mid [x(1) = 0\ \wedge\ 8\cdot x(2) = 10\cdot x(3)\ \wedge\ x(4) = 0\ \wedge\ 10\cdot x(5) = 8\cdot x(6)\ \wedge\ x(7) = 0 \wedge\right.\right.$
  $\left.\left.\forall\, n \in \mathbb{N} \text{ such that } 8 \leqslant n,\ x(n+1) = \frac{n\cdot(n-3)\cdot(n-6)}{(n-1)\cdot(n-4)\cdot(n-7)}\cdot x(n)]\right\}\right) =$
  $\dim_K(K\langle\{\chi_{\{0\}}(n), \chi_{\{2\}}(n) + \frac{4}{5}\cdot\chi_{\{3\}}(n), \chi_{\{5\}}(n) + \frac{5}{4}\cdot\chi_{\{6\}}(n), (n-1)\cdot(n-4)\cdot(n-7)\}\rangle) = 4.$

However, in Section 1.3 it will be shown a special case in which dimension and order coincide, and its conditions will constitute the general framework in the sequel.

1.3 ALMOST-EVERYWHERE EQUALITY

In order to reach the ideal situation in which dimension and order coincide, one way is to fold the space of sequences by means of certain equivalence relation, the so-called almost-everywhere equality, which will be defined now.

Definition 1.3.1. Let $a, b \in K^{\mathbb{N}}$. Then a and b are said to be **almost-everywhere equal** if $\{n \in \mathbb{N} \mid a(n) \neq b(n)\}$ is finite, fact which is denoted by $a = b$ a.e.

Note that, given $a, b \in K^{\mathbb{N}}$, if $a \neq b$ a.e. then $\neg\ [a = b$ a.e.$]$; but the reciprocal does not necessarily hold. Indeed, $a \neq b$ a.e. is equivalent to the fact that $\{n \in \mathbb{N} \mid a(n) = b(n)\}$ is finite, i.e. to the existence of $m \in \mathbb{N}$ such that $a(n) \neq b(n)$, for all $n \in \mathbb{N}$ such that $m \leqslant n$. And $\neg\ [a = b$ a.e.$]$ is equivalent to the fact that $\{n \in \mathbb{N} \mid a(n) \neq b(n)\}$ is infinite, i.e. to the fact that, for all $\tilde{m} \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$ and $a(n) \neq b(n)$. But from the fact that, for all $\tilde{m} \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$ and $a(n) \neq b(n)$ it is not possible to conclude the existence of $m \in \mathbb{N}$ such that $a(n) \neq b(n)$, for all $n \in \mathbb{N}$ such that $m \leqslant n$.

From now on, $\bigcup_{r\in\mathbb{N}}(\mathrm{Ker}(\mathcal{N}^r))$ will be denoted by $\mathbf{J}_K$ (or even simply by $\mathbf{J}$, if the field is clear from the context).

Proposition 1.3.2. Characterization of almost-everywhere equality
Let $a, b \in K^{\mathbb{N}}$. $a = b$ a.e. if, and only if, $a - b \in \mathbf{J}$.

Proof $a = b$ a.e. $\Leftrightarrow$
$a(n) = b(n)$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, for some $m \in \mathbb{N}$ $\Leftrightarrow$
$(a-b)(n) = 0$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, for some $m \in \mathbb{N}$ $\Leftrightarrow$
$(a-b)(n+m) = 0$, for all $n \in \mathbb{N}$, for some $m \in \mathbb{N}$ $\Leftrightarrow$
$\mathcal{N}^m(a-b) = 0$, for some $m \in \mathbb{N}$ $\Leftrightarrow$
$a - b \in \mathrm{Ker}(\mathcal{N}^m)$, for some $m \in \mathbb{N}$ $\Leftrightarrow$
$a - b \in \mathbf{J}$. □

In particular, given $a \in K^{\mathbb{N}}$, recall that the following conditions hold:
- $\{n \in \mathbb{N} \mid a(n) = 0\}$ is finite if, and only if, $a \neq 0$ a.e.,
- $\{n \in \mathbb{N} \mid a(n) = 0\}$ is infinite if, and only if, $\neg\ [a \neq 0$ a.e.$]$,
- $\{n \in \mathbb{N} \mid a(n) \neq 0\}$ is finite if, and only if, $a = 0$ a.e. (i.e. if, and only if, $a \in \mathbf{J}$),
- $\{n \in \mathbb{N} \mid a(n) \neq 0\}$ is infinite if, and only if, $\neg\ [a = 0$ a.e.$]$ (i.e. if, and only if, $a \notin \mathbf{J}$).

For example, let $a \in K^{\mathbb{N}}$ such that $a(n) = \frac{1+(-1)^n}{2}$. Then $a \notin \mathbf{J}$, but note that also $\neg\ [a \neq 0$ a.e.$]$.

Lemma 1.3.3. $\mathbf{J} \trianglelefteq_{\text{K-alg}} K^{\mathbb{N}}$ (cf. Section 8.2 of [Petkovšek et al.]).

Proposition 1.3.4. Characterization of the zero divisors of ${}^{K^{\mathbb{N}}}/_{J}$
Let $a \in K^{\mathbb{N}} \setminus J$. Then the following conditions hold:

- $a + J$ is a zero divisor of ${}^{K^{\mathbb{N}}}/_{J}$,
- $\neg\ [a \neq 0$ a.e.$]$,
- $a + J$ is not a unit of ${}^{K^{\mathbb{N}}}/_{J}$.

Proof $a + J$ is a zero divisor of ${}^{K^{\mathbb{N}}}/_{J} \overset{a \notin J}{\Leftrightarrow}$
$(a + J) \cdot (b + J) = 0$, for some $b \in K^{\mathbb{N}}$ such that $b + J \neq 0 \Leftrightarrow$
$a \cdot b \notin J$, for some $b \in K^{\mathbb{N}} \setminus J \overset{1.3.2}{\Leftrightarrow}$
$a \cdot b = 0$ a.e., for some $b \in K^{\mathbb{N}}$ such that $\neg\ [b = 0$ a.e.$] \Leftrightarrow$
$\{n \in \mathbb{N} \mid (a \cdot b)(n) \neq 0\}$ is finite, for some $b \in K^{\mathbb{N}}$ such that $\{n \in \mathbb{N} \mid b(n) \neq 0\}$ is infinite $\Leftrightarrow$
$\{n \in \mathbb{N} \mid a(n) = 0\}$ is infinite $\Leftrightarrow$
$\neg\ [a \neq 0$ a.e.$] \Leftrightarrow$
$\{n \in \mathbb{N} \mid a(n) = 0\}$ is infinite $\Leftrightarrow$
$\{n \in \mathbb{N} \mid (a \cdot b)(n) \neq 1\}$ is infinite, for all $b \in K^{\mathbb{N}}$ such that $\{n \in \mathbb{N} \mid b(n) \neq 0\}$ is infinite $\Leftrightarrow$
$\neg\ [a \cdot b = 1$ a.e.$]$, for all $b \in K^{\mathbb{N}}$ such that $\neg\ [b = 0$ a.e.$] \Leftrightarrow$
$a \cdot b - 1 \notin J$, for all $b \in K^{\mathbb{N}} \setminus J \overset{1.3.2}{\Leftrightarrow}$
$(a \cdot b - 1) + J \neq 0$, for all $b \in K^{\mathbb{N}}$ such that $b + J \neq 0 \Leftrightarrow$
$(a + J) \cdot (b + J) \neq 1$, for all $b \in K^{\mathbb{N}}$ such that $b + J \neq 0 \overset{a \notin J}{\Leftrightarrow}$
$a + J$ is not a unit of ${}^{K^{\mathbb{N}}}/_{J}$. □

Proposition 1.3.5. For solutions holding a.e., dimension and order coincide
Let $r \in \mathbb{N}$, $a_0, \ldots, a_r \in K^{\mathbb{N}}$ such that $a_0 \neq 0$ a.e. and $a_r \neq 0$ a.e., $L = \sum_{i=0}^{r}(a_i \bullet \mathcal{N}^i)$ and $S = \{x + J \mid [x \in K^{\mathbb{N}} \wedge L(x) = 0$ a.e.$]\}$. Then $S \leqslant_{\mathbf{Vect}_K} {}^{K^{\mathbb{N}}}/_{J}$ and $\dim_K(S) = \operatorname{order}(L)$ (cf. Theorem 8.2.1 of [Petkovšek et al.]).

1.4 SOME IMPORTANT TYPES OF SEQUENCES

This section constitutes essentially a brief list of important classes of sequences, that will be widely used in the sequel, and of some of the relations that hold between them. Remarkably, it is introduced the concept of hypergeometric closed form, which is maybe the most interesting concept in this work. It allows one to formalize what it is understood by solving a difference equation but, moreover, it is a landmark in the constructive mathematics, because it suppose a very reasonable definition of "explicit expression" of a sequence.

Definition 1.4.1. Let $a \in K^{\mathbb{N}}$. Then a is said to be **constant** if there exist $\alpha \in K$ and $m \in \mathbb{N}$ such that $a(n) = \alpha$, for all $n \in \mathbb{N}$ such that $m \leqslant n$.

For example, given $a \in K^{\mathbb{N}}$ such that $a(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ 2 & \text{otherwise} \end{cases}$, for all $n \in \mathbb{N}$, a is constant.

Definition 1.4.2. Let $a \in K^{\mathbb{N}}$ and $\phi : K[t] \longrightarrow K^{\mathbb{N}}$ such that $\phi(p(t))(n) = p(n)$, for all $p(t) \in K[t]$ and $n \in \mathbb{N}$. Then a is said to be **polynomial** if there exists $p(t) \in K[t]$ such that $\phi(p(t)) = a$ a.e.

From now on, $\{a \in K^{\mathbb{N}} \mid a \text{ is polynomial}\}$ will be denoted by $\operatorname{pol}(K)$.

For example, given $a \in K^{\mathbb{N}}$ such that $a(n) = \begin{cases} (-1)^n & \text{if } n < 15 \\ 2 \cdot n + 1 & \text{otherwise} \end{cases}$, for all $n \in \mathbb{N}$, $a \in \operatorname{pol}(K)$.

Definition 1.4.3. Let $a \in K^{\mathbb{N}}$. Then a is said to be **rational** if there exists $r(t) \in K(t)$ such that $\{n \in \mathbb{N} \mid [n \text{ is not a pole of } r(t) \wedge r(n) \neq a(n)]\}$ is finite.

From now on, $\{a \in K^{\mathbb{N}} \mid a \text{ is rational}\}$ will be denoted by $\mathrm{rat}(K)$.

For example, given $a \in K^{\mathbb{N}}$ such that $a(n) = \begin{cases} (-1)^n & \text{if } n < 15 \\ \frac{2 \cdot n+1}{n^2+3} & \text{otherwise} \end{cases}$, for all $n \in \mathbb{N}$, $a \in \mathrm{rat}(K)$.

Note that every polynomial sequence is also rational.

Every $a \in K^{\mathbb{N}}$ such that $a \neq 0$ a.e. is annihilated a.e. by some linear recurrence K-operator of order 1. Indeed, $\mathcal{N}(a) + b \cdot a = 0$ a.e., for all $b \in K^{\mathbb{N}}$ such that there exists $m \in \mathbb{N}$ such that $b(n) = -\frac{a(n+1)}{a(n)}$, for all $n \in \mathbb{N}$ such that $m \leqslant n$. It is however more interesting to treat the following situations.

Definition 1.4.4. Let $r \in \mathbb{N}$, $a_0, ..., a_r \in K^{\mathbb{N}}$ and $L = \sum_{i=0}^{r}(a_i \bullet \mathcal{N}^i)$. Then L is said to be a **linear recurrence** K-**operator with constant** (resp. **polynomial**) **coefficients** if $a_0, ..., a_r$ are constant (resp. polynomial).

For example, $(2 \cdot n^3) \bullet \mathcal{N}^2 + 0 \bullet \mathcal{N}^1 + 5 \bullet \mathcal{N}^0$ is a linear recurrence K-operator with polynomial coefficients.

Definition 1.4.5. Let $a \in K^{\mathbb{N}} \backslash \mathbf{J}$. Then a is said to be **hypergeometric** if there exists a linear recurrence K-operator with polynomial coefficients L of order 1 such that $L(a) = 0$ a.e.

From now on, $\{a \in K^{\mathbb{N}} \mid a \text{ is hypergeometric}\}$ will be denoted by $\mathrm{hyp}(K)$.

For example, given $a \in K^{\mathbb{N}}$ such that $a(n) = 3^n$, for all $n \in \mathbb{N}$, $a \in \mathrm{hyp}(K)$, since $(\mathcal{N} - 3)(a)(n) = (\mathcal{N}(a) - 3(a))(n) = a(n+1) - 3 \cdot a(n) = 3^{n+1} - 3 \cdot 3^n = 0$, for all $n \in \mathbb{N}$.

Proposition 1.4.6. Characterization of hypergeometric sequence
Let $a \in K^{\mathbb{N}}$. Then $a \in \mathrm{hyp}(K)$ if, and only if, there exist $r(t) \in K(t)$ and $m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, n is not a pole of $r(t)$, $a(n) \neq 0$ and $r(n) = \frac{a(n+1)}{a(n)}$.

Proof

$\Rightarrow$) $a \in \mathrm{hyp}(K)$, in particular, $a \notin \mathbf{J}$. So, applying Proposition 1.3.2, $\neg\, [a = 0$ a.e.$]$, i.e. $\{n \in \mathbb{N} \mid a(n) \neq 0\}$ is infinite. [0]

Moreover, again because a is hypergeometric, there exist $a_0, a_1 \in \mathrm{pol}(K)$ such that $a_0 \cdot a_1 \neq 0$ and $a_1 \cdot \mathcal{N}(a) + a_0 \cdot a = 0$ a.e., so there exist $p(t), q(t) \in K[t]\backslash\{0\}$ and $m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, $a_0(n) = p(n) \neq 0$, $a_1(n) = q(n) \neq 0$ and $q(n) \cdot a(n+1) + p(n) \cdot a(n) = 0$. [1]

$a(n+1) = -\frac{p(n)}{q(n)} \cdot a(n)$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, by [1]. [2]

$a(n) \neq 0$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, applying [0], [1] and [2]. Hence, $\frac{a(n+1)}{a(n)} = -\frac{p(n)}{q(n)}$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, by [2]. So there exists $r(t) \in K(t)$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, n is not a pole of $r(t)$, $a(n) \neq 0$ and $r(n) = \frac{a(n+1)}{a(n)}$; by [1] and [2].

$\Leftarrow$) There exist $r(t) \in K(t)$ and $m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, n is not a pole of $r(t)$, $a(n) \neq 0$ and $r(n) = \frac{a(n+1)}{a(n)}$; which yields the existence of $p(t), q(t) \in K[t]\backslash\{0\}$ such that $\frac{a(n+1)}{a(n)} = -\frac{p(n)}{q(n)}$, for all $n \in \mathbb{N}$ such that $m \leqslant n$. Hence, there exist $a_0, a_1 \in \mathrm{pol}(K)$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, $0 \notin \{a(n), a_0(n), a_1(n)\}$ and $\frac{a(n+1)}{a(n)} = -\frac{a_0(n)}{a_1(n)}$. Consequently, $a \notin \mathbf{J}$, $a_0, a_1 \in \mathrm{pol}(K)$, $a_0 \cdot a_1 \neq 0$ and $a_1 \cdot \mathcal{N}(a) + a_0 \cdot a = 0$ a.e. Thus $a \notin \mathbf{J}$ and there exists a linear recurrence K-operator with polynomial coefficients L of order 1 such that $L(a) = 0$ a.e., i.e. a is hypergeometric. □

From Proposition 1.4.6 it is clear that every nonzero rational sequence is also hypergeometric. In Section 1.5 it will be shown when the converse holds.

Note that $(\text{hyp}(K),\cdot)$ is an abelian group, but $\text{hyp}(K)$ is not closed under addition; eg. considering $a, b \in K^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a(n) = 2$ and $b(n) = \frac{-n!}{(2\cdot n+1)!}$, a and b are hypergeometric, but $a+b$ is not. However, in Section 1.7 it will be shown that it is possible to ensure, under certain conditions, that a finite sum of hypergeometric sequences is still hypergeometric.

Lemma 1.4.7. Let $a \in \text{hyp}(K)$ such that $\Delta(a) \neq 0$ a.e. Then $\Delta(a) \in \text{hyp}(K)$.

Proof $a \in \text{hyp}(K)$, so, applying Proposition 1.4.6, there exist $r(t) \in K(t)$ and $n_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $n_0 \leqslant n$, n is not a pole of $r(t)$, $a(n) \neq 0$ and $r(n) = \frac{a(n+1)}{a(n)}$. Let $q(t) = \frac{r(t+1)-1}{1-\frac{1}{r(t)}}$. It is clear that $q(t) \in K(t)$. $\Delta(a) \neq 0$ a.e., so there exists $n_1 \in \mathbb{N}$ such that $\Delta(a)(n) \neq 0$, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$, and then $\frac{\Delta(a)(n+1)}{\Delta(a)(n)} = \frac{a(n+2)-a(n+1)}{a(n+1)-a(n)} = \frac{\frac{a(n+2)}{a(n+1)}-1}{1-\frac{a(n)}{a(n+1)}} = \frac{r(n+1)-1}{1-\frac{1}{r(n)}} = q(n)$, for all $n \in \mathbb{N}$ such that $\max(\{n_0, n_1\}) \leqslant n$. Therefore, applying again Proposition 1.4.6, $\Delta(a) \in \text{hyp}(K)$. $\square$

Proposition 1.4.8. No zero divisor of $K^{\mathbb{N}}/_J$ is hypergeometric
Let $a \in \text{hyp}(K)$. Then $a + J$ is not a zero divisor of $K^{\mathbb{N}}/_J$.

Proof a is hypergeometric, so $a \notin J$ and there exist $a_0, a_1 \in \text{pol}(K)$ such that $a_0 \cdot a_1 \neq 0$ and $a_1 \cdot \mathcal{N}(a) + a_0 \cdot a = 0$ a.e. [0]

$a, a_0, a_1 \in K^{\mathbb{N}} \setminus J$, by [0], so $\{n \in \mathbb{N} \mid b(n) \neq 0\}$ is infinite, for all $b \in \{a, a_0, a_1\}$. [1]

And, since $a_0, a_1 \in \text{pol}(K)$ and $a_0 \cdot a_1 \neq 0$, $\{n \in \mathbb{N} \mid a_i(n) = 0\}$ is finite, for all $i \in \{0, 1\}$. [2]

The fact $a_1 \cdot \mathcal{N}(a) + a_0 \cdot a = 0$ a.e. yields the existence of $m \in \mathbb{N}$ such that $a_1(n) \cdot a(n+1) = -a_0(n) \cdot a(n)$, for all $n \in \mathbb{N}$ such that $m \leqslant n$. Hence, there exists $\tilde{m} \in \mathbb{N}$ such that $a(n+1) = -\frac{a_0(n)}{a_1(n)} \cdot a(n)$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$, by [2]. Therefore, applying [1] and [2], $\{n \in \mathbb{N} \mid a(n) = 0\}$ is finite, i.e. $a \neq 0$ a.e. Applying then Proposition 1.3.4, $a + J$ is not a zero divisor of $K^{\mathbb{N}}/_J$. $\square$

Definition 1.4.9. Let $a \in K^{\mathbb{N}} \setminus J$. Then a is said to be **d'Alembertian** if there exist $r \in \mathbb{N}$ and linear recurrence K-operators with polynomial coefficients $L_0, \ldots, L_r$ of order 1 such that $(L_r \circ \ldots \circ L_0)(a) = 0$ a.e.

For example, $a \in K^{\mathbb{N}}$ such that $a(n) = 2^n + n!$, for all $n \in \mathbb{N}$, a is d'Alembertian, since
$(((n-1) \bullet \mathcal{N}^1 - (n \cdot (n+1)) \bullet \mathcal{N}^0) \circ (\mathcal{N} - 2))(a)(m) =$
$((n-1) \bullet \mathcal{N}^1 - (n \cdot (n+1)) \bullet \mathcal{N}^0)(\mathcal{N}(a) - 2 * a)(m) =$
$(m-1) \cdot \mathcal{N}(\mathcal{N}(a) - 2 * a)(m) - m \cdot (m+1) \cdot (\mathcal{N}(a) - 2 * a)(m) =$
$(m-1) \cdot (\mathcal{N}^2(a) - 2 * \mathcal{N}(a))(m) - m \cdot (m+1) \cdot (\mathcal{N}(a) - 2 \cdot a)(m) =$
$(m-1) \cdot (a(m+2) - 2 \cdot a(m+1)) - m \cdot (m+1) \cdot (a(m+1) - 2 \cdot a(m)) =$
$(m-1) \cdot a(m+2) - (m^2 + 3 \cdot m - 2) \cdot a(m+1) + 2 \cdot m \cdot (m+1) \cdot a(m) =$
$(m-1) \cdot (2^{m+2} + (m+2)!) - (m^2 + 3 \cdot m - 2) \cdot (2^{m+1} + (m+1)!) + 2 \cdot m \cdot (m+1) \cdot (2^m + m!) =$
$((m-1) \cdot 4 - (m^2 + 3 \cdot m - 2) \cdot 2 + 2 \cdot m \cdot (m+1)) \cdot 2^m +$
$((m-1) \cdot (m+2) \cdot (m+1) - (m^2 + 3 \cdot m - 2) \cdot (m+1) + 2 \cdot m \cdot (m+1)) \cdot m! =$
$0 \cdot 2^m + 0 \cdot m! = 0$, for all $m \in \mathbb{N}$.

Note that every hypergeometric sequence is also d'Alembertian.

Proposition 1.4.10. D'Alembertian sequences form a unital ring
$(\{a \in K^{\mathbb{N}} \mid a \text{ is d'Alembertian}\}, +, \cdot)$ is a unital ring (cf. Section 8.6 of [Petkovšek et al.]).

Definition 1.4.11. Let $a \in K^{\mathbb{N}} \backslash J$. Then a is said to be **C-recursive** (resp. **P-recursive**, or **holonomic**) if there exists a linear recurrence K-operator with constant (resp. polynomial) coefficients L such that $L(a) = 0$ a.e.

For example, given $a \in K^{\mathbb{N}}$ such that $a(n) = 3^n$, for all $n \in \mathbb{N}$, a is C-recursive, since $(\mathcal{N} - 3)(a) = 0$.

Note that every d'Alembertian sequence is also holonomic.

Proposition 1.4.12. Every C-recursive sequence on an algebraically closed field is d'Alembertian Let $a \in K^{\mathbb{N}}$ such that a is C-recursive. If K is algebraically closed, then a is d'Alembertian (cf. Section 3 of [Petkovšek & Zakrajšek]).

It is interesting to remark that, for every possible order being greater than 1, there exist homogeneous linear difference K-equations with polynomial coefficients of such an order having nontrivial nonhypergeometric solutions, eg. the difference K-equation given by $y(n+d) = \sum_{j=1}^{d}(\prod_{i=1}^{j-1}(n+d-i) \cdot y(n+d-j))$, for all $d \in \mathbb{N}$ such that $2 \leqslant n$ (cf. Example 8.4.3 of [Petkovšek et al.]; this example is interesting also because $y \in K^{\mathbb{N}}$ such that $y(0) = 0$ and $y(n)$ is the number of n-permutations that contain no cycle longer than d, for all $n \in \mathbb{N}^+$, is a solution of such equation). In Section 1.6 an algorithm for deciding constructively if a given homogeneous linear difference K-equation with polynomial coefficients has hypergeometric solutions, the so-called Petkovšek's Hyper algorithm, will be explained.

Definition 1.4.13. Let $a \in K^{\mathbb{N}}$. Then it is said that a has **hypergeometric closed form** if $a \in K\langle \mathrm{hyp}(K) \rangle$.

For example, given $a, b, c \in K^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a(n) = 3^n$, $b(n) = (n+2)!$ and $c(n) = 7 \cdot 3^n + (n+2)!$, c has hypergeometric closed form, since $c = 7 \cdot a + 1 \cdot b$ and $a, b \in \mathrm{hyp}(K)$.

Proposition 1.4.14. Every sequence having hypergeometric closed form is d'Alembertian Let $a \in K^{\mathbb{N}}$ such that a has hypergeometric closed form. Then a is d'Alembertian.

Proof (To be read after Section 1.7) Let $\alpha_0, \alpha_2 \in K$ and $a_0, a_1 \in \mathrm{hyp}(K)$ such that $a_0 \neq a_1$ a.e. Then there exists a linear recurrence K-operator with polynomial coefficients L_0 of order 1 such that $L_0(a_0) = 0$ a.e. Applying that $a_0 \neq a_1$ a.e. and Lemma 1.7.7, $L_0(a_1) \in \mathrm{hyp}(K)$; which yields the existence of a linear recurrence K-operator with polynomial coefficients $\tilde{L}$ of order 1 such that $\tilde{L}(L_0(a_1)) = 0$ a.e. Thus, $(\tilde{L} \circ L_0)(\alpha_0 * a_0 + \alpha_1 * a_1) = \alpha_0 * \tilde{L}(L_0(a_0)) + \alpha_1 * \tilde{L}(L_0(a_1)) = \alpha_0 * \tilde{L}(0) + \alpha_1 * 0 = 0$ a.e.; so $\alpha_0 * a_0 + \alpha_1 * a_1$ is d'Alembertian. □

Hence, it is clear that no nonholonomic sequence has hypergeometric closed form. Unfortunately, this is the case of several sequences of central importance in mathematics, eg.:

- $a_0 \in \mathbb{C}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a_0(n)$ is the number of rooted $(n+1)$-labeled trees (recall that $a_0(n) = \frac{(n+1)^n}{(n+1)!}$, for all $n \in \mathbb{N}$),
- $a_1 \in \mathbb{C}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a_1(n)$ is the number of partitions of $n+1$ (however, even a_1 not being annihilated a.e. by any linear recurrence $\mathbb{C}$-operator with polynomial coefficients, at least $\sum_{k \in \mathbb{Z}}\left((-1)^k \cdot a_1\left(n - \frac{k \cdot (3 \cdot k+1)}{2}\right)\right) = 0$, for all $n \in \mathbb{N}$; and, considering $r \in \mathbb{N}^+$ and $b_r \in \mathbb{C}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $b_r(n)$ is the number of partitions of $n+1$ of at most r parts, then b_r is holonomic),
- $a_2 \in \mathbb{C}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a_2(n)$ is the number of $(n+1)$-derangements (i.e. $(n+1)$-permutations without fixed points) (however, a_2 has "nonhypergeometric closed form", viz. $a_2(n) = \lfloor \frac{(n+1)!}{e} \rfloor$, for all $n \in \mathbb{N}$),
- $a_3 \in \mathbb{C}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a_3(n)$ is the n^{th} central trinomial coefficient (i.e. given an indeterminate t over $\mathbb{C}$, $a_3(n) = \mathrm{coeff}_n((1+t+t^2)^n)$, for all $n \in \mathbb{N}$),

- $a_4 \in \mathbb{C}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a_4(n)$ is the n^{th} Bell number (however, even a_4 not being annihilated a.e. by any linear recurrence $\mathbb{C}$-operator with polynomial coefficients, at least $\sum_{k=0}^{n} \left(\binom{n}{k} \cdot a_4(k)\right) = a_4(n+1)$, for all $n \in \mathbb{N}$),
- $a_5 \in \mathbb{C}^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a_5(n)$ is the n^{th} prime number (cf. Section 4 of [Flajolet et al.]).

However, some generalizations of nonholonomic sequences are sometimes holonomic, eg. considering $a \in K^{\mathbb{N}}$ and $b \in K(t)^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a(n) = (n+1)^n$ and $b(n) = (t+1)^n$, a is not holonomic, but b is. Section 9.11 of [Petkovšek et al.] contains some comments about how to determine if there exists a holonomic generalization of a given sequence. A general study of the holonomic theory can be found in [Parnes].

Definition 1.4.15. Let $a \in K^{\mathbb{N}}$. Then a is said to be **geometric** if there exist $\alpha \in K$ and $m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, $a(n) \neq 0$ and $\alpha = \frac{a(n+1)}{a(n)}$.

For example, given $a \in K^{\mathbb{N}}$ such that $a(n) = \begin{cases} 5 & \text{if } n = 9 \\ 2^n & \text{otherwise} \end{cases}$, for all $n \in \mathbb{N}$, a is geometric, since $\frac{a(n+1)}{a(n)} = 2$, for all $n \in \mathbb{N}$ such that $n \leqslant 10$.

Note that every geometric sequence is also hypergeometric, by Proposition 1.4.6, and C-recursive.

1.5 GOSPER'S FACTORIZATION

One of the main results that underpins the theory developed in this thesis is the so-called Gosper's Factorization Theorem. It is purely algebraic, and provides a very useful way of expressing rational functions.

Proposition 1.5.1. Gosper's Factorization Theorem
Let $r(t) \in K(t)\backslash\{0\}$. Then there exist unique $a(t), b(t), c(t) \in K[t]\backslash\{0\}$ such that the following conditions hold:
1. $b(t)$ and $c(t)$ are monic,
2. $r(t) = \frac{a(t) \cdot c(t+1)}{b(t) \cdot c(t)}$,
3. $\text{g.c.d.}(\{a(t), b(t+h)\}) = \text{g.c.d.}(\{a(t), c(t)\}) = \text{g.c.d.}(\{b(t), c(t+1)\}) = 1$, for all $h \in \mathbb{N}$

(cf. Theorem 5.3.1 and Corollary 5.3.1 of [Petkovšek et al.]).

An algorithm computing Gosper's factorization of a given rational expression can be found in Section 5.3 of [Petkovšek et al.].

Lemma 1.5.2. Let $a(t), b(t), c(t), A(t), B(t), C(t) \in K[t]\backslash\{0\}$ such that the following conditions hold:
1. $\frac{a(t) \cdot c(t+1)}{b(t) \cdot c(t)} = \frac{A(t) \cdot C(t+1)}{B(t) \cdot C(t)}$,
2. $\text{g.c.d.}(\{A(t), B(t+h)\}) = \text{g.c.d.}(\{a(t), c(t)\}) = \text{g.c.d.}(\{b(t), c(t+1)\}) = 1$, for all $h \in \mathbb{N}$.

Then $c(t) \mid C(t)$ (cf. Lemma 5.3.1 of [Petkovšek et al.]).

Proposition 1.5.3. Rationality criterion for hypergeometric sequences
Let $r(t) \in K(t)\backslash\{0\}$, $a(t), b(t), c(t), A(t), B(t), C(t) \in K[t]\backslash\{0\}$ such that the following conditions hold:

1. $b(t), c(t), B(t), C(t)$ are monic,
2. g.c.d.$(\{a(t), b(t+h)\})$ = g.c.d.$(\{a(t), c(t)\})$ = g.c.d.$(\{b(t), c(t+1)\}) = 1$, for all $h \in \mathbb{N}$,
3. g.c.d.$(\{A(t), B(t+h)\})$ = g.c.d.$(\{A(t), C(t)\})$ = g.c.d.$(\{B(t), C(t+1)\}) = 1$, for all $h \in \mathbb{N}$,
4. $r(t) = \frac{A(t) \cdot C(t+1)}{B(t) \cdot C(t)}$,
5. $\frac{B(t)}{A(t)} = \frac{a(t) \cdot c(t+1)}{b(t) \cdot c(t)}$,

and $y \in hyp(K)$ such that there exists $m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, n is not a pole of $r(t)$, $t(n) \neq 0$ and $\frac{y(n+1)}{y(n)} = r(n)$. Then $y \in rat(K)$ if, and only if, [$A(t)$ is monic and $a(t) = b(t) = 1$].

Proof

$\Rightarrow$) $y \in rat(K) \cap hyp(K)$, so there exist $q(t) \in K(t)\backslash\{0\}$ and $\tilde{m} \in \mathbb{N}$ such that $m \leqslant \tilde{m}$ and, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$, n is not a pole of $q(t)$ and $y(n) = q(n)$. Thus, there exist $p(t), \tilde{p}(t) \in K[t]\backslash\{0\}$ such that g.c.d.$(\{p(t), \tilde{p}(t)\}) = 1$, $\tilde{p}(t)$ is monic and $\frac{p(t)}{\tilde{p}(t)} = q(t)$, which, applying Proposition 1.4.6 and condition 4, yields that $\frac{\tilde{p}(t)}{\tilde{p}(t+1)} \cdot \frac{p(t+1)}{p(t)} = \frac{q(t+1)}{q(t)} = r(t) = \frac{A(t) \cdot C(t+1)}{B(t) \cdot C(t)}$. [0]

[0] yields that $\tilde{p}(t) \cdot p(t+1) \cdot B(t) \cdot C(t) = A(t) \cdot C(t+1) \cdot \tilde{p}(t+1) \cdot p(t)$; so, as $B(t), C(t), \tilde{p}(t)$ are monic, l.c.$(p(t))$ = l.c.$(A(t)) \cdot$ l.c.$(p(t))$. And, as l.c.$(p(t)) \neq 0$, l.c.$(A(t)) = 1$; i.e. $A(t)$ is monic.

From [0], condition 3 and Lemma 1.5.2 follows that $p(t) \mid C(t)$, i.e. that there exists $s(t) \in K[t]$ such that $s(t) \cdot p(t) = C(t)$, which, again by [0], implies that $\frac{\tilde{p}(t)}{\tilde{p}(t+1)} = \frac{A(t) \cdot s(t+1)}{B(t) \cdot s(t)}$ holds, i.e. $\frac{B(t)}{A(t)} = \frac{1 \cdot \tilde{p}(t+1) \cdot s(t+1)}{1 \cdot \tilde{p}(t) \cdot s(t)}$. [1]

Applying Proposition 1.5.1 and conditions 1, 2 and 5, $\tilde{a}(t) = a(t), \tilde{b}(t) = b(t)$ and $\tilde{c}(t) = c(t)$, for all $\tilde{a}(t), \tilde{b}(t), \tilde{c}(t) \in K[t]$ such that $\tilde{b}(t), \tilde{c}(t)$ are monic, $\frac{B(t)}{A(t)} = \frac{\tilde{a}(t) \cdot \tilde{c}(t+1)}{\tilde{b}(t) \cdot \tilde{c}(t)}$ and g.c.d.$(\{\tilde{a}(t), \tilde{b}(t+h)\})$ = g.c.d.$(\{\tilde{a}(t), \tilde{c}(t)\})$ = g.c.d.$(\{\tilde{b}(t), \tilde{c}(t+1)\}) = 1$, for all $h \in \mathbb{N}$. Therefore, applying [1] and condition 5, $c(t) = \tilde{p}(t) \cdot s(t)$ and $a(t) = b(t) = 1$.

$\Leftarrow$) $r(t) \cdot \frac{C(t)}{C(t+1)} = \frac{A(t)}{B(t)} = \frac{b(t) \cdot c(t)}{a(t) \cdot c(t+1)}$, by conditions 4 and 5. Hence, applying that $a(t) = b(t) = 1$, $r(t) = \frac{c(t) \cdot C(t+1)}{c(t+1) \cdot C(t)}$ holds, i.e. there exist $\alpha \in K\backslash\{0\}$ and $\tilde{m} \in \mathbb{N}$ such that $m \leqslant \tilde{m}$ and, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$, $c(n) \neq 0$ and $y(n) = \alpha \cdot \frac{C(n)}{c(n)}$. Therefore, $y \in rat(K)$. □

1.6 POLY AND HYPER ALGORITHMS

With the concepts and results appeared so far, there has been reached the conditions to build a first couple of algorithms from which it will be later constructed Petkovšek's complete Hyper algorithm: Abramov - Petkovšek Poly algorithm and Petkovšek's Hyper algorithm. They are explained now.

From now on, given $a \in rat(K)$ and $r(t) \in K(t)$ such that $\{n \in \mathbb{N} \mid [n \text{ is not a pole of } r(t) \wedge r(n) \neq a(n)]\}$ is finite, $r(t)$ will be denoted by $a(t)$.

In addition, from now on, given $a \in pol(K)$, an indeterminate t over K and $m \in \mathbb{N}$, $\deg(a(t))$ (i.e. degree of $a(t)$) (resp. l.c.$(a(t))$ (i.e. leading coefficient of $a(t)$), $\text{coeff}_m(a(t))$ (i.e. m^{th} coefficient of $a(t)$)) will be denoted by $\deg(a)$ (resp. l.c.(a), $\text{coeff}_m(a)$); and recall that $\deg(0) = -\infty$ and that, if $\deg(a) < m$, then $\text{coeff}_m(a) = 0$.

Proposition 1.6.1. Foundation of Abramov - Petkovšek Poly algorithm
Let $r \in \mathbb{N}$, $p_0, ..., p_r, f, y \in \mathrm{pol}(K)$ such that $p_0 \cdot p_r \neq 0$ a.e., $L = \sum_{i=0}^{r}(p_i \bullet \mathcal{N}^i)$, $q : \{0, ..., r\} \longrightarrow \mathrm{pol}(K)$ such that $q(j) = \sum_{i=j}^{r}\left(\binom{i}{j} * p_i\right)$, for all $j \in \mathrm{dom}(q)$, $b = \max(\{\deg(q(i)) - i\}_{i=0}^{r})$, $S = \{i \in \{0, ..., r\} \mid \deg(q(i)) - i = b\}$, $a \in K^{\mathbb{N}}$ such that $a(n) = \sum_{s \in S}\left(\mathrm{l.c.}(q(s)) \cdot \prod_{i=0}^{s-1}(n-i)\right)$, for all $n \in \mathbb{N}$, $d_1 = \max(\{n \in \mathbb{N} \mid a(n) = 0\})$ and $d = \max(\{\deg(f) - b, -b-1, d_1\})$. If $L(y) = f$ a.e., then $\deg(y) \leqslant d$ (cf. Section 8.3 of [Petkovšek et al.]).

The following algorithm, called Abramov - Petkovšek Poly algorithm, computes all the polynomial solutions of a given linear difference K-equation with polynomial coefficients and polynomial independent term.

Abramov - Petkovšek Poly algorithm	Example
Input: $r \in \mathbb{N}$, and $p_0, ..., p_r, f \in \mathrm{pol}(K)$ such that $p_0 \cdot p_r \neq 0$ a.e.	Input: $p_0(n) := -3 \cdot (2 \cdot n + 1)$, $p_1(n) := 13 \cdot n + 5$, $p_2(n) := -7 \cdot n$, $f(n) := 0$.
1. Compute $q : \{0, ..., r\} \longrightarrow \mathrm{pol}(K)$ such that $q(j) = \sum_{i=j}^{r}\left(\binom{i}{j} * p_i\right)$, for all $j \in \mathrm{dom}(q)$.	1. $q(0)(n) := \sum_{i=0}^{2}\left(\binom{i}{0} \cdot p_i(n)\right) = \sum_{i=0}^{2}(1 \cdot p_i(n)) = -3 \cdot (2 \cdot n + 1) + 13 \cdot n + 5 - 7 \cdot n = 2$, $q(1)(n) := \sum_{i=1}^{2}\left(\binom{i}{1} \cdot p_i(n)\right) = \sum_{i=1}^{2}(i \cdot p_i(n)) = 13 \cdot n + 5 + 2 \cdot (-7 \cdot n) = 5 - n$, $q(2)(n) := \sum_{i=2}^{2}\left(\binom{i}{2} \cdot p_i(n)\right) = 1 \cdot p_2(n) = -7 \cdot n$.
2. Compute $d := \max(\{\deg(f) - b, -b-1, d_1\})$, being: • $d_1 = \max(\{n \in \mathbb{N} \mid a(n) = 0\})$, • $a \in K^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a(n) = \sum_{s \in S}\left(\mathrm{l.c.}(q(s)) \cdot \prod_{i=0}^{s-1}(n-i)\right)$, • $S = \{i \in \{0, ..., r\} \mid \deg(q(i)) - i = b\}$, • $b = \max(\{\deg(q(i)) - i\}_{i=0}^{r})$.	2. $b := \max(\{\deg(q(i)) - i\}_{i=0}^{2}) = \max(\{0-0, 1-1, 1-2\}) = 0$, $S := \max(\{i \in \{0,1,2\} \mid \deg(q(i)) - i = 0\}) = \{0, 1\}$, $a(n) := \sum_{s=0}^{1}\left(\mathrm{l.c.}(q(s)) \cdot \prod_{i=0}^{s-1}(n-i)\right) = 2 \cdot 1 + (-1) \cdot (n - 0) = 2 - n$, $d_1 := \max(\{n \in \mathbb{N} \mid 2 - n = 0\}) = 2$, $d := \max(\{\deg(0) - 0, -0-1, 2\}) = \max(\{-\infty, -1, 2\}) = 2$.
3. Using the so-called **method of indeterminate coefficients** (i.e. setting up a general polynomial of degree d, plugging it into the recurrence relation, equating the coefficients of like powers of the variable and solving the resulting system of linear algebraic K-equations for $d+1$ unknown coefficients), compute and return the set of $y \in \mathrm{pol}(K)$ such that there exist $\vec{c} \in K^{d+1}$ and $m \in \mathbb{N}$ such that $\sum_{i=0}^{d}(\vec{c}(i) \cdot n^i) = y(n)$, for all $n \in \mathbb{N}$ such that $m \leqslant n$ and that $\sum_{i=0}^{r}(p_i \cdot \mathcal{N}^i(y)) = f$ a.e., and STOP.	3. $(-3 \cdot (2 \cdot n + 1)) \cdot \mathcal{N}^0(y) + (13 \cdot n + 5) \cdot \mathcal{N}^1(y) + (-7 \cdot n) \cdot \mathcal{N}^2(y) = 0 \Leftrightarrow$ $(-3 \cdot (2 \cdot n + 1)) \cdot (c_0 + c_1 \cdot n + c_2 \cdot n^2) + (13 \cdot n + 5) \cdot (c_0 + c_1 \cdot (n+1) + c_2 \cdot (n+1)^2) + (-7 \cdot n) \cdot (c_0 + c_1 \cdot (n+2) + c_2 \cdot (n+2)^2) = 0 \Leftrightarrow$ $(c_1 - 5 \cdot c_2) \cdot n + 2 \cdot c_0 + 5 \cdot (c_1 + c_2) = 0 \Leftrightarrow$ $[c_1 - 5 \cdot c_2 = 0 \ \wedge \ 2 \cdot c_0 + 5 \cdot (c_1 + c_2) = 0] \Leftrightarrow$ $[c_0 = -15 \cdot c_2 \ \wedge \ c_1 = 5 \cdot c_2 \ \wedge \ c_2 = c_2]$.

Note that, if the method of coefficients fails (i.e. if the returned set is empty), then the initial difference K-equation has certainly no polynomial solution, by Proposition 1.6.1.

Output: $\left\{y \in \mathrm{pol}(K) \mid \sum_{i=0}^{r}(p_i \cdot \mathcal{N}^i)(y) = f \text{ a.e.}\right\}$.	Output: $\{y \in \mathrm{pol}(K) \mid \exists\, c \in K$ such that $\exists\, m \in \mathbb{N}$ such that, $\forall\, n \in \mathbb{N}$ such that $m \leqslant n$, $y(n) = c \cdot (n^2 + 5 \cdot n - 15)\}$.

The solutions are easy to check, eg. following the previous example, it would be simply a matter of evaluating $y(n) = c \cdot (n^2 + 5 \cdot n - 15)$ in the expression $-(3 \cdot (2 \cdot n + 1)) \cdot y(n) + (13 \cdot n + 5) \cdot y(n+1) - 7 \cdot n \cdot y(n+2)$ and check that indeed such an expression equals zero.

Proposition 1.6.2. Foundation of Petkovšek's Hyper algorithm

Let F be a field such that $K \leqslant_{\textbf{Field}} F$, x an indeterminate over F, $d \in \mathbb{N}$, $p_0, ..., p_d \in \mathrm{pol}(K)$ such that $p_0 \cdot p_d \neq 0$ a.e., $y \in F^{\mathbb{N}}$ such that $\sum_{i=0}^{d}\left(p_i \cdot \mathcal{N}^i(y)\right) = 0$ a.e. and that there exist $s \in \mathrm{rat}(F)$, $z \in F$ and $a, b, c \in \mathrm{pol}(F)$ such that the following conditions hold:

- $\mathcal{N}(y) = s \cdot y$ a.e.,
- $s(x) = z \cdot \frac{a(x) \cdot c(x+1)}{b(x) \cdot c(x)}$,
- g.c.d.$(\{a(x), b(x+h)\})$ = g.c.d.$(\{a(x), c(x)\})$ = g.c.d.$(\{b(x), c(x+1)\}) = 1$, for all $h \in \mathbb{N}$,

$P : \{0, ..., d\} \longrightarrow \mathrm{pol}(F)$ such that $P(i)(x) = p_i(x) \cdot \prod_{j=0}^{i-1}(a(x+j)) \cdot \prod_{j=i}^{d-1}(b(x+j))$, for all $i \in \{0, ..., d\}$, $m = \max(\{\deg(P(i))\}_{i=0}^{d})$ and $\alpha : \{0, ..., d\} \longrightarrow F$ such that $\alpha(i) = \mathrm{coeff}_m(P(i))$, for all $i \in \{0, ..., d\}$. Then the following conditions hold:

1. $\sum_{i=0}^{d}(\alpha(i) \cdot z^i) = 0$,
2. $a(x) \mid p_0(x)$,
3. $b(x) \mid p_d(x - d + 1)$,
4. $\sum_{i=0}^{d}(z^i \cdot P(i)(x) \cdot c(x+i)) = 0$.

Proof $\sum_{i=0}^{d}\left(p_i \cdot \mathcal{N}^i(y)\right) = 0$ a.e., so there exists $n_1 \in \mathbb{N}$ such that $\sum_{i=0}^{d}\left(p_i(n) \cdot y(n+i)\right) = 0$, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$. Applying Proposition 1.4.6, there exists $n_2 \in \mathbb{N}$ such that $n_1 \leqslant n_2$ and, for all $n \in \mathbb{N}$ such that $n_2 \leqslant n$, $y(n) \neq 0$ and $\sum_{i=0}^{d}\left(p_i(n) \cdot \prod_{j=0}^{i-1}\left(\frac{y(n+j+1)}{y(n+j)}\right)\right) \cdot y(n) = 0$. [0]

$\mathcal{N}(y) = s \cdot y$ a.e., so there exists $n_3 \in \mathbb{N}$ such that $n_2 \leqslant n_3$ and $y(n+1) = s(n) \cdot y(n)$, for all $n \in \mathbb{N}$ such that $n_3 \leqslant n$. Thus, applying [0], $\sum_{i=0}^{d}\left(p_i(n) \cdot \prod_{j=0}^{i-1}(s(n+j))\right) \cdot y(n) = 0$ holds, for all $n \in \mathbb{N}$ such that $n_3 \leqslant n$, ergo $\sum_{i=0}^{d}\left(p_i(x) \cdot \prod_{j=0}^{i-1}(s(x+j))\right) = 0$.

As $s(x) = z \cdot \frac{a(x) \cdot c(x+1)}{b(x) \cdot c(x)}$, $\sum_{i=0}^{d}\left(p_i(x) \cdot z^i \cdot \prod_{j=0}^{i-1}\left(\frac{a(x+j)}{b(x+j)}\right) \cdot \frac{c(x+i)}{c(x)}\right) = 0$ holds,

so $\sum_{i=0}^{d}\left(p_i(x) \cdot z^i \cdot \prod_{j=0}^{i-1}(a(x+j)) \cdot c(x+i) \cdot \prod_{j=i}^{d-1}(b(x+j))\right) = 0$

and then $\sum_{i=0}^{d}(z^i \cdot P(i)(x) \cdot c(x+i)) = 0$. [1]

$\sum_{i=0}^{d}(\alpha(i) \cdot z^i) = 0$, by [1].

Moreover, again applying [1], $a(x) \mid z^0 \cdot P(0)(x) \cdot c(x+0) = p_0(x) \cdot c(x) \cdot \prod_{j=0}^{d-1}(b(x+j))$ and hence $a_0(x) \mid p_0(x)$, since g.c.d.$(\{a(x), b(x+h)\})$ = g.c.d.$(\{a(x), c(x)\})$ = g.c.d.$(\{b(x), c(x+1)\}) = 1$, for all $h \in \mathbb{N}$.

Similarly, it can be proven that $b(x+d-1) \mid z^d \cdot p_d(x) \cdot c(x+d) \cdot \prod_{j=0}^{d-1}(a(x+j))$, so, again by the fact that g.c.d.$(\{a(x), b(x+h)\})$ = g.c.d.$(\{a(x), c(x)\})$ = g.c.d.$(\{b(x), c(x+1)\}) = 1$, for all $h \in \mathbb{N}$, $b(x+d-1) \mid p_d(x)$ holds, i.e. $b(x) \mid p_d(x-d+1)$ (changing the variables). □

The following algorithm, called Petkovšek's Hyper algorithm, decides constructively if there exists a hypergeometric solution of a given homegeneous linear difference K-equation with polynomial coefficients.

Petkovšek's Hyper algorithm	Example
Input: a field F such that $K \leqslant_{\mathbf{Field}} F$, $d \in \mathbb{N}$ and $p_0, ..., p_d \in \mathrm{pol}(K)$ such that $p_0 \cdot p_d \neq 0$ a.e.	Input: $p_0(n) := 4 \cdot (n+1)^2 \cdot (2 \cdot n+1) \cdot (2 \cdot n+3)$, $p_1(n) := -2 \cdot (2 \cdot n+3)^2$, $p_2(n) := 1$, $F := K$.
1. For all $a(x) \in F[x]$ such that $a(x)$ is monic and $a(x) \mid p_0(x)$, and $b(x) \in F[x]$ such that $b(x)$ is monic and $b(x) \mid p_d(x-d+1)$, do:	1. $\{a(x) \in K[x] \mid [a(x) \text{ is monic } \wedge\ a(x) \mid p_0(x)]\} = \{1, x+1, x+\frac{1}{2}, x+\frac{3}{2}, x^2+2 \cdot x+1, x^2+2 \cdot x+\frac{3}{4}, x^2+\frac{3}{2} \cdot x+\frac{1}{2}, x^2+\frac{5}{2} \cdot x+\frac{3}{2}, x^3+3 \cdot x^2+\frac{11}{4} \cdot x+\frac{3}{4}, x^3+\frac{5}{2} \cdot x^2+2 \cdot x+\frac{1}{2}, x^3+\frac{7}{2} \cdot x^2+4 \cdot x+\frac{3}{2}, x^4+4 \cdot x^3+\frac{23}{4} \cdot x^2+\frac{7}{2} \cdot x+\frac{3}{4}\}$, $\{b(x) \in K[x] \mid [b(x) \text{ is monic } \wedge\ b(x) \mid p_2(x-2+1)]\} = 1$. The case $a(x) = x^2+\frac{3}{2} \cdot x+\frac{1}{2}$ and $b(x) = 1$ satisfies the condition of the step 1.4.1, so from this point it is described how the algorithm work for these values.
1.1. compute $P : \{0, ..., d\} \longrightarrow \mathrm{pol}(F)$ such that $P(i)(x) = p_i(x) \cdot \prod_{j=0}^{i-1}(a(x+j)) \cdot \prod_{j=i}^{d-1}(b(x+j))$, for all $i \in \{0, ..., d\}$,	1.1. $P(0)(x) := p_0(x) \cdot \prod_{j=0}^{0-1}(a(x+j)) \cdot \prod_{j=0}^{1}(1) = p_0(x) = 16 \cdot x^4+64 \cdot x^3+92 \cdot x^2+56 \cdot x+12$, $P(1)(x) := p_1(x) \cdot \prod_{j=0}^{1-1}(a(x+j)) \cdot \prod_{j=1}^{1}(1) = p_1(x) \cdot a(x) = -8 \cdot x^4-36 \cdot x^3-58 \cdot x^2-39 \cdot x-9$, $P(2)(x) := p_2(x) \cdot \prod_{j=0}^{2-1}(a(x+j)) \cdot \prod_{j=2}^{1}(1) = a(x) \cdot a(x+1) = x^4+5 \cdot x^3+\frac{35}{4} \cdot x^2+\frac{25}{4} \cdot x+\frac{3}{2}$.
1.2. $m := \max(\{\deg(P(i))\}_{i=0}^{d})$,	1.2. $m := \max(\{\deg(P(i))\}_{i=0}^{2}) = 4$.
1.3. compute $\alpha : \{0, ..., d\} \longrightarrow F$ such that $\alpha(i) = \mathrm{coeff}_m(P(i))$, for all $i \in \{0, ..., d\}$,	1.3. $\alpha(0) := 16$, $\alpha(1) := -8$, $\alpha(2) := 1$.
1.4. for all $z \in F \backslash \{0\}$ such that $\sum_{i=0}^{d}(\alpha(i) \cdot z^i) = 0$, do:	1.4. $\{z \in F \backslash \{0\} \mid 16-8 \cdot z+z^2 = 0\} = \{4\}$.
1.4.1. if there exists $c \in \mathrm{pol}(F) \backslash \{0\}$ such that $\sum_{i=0}^{d}(z^i \cdot P(i)(x) \cdot c(x+i)) = 0$ (this question can be solved by applying Abramov - Petkovšek Poly algorithm), then:	1.4.1. Note that $1 \in \{c \in \mathrm{pol}(K) \backslash \{0\} \mid P(0)(x) \cdot c(x)+4 \cdot P(1)(x) \cdot c(x+1)+16 \cdot P(2)(x) \cdot c(x+2) = 0\}$.
1.4.1.1. compute $s \in \mathrm{rat}(F)$ such that $s(x) = z \cdot \frac{a(x) \cdot c(x+1)}{b(x) \cdot c(x)}$,	1.4.1.1. $s(x) := 4 \cdot a(x)$.
1.4.1.2. compute $y \in \mathrm{hyp}(F)$ such that $\mathcal{N}(y) = s \cdot y$ a.e. and STOP;	1.4.1.2. $y(n) := (2 \cdot n)!$ (note that $(2 \cdot (n+1))! = 4 \cdot (n^2+3/2 \cdot n+1/2) \cdot (2 \cdot n)!$, for all $n \in \mathbb{N}$).
1.4.2. otherwise, return "There exists no hypergeometric solution over F." and STOP.	1.4.2. ———
Output: $y \in \mathrm{hyp}(F)$ such that $\sum_{k=0}^{d}\left(p_k \cdot \mathcal{N}^k(y)\right) = 0$ a.e., if it exists; "There exists no hypergeometric solution over F." otherwise.	Output: $(2 \cdot n)!$.

In Section 1.8, the so-called Petkovšek's complete Hyper algorithm, which computes all the solutions having "closed form" (in a sense that will be formalized) of a given homogeneous linear difference K-equation with polynomial coefficients, will be explained.

Proposition 1.6.3. A partial converse to the foundation of Petkovšek's Hyper algorithm
Let F be a field such that $K \leqslant_{\mathbf{Field}} F$, $d \in \mathbb{N}$, $p_0, \ldots, p_d \in \mathrm{pol}(K)$ such that $p_0 \cdot p_d \neq 0$ a.e., $z \in F$, $a, b, c, s \in F^{\mathbb{N}}$ such that there exists $n_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $n_0 \leqslant n$, $b(n) \cdot c(n) \neq 0$ and $s(n) = z \cdot \frac{a(n) \cdot c(n+1)}{b(n) \cdot c(n)}$, $P : \{0, \ldots, d\} \longrightarrow F^{\mathbb{N}}$ such that there exists $n_1 \in \mathbb{N}$ such that $\sum_{i=0}^{d} (z^i \cdot P(i)(n) \cdot c(n+i)) = 0$ and $P(i)(n) = p_i(n) \cdot \prod_{j=0}^{i-1} (a(n+j)) \cdot \prod_{j=i}^{d-1} (b(n+j))$, for all $i \in \{0, \ldots, d\}$ and $n \in \mathbb{N}$ such that $n_1 \leqslant n$, and $y \in F^{\mathbb{N}}$ such that $\mathcal{N}(y) = s \cdot y$ a.e. Then $\sum_{k=0}^{d} \left(p_k \cdot \mathcal{N}^k(y)\right) = 0$ a.e. (cf. Section 8.4 of [Petkovšek et al.]).

1.7 SIMILARITY

The next essential notion is a equivalence relation between hypergeometric sequences called similarity. This relation is especially useful, since it provides a criterion to determine if a finite sum of hypergeometric sequences remains hypergeometric, as it will be shown below.

Definition 1.7.1. Let $a, b \in \mathrm{hyp}(K)$. Then a and b are said to be **similar** if there exist $r(t) \in K(t)$ and $m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, n is not a pole of $r(t)$, $b(n) \neq 0$ and $r(n) = \frac{a(n)}{b(n)}$, fact which is denoted by $a \underset{\mathrm{hyp}}{\sim} b$.

For example, given $a, b \in K^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $b(n) = \begin{cases} 3 & \text{if } n < 15 \\ (n+2)! & \text{otherwise} \end{cases}$ and $a(n) = n!$, $a \underset{\mathrm{hyp}}{\sim} b$, since $\frac{a(n)}{b(n)} = \frac{1}{(n+1) \cdot (n+2)}$, for all $n \in \mathbb{N}$ such that $15 \leqslant n$.

Note that $\left\{ (a, b) \in \mathrm{hyp}(K)^2 \mid a \underset{\mathrm{hyp}}{\sim} b \right\}$ is an equivalence relation on $\mathrm{hyp}(K)$.

Lemma 1.7.2. Let $a \in \mathrm{hyp}(K)$ such that $\Delta(a) \neq 0$ a.e. Then $a \underset{\mathrm{hyp}}{\sim} \Delta(a)$.

Proof $a \in \mathrm{hyp}(K)$, so, applying Proposition 1.4.6, there exist $r(t) \in K(t)$ and $m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, n is not a pole of $r(t)$, $a(n) \neq 0$ and $r(n) = \frac{a(n+1)}{a(n)}$. Let $q(t) = r(t) - 1$. It is clear that $q(t) \in K(t)$, and $\frac{\Delta(a)(n)}{a(n)} = \frac{a(n+1) - a(n)}{a(n)} = \frac{a(n+1)}{a(n)} - 1 = r(n) - 1 = q(n)$, for all $n \in \mathbb{N}$ such that $m \leqslant n$. Therefore, as $\Delta(a) \neq 0$ a.e., applying Lemma 1.4.7, $a \underset{\mathrm{hyp}}{\sim} \Delta(a)$. □

Proposition 1.7.3. Criterion for checking the hypergeometric character of a finite sum
Let $a, b \in \mathrm{hyp}(K)$ such that $a + b \neq 0$ a.e.. Then $a + b \in \mathrm{hyp}(K)$ if, and only if, $a \underset{\mathrm{hyp}}{\sim} b$.

Proof $a, b \in \mathrm{hyp}(K)$; so, applying Proposition 1.4.6, there exist $n_0 \in \mathbb{N}$, $A, B \in \mathrm{rat}(K)$ and $D \in K^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$ such that $n_0 \leqslant n$, n is not a pole neither of A nor of B, $0 \notin \{a(n), b(n)\}$, $A(n) = \frac{a(n+1)}{a(n)}$, $B(n) = \frac{b(n+1)}{b(n)}$ and $D(n) = \frac{a(n)}{b(n)}$. In addition, as $a + b \neq 0$ a.e., $(a+b)(n) \neq 0$, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$, for some $n_1 \in \mathbb{N}$, so there exists $C \in K^{\mathbb{N}}$ such that $C(n) = \frac{(a+b)(n+1)}{(a+b)(n)}$, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$. Thus, $\frac{(a+b)(n+1)}{(a+b)(n)} = \frac{\frac{a(n+1)}{a(n)} \cdot \frac{a(n)}{b(n)} + \frac{b(n+1)}{b(n)}}{\frac{a(n)}{b(n)} + 1}$, for all $n \in \mathbb{N}$ such that $n_0, n_1 \leqslant n$; that is to say, $C(n) = \frac{A(n) \cdot D(n) + B(n)}{D(n) + 1}$, for all $n \in \mathbb{N}$ such that $n_0, n_1 \leqslant n$. It is clear that, if $D \in \mathrm{rat}(K)$, then $C \in \mathrm{rat}(K)$. Recall that, given $r(t), s(t) \in K(t)$, $r(t) = s(t)$ or $\{n \in \mathbb{N} \mid r(n) = s(n)\}$ is finite. Hence, if $C \in \mathrm{rat}(K)$, then the possibilities are only the following:

- If $A = C$ a.e., then there exists $m \in \mathbb{N}$ such that $n_0, n_1 \leqslant m$ and $A(n) = \frac{A(n) \cdot D(n) + B(n)}{D(n)+1}$, for all $n \in \mathbb{N}$ such that $m \leqslant n$. Thus, simplifying, $A(n) = B(n)$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, ergo $\frac{a(n+1)}{a(n)} = \frac{b(n+1)}{b(n)}$, for all $n \in \mathbb{N}$ such that $m \leqslant n$. This yields the existence of $\alpha \in K \backslash \{0\}$ such that $a = \alpha \cdot b$ a.e. and, hence, that $D \in \mathrm{rat}(K)$.
- If $A \neq C$ a.e., then, as $D(n) - \frac{A(n)}{C(n)} \cdot D(n) = \frac{B(n)}{C(n)} - 1$, for all $n \in \mathbb{N}$ such that $n_0, n_1 \leqslant n$, there exists $\tilde{m} \in \mathbb{N}$ such that $n_0, n_1 \leqslant \tilde{m}$ and $D(n) = \frac{\frac{B(n)}{C(n)} - 1}{1 - \frac{A(n)}{C(n)}} = \frac{B(n) - C(n)}{C(n) - A(n)}$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$.

Therefore, whenever $C \in \mathrm{rat}(K)$, $D \in \mathrm{rat}(K)$ too. Thus $C \in \mathrm{rat}(K)$ if, and only if, $D \in \mathrm{rat}(K)$, which, applying again Proposition 1.4.6, yields that $a + b \in \mathrm{hyp}(K)$ if, and only if, $a \underset{\mathrm{hyp}}{\sim} b$. □

Proposition 1.7.4. No finite sum of pairwise dissimilar hypergeometric sequences vanishes
Let $k \in \mathbb{N}^+$ and $a_0, ..., a_k \in \mathrm{hyp}(K)$ such that $\sum_{i=0}^{k}(a_i) = 0$ a.e. Then $a_i \underset{\mathrm{hyp}}{\sim} a_j$, for some $i, j \in \{0, ..., k\}$ such that $i < j$.

Proof (Induction on k) $a_0, ..., a_k \in \mathrm{hyp}(K)$, so, by Proposition 1.4.6, there exist $r : \{0, ..., k\} \longrightarrow K(t)$ and $n_0 \in \mathbb{N}$ such that, for all $i \in \{0, ..., k\}$ and $n \in \mathbb{N}$ such that $n_0 \leqslant n$, n is not a pole of $r(i)(t)$, $a_i(n) \neq 0$ and $r(i)(n) = \frac{a_i(n+1)}{a_i(n)}$.

Case 1 $a_1 + a_0 = 0$ a.e., i.e. $a_1 = -a_0$ a.e. yields the existence of $\tilde{n}_0 \in \mathbb{N}$ such that $n_0 \leqslant \tilde{n}_0$ and $\frac{a_0(n)}{a_1(n)} = \frac{a_0(n)}{-a_0(n)} = -1$, for all $n \in \mathbb{N}$ such that $\tilde{n}_0 \leqslant n$, so obviously $a_0 \underset{\mathrm{hyp}}{\sim} a_1$.

Case $k-1$ Induction Hypothesis (I.H.).

Case k $\sum_{i=0}^{k}(a_i) = 0$ a.e., so there exists $n_1 \in \mathbb{N}$ such that $n_0 \leqslant n_1$ and $\sum_{i=0}^{k}(a_i(n)) = 0$, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$. Thus, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$, $\sum_{i=0}^{k}(a_i(n) \cdot r(k)(n)) = 0$ and $\sum_{i=0}^{k}(a_i(n) \cdot r(i)(n)) = 0$, so $\sum_{i=0}^{k-1}(a_i(n) \cdot (r(k)(n) - r(i)(n))) = 0$, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$. The possibilities are only the following:

- If there exists $i \in \{0, ..., k-1\}$ such that $r(i)(t) = r(k)(t)$, then $\frac{a_i(n+1)}{a_i(n)} = \frac{a_k(n+1)}{a_k(n)}$, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$. This yields the existence of $\alpha \in K \backslash \{0\}$ such that $a_i = \alpha \cdot a_k$ a.e., ergo $a_i \underset{\mathrm{hyp}}{\sim} a_k$.
- Otherwise, let $b : \{0, ..., k-1\} \to K^{\mathbb{N}}$ such that $b(i)(n) = a_i(n) \cdot (r(k)(n) - r(i)(n))$, for all $i \in \{0, ..., k-1\}$ and $n \in \mathbb{N}$ such that $n_1 \leqslant n$. By Proposition 1.4.6, $b(i) \in \mathrm{hyp}(K)$, for all $i \in \{0, ..., k-1\}$, so, applying I.H., $b(i) \underset{\mathrm{hyp}}{\sim} b(j)$, for some $i, j \in \{0, ..., k-1\}$ such that $i < j$, and hence $a_i \underset{\mathrm{hyp}}{\sim} a_j$ for such i, j. □

Proposition 1.7.5. No nonzero sum of hypergeometric sequences has infinitely many zeros
Let $a, b \in \mathrm{hyp}(K)$. Then $a + b = 0$ a.e. or $a + b \neq 0$ a.e.

Proof Suppose the contrary. Then $\neg[a + b = 0$ a.e.$]$; so $\{n \in \mathbb{N} \mid a(n) = -b(n)\}$ is infinite. [0]

$a, b \in \mathrm{hyp}(K)$; so, applying Proposition 1.4.6, there exist $n_0 \in \mathbb{N}$ and $A, B \in \mathrm{rat}(K)$ such that, for all $n \in \mathbb{N}$ such that $n_0 \leqslant n$, n is not a pole neither of A nor of B, $0 \notin \{a(n), b(n)\}$, $A(n) = \frac{a(n+1)}{a(n)}$ and $B(n) = \frac{b(n+1)}{b(n)}$. In particular, $b \neq 0$ a.e. [1]

[0] yields that $\left\{n \in \mathbb{N} \mid \frac{a(n+1)}{a(n)} = \frac{b(n+1)}{b(n)}\right\}$ is infinite; i.e. that $\neg[A \neq B$ a.e.$]$. [2]

Recall that, given $r(t), s(t) \in K(t)$, $r(t) = s(t)$ or $\{n \in \mathbb{N} \mid r(n) = s(n)\}$ is finite; so the possibilities are only the following:

- If $A = B$ a.e., then $\mathcal{N}(a) \cdot b = \mathcal{N}(b) \cdot a$ a.e. Applying [0], $\{n \in \mathbb{N} \mid (-b(n+1)) \cdot b(n) = b(n+1) \cdot (-b(n))\}$ is infinite, i.e. $\{n \in \mathbb{N} \mid b(n) = 0\}$ is infinite. Contradiction with [1].
- If $A \neq B$ a.e., then there is a contradiction with [2]. □

Proposition 1.7.6. Reducibility criterion for sums

Let $k \in \mathbb{N}^+$ and $a_0, \ldots, a_k \in \operatorname{hyp}(K)$. Then there exist $r \in \{0, \ldots, k\}$ and $b_0, \ldots, b_r \in \operatorname{hyp}(K)$ such that the following conditions hold:

1. $b_i \underset{\text{hyp}}{\nsim} b_j$, for all $i, j \in \{0, \ldots, r\}$ such that $i \neq j$,
2. $\sum_{i=0}^{k}(a_i) = \sum_{i=0}^{r}(b_i)$ a.e.,
3. for all $s \in \mathbb{N}$ and $c_0, \ldots, c_s \in \operatorname{hyp}(K)$ such that the following conditions hold:
   - $c_i \underset{\text{hyp}}{\nsim} c_j$, for all $i, j \in \{0, \ldots, s\}$ such that $i \neq j$,
   - $\sum_{i=0}^{k}(a_i) = \sum_{i=0}^{s}(c_i)$ a.e.,

   $s = r$ and $b_i = c_i$ a.e., for all $i \in \{0, \ldots, r\}$.

<u>Proof</u>

1 and 2 follow from the application of the following algorithm to $k, a_0, \ldots, a_k$.

Input: $l \in \mathbb{N}$, $h_0, \ldots, h_l \in \operatorname{hyp}(K)$.

1. $\tilde{k} := l$, $\tilde{a}_0 := h_0$, ... , $\tilde{a}_{\tilde{k}} := h_l$.
2. Compute $r, R \in \mathbb{N}$ and $S_0, \ldots, S_R \subseteq \{0, \ldots, \tilde{k}\}$ such that the following conditions hold:
   - $r \leqslant R \leqslant \tilde{k}$,
   - $S_u \neq \varnothing$, for all $u \in \{0, \ldots, R\}$,
   - $\bigcup_{u=0}^{R}(S_u) = \{0, \ldots, \tilde{k}\}$,
   - $\tilde{a}_i \underset{\text{hyp}}{\sim} \tilde{a}_j$, for all $i, j \in S_u$, for all $u \in \{0, \ldots, R\}$,
   - $\tilde{a}_i \underset{\text{hyp}}{\nsim} \tilde{a}_j$, for all $i \in S_u$ and $j \in S_v$, for all $u, v \in \{0, \ldots, R\}$ such that $u \neq v$,
   - $\sum_{i \in S_u}(\tilde{a}_i) \in \operatorname{hyp}(K)$, for all $u \in \{0, \ldots, r\}$,
   - $\sum_{i \in S_u}(\tilde{a}_i) = 0$ a.e., for all $u \in \{r+1, \ldots, R\}$.

Note that $r, R, S_0, \ldots S_r$ exist by Proposition 1.7.3; and that, if $r = R$, then $\{r+1, \ldots, R\} = \varnothing$.

3. For all $u \in \{0, \ldots, r\}$, let $b_u = \sum_{i \in S_u}(\tilde{a}_i)$.
4. If $b_u \underset{\text{hyp}}{\nsim} b_v$, for all $u, v \in \{0, \ldots, r\}$ such that $u \neq v$, then return $r, b_0, \ldots, b_r$ and STOP; otherwise, let $\tilde{k} = r$, $\tilde{a}_0 := b_0$, ... , $\tilde{a}_{\tilde{k}} := b_r$ and go to 2.

Output: $r \in \{0, \ldots, k\}$ and $b_0, \ldots, b_r \in \operatorname{hyp}(K)$ such that the following conditions hold:
- $b_i \underset{\text{hyp}}{\nsim} b_j$, for all $i, j \in \{0, \ldots, r\}$ such that $i \neq j$,
- $\sum_{i=0}^{l}(h_i) = \sum_{i=0}^{r}(b_i)$ a.e.

In order to prove 3, let $s \in \mathbb{N}$ and $c_0, \ldots, c_s \in \operatorname{hyp}(K)$ such that the following conditions hold:
- $c_i \underset{\text{hyp}}{\nsim} c_j$, for all $i, j \in \{0, \ldots, s\}$ such that $i \neq j$,
- $\sum_{0=u}^{r}(b_u) = \sum_{i=0}^{s}(c_i)$ a.e.

Now it is applied induction on $r + s$.

<u>Case 0</u> Immediate.

<u>Case $r + s - 1$</u> Induction Hypothesis (I.H.).

<u>Case $r + s$</u> $\sum_{0=u}^{r}(b_u) = \sum_{i=0}^{s}(c_i)$ a.e., i.e. $\sum_{0=u}^{r}(b_u) - \sum_{i=0}^{s}(c_i) = 0$ a.e., which, applying Proposition 1.7.4, yields the existence of $u_0 \in \{0, \ldots, r\}$ and $i_0 \in \{0, \ldots, s\}$ such that $b_{u_0} \underset{\text{hyp}}{\sim} c_{i_0}$. Let $\tilde{\ }$ be a permutation of $\{b_0, \ldots, b_r\}$ such that $\tilde{b_r} = b_{u_0}$, $\hat{\ }$ a permutation of $\{c_0, \ldots, c_s\}$ such that $\hat{c_s} = c_{i_0}$ and $h = \tilde{b_r} - \hat{c_s}$. By Proposition 1.7.5, $h \neq 0$ a.e. or $h = 0$ a.e. If $h \neq 0$ a.e., as $\sum_{0=u}^{r-1}(\tilde{b_u}) + h = \sum_{i=0}^{s-1}(\hat{c_i})$ a.e. and $\sum_{0=u}^{r-1}(\tilde{b_u}) = \sum_{i=0}^{s-1}(\hat{c_i}) - h$ a.e., then, by I.H., $r = s - 1$ and $r - 1 = s$ would hold, which is impossible. Therefore, $h = 0$ a.e., i.e. $\tilde{b_r} = \hat{c_s}$ a.e. and so $\sum_{0=u}^{r-1}(\tilde{b_u}) = \sum_{i=0}^{s-1}(\hat{c_i})$ a.e. Applying I.H., $r - 1 = s - 1$ and $\tilde{b_u} = \hat{c_u}$ a.e., for all $u \in \{0, \ldots, r-1\}$, which implies that $r = s$ and $\tilde{b_u} = \hat{c_u}$ a.e., for all $u \in \{0, \ldots, r\}$.

$\square$

Lemma 1.7.7. Let $r \in \mathbb{N}$, $p_0, \ldots, p_r \in \mathrm{pol}(K)$ such that $p_0 \cdot p_r \neq 0$ a.e., $h \in \mathrm{hyp}(K)$ and $H = \sum_{k=0}^{r}(p_k \cdot \mathcal{N}^k(h))$. If $H \neq 0$ a.e., then $H \in \mathrm{hyp}(K)$ and $H \underset{\mathrm{hyp}}{\sim} h$.

Proof $h \in \mathrm{hyp}(K)$, so, applying Proposition 1.4.6, there exist $r(t) \in K(t)\backslash\{0\}$ and $m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, n is not a pole of $r(t)$, $h(n) \neq 0$ and $r(n) = \frac{h(n+1)}{h(n)}$. Then $\mathcal{N}^i(h) = \mathcal{N}^{i-1}(\mathcal{N}(h)) = \mathcal{N}^{i-1}(r \cdot h) = \mathcal{N}^{i-1}(r) \cdot \mathcal{N}^{i-1}(h)$ holds, for all $i \in \{0, \ldots, r\}$, so $\mathcal{N}^i(h) = \mathcal{N}^{i-1}(r) \cdot \mathcal{N}^{i-2}(r) \cdot \mathcal{N}^{i-2}(h)$, for all $i \in \{0, \ldots, r\}$ and thus, iterating, $\mathcal{N}^i(h) = \prod_{j=0}^{i-1}(\mathcal{N}^j(r)) \cdot h$, for all $i \in \{0, \ldots, r\}$. Hence, $\sum_{k=0}^{r}(p_k \cdot \mathcal{N}^k(h)) = \sum_{k=0}^{r}\left(p_k \cdot \prod_{j=0}^{k-1}(\mathcal{N}^j(r)) \cdot h\right) = \sum_{k=0}^{r}\left(p_k \cdot \prod_{j=0}^{k-1}(\mathcal{N}^j(r))\right) \cdot h \neq 0$ a.e., which, applying that $\sum_{k=0}^{r}\left(p_k \cdot \prod_{j=0}^{k-1}(\mathcal{N}^j(r))\right) \in \mathrm{rat}(K)$, yields that $\sum_{k=0}^{r}(p_k \cdot \mathcal{N}^k(h)) \in \mathrm{hyp}(K)$ and $\sum_{k=0}^{r}(p_k \cdot \mathcal{N}^k(h)) \underset{\mathrm{hyp}}{\sim} h$.

□

Lemma 1.7.8. Let $k, r \in \mathbb{N}$, $p_0, \ldots, p_r \in \mathrm{pol}(K)$ such that $p_0 \cdot p_r \neq 0$ a.e., and $h_0, \ldots, h_k \in \mathrm{hyp}(K)$ such that, for all $i, j \in \{0, \ldots, k\}$ such that $i \neq j$, $\sum_{i=0}^{r}\left(p_i \cdot \mathcal{N}^i\left(\sum_{j=0}^{k}(h_j)\right)\right) = 0$ a.e. and $h_i \underset{\mathrm{hyp}}{\nsim} h_j$. Then $\sum_{i=0}^{r}(p_i \cdot \mathcal{N}^i(h_j)) = 0$ a.e., for all $j \in \{0, \ldots, k\}$.

Proof For all $j \in \{0, \ldots, k\}$, there exists $r_j \in \mathrm{rat}(K)$ such that $\sum_{i=0}^{r}(p_i \cdot \mathcal{N}^i(h_j)) = r_j \cdot h_j$; indeed, given $j \in \{0, \ldots, k\}$:

- if $\sum_{i=0}^{r}(p_i \cdot \mathcal{N}^i(h_j)) = 0$, then $\sum_{i=0}^{r}(p_i \cdot \mathcal{N}^i(h_j)) = 0 \cdot h_j$;
- otherwise, applying Lemma 1.7.7, $\sum_{i=0}^{r}(p_i \cdot \mathcal{N}^i(h_j)) \in \mathrm{hyp}(K)$ and $\sum_{i=0}^{r}(p_i \cdot \mathcal{N}^i(h_j)) \underset{\mathrm{hyp}}{\sim} h_j$.

Thus, $0 = \sum_{i=0}^{r}\left(p_i \cdot \mathcal{N}^i\left(\sum_{j=0}^{k}(h_j)\right)\right) = \sum_{j=0}^{k}\left(\sum_{i=0}^{r}\left(p_i \cdot \mathcal{N}^i(h_j)\right)\right) = \sum_{j=0}^{k}(r_j \cdot h_j)$ a.e. And note that $r_j = 0$, for all $j \in \{0, \ldots, k\}$; indeed, the existence of $i, j \in \{0, \ldots, k\}$ such that $i \neq j$ and $r_i \neq 0 \neq r_j$ would imply, by Proposition 1.7.4, that $r_i \cdot h_i \underset{\mathrm{hyp}}{\sim} r_j \cdot h_j$, i.e. that $h_i \underset{\mathrm{hyp}}{\sim} h_j$, contradicting the hypotheses. Therefore $\sum_{i=0}^{r}(p_i \cdot \mathcal{N}^i(h_j)) = 0$ a.e., for all $j \in \{0, \ldots, k\}$. □

1.8 PETKOVŠEK'S COMPLETE HYPER ALGORITHM

With the results and techniques of the previous sections, it is finally possible to build Petkovšek's complete Hyper algorithm.

Proposition 1.8.1. Foundation of Petkovšek's complete Hyper algorithm
Let $r \in \mathbb{N}$, $p_0, \ldots, p_r \in \mathrm{pol}(K)$ such that $p_0 \cdot p_r \neq 0$ a.e., and $L = \sum_{k=0}^{r}(p_k \bullet \mathcal{N}^k)$. Then $\{y + \mathbf{J} \mid [y \in K^{\mathbb{N}} \wedge L(y) = 0 \text{ a.e.}]\} \cap K\langle\{y + \mathbf{J} \mid y \in \mathrm{hyp}(K)\}\rangle = K\langle\{y + \mathbf{J} \mid [y \in \mathrm{hyp}(K) \wedge L(y) = 0 \text{ a.e.}]\}\rangle$.

Proof Let $h \in K^{\mathbb{N}}$ such that $h + \mathbf{J} \in \{y + \mathbf{J} \mid [y \in K^{\mathbb{N}} \wedge L(y) = 0 \text{ a.e.}]\} \cap K\langle\{y + \mathbf{J} \mid y \in \mathrm{hyp}(K)\}\rangle$. Then $h + \mathbf{J} \in K\langle\{y + \mathbf{J} \mid y \in \mathrm{hyp}(K)\}\rangle$, which, applying Proposition 1.7.6, yields the existence of $h_1, \ldots, h_k \in \mathrm{hyp}(K)$ such that, for all $i, j \in \{1, \ldots, k\}$ such that $i \neq j$, $\sum_{i=1}^{k}(h_i) = h$ a.e. and $h_i \underset{\mathrm{hyp}}{\nsim} h_j$. Then, applying that $L(h) = 0$ a.e. and Lemma 1.7.8, $L(h_i) = 0$ a.e., for all $i \in \{1, \ldots, k\}$, which yields that $\{y + \mathbf{J} \mid [y \in K^{\mathbb{N}} \wedge L(y) = 0 \text{ a.e.}]\} \cap K\langle\{y + \mathbf{J} \mid y \in \mathrm{hyp}(K)\}\rangle = K\langle\{y + \mathbf{J} \mid [y \in \mathrm{hyp}(K) \wedge L(y) = 0 \text{ a.e.}]\}\rangle$.
□

The following algorithm, called Petkovšek's complete Hyper algorithm, computes all the solutions (or more concretely, a K-basis generating them) having hypergeometric closed form (a.e.) of a given homogeneous linear difference K-equation with polynomial coefficients (cf. Section 8.6 of [Petkovšek et al.] and [Abramov] for two faster algorithms doing the same, the last one finding only the rational solutions though). Note that it is just a modification of Petkovšek's Hyper algorithm.

Petkovšek's complete Hyper algorithm
Input: a field F such that $K \leqslant_{\mathbf{Field}} F$, $d \in \mathbb{N}$ and $p_0, ..., p_d \in \mathrm{pol}(K)$ such that $p_0 \cdot p_d \neq 0$ a.e.
1. $S := \varnothing$.
2. For all $a(x) \in F[x]$ such that $a(x)$ is monic and $a(x) \mid p_0(x)$, and $b(x) \in F[x]$ such that $b(x)$ is monic and $b(x) \mid p_d(x-d+1)$, do:
2.1. compute $P : \{0, ..., d\} \longrightarrow \mathrm{pol}(F)$ such that $P(i)(x) = p_i(x) \cdot \prod_{j=0}^{i-1}(a(x+j)) \cdot \prod_{j=i}^{d-1}(b(x+j))$, for all $i \in \{0, ..., d\}$,
2.2. $m := \max(\{\deg(P(i))\}_{i=0}^{d})$,
2.3. compute $\alpha : \{0, ..., d\} \longrightarrow F$ such that $\alpha(i) = \mathrm{coeff}_m(P(i))$, for all $i \in \{0, ..., d\}$,
2.4. for all $z \in F \backslash \{0\}$ such that $\sum_{i=0}^{d}(\alpha(i) \cdot z^i) = 0$, do:
2.4.1. if there exists $c \in \mathrm{pol}(F) \backslash \{0\}$ such that $\sum_{i=0}^{d}(z^i \cdot P(i)(x) \cdot c(x+i)) = 0$ (this question can be solved by applying Abramov - Petkovšek Poly algorithm), then:
2.4.1.1. compute $s \in \mathrm{rat}(F)$ such that $s(x) = z \cdot \frac{a(x) \cdot c(x+1)}{b(x) \cdot c(x)}$,
2.4.1.2. compute $y \in \mathrm{hyp}(F)$ such that $\mathcal{N}(y) = s \cdot y$ a.e. and store $y + \mathbf{J}$ in S.
3. If $S \neq \varnothing$, then reduce it to a K-free (i.e. linearly independent) set.
4. Return S and STOP.
Output: a K-basis S for $K\langle\{y + \mathbf{J} \mid [y \in \mathrm{hyp}(K) \ \wedge \ \sum_{i=0}^{d}(p_i \cdot \mathcal{N}^i(y)) = 0 \text{ a.e.}]\}\rangle$.

Note that, by Proposition 1.8.1, the output of this algorithm is indeed a K-basis generating the solutions.

The following proposition ensures that, if the independent term of a given linear difference K-equation with polynomial coefficients does not have hypergeometric closed form, then the equation can not have a solution having hypergeometric closed form either.

Proposition 1.8.2. Let $d \in \mathbb{N}$, $p_0, ..., p_d \in \mathrm{pol}(K)$ such that $p_0 \cdot p_d \neq 0$ a.e. and $f \in K^{\mathbb{N}}$. If $\sum_{i=0}^{d}(p_i \cdot \mathcal{N}^i(y)) = f$ a.e., for some $y \in K^{\mathbb{N}}$ such that y has hypergeometric closed form, then f has hypergeometric closed form.

Proof Immediate from Lemma 1.7.7. □

Finally, the conditions for describing an algorithm computing all the solutions having hypergeometric closed form (a.e.) of a given (not necessarily homogeneous) linear difference K-equation with polynomial coefficients have been reached.

Input: $d \in \mathbb{N}$, $p_0, ..., p_d \in \mathrm{pol}(K)$ such that $p_0 \cdot p_d \neq 0$ a.e. and $f \in K^{\mathbb{N}}$.
1. If $f \in \mathrm{hyp}(K)$, then continue; otherwise return "The associated difference K-equation has no solution having hypergeometric closed form." and STOP.

Note that Proposition 1.8.2 has been applied here.

2. Compute $r \in \{0, ..., p\}$ and $f_0, ..., f_r \in \mathrm{hyp}(K)$ such that $f_i \underset{\mathrm{hyp}}{\not\sim} f_j$, for all $i, j \in \{0, ..., r\}$ such that $i \neq j$, and $f = \sum_{i=0}^{r}(f_i)$ a.e. (this step requires to check pairwise similarity, and thus to test rationality of hypergeometric sequences, task which can be accomplished by applying Proposition 1.5.3).

Note that Proposition 1.7.6 has been applied here.

3. For all $i \in \{0, ..., r\}$ do:
3.1. compute $s_i(t) \in K(t) \backslash \{0\}$ such that there exists $m_i \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $m_i \leqslant n$, n is not a pole of $s_i(t)$, $f_i(n) \neq 0$ and $s_i(n) = \frac{f_i(n+1)}{f_i(n)}$.

Note that Proposition 1.4.6 has been applied here.

4. For all $i \in \{0, ..., r\}$ do:
4.1. let $L_1 = \mathcal{N} - 1$ and, using of Petkovšek's Hyper algorithm, compute, if possible, $r_i \in \mathrm{rat}(K)\backslash\{0\}$ such that $L_1 \left(\sum_{j=0}^{d} \left(p_j \cdot \prod_{k=0}^{j-1} (\mathcal{N}^k(s_k)) \cdot \mathcal{N}^j(r_i) \right) \right) = 0$ a.e.: if it exists, then continue; otherwise, return "The associated difference K-equation has no solution having hypergeometric closed form." and STOP (cf. Section 8.9 of [Petkovšek et al.] for a more efficient way of doing this step, with the help of Abramov's algorithm).
5. Let $V = K\langle\{y + \mathbf{J} \mid [y \in \mathrm{hyp}(K) \ \wedge \ \sum_{i=0}^{d}(p_i \cdot \mathcal{N}^i(y)) = 0 \text{ a.e.}]\}\rangle$, let $m = \dim_K(V)$, compute, by using Petkovšek's complete Hyper algorithm, a K-basis $(y_0, ..., y_m)$ for V, return $\left\{ \sum_{j=0}^{m}(\alpha_j * y_j) + \sum_{j=0}^{r}(r_j \cdot f_j) \mid \alpha_0, ..., \alpha_m \in K \right\}$ and STOP.
Output: a nonempty finite set S such that, for all $y \in K^{\mathbb{N}}$ such that y has hypergeometric closed form and $\sum_{i=0}^{d}(p_i \cdot \mathcal{N}^i(y)) = f$ a.e., there exists $s \in S$ such that $s = y$ a.e., if S exists; "The associated difference K-equation has no solution having hypergeometric closed form." otherwise.

1.9 FACTORIZATION OF LINEAR RECURRENCE OPERATORS

Sometimes it is interesting to compute a linear recurrence operator with polynomial coefficients and minimal order making a given holonomic sequence vanish. Clearly, this can be accomplished by factoring an already known linear recurrence operator with polynomial coefficients annihilating such a sequence, so now it will be commented how this problem can be handled.

Definition 1.9.1. Let $r \in \mathbb{N}$, t be an indeterminate over K and $p_0, ..., p_r \in \mathrm{pol}(K)$ such that $p_0 \cdot p_r \neq 0$ a.e. Then $\sum_{i=0}^{r}(p_i \bullet \mathcal{N}^i)$ is said to be **monic** if $p_r(t)$ is monic.

For example, $n^3 \bullet \mathcal{N}^2 + 0 \bullet \mathcal{N}^1 + 5 \bullet \mathcal{N}^0$ is monic, but $(2 \cdot n^3) \bullet \mathcal{N}^2 + 0 \bullet \mathcal{N}^1 + 5 \bullet \mathcal{N}^0$ is not.

From now on, $\{L \in \mathrm{End}_{\mathbf{Vect}_K}(K^{\mathbb{N}}) \mid L$ is a linear recurrence K-operator with polynomial coefficients$\}$ will be denoted by $\mathfrak{O}_K$; and given $r, s \in \mathbb{N}$ and $p_0, ..., p_r, q_0, ..., q_s \in \mathrm{pol}(K)$, the fact $r = s$ and $p_i = q_i$ a.e., for all $i \in \{0, ..., r\}$, will be abbreviated as $\sum_{i=0}^{r}(p_i \bullet \mathcal{N}^i) \equiv \sum_{i=0}^{s}(q_i \bullet \mathcal{N}^i)$.

Note that $\left\{(L, M) \in \mathfrak{O}_K^2 \mid L \equiv M\right\}$ is an equivalence relation on $\mathfrak{O}_K$.

Lemma 1.9.2. Let $y \in K^{\mathbb{N}}\backslash\mathbf{J}$ such that y is holonomic. Then there exists a unique $R \in \mathfrak{O}_K$ such that $0 < \mathrm{order}(R)$, R is monic, $R(y) = 0$ a.e. and that, for all $L \in \mathfrak{O}_K$ such that $L(y) = 0$ a.e., the following conditions hold:

- if $\mathrm{order}(R) = \mathrm{order}(L)$, then $R \equiv L$,
- if $\mathrm{order}(R) < \mathrm{order}(L)$, then there exists $P \in \mathfrak{O}_K$ such that $P \circ R = L$

(cf. Section 8.10 of [Petkovšek et al.]).

Definition 1.9.3. Let $y \in K^{\mathbb{N}}\backslash\mathbf{J}$ such that y is holonomic and $R \in \mathfrak{O}_K$. Then R is said to be the **right minimal linear recurrence** K-**operator** annihilating y, fact which is denoted by $R = \mathrm{rmlr}(y)$, if $0 < \mathrm{order}(R)$, R is monic, $R(y) = 0$ a.e. and that, for all $L \in \mathfrak{O}_K$ such that $L(y) = 0$ a.e., the following conditions hold:

- if $\mathrm{order}(R) = \mathrm{order}(L)$, then $R \equiv L$,
- if $\mathrm{order}(R) < \mathrm{order}(L)$, then there exists $P \in \mathfrak{O}_K$ such that $P \circ R = L$.

For example, given $a, b \in K^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a(n) = 2^n$ and $b(n) = n!$, $\mathrm{rmlr}(a) = \mathcal{N} - 2$ and $\mathrm{rmlr}(b) = 1 \bullet \mathcal{N}^1 + (-n - 1) \bullet \mathcal{N}^0$.

Note that, given $y \in K^{\mathbb{N}}\backslash\mathbf{J}$ such that y is holonomic, $\mathrm{order}(\mathrm{rmlr}(y)) = 1$ if, and only if, $y \in \mathrm{hyp}(K)$.

From now on, given $L \in \mathfrak{O}_K$ such that $\mathrm{order}(L) > 0$, $\{y + \mathbf{J} \mid [y \in \mathrm{hyp}(K) \ \wedge \ L(y) = 0 \text{ a.e.}]\}$ (resp. $\{[L_1]_{\equiv} \mid [L_1 \in \mathfrak{O}_K \ \wedge \ L_1$ is monic $\wedge \ \mathrm{order}(L_1) = 1 \ \wedge \ \exists\, P \in \mathfrak{O}_K$ such that $L_1 \circ P = L]\}$,

$\{[L_1]_\equiv \mid [L_1 \in \mathfrak{O}_K \wedge L_1$ is monic $\wedge$ order$(L_1) = 1 \wedge \exists\, P \in \mathfrak{O}_K$ such that $P \circ L_1 = L]\})$ will be denoted by HypSol(L) (resp. Left(L), Right(L)).

Proposition 1.9.4. Relation between hypergeometric solutions and first-order right factors
Let $L \in \mathfrak{O}_K$ such that order$(L) > 0$ and ϕ : HypSol$(L) \longrightarrow$ Right(L) such that $\phi(y + \mathbf{J}) = [\mathrm{rmlr}(y)]_\equiv$, for all $y \in \mathrm{hyp}(K)$ such that $L(y) = 0$ a.e. Then ϕ is a bijection (cf. Section 8.10 of [Petkovšek et al.]).

For example, considering $L = (n-1) \bullet \mathcal{N}^2 + (2 - n^2 - 3 \cdot n) \bullet \mathcal{N}^1 + (2 \cdot n \cdot (n+1)) \bullet \mathcal{N}^0$, then it can be checked that HypSol$(L) = K\langle\{2^n + \mathbf{J}, n! + \mathbf{J}\}\rangle$ (by applying for instance Petkovšek's complete Hyper algorithm) and Right$(L) = \{[\mathcal{N} - 2]_\equiv, [1 \bullet \mathcal{N}^1 + (-n-1) \bullet \mathcal{N}^0]_\equiv\}$ (indeed, $L = ((n-1) \bullet \mathcal{N}^1 + (-n \cdot (n+1)) \bullet \mathcal{N}^0) \circ (\mathcal{N} - 2) = ((n-1) \bullet \mathcal{N}^1 + (-2 \cdot n) \bullet \mathcal{N}^0) \circ (\mathcal{N}^1 + (-n-1) \bullet \mathcal{N}^0)$).

Definition 1.9.5. Let $d \in \mathbb{N}$, $p_0, \ldots, p_d \in \mathrm{pol}(K)$, $L = \sum_{i=0}^{d} (p_i \bullet \mathcal{N}^i)$ and $M \in \mathfrak{O}_K$. Then M is said to be the **adjoint operator** of L if $M = \sum_{i=0}^{d} (\mathcal{N}^i(p_{d-i}) \bullet \mathcal{N}^i)$, fact which is denoted by $M = L^*$.

Proposition 1.9.6. Some properties of the adjoint operator
Let $L, M \in \mathfrak{O}_K$ and $d = \mathrm{order}(L)$. Then the following conditions hold:

1. order$(L^*) =$ order(L),
2. $(L \circ M)^* = \mathcal{N}^d \circ M^* \circ \mathcal{N}^{-d} \circ L^*$,
3. $L^{**} = \mathcal{N}^d \circ L \circ \mathcal{N}^{-d}$

(cf. Section 8.10 of [Petkovšek et al.]).

Proposition 1.9.7. Criterion for obtaining the first-order left factors
Let $L \in \mathfrak{O}_K$ such that order$(L) > 0$ and ϕ : Right$(L) \longrightarrow$ Left(L) such that $\phi([L_1]_\equiv) = [\mathcal{N}^{-1} \circ L_1^* \circ \mathcal{N}]_\equiv$, for all $L_1 \in \mathfrak{O}_K$ such that $[L_1]_\equiv \in$ Right(L). Then ϕ is a bijection (cf. Section 8.10 of [Petkovšek et al.]).

Propositions 1.9.4 and 1.9.7 yield the possibility of computing, from the output of Petkovšek's complete Hyper algorithm, all the monic first-order linear recurrence K-operators with polynomial coefficients dividing (from the left or from the right) the input linear recurrence K-operator.

Hence, linear recurrence K-operators of order 2 and 3 can be factored completely by using Petkovšek's complete Hyper algorithm. An algorithm for factoring linear recurrence K-operators of any order is described in [Bronstein & Petkovšek].

2 | INDEFINITE HYPERGEOMETRIC SUMMATION

This chapter introduces an algorithm, called Gosper's algorithm, capable to solve the so-called problem of indefinite hypergeometric summation, which will also be formalized within this chapter.

During this chapter, let K be a field of characteristic zero.

2.1 GOSPER-SUMMABILITY

The main notion in this chapter is that of Gosper-summable sequence. Roughly speaking, it involves a telescoping property which cancels the summation symbol, transforming the problem of determining if an indefinite sum has hypergeometric closed form into the problem of determining if its summand sequence belongs to this class of sequences, as it will be soon explained.

Definition 2.1.1. Let $t \in \mathrm{hyp}(K)$. Then t is said to be **Gosper-summable** if there exists $z \in \mathrm{hyp}(K)$ such that $\Delta(z) = t$ a.e.

For example, given $t \in K^{\mathbb{N}}$ such that $t(n) = (4 \cdot n + 1) \cdot \frac{n!}{(2 \cdot n+1)!}$, for all $n \in \mathbb{N}$, t is Gosper-summable, since $t \in \mathrm{hyp}(K)$ and, given $z \in K^{\mathbb{N}}$ such that $z(n) = -2 \cdot \frac{n!}{(2 \cdot n)!}$, for all $n \in \mathbb{N}$, $\Delta(z) = t$ and $z \in \mathrm{hyp}(K)$.

Note that deciding constructively if a given $t \in \mathrm{hyp}(K)$ is Gosper-summable yields, in particular, solving the first-order linear difference K-equation with constant coefficients given by $y(n+1) - y(n) = t(n)$.

Proposition 2.1.2. Characterization of Gosper-summability
Let $t \in \mathrm{hyp}(K)$ and $f \in K^{\mathbb{N}}$ such that $f(n) = \sum_{j=0}^{n-1}(t(j))$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, for some $m \in \mathbb{N}$. Then f has hypergeometric closed form if, and only if, t is Gosper-summable.

Proof

$\Rightarrow$) f has hypergeometric closed form, so there exist $k \in \mathbb{N}^+$ and $a_0, \ldots, a_k \in \mathrm{hyp}(K)$ such that $\sum_{i=1}^{k}(a_i) = f$ a.e. By Proposition 1.7.6, there exist $r \in \{0, \ldots, k\}$ and $b_0, \ldots, b_r \in \mathrm{hyp}(K)$ such that $b_i \underset{\mathrm{hyp}}{\nsim} b_j$, for all $i, j \in \{0, \ldots, r\}$ such that $i \neq j$, and $f = \sum_{i=0}^{r}(b_i)$ a.e. Hence, $\Delta(f) = \Delta(\sum_{i=0}^{r}(b_i))$ a.e., i.e. $t = \sum_{i=0}^{r}(\Delta(b_i))$ a.e.; and, applying Lemma 1.7.2, $\Delta(b_i) \underset{\mathrm{hyp}}{\nsim} \Delta(b_j)$, for all $i, j \in \{0, \ldots, r\}$ such that $i \neq j$. From Proposition 1.7.4 follows that $r = 0$, so $t = \Delta(b_0)$ a.e. and consequently t is Gosper-summable.

$\Leftarrow$) t is Gosper-summable, so there exists $z \in \mathrm{hyp}(K)$ such that $\Delta(z) = t$ a.e. Hence, $f(n) = \sum_{j=0}^{n-1}(t(j)) = \sum_{j=0}^{n-1}(\Delta(z)(j)) = \sum_{j=0}^{n-1}(z(j+1) - z(j)) = z(n) - z(0)$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, for some $m \in \mathbb{N}$. Therefore, f has hypergeometric closed form. □

2.2 GOSPER'S ALGORITHM

In this section Gosper's algorithm is presented. It decides constructively if a sequence is Gosper-summable; and it is based on a result called Gosper's theorem, which reduces the question to a problem of resolution of polynomial equations.

Proposition 2.2.1. Gosper's Theorem
Let $t \in \text{hyp}(K)$. Then there exist unique $a(x), b(x), c(x) \in K[x]\backslash\{0\}$ such that, for all $z \in \text{hyp}(K)$, the following conditions are equivalent:

- $\Delta(z) = t$ a.e.,
- there exists $u \in K[x]\backslash\{0\}$ such that $a(x) \cdot u(x+1) - b(x-1) \cdot u(x) = c(x)$ and there exists $n_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $n_0 \leqslant n$, $c(n) \neq 0$ and $z(n) = \frac{b(n-1) \cdot u(n)}{c(n)} \cdot t(n)$.

Proof Let $z \in K^{\mathbb{N}}$. $t \in \text{hyp}(K)$ yields, by Proposition 1.4.6, the existence of $m \in \mathbb{N}$ and $r(x) \in K(x)\backslash\{0\}$ such that, for all $n \in \mathbb{N}$ such that $m \leqslant n$, n is not a pole of $r(x)$, $t(n) \neq 0$ and $r(n) = \frac{t(n+1)}{t(n)}$. And applying Proposition 1.5.1, there exist unique $a(x), b(x), c(x) \in K[x]\backslash\{0\}$ such that the following conditions hold:

1. $b(x)$ and $c(x)$ are monic,
2. $r(x) = \frac{a(x) \cdot c(x+1)}{b(x) \cdot c(x)}$,
3. $\text{g.c.d.}(\{a(x), b(x+h)\}) = \text{g.c.d.}(\{a(x), c(x)\}) = \text{g.c.d.}(\{b(x), c(x+1)\}) = 1$, for all $h \in \mathbb{N}$.

It is clear that $c(n) \neq 0$, for all $n \in \mathbb{N}$ such that $m \leqslant n$.

$\Rightarrow$) The hypothesis together with the condition that $z \in \text{hyp}(K)$ and Proposition 1.4.6 yield the existence of $R(x) \in K(x)\backslash\{0\}$ such that there exists $\tilde{m} \in \mathbb{N}$ such that $m \leqslant \tilde{m}$ and, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$, n is not a pole of $R(x)$, $z(n) \neq 0$, $R(n) = \frac{z(n+1)}{z(n)} \neq 0$ and $\frac{z(n)}{t(n)} = \frac{z(n)}{z(n+1)-z(n)}$. Hence, $\frac{z(n)}{t(n)} = \frac{1}{R(n)-1} =: y(n)$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$. [0]

From [0] follows that $z(n) = y(n) \cdot t(n)$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$, which, applying the hypotheses, implies that $y(n+1) \cdot t(n+1) - y(n) \cdot t(n) = t(n)$, i.e. $y(n+1) \cdot r(n) - y(n) = 1$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$. [1]

Clearly, $\frac{1}{R(x)-1} = \frac{f(x)}{g(x)}$, for some $f(x), g(x) \in K[x]\backslash\{0\}$ such that $\text{g.c.d.}(\{f(x), g(x)\}) = 1$. [2]

By [2], $\text{g.c.d.}(\{f(x)+g(x), g(x)\}) = \text{g.c.d.}(\{f(x+1), g(x+1)\}) = 1$, for all $h \in \mathbb{N}$. [3]

$r(n) \overset{[1]}{=} \frac{1+y(n)}{y(n+1)} \overset{[0],[2]}{=} \frac{(f(n)+g(n)) \cdot g(n+1)}{g(n) \cdot f(n+1)}$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$. Thus, applying [3] and Lemma 1.5.2, $g(x) \mid c(x)$; ergo there exists $v(x) \in K[x]\backslash\{0\}$ such that $y(n) = \frac{v(n)}{c(n)}$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$. [4]

$1 \overset{[1]}{=} y(n+1) \cdot r(n) - y(n) \overset{[4]}{=} \frac{v(n+1)}{c(n+1)} \cdot r(n) - \frac{v(n)}{c(n)} = \frac{a(n) \cdot v(n+1)}{b(n) \cdot c(n)} - \frac{v(n)}{c(n)}$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$; so $a(x) \cdot v(x+1) = (v(x) + c(x)) \cdot b(x)$. Consequently, $b(x) \mid a(x) \cdot v(x+1)$, which yields that $b(x) \mid a(x)$ or $b(x) \mid v(x+1)$. But, as $\text{g.c.d.}(\{a(x), b(x+0)\}) = 1$, $b(x) \mid v(x+1)$ is the only option, so $b(x) \cdot u(x+1) = v(x+1)$, for some $u(x) \in K[x]\backslash\{0\}$. Hence, applying [4], $y(n) = \frac{b(n-1) \cdot u(n)}{c(n)}$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$. [5]

$1 \overset{[1]}{=} y(n+1) \cdot r(n) - y(n) \overset{[5]}{=} \frac{b(n) \cdot u(n+1)}{c(n+1)} \cdot r(n) - \frac{b(n-1) \cdot u(n)}{c(n)} = \frac{u(n+1) \cdot a(n)}{c(n)} - \frac{b(n-1) \cdot u(n)}{c(n)}$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$; so $a(x) \cdot u(x+1) - b(x-1) \cdot u(x) = c(x)$. Therefore, applying [0] and [5], $z(n) = \frac{b(n-1) \cdot u(n)}{c(n)} \cdot t(n)$, for all $n \in \mathbb{N}$ such that $\tilde{m} \leqslant n$.

$\Leftarrow$) From the hypotheses follows that $a(n) \cdot \frac{z(n+1) \cdot c(n+1)}{b(n) \cdot t(n+1)} - \frac{z(n) \cdot c(n)}{t(n)} = c(n)$, i.e. $\frac{z(n+1)}{t(n+1)} \cdot r(n) - \frac{z(n)}{t(n)} = 1$ or, equivalently, $z(n+1) - z(n) = t(n)$, for all $n \in \mathbb{N}$ such that $n_0, m \leqslant n$. Therefore, $\Delta(z) = t$ a.e. $\square$

The problem "Given a field F of characteristic zero, $t \in \text{hyp}(F)$ and $f \in F^{\mathbb{N}}$ such that $f(n) = \sum_{k=0}^{n-1}(t(k))$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, for some $m \in \mathbb{N}$, decide constructively if f has hypergeometric closed form." is called **problem of indefinite hypergeometric summation**.

The next algorithm decides constructively if a given hypergeometric sequence is Gosper-summable, solving therefore, by Proposition 2.1.2, the problem of indefinite hypergeometric summation.

Gosper's algorithm	Example
Input: $t \in \text{hyp}(K)$.	Input: $t(n) := n^2 \cdot 5^n$.
1. Compute $r(x) \in K(x) \backslash \{0\}$ such that there exists $n_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $n_0 \leqslant n$, n is not a pole of $r(x)$, $t(n) \neq 0$ and $r(n) = \frac{t(n+1)}{t(n)}$.	1. $r(x) := 5 \cdot \left(\frac{x+1}{x}\right)^2$.

Note that Proposition 1.4.6 has been applied here.

2. Compute $a(x), b(x), c(x) \in K[x] \backslash \{0\}$ such that the following conditions hold: • $b(x)$ and $c(x)$ are monic, • $r(x) = \frac{a(x) \cdot c(x+1)}{b(x) \cdot c(x)}$, • g.c.d. $(\{a(x), b(x+h)\})$ = g.c.d.$(\{a(x), c(x)\})$ = g.c.d. $(\{b(x), c(x+1)\}) = 1$, for all $h \in \mathbb{N}$.	2. $a(x) := 5$, $b(x) := 1$, $c(x) := x$.

Note that Proposition 1.5.1 has been applied here; and recall that an algorithm performing the computation of the step 2 can be found in Section 5.3 of [Petkovšek et al.].

3. If there exists $u(x) \in K[x] \backslash \{0\}$ such that $a(x) \cdot u(x+1) - b(x) \cdot u(x) = c(x)$ (this question can be solved by applying Abramov - Petkovšek Poly algorithm), then return $z \in \text{hyp}(K)$ such that there exists $n_1 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$, $c(n) \neq 0$ and $z(n) = \frac{b(n-1) \cdot u(n)}{c(n)} \cdot t(n)$, and STOP; otherwise return "t is not Gosper-summable" and STOP.	3. $u(x) := \frac{1}{4} \cdot x^2 - \frac{5}{8} \cdot x + \frac{15}{32}$, $z(n) := \left(\frac{1}{4} \cdot n^2 - \frac{5}{8} \cdot n + \frac{15}{32}\right) \cdot 5^n$.

Note that Proposition 2.2.1 has been applied here.

Output: $z \in \text{hyp}(K)$ such that $\Delta(z) = t$ a.e. if it exists; "t is not Gosper-summable." otherwise.	Output: $\left(\frac{1}{4} \cdot n^2 - \frac{5}{8} \cdot n + \frac{15}{32}\right) \cdot 5^n$.

The problem of indefinite hypergeometric summation can be extended to a more general question in which, instead of asking for a hypergeometric input, one asks for an input having hypergeometric closed form, viz. "Given a field F of characteristic zero, $p \in \mathbb{N}$, $a_0, ..., a_p \in \text{hyp}(F)$ and $f \in F^{\mathbb{N}}$ such that $f(n) = \sum_{k=0}^{n-1}(\sum_{i=0}^{p}(a_i(k)))$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, for some $m \in \mathbb{N}$, decide constructively if f has hypergeometric closed form.". By using Gosper's algorithm as subroutine, it is constructed the so-called extended Gosper's algorithm, which solves this problem. It works as follows.

Extended Gosper's algorithm
Input: $p \in \mathbb{N}$ and $a_0, ..., a_j \in \text{hyp}(K)$.
1. Compute $r \in \{0, ..., p\}$ and $b_0, ..., b_r \in \text{hyp}(K)$ such that $b_i \underset{\text{hyp}}{\nsim} b_j$, for all $i, j \in \{0, ..., r\}$ such that $i \neq j$, and $\sum_{j=0}^{p}(a_j) = \sum_{i=0}^{r}(b_i)$ a.e. (this step requires to check pairwise similarity, and thus to test rationality of hypergeometric sequences, task which can be accomplished by applying Proposition 1.5.3).

Note that Proposition 1.7.6 has been applied here.

2. For all $i \in \{0, ..., r\}$ do:
2.1. apply Gosper's algorithm to b_i: if it succeeds with output z_i, then continue; otherwise, return "$\sum_{j=0}^{p}(a_j)$ is not Gosper-summable." and STOP.
2. Let $z = \sum_{i=0}^{r}(z_i)$, return z and STOP.
Output: $z \in \text{hyp}(K)$ such that $\Delta(z) = \sum_{j=0}^{p}(a_j)$ a.e. if it exists; "$\sum_{j=0}^{p}(a_j)$ is not Gosper-summable." otherwise.

Proposition 2.2.2. Let $t \in \text{hyp}(K)$, $r(x) \in K(x)\backslash\{0\}$, $m \in \mathbb{N}$ and $a(x), b(x), c(x) \in K[x]\backslash\{0\}$ such that the following conditions hold:

1. for all $n \in \mathbb{N}$ such that $m \leqslant n$, n is not a pole of $r(x)$, $t(n) \neq 0$ and $r(n) = \frac{t(n+1)}{t(n)}$,
2. $b(x)$ and $c(x)$ are monic,
3. $r(x) = \frac{a(x) \cdot c(x+1)}{b(x) \cdot c(x)}$,
4. g.c.d.$(\{a(x), b(x+h)\})$ = g.c.d.$(\{a(x), c(x)\})$ = g.c.d.$(\{b(x), c(x+1)\}) = 1$, for all $h \in \mathbb{N}$.

If $t \notin \text{rat}(K)$, then $\#(\{u(x) \in K[x]\backslash\{0\} \mid a(x) \cdot u(x+1) - b(x-1) \cdot u(x) = c(x)\}) \leqslant 1$.

Proof (Contrapositive argument) If $1 < \#(\{u(x) \in K[x]\backslash\{0\} \mid a(x) \cdot u(x+1) - b(x-1) \cdot u(x) = c(x)\})$, then there exist $u_1(x), u_2(x) \in K[x]\backslash\{0\}$ such that $a(x) \cdot u_1(x+1) - b(x-1) \cdot u_1(x) = c(x)$, $a(x) \cdot u_1(x+1) - b(x-1) \cdot u_1(x) = c(x)$ and $u_1(x) \neq u_2(x)$. Let $z_1, z_2 \in \text{hyp}(K)$ such that there exists $n_0 \in \mathbb{N}$ such that $m \leqslant n_0$ and, for all $n \in \mathbb{N}$ such that $n_0 \leqslant n$, $c(n) \neq 0$, $z_1(n) = \frac{b(n-1) \cdot u_1(n)}{c(n)} \cdot t(n)$ and $z_2(n) = \frac{b(n-1) \cdot u_2(n)}{c(n)} \cdot t(n)$. By Proposition 2.2.1, $\Delta(z_1) = \Delta(z_2) = t$ a.e. Thus, $(z_1 - z_2)(n+1) - (z_1 - z_2)(n) = 0$, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$, for some $n_1 \in \mathbb{N}$ such that $n_0 \leqslant n_1$, which yields the existence of $\alpha \in K\backslash\{0\}$ such that $(z_1 - z_2)(n) = \alpha$, for all $n \in \mathbb{N}$ such that $n_1 \leqslant n$. Consequently, $z_1 - z_2 \in \text{hyp}(K)$. Applying Proposition 1.5.3, $z_1 \underset{\text{hyp}}{\sim} -z_2 \underset{\text{hyp}}{\sim} z_2$, so there exist $\beta \in K\backslash\{0\}$ and $s \in \text{hyp}(K)$ such that $z_1 \underset{\text{hyp}}{\sim} s$ and $s(n) = \beta$, for all $n \in \mathbb{N}$ such that $n_2 \leqslant n$, for some $n_2 \in \mathbb{N}$ such that $n_1 \leqslant n_2$. It follows the existence of $q(x) \in K(x)\backslash\{0\}$ such that, for all $n \in \mathbb{N}$ such that $n_2 \leqslant n$, n is not a pole of $q(x)$ and $q(n) = \frac{z_1(n)}{\beta}$. Hence, z_1 is a rational sequence; and therefore t is so. □

As a direct consequence of Proposition 2.2.1 and Proposition 2.2.2, given $t \in \text{hyp}(K)\backslash\text{rat}(K)$ such that t is Gosper-summable, $\#(\{z + \mathbf{J} \mid [z \in \text{hyp}(K) \wedge \Delta(z) = t \text{ a.e.}]\}) = 1$.

Finally, it is remarkable that, sometimes, generalizations of sequences that are not Gosper-summable are however Gosper-summable. For example, considering $a \in K^{\mathbb{N}}$ and $b \in K(t)^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $a(n) = (n+1)^n$ and $b(n) = (t+1)^n$, it is easy to verify, by using Gosper's algorithm, that a is not Gosper-summable, but b is.

3 DEFINITE HYPERGEOMETRIC SUMMATION

Similarly with the definite integration in the Liouvillian sense, in the discrete case it is possible to express and solve, with the help of an algorithm, called Zeilberger's creative telescoping, the so-called problem of definite hypergeometric summation. This will be tackled within this chapter, showing also several important results.

During this chapter, let K be a field of characteristic zero such that $K \subseteq \mathbb{C}$.

3.1 VERBAETEN'S FUNDAMENTAL THEOREM

Zeilberger's creative telescoping is based on the result known as Verbaeten's Fundamental Theorem for proper hypergeometric terms. Now these concepts will be shown, proving also an interesting proposition.

Definition 3.1.1. An expression is said to be a **proper hypergeometric term** if it is of the form $P(n,k) \cdot \dfrac{\prod_{i=0}^{m}(\Gamma(\alpha_i \cdot n + \beta_i \cdot k + \gamma_i))}{\prod_{i=0}^{r}(\Gamma(\delta_i \cdot n + \varepsilon_i \cdot k + \varphi_i))} \cdot x^k$, being:

- $P(u,v) \in K[u,v]$,
- n, k integer parameters,
- $m, r \in \mathbb{N}$,
- $\alpha_i, \beta_i, \delta_j, \varepsilon_j \in \mathbb{Z}$, for all $i \in \{0, ..., m\}$ and $j \in \{0, ..., r\}$,
- for all $i \in \{0, ..., m\}$ and $j \in \{0, ..., r\}$, γ_i, φ_j, x are expressions which do not depend on n, k and such that, when all their parameters take concrete values, the result lies in K.

The term will be considered as well-defined also in $\{(n_0, k_0) \in \mathbb{Z}^2 \mid \delta_i \cdot n_0 + \varepsilon_i \cdot k_0 + \varphi_i \in \mathbb{Z}^-\}$, by extending it to 0.

For example, $\binom{n}{k} \cdot (4 \cdot y + z)^k$, i.e. $1 \cdot \dfrac{\Gamma(1 \cdot n + 0 \cdot k + 1)}{\Gamma(0 \cdot n + 1 \cdot k + 1) \cdot \Gamma(1 \cdot n + (-1) \cdot k + 1)} \cdot (4 \cdot y + z)^k$, is a proper hypergeometric term.

Proper hypergeometric terms will be often denoted by indicating the integer parameters only (eg. with notations as $F(n,k)$), but recall that their inner expressions can involve additional parameters.

Note that the concept of proper hypergeometric term could have been formalized as a piecewise partial function over $\mathbb{Z}^2$ whose images could involve several indeterminates, instead of talking about *expressions* and *parameters*. However, Definition 3.1.1 has been chosen in order to avoid the (unworthy) difficulty of analyzing the domains in the sequel.

Definition 3.1.2. Let $F(n,k)$ be an expression involving n, k as integer parameters (it can involve more parameters) and such that, when all its parameters take concrete values, the result lies in K. Then $F(n,k)$ is said to be a **doubly hypergeometric term** if $\frac{F(n+1,k)}{F(n,k)}, \frac{F(n,k+1)}{F(n,k)}$ are rational expressions in n, k.

Proposition 3.1.3. Every proper hypergeometric term is a doubly hypergeometric term

Proof Call $F(n,k)$ the expression defined in Definition 3.1.1. It must be checked, for instance, if $\frac{F(n,k+1)}{F(n,k)}$ is a rational expression in n, k $\left(\frac{F(n+1,k)}{F(n,k)} \text{ can be checked similarly}\right)$. There are four possible cases depending on if the values β_i, ε_i are nonnegative or not. For example, suppose that

β_i is negative and ε_i is nonnegative (the remaining three cases can be checked similarly). Then $\dfrac{P(n,k+1)\cdot\frac{\prod_{i=0}^{m}(\Gamma(\alpha_i\cdot n+\beta_i\cdot(k+1)+\gamma_i))}{\prod_{i=0}^{r}(\Gamma(\delta_i\cdot n+\varepsilon_i\cdot(k+1)+\varphi_i))}\cdot x^{k+1}}{P(n,k)\cdot\frac{\prod_{i=0}^{m}(\Gamma(\alpha_i\cdot n+\beta_i\cdot k+\gamma_i))}{\prod_{i=0}^{r}(\Gamma(\delta_i\cdot n+\varepsilon_i\cdot k+\varphi_i))}\cdot x^{k}}$, i.e. $\dfrac{P(n,k+1)}{P(n,k)}\cdot\dfrac{1}{\prod_{i=0}^{m}(\prod_{j=1}^{-\beta_i}(\alpha_i\cdot n+\beta_i\cdot k+\gamma_i-j))\cdot\prod_{i=0}^{r}(\prod_{j=1}^{\varepsilon_i}(\delta_i\cdot n+\varepsilon_i\cdot k+\varphi_i+j-1))}\cdot x$ is a rational expression in n,k. □

Note that the converse is not always true, eg. $\frac{1}{n^2+k^2+1}$ is a doubly hypergeometric term, but not a proper hypergeometric term.

Proposition 3.1.4. Verbaeten's Fundamental Theorem

Let $F(n,k)$ a proper hypergeometric term. Then $\sum_{i=0}^{I}\left(\sum_{j=0}^{J}(A(i+1,j+1)(n)\cdot F(n+j,k+i))\right)$ $=0$, for some nonzero matrix A of dimension $(I+1)\times(J+1)$ whose entries are polynomial expressions in n which do not depend on k, for some $I,J\in\mathbb{N}$ (cf. Section 4.4 of [Petkovšek et al.]).

Moreover, the values I,J in Proposition 3.1.4 can be computed. Following the lines of Definition 3.1.1, it suffices to define $J=\sum_{i=0}^{m}(|\beta_i|)+\sum_{i=0}^{r}(|\varepsilon_i|)$ and $I=1+\deg(P(u,v))+J\cdot(\sum_{i=0}^{m}(|\alpha_i|)+\sum_{i=0}^{r}(|\delta_i|)-1)$ (cf. Section 4.4 of [Petkovšek et al.]).

The following and surprising result guarantees that, given a combinatorial identity (of the kind considered here), it can be proven just by checking finitely many values.

Proposition 3.1.5. Let $F(n,k)$ a proper hypergeometric term which is well-defined in $\{(a,b)\mid [n_0\leqslant a\ \wedge\ b\in\mathbb{Z}]\}$, for some $n_0\in\mathbb{N}$. Then there exists $m\in\mathbb{N}$ such that $n_0\leqslant m$ and, if $\sum_{b\in\mathbb{Z}}(F(a,b))=1$, for all $a\in\{n_0,...,m\}$, then $\sum_{b\in\mathbb{Z}}(F(a,b))=1$, for all $a\in\mathbb{N}$ such that $n_0\leqslant a$.

Proof $F(n,k)$ is a proper hypergeometric term, so, applying Proposition 3.1.4, $\sum_{i=0}^{I}\left(\sum_{j=0}^{J}(A(i+1,j+1)(n)\cdot F(n+j,k+i))\right)=0$, for some nonzero matrix A of dimension $(I+1)\times(J+1)$ whose entries are polynomial expressions in n which do not depend on k, for some $I,J\in\mathbb{N}$ such that $n_0\leqslant J$ (note that I,J can be taken arbitrarily large). Hence, as $F(n,k)$ is well-defined in $\{(a,b)\mid [n_0\leqslant a\ \wedge\ b\in\mathbb{Z}]\}$, $\sum_{b\in\mathbb{Z}}\left(\sum_{i=0}^{I}\left(\sum_{j=0}^{J}(A(i+1,j+1)(n)\cdot F(n+j,b+i))\right)\right)=0$ holds, i.e. $\sum_{i=0}^{I}\left(\sum_{j=0}^{J}(A(i+1,j+1)(n)\cdot\sum_{b\in\mathbb{Z}}(F(n+j,b+i)))\right)=0$. Thus, there exist $L\in\mathbb{N}$ such that $J\leqslant L$ and a nonzero matrix B of dimension $(I+1)\times(L+1)$ whose elements do not depend on n,k, and such that $\sum_{i=0}^{I}\left(\sum_{j=0}^{L}(B(i+1,j+1)\cdot n^j\cdot\sum_{k\in\mathbb{Z}}(F(n+j,b+i)))\right)=0$ and
$\sum_{j=0}^{L}(B(1,j+1))\neq 0.$ [0]

Let $M_1=\max\left(\left\{j\in\{0,...,J\}\mid \sum_{i=0}^{I}(A(i+1,j+1)(n)\cdot\sum_{b\in\mathbb{Z}}(F(n+j,b+i)))\neq 0\right\}\right)$.

Let $m\in\mathbb{N}$ such that the following conditions hold:

- $L\leqslant m$,
- if $\{a\in\mathbb{N}\mid A(i+1,M_1+1)(a)\neq 0\}\neq\varnothing$, then $M_1+\max(\{a\in\mathbb{N}\mid A(i+1,M_1+1)(a)\neq 0\})\leqslant m$ (note that this condition is necessary for having enough initial values for being able to calculate the following ones iteratively, for instance in the recurrence given by $(n-8)\cdot f(n+2)=(n+2)\cdot f(n+1)-10\cdot f(n)$ it is necessary to know all the values in $\{f(0),...,f(10)\}$).

If $\sum_{b\in\mathbb{Z}}(F(a,b))=1$, for all $a\in\{n_0,...,m\}$, then [0] implies that
$\sum_{i=0}^{I}\left(\sum_{j=0}^{L}(B(i+1,j+1)\cdot a^j)\right)=0$, for all $n\in\{n_0,...,m\}$, i.e.
$\sum_{j=0}^{L}\left(\sum_{i=0}^{I}(B(i+1,j+1)\cdot a^j)\right)=0$, for all $a\in\{n_0,...,m\}$ or, equivalently,
$\sum_{j=0}^{L}\left(\sum_{i=0}^{I}(B(i+1,j+1))\cdot a^j\right)=0$, for all $a\in\{n_0,...,m\}$. In particular, applying that $L\leqslant m$, $\sum_{j=0}^{L}\left(\sum_{i=0}^{I}(B(i+1,j+1))\cdot a^j\right)=0$, for all $a\in\{n_0,...,L\}$, i.e.

$\begin{bmatrix} n_0^0 & n_0^1 & \dots & n_0^L \\ (n_0+1)^0 & (n_0+1)^1 & \dots & (n_0+1)^L \\ \vdots & \vdots & \ddots & \vdots \\ L^0 & L^1 & \dots & L^L \end{bmatrix} \cdot \begin{bmatrix} \sum_{i=0}^{I}(B(i+1,n_0+1)) \\ \sum_{i=0}^{I}(B(i+1,n_0+2)) \\ \vdots \\ \sum_{i=0}^{I}(B(i+1,L+1)) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Note that $\begin{bmatrix} n_0^0 & n_0^1 & \dots & n_0^L \\ (n_0+1)^0 & (n_0+1)^1 & \dots & (n_0+1)^L \\ \vdots & \vdots & \ddots & \vdots \\ L^0 & L^1 & \dots & L^L \end{bmatrix}$ is a Vandermonde matrix, and recall that Vandermonde matrices are regular, ergo $\begin{bmatrix} \sum_{i=0}^{I}(B(i+1,n_0+1)) \\ \sum_{i=0}^{I}(B(i+1,n_0+2)) \\ \vdots \\ \sum_{i=0}^{I}(B(i+1,L+1)) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Hence, $\sum_{i=0}^{I}\left(\sum_{j=0}^{L}\left(B(i+1,j+1)\cdot a^j\right)\right) = 0$, for all $a \in \mathbb{N}$ such that $n_0 \leqslant a$. Consequently, if $\sum_{b\in\mathbb{Z}}(F(a,b)) = 1$, for all $a \in \mathbb{N}$ such that $n_0 \leqslant a$, then $\sum_{i=0}^{I}\left(\sum_{j=0}^{L}\left(B(i+1,j+1)\cdot a^j \cdot \sum_{b\in\mathbb{Z}}(F(a+j,b+i))\right)\right) = 0$, for all $a \in \mathbb{N}$ such that $n_0 \leqslant a$. And, as $\sum_{j=0}^{L}(B(1,j+1)) \neq 0$, the condition "$\sum_{b\in\mathbb{Z}}(F(a,b)) = 1$, for all $a \in \mathbb{N}$ such that $n_0 \leqslant a$" is in fact equivalent to "$\sum_{i=0}^{I}\left(\sum_{j=0}^{L}\left(B(i+1,j+1)\cdot a^j \cdot \sum_{b\in\mathbb{Z}}(F(a+j,b+i))\right)\right) = 0$, for all $a \in \mathbb{N}$ such that $n_0 \leqslant a$". Therefore, applying [0], $\sum_{b\in\mathbb{Z}}(F(a,b)) = 1$, for all $a \in \mathbb{N}$ such that $n_0 \leqslant a$.

□

Alas, the currently known upper bounds of the value m in Proposition 3.1.5 are still extremely large. It is hence a research problem to optimize them.

3.2 ZEILBERGER'S CREATIVE TELESCOPING

Having shown the theoretic framework of the previous section, the conditions to explain Zeilberger's creative telescoping have been reached.

Proposition 3.2.1. There exist "telescoped" recurrences
Let $F(n,k)$ be a proper hypergeometric term. Then there exist $J \in \mathbb{N}^+$, $\alpha_0(n), \dots, \alpha_J(n)$ polynomial expressions in n which do not depend on k, and an expression $R(n,k)$ such that $\{\alpha_0(n), \dots, \alpha_J(n)\} \neq \{0\}$, $R(n,k)$ is rational in n,k and, considering $G(n,k) = R(n,k)\cdot F(n,k)$, $\sum_{j=0}^{J}\left(\alpha_j(n)\cdot F(n+j,k)\right) = G(n,k+1) - G(n,k)$.

Proof $F(n,k)$ is a proper hypergeometric term, so, applying Proposition 3.1.4, $\sum_{i=0}^{I}\left(\sum_{j=0}^{J}(A(i+1,j+1)(n)\cdot F(n+j,k+i))\right) = 0$, for some nonzero matrix A of dimension $(I+1)\times(J+1)$ whose entries are polynomial expressions in n which do not depend on k, for some $I,J \in \mathbb{N}$ such that $0 < J$ (note that I,J can be taken arbitrarily large). [0]

Let $H = \{L(n,k) \mid L(n,k)$ is a proper hypergeometric term$\}$ and $\underset{n}{\mathcal{N}}, \underset{k}{\mathcal{N}}, \underset{n}{\mathcal{M}} \in H^H$ such that $\underset{n}{\mathcal{N}}(L(n,k)) = L(n+1,k)$, $\underset{k}{\mathcal{N}}(L(n,k)) = L(n,k+1)$, $\underset{n}{\mathcal{M}}(L(n,k)) = n\cdot L(n,k)$, for all $L(n,k) \in H$. Then from [0] follows that $A(i+1,j+1)(n)\cdot F(n+j,k+i) = \left(A(i+1,j+1)\left(\underset{n}{\mathcal{M}}\right)\circ \underset{n}{\mathcal{N}}^j \circ \underset{k}{\mathcal{N}}^i\right)(F(n,k))$, for all $i \in \{0,\dots,I\}$ and $j \in \{0,\dots,J\}$. As A is a matrix whose entries are polynomial expressions, there exists a polynomial expression P such that $P\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right) = \sum_{i=0}^{I}\left(\sum_{j=0}^{J}\left(A(i+1,j+1)\left(\underset{n}{\mathcal{M}}\right)\circ \underset{n}{\mathcal{N}}^j \circ \underset{k}{\mathcal{N}}^i\right)\right)$. Thus, $P\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right)(F(n,k)) = 0$. [1]

In addition, there exists a polynomial expression Q such that

$$P\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right) = P\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, 1\right) + \left(1 - \underset{k}{\mathcal{N}}\right) \circ Q\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right)$$

(expanding P in a power series in the third considered parameter about the point 1). Applying [1] and considering $G(n,k) = Q\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right)(F(n,k))$, $P\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, 1\right)(F(n,k)) = \left(\underset{k}{\mathcal{N}} - 1\right)(G(n,k))$ holds. [2]

Call $\alpha_j(n) = \sum_{i=0}^{I}(A(i+1,j+1)(n))$, for all $j \in \{0,...,J\}$. Then $P\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, 1\right) =$ $\sum_{i=0}^{I}\left(\sum_{j=0}^{J}\left(A(i+1,j+1)\left(\underset{n}{\mathcal{M}}\right) \circ \underset{n}{\mathcal{N}^j}\right)\right) = \sum_{j=0}^{J}\left(\sum_{i=0}^{I}\left(A(i+1,j+1)\left(\underset{n}{\mathcal{M}}\right) \circ \underset{n}{\mathcal{N}^j}\right)\right) =$ $\sum_{j=0}^{J}\left(\sum_{i=0}^{I}\left(A(i+1,j+1)\left(\underset{n}{\mathcal{M}}\right)\right) \circ \underset{n}{\mathcal{N}^j}\right) = \sum_{j=0}^{J}\left(\alpha_j\left(\underset{n}{\mathcal{M}}\right) \circ \underset{n}{\mathcal{N}^j}\right)$. [3]

Any number of shift operators, when applied to a hypergeometric sequence, only multiply it by a rational function, so $G(n,k) = R(n,k) \cdot F(n,k)$, for some $R(n,k)$ which is a rational expression in n,k. [4]

[2] and [3] yield that $\sum_{j=0}^{J}(\alpha_j(n) \cdot F(n+j,k)) = G(n,k+1) - G(n,k)$. Let P_0 a nonzero polynomial expression such that $P_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right)(F(n,k)) = 0$ and that, for every nonzero polynomial expression $\tilde{P}$ such that $\tilde{P}\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right)(F(n,k)) = 0$, the degree of P_0 in its third considered variable is less or equal than the degree of $\tilde{P}$ with in its third considered variable. Let also Q_0 be a polynomial expression such that $P_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right) = P_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, 1\right) + \left(1 - \underset{k}{\mathcal{N}}\right) \circ Q_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right)$. Assume that $P_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, 1\right) = 0$. Then $P_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right) = \left(1 - \underset{k}{\mathcal{N}}\right) \circ Q_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right)$. Hence, as $P_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right)(F(n,k)) = 0$, $\left(1 - \underset{k}{\mathcal{N}}\right) \cdot Q_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right) = 0$ holds. Consequently, considering $G_0(n,k) = Q_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right)(F(n,k))$, $G_0(n,k+1) = G_0(n,k)$ holds; so $G_0(n,k)$ does not depend on k and it can be denoted by simply $G_0(n)$. $F(n,k)$ is a proper hypergeometric term, in particular, by Proposition 3.1.3, a doubly hypergeometric term, which, applying [4], implies that $\frac{G(n+1,k)}{G(n,k)}$ is a rational expression in n,k. Hence, so is $\frac{G_0(n+1)}{G_0(n)}$. Applying Proposition 1.4.6, $G_0(n)$ is a hypergeometric expression in n; from which follows the existence of polynomial expressions $p(n), q(n)$ in n which do not depend in k, and such that $G_0(n+1) \cdot p(n) + G_0(n) \cdot q(n) = 0$. It is clear then that there exists a nonzero polynomial expression M such that $M\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}\right)(G_0(n,k)) = 0$ and that its degree in the first considered variable is 1. There are two possible cases:

- Q_0 is zero: if so, then P_0 is zero too. Impossible.
- Q_0 is nonzero: then $M\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}\right)(G_0(n,k)) = 0$, i.e. $M\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}\right)\left(Q_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, \underset{k}{\mathcal{N}}\right)(F(n,k))\right) = 0$, and the following happens to the degrees in the third considered variable: the one of $M \cdot Q_0$ equals the one of Q_0, which is less than the one of P_0, in contradiction with the minimality. So $M \cdot Q_0$ should be zero. Also impossible.

Therefore, $P_0\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, 1\right) \neq 0$; so $0 \neq P\left(\underset{n}{\mathcal{N}}, \underset{n}{\mathcal{M}}, 1\right) \overset{[3]}{=} \sum_{j=0}^{J}\left(\alpha_j\left(\underset{n}{\mathcal{M}}\right) \circ \underset{n}{\mathcal{N}^j}\right)$ and thus $\{\alpha_0(n), ..., \alpha_J(n)\} \neq \{0\}$. □

Definition 3.2.2. Let $S \in \wp(\mathbb{Z})^{\mathbb{N}}$. Then S is said to be a **compact support** if there exist $s,t \in \mathbb{Z}^{\mathbb{N}}$ such that s or t are nonconstant and $\{s(a), ..., t(a)\} = S(a)$, for all $a \in \mathbb{N}$.

The problem "Given a field E of characteristic zero such that $E \subseteq \mathbb{C}$ and a proper hypergeometric term (defined with respect to E) $F(n,k)$ such that there exist a compact support S and $n_0 \in \mathbb{N}$ such that $F(n,k)$ is well-defined in (a,b), for all $b \in S(a)$, for all $a \in \mathbb{N}$ such that $n_0 \leqslant a$, decide constructively if $\sum_{b \in S(n)}(F(n,b))$ represent a sequence in n having hypergeometric closed form." is called **problem of definite hypergeometric summation**.

So if, following the lines of Proposition 3.2.1, there exist a compact support S, $n_0 \in \mathbb{N}$ and a field $\tilde{F}$ of characteristic zero such that $F(a,b) \in \tilde{F}$, for all $b \in S(a)$, for all $a \in \mathbb{N}$ such that $n_0 \leqslant a$, then, considering $f \in \tilde{F}^{\mathbb{N}}$ such that $f(a) = \sum_{b \in S(n)} (F(a,b))$, for all $a \in \mathbb{N}$ such that $n_0 \leqslant a$, from $\sum_{j=0}^{J} \left(\alpha_j(n) \cdot F(n+j,k)\right) = G(n,k+1) - G(n,k)$ follows that $\sum_{b \in S(a)} \left(\sum_{j=0}^{J} \left(\alpha_j(a) \cdot F(a+j,b)\right)\right) = \sum_{b \in S(a)} (G(a,b+1) - G(a,b))$, i.e. $\sum_{j=0}^{J} \left(\alpha_j(a) \cdot \mathcal{N}^j(f)(a)\right) = G(a,t(a)+1) - G(a,s(a))$, for all $a \in \mathbb{N}$ such that $n_0 \leqslant a$. Thus, the algorithm from Chapter 1 for solving linear difference equations with polynomial coefficients is applicable (note that the order of the corresponding linear recurrence operator may be lower than J, since it is only known that $\{\alpha_0(n), ..., \alpha_J(n)\} \neq \{0\}$, but not that $\alpha_0(n) \cdot \alpha_J(n) \neq 0$), solving therefore the problem of definite hypergeometric summation.

What is remaining then, is to construct an algorithm computing the elements $\alpha_0(n), ..., \alpha_J(n)$, $G(n,k)$. Such an algorithm is precisely the so-called Zeilberger's creative telescoping algorithm, and it works as follows.

Zeilberger's creative telescoping algorithm	Example
Input: a proper hypergeometric term $F(n,k)$.	Input: $F(n,k) := \binom{2\cdot k}{k} \cdot \binom{2\cdot n-2\cdot k}{n-k}$.
1. $J := 0$.	1. $J := 0$.
2. $J := J+1$.	2. $J := 1$.
3. Compute $p_0(n,k), q_0(n,k), p_1(n,k), q_1(n,k)$, polynomial expressions in n,k such that $\frac{p_0(n,k)}{q_0(n,k)} = \frac{F(n,k+1)}{F(n,k)}$ and $\frac{p_1(n,k)}{q_1(n,k)} = \frac{F(n,k)}{F(n-1,k)}$.	3. $p_0(n,k) := (2\cdot k+1)\cdot(k-n)$, $q_0(n,k) := (1-2\cdot n+2\cdot k)\cdot(k+1)$, $p_1(n,k) := 2\cdot(1-2\cdot n+2\cdot k)$, $q_1(n,k) := k-n$.

Note that Proposition 3.1.3 has been applied here.

4. $p(k) := \sum_{j=0}^{J}(a_j \cdot \prod_{i=1}^{j}(p_1(n+i,k)) \cdot \prod_{i=j+1}^{J}(q_1(n+i,k)))$ ($a_0, ..., a_J$ are new parameters), $r(k) := p_0(n,k) \cdot \prod_{j=1}^{J}(q_1(n+j,k))$, $s(k) := q_0(n,k) \cdot \prod_{j=1}^{J}(q_1(n+j,k+1))$, $T(n,k) := \sum_{j=0}^{J}(a_j \cdot F(n+j,k))$.	4. $p(k) :=$ $a_0\cdot(k-n-1) + a_1\cdot 2\cdot(2\cdot k-1-2\cdot n)$, $r(k) := (2\cdot k+1)\cdot(k-n)\cdot(k-n-1)$, $s(k) := (1-2\cdot n+2\cdot k)\cdot(k+1)\cdot(k-n)$, $T(n,k) := \frac{2}{n+1-k}\cdot\binom{2\cdot k}{k}\cdot\binom{2\cdot n-2\cdot k}{n-k}$.
5. Compute nonzero polynomial expressions $a(k), b(k), c(k)$ in k such that the following conditions hold: • $b(k)$ and $c(k)$ are monic, • $\frac{r(k)}{s(k)} = \frac{a(k)\cdot c(k+1)}{b(k)\cdot c(k)}$, • g.c.d. $(\{a(k), b(k+h)\})$ = g.c.d.$(\{a(k), c(k)\})$ = g.c.d. $(\{b(k), c(k+1)\}) = 1$, for all $h \in \mathbb{N}$.	5. $a(k) := (k+1/2)\cdot(k-(n+1))$, $b(k) := (k+1)\cdot(k-(n-1/2))$, $c(t) := 1$.

Note that Proposition 1.5.1 has been applied here; and recall that an algorithm performing the computation of the step 5 can be found in Section 5.3 of [Petkovšek et al.].

6. $P(k) := c(k)\cdot p(k)$.	6. $P(k) :=$ $a_0\cdot(k-n-1) + a_1\cdot 2\cdot(2\cdot k-1-2\cdot n)$.

At this point, it is interesting to remark that $\frac{a(k)\cdot P(k+1)}{b(k)\cdot P(k)} = \frac{a(k)\cdot c(k+1)\cdot p(k+1)}{b(k)\cdot c(k)\cdot p(k)} = \frac{r(k)\cdot p(k+1)}{s(k)\cdot p(k)} =$

$$\frac{p_0(n,k)}{q_0(n,k)} \cdot \frac{\prod_{j=1}^{J}(q_1(n+j,k)) \cdot \sum_{j=0}^{J}(a_j \cdot \prod_{i=1}^{j}(p_1(n+i,k+1)) \cdot \prod_{i=j+1}^{J}(q_1(n+i,k+1)))}{\prod_{j=1}^{J}(q_1(n+j,k+1)) \cdot \sum_{j=0}^{J}(a_j \cdot \prod_{i=1}^{j}(p_1(n+i,k)) \cdot \prod_{i=j+1}^{J}(q_1(n+i,k)))} =$$

$$\frac{p_0(n,k)}{q_0(n,k)} \cdot \frac{\sum_{j=0}^{J}\left(a_j \cdot \frac{\prod_{i=1}^{j}(p_1(n+i,k+1)) \cdot \prod_{i=j+1}^{J}(q_1(n+i,k+1))}{\prod_{i=1}^{J}(q_1(n+i,k+1))}\right)}{\sum_{j=0}^{J}\left(a_j \cdot \frac{\prod_{i=1}^{j}(p_1(n+i,k)) \cdot \prod_{i=j+1}^{J}(q_1(n+i,k))}{\prod_{i=1}^{J}(q_1(n+i,k))}\right)} =$$

$$\frac{p_0(n,k)}{q_0(n,k)} \cdot \frac{\sum_{j=0}^{J}\left(a_j \cdot \prod_{i=1}^{j}\left(\frac{p_1(n+i,k+1)}{q_1(n+i,k+1)}\right)\right)}{\sum_{j=0}^{J}\left(a_j \cdot \prod_{i=1}^{j}\left(\frac{p_1(n+i,k)}{q_1(n+i,k)}\right)\right)} =$$
$$\frac{F(n,k+1)}{F(n,k)} \cdot \frac{\sum_{j=0}^{J}\left(a_j \cdot \prod_{i=1}^{j}\left(\frac{F(n+i,k+1)}{F(n+i-1,k+1)}\right)\right)}{\sum_{j=0}^{J}\left(a_j \cdot \prod_{i=1}^{j}\left(\frac{F(n+i,k)}{F(n+i-1,k)}\right)\right)} = \frac{\sum_{j=0}^{J}\left(a_j \cdot F(n+j,k+1)\right)}{\sum_{j=0}^{J}\left(a_j \cdot F(n+j,k)\right)} = \frac{T(n,k+1)}{T(n,k)}.$$

7. If $\deg(a(k)) \neq \deg(b(k))$ or $\mathrm{l.c.}(a(k)) \neq \mathrm{l.c.}(b(k))$, then let $d = \deg(P(k)) - \max(\{\deg(a(k)), \deg(b(k))\}$; otherwise:	7. ————————
7.1. if $\deg(a(k) \cdot B(k+1) - b(k-1) \cdot B(k)) < \deg(a(k)) + \deg(B(k)) - 1$, for every $B(k)$ nonzero polynomial expression in k, then let $\lambda = \mathrm{l.c.}(a(k))$, $m = \deg(a(k))$, $A = \mathrm{coeff}_{m-1}(a(k))$, $B = \mathrm{coeff}_{m-1}(b(k-1))$ and $d = \frac{B-A}{\lambda}$;	7.1. ————————
7.2. otherwise, let $d = \deg(P(k)) - \deg(a(k)) + 1$.	7.2. $a(k) \cdot (C_0 \cdot k^r + (C_0 \cdot r + C_1) \cdot k^{r-1} + \mathcal{O}(k^{r-2})) - b(k-1) \cdot (C_0 \cdot k^r + C_1 \cdot k^{r-1} + \mathcal{O}(k^{r-2})) = C_0 \cdot (r-2) \cdot k^{r+1} + \mathcal{O}(k^r)$, for all $r \in \mathbb{N}$ and C_0, C_1 expressions which do not depend on k and such that C_0 is nonzero, so $d := \deg(P(k)) - \deg(a(k)) + 1 = 0$.
8. $B(k) := \sum_{i=0}^{d}(b_i \cdot k^i)$ ($b_0, ..., b_d$ are new parameters).	8. $B(k) := b_0$.
9. Compute, if possible, a solution of the polynomial equation given by $a(k) \cdot B(k+1) - b(k-1) \cdot B(k) = P(k)$ with $a_0, ..., a_J, b_0, ..., b_d$ as unknowns (which reduces to a system of linear algebraic equations by matching the coefficients of like powers of k): if it exists, then continue; otherwise, go to 2.	9. $a(k) \cdot B(k+1) - b(k-1) \cdot B(k) = P(k) \Leftrightarrow$ $-(a_0 + 4 \cdot a_1) \cdot k - b_0 \cdot n/2 - b_0/2 + a_0 \cdot n + a_0 + 4 \cdot a_1 \cdot n + 2 \cdot a_1 = 0 \Leftrightarrow$ $[a_0 + 4 \cdot a_1 = 0 \wedge a_0 \cdot (n+1) + 2 \cdot a_1 \cdot (2 \cdot n + 1) - b_0 \cdot (n+1)/2 = 0] \Leftrightarrow$ $[a_0 = (n+1) \cdot x \wedge a_1 = -(n+1) \cdot x/4 \wedge b_0 = x]$, for every nonzero expression x which does not depend on k. So, for instance, call $b_0 = \frac{4}{n+1}$, $a_0 = 4$, $a_1 = -1$.
10. Let $G(n,k) = \frac{b(k-1) \cdot B(k)}{P(k)} \cdot T(n,k)$, return $G(n,k)$ and the found values of $a_0, ..., a_J$, and STOP.	10. $G(n,k) := \frac{2 \cdot k \cdot (2 \cdot n + 1 - 2 \cdot k)}{(n+1-k) \cdot (n+1)} \cdot \binom{2 \cdot k}{k} \cdot \binom{2 \cdot n - 2 \cdot k}{n-k}$.
Output: for some $J \in \mathbb{N}^+$, $\alpha_0(n), ..., \alpha_J(n)$ polynomial expressions in n which do not depend on k and such that $\{\alpha_0(n), ..., \alpha_J(n)\} \neq \{0\}$, and an expression $G(n,k)$ such that $\frac{G(n,k)}{F(n,k)}$ is rational in n, k and $\sum_{j=0}^{J}\left(\alpha_j(n) \cdot F(n+j,k)\right) = G(n,k+1) - G(n,k)$.	Output: $4, -1, \frac{2 \cdot k \cdot (2 \cdot n + 1 - 2 \cdot k)}{(n+1-k) \cdot (n+1)} \cdot \binom{2 \cdot k}{k} \cdot \binom{2 \cdot n - 2 \cdot k}{n-k}$.

Note that this procedure is based on Proposition 2.2.1; and Proposition 3.2.1 guarantees that it stops at some point.

Although Zeilberger's algorithm has been stated here for proper hypergeometric terms, it seems to admit doubly hypergeometric terms.

As a last remark, note that sometimes the problem of definite hypergeometric summation has particular positive answers where the problem of indefinite hypergeometric summation has not; eg. considering $n_0 \in \mathbb{N}$ and $f, g \in \mathbb{Q}^{\mathbb{N}}$ such that $f(n) = \sum_{k=0}^{n} \left(\binom{n}{k}\right)$ and $g(n) = \sum_{k=0}^{n_0} \left(\binom{n}{k}\right)$, for all $n \in \mathbb{N}$, it is easy to verify that g has no hypergeometric closed form, but f has (indeed, $f(n) = 2^n$, for all $n \in \mathbb{N}$). This phenomenon is similar to what happens in the comparison between definite integration and indefinite integration, eg. considering $h, i \in \mathbb{R}^{\mathbb{R}}$ such that $h(v) = \int_0^v \exp(-\frac{u^2}{2}) \cdot du$ and $i(v) = \int_{-\infty}^{\infty} \exp(-\frac{u^2}{2}) \cdot du$, for all $v \in \mathbb{R}$, h has no elementary primitive (cf. Section 5 of [Ivorra Castillo], in Spanish), but i has (indeed, $i(v) = \sqrt{2 \cdot \pi}$).

3.3 THE WZ METHOD

The aim of this section is to show the so-called WZ method, which can be used as alternative to Zeilberger's creative telescoping in some cases. A precise comparative between them will be provided.

Definition 3.3.1. Let $F(n,k)$ be a doubly hypergeometric term, $R(n,k)$ a rational expression in n,k and $G(n,k) = R(n,k) \cdot F(n,k)$. Then $(F(n,k), G(n,k))$ is said to be a **WZ pair**, $G(n,k)$ is said to be a **WZ mate** of $F(n,k)$ and $R(n,k)$ is said to be a **WZ certificate** for $F(n,k)$ if $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$.

For example, $\left(\frac{k}{2^{n-1} \cdot n} \cdot \binom{n}{k}, -\frac{1}{2^n} \cdot \binom{n-1}{k-2}\right)$ is a WZ pair.

Lemma 3.3.2. Let $F(n,k)$ be a doubly hypergeometric term which is well-defined in $\{(n,b)\}_{b \in \mathbb{Z}}$ and which has a WZ mate $G(n,k)$ which is well-defined in $\{(n,b)\}_{b \in \mathbb{Z}}$ and such that $\lim_{t \to \infty} (G(n, \pm t)) = 0$. Then $\sum_{b \in \mathbb{Z}} (F(n,b)) = x$, for some expression x which does not depend on n, k.

Proof $G(n,k)$ is a WZ mate of $F(n,k)$, so $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$ and then, applying that $\lim_{t \to \infty} (G(n, \pm t)) = 0$, $\sum_{b \in \mathbb{Z}} (F(n+1,b)) - \sum_{b \in \mathbb{Z}} (F(n,b)) = \sum_{b \in \mathbb{Z}} (F(n+1,b) - F(n,b)) = \sum_{b \in \mathbb{Z}} (G(n,b+1) - G(n,b)) = \lim_{t \to \infty} \left(\sum_{b=-t}^{t} (G(n,b+1) - G(n,b)) \right) = \lim_{t \to \infty} (G(n,t+1) - G(n,-t)) = 0$. Hence, $\sum_{b \in \mathbb{Z}} (F(n,b)) = x$, for some expression x which does not depend on n, k. □

To compute WZ mates can be extremely useful for proving combinatorial identities. For instance, given a doubly hypergeometric term $f(n,k)$ such that $f(n,k)$ is well-defined in $\{(n,b)\}_{b \in \mathbb{Z}}$ and an expression $r(n)$ which does not depend on k and such that, if $r(n) \neq 0$, then $\frac{f(n,k)}{r(n)}$ is a doubly hypergeometric term, to prove that $\sum_{b \in \mathbb{Z}} (f(n,b)) = r(n)$ it suffices to:

1. let $F(n,k) = \begin{cases} f(n,k) & \text{if } r(n) = 0 \\ \frac{f(n,k)}{r(n)} & \text{otherwise} \end{cases}$,
2. compute, if possible, a WZ mate $G(n,k)$ of $F(n,k)$,
3. check if $\lim_{t \to \infty} (G(n, \pm t)) = 0$ (if so, by Lemma 3.3.2, $\sum_{b \in \mathbb{Z}} (F(n,b)) = x$, for some expression x which does not depend on n, k),
4. compute, if possible, $\sum_{k \in \mathbb{Z}} (f(n_0,k))$ and $r(n_0)$, for some $n_0 \in \mathbb{Z}$ (if $r(n) \neq 0$, then n_0 must be chosen so that $r(n_0) \neq 0$),
5. check if $\sum_{b \in \mathbb{Z}} (f(n_0,b)) = r(n_0)$.

Note that:

- If $r(n) = 0$, then $\sum_{k \in \mathbb{Z}} (f(n,k)) = \sum_{k \in \mathbb{Z}} (F(n,k)) = x = \sum_{k \in \mathbb{Z}} (F(n_0,k)) = \sum_{k \in \mathbb{Z}} (f(n_0,k)) = r(n_0) = 0$.
- if $r(n) \neq 0$, then $\sum_{b \in \mathbb{Z}} \left(\frac{f(n,b)}{r(n)} \right) = x$. Hence, $\sum_{b \in \mathbb{Z}} (f(n,b)) = x \cdot r(n)$; in particular, $\sum_{b \in \mathbb{Z}} (f(n_0,b)) = x \cdot r(n_0)$. And, applying that $\sum_{b \in \mathbb{Z}} (f(n_0,b)) = r(n_0)$ and $r(n_0) \neq 0$, $x = 1$ holds. Therefore, $\sum_{k \in \mathbb{Z}} (f(n,k)) = r(n)$.

There is an open problem which reads "Let $a(k)$ be an expression involving k as integer parameter (it can involve more parameters) and such that $\frac{a(k+1)}{a(k)}$ is a rational expression in k and, when all the parameters of $a(k)$ take concrete values, the result lies in $\mathbb{K}$. What additional conditions should $a(k)$ satisfy in order to ensure the existence of a doubly hypergeometric term $A(n,k)$ having a WZ mate and such that $A(0,k) = a(k)$?". Roughly speaking, its interest lies in the fact that it often works better to have a doubly hypergeometric summand than a hypergeometric one, due to the possibility of using the WZ method. This idea has been successfully applied in several particular cases, for example for proving Ramanujan's series of π (cf. [Ekhad & Zeilberger]).

Lemma 3.3.3. Let $F(n,k)$ a doubly hypergeometric term and $a(k) = F(n+1,k) - F(n,k)$. Then $\frac{a(k+1)}{a(k)}$ is a rational expression in k.

Proof The fact that $F(n,k)$ is a doubly hypergeometric term yields that $\frac{a(k+1)}{a(k)}$, i.e. $\frac{F(n+1,k+1)-F(n,k+1)}{F(n+1,k)-F(n,k)}$ or, equivalently, $\frac{\frac{F(n+1,k+1)}{F(n,k+1)}-1}{\frac{F(n+1,k)}{F(n,k)}-1} \cdot \frac{F(n,k+1)}{F(n,k)}$, is a rational expression in k. □

The so-called WZ method, whose aim is precisely to decide constructively if there are WZ mates, is now constructed, by using Gosper's algorithm as subroutine.

WZ method	Example
Input: a doubly hypergeometric term $F(n,k)$.	Input: $\frac{k}{2^{n-1} \cdot n} \cdot \binom{n}{k}$.
1. $a(k) := F(n+1,k) - F(n,k)$.	1. $a(k) := \frac{k}{2^n \cdot (n+1)} \cdot \binom{n+1}{k} - \frac{k}{2^{n-1} \cdot n} \cdot \binom{n}{k}$.
2. Apply Gosper's algorithm to the sequence induced by $a(k)$: if it succeeds with output b, then let $G(n,k) = b(k)$, return $G(n,k)$ and STOP; otherwise, return "The WZ method is not applicable." and STOP.	2. $G(n,k) := -\frac{1}{2^n} \cdot \binom{n-1}{k-2}$.

Note that Lemma 3.3.3 has been applied here.

Output: a WZ mate $G(n,k)$ of $F(n,k)$, if it exists; "The WZ method is not applicable." otherwise.	Output: $-\frac{1}{2^n} \cdot \binom{n-1}{k-2}$.

From Proposition 2.2.1 it is derived that, given a doubly hypergeometric term $F(n,k)$, $F(n,k)$ has a WZ mate if, and only if, the WZ method succeeds for $F(n,k)$.

Zeilberger's creative telescoping algorithm already does the work of the WZ method, under certain conditions. Concretely, given an expression $f(n)$ whose only parameter is n (and it is integer), and such that $\frac{f(n+1)}{f(n)}$ is a rational expression in n, and a proper hypergeometric term $F(n,k)$ such that $F(n,k)$ is well-defined in $\{(n,b)\}_{b \in \mathbb{Z}}$ and $\sum_{b \in \mathbb{Z}}(F(n,b)) = f(n)$, then the WZ method succeeds for $\frac{F(n,k)}{f(n)}$ if, and only if, Zeilberger's creative telescoping algorithm computes a first-order recurrence for $F(n,k)$ (cf. Proposition 8.1.1 of [Petkovšek et al.]).

3.4 GETTING MORE COMBINATORIAL IDENTITIES FROM A WZ PAIR

Lemma 3.3.2 and its following explanation exhibit how a WZ pair induces a combinatorial identity. As it will be shown now, other combinatorial identities (the so-called companion WZ identities, dual WZ identities and Zeilberger's definite-sum-made-indefinite identity) can be derived from a WZ pair (cf. [Gessel], Section 7.3 of [Petkovšek et al.] and Section 1 of [Guillera Goyanes] for more derived identities).

Proposition 3.4.1. There exist companion WZ identities

Let $F(n,k)$ be a doubly hypergeometric term such that the following conditions hold:

- there exists $k_0 \in \mathbb{Z}$ such that $F(n,k)$ is well-defined in $\{(a,b) \mid [a \in \mathbb{N} \wedge b \in \mathbb{Z} \wedge b \leqslant k_0]\}$ and the sequence given by $F(n,i)$ is convergent, for all $i \in \mathbb{Z}$ such that $i \leqslant k_0$,
- $F(n,k)$ has a WZ mate $G(n,k)$ which is well-defined in $\{(a,k)\}_{a\in\mathbb{N}}$ and such that $\lim\limits_{b\to\infty}\left(\sum_{a\in\mathbb{N}}(G(a,-b))\right) = 0$.

Then $\sum_{a\in\mathbb{N}}(G(a,k)) = \sum_{i=-\infty}^{k-1}\left(\lim\limits_{a\to\infty}(F(a,i)) - F(0,i)\right)$.

Proof $\sum_{a\in\mathbb{N}}(G(a,k)) =$
$\sum_{a\in\mathbb{N}}(G(a,k)) - \lim\limits_{b\to\infty}\left(\sum_{a\in\mathbb{N}}(G(a,-b))\right) =$
$\lim\limits_{b\to\infty}\left(\sum_{a\in\mathbb{N}}(G(a,k)) - \sum_{a\in\mathbb{N}}(G(a,-b))\right) =$
$\lim\limits_{b\to\infty}\left(\sum_{i=-b}^{k-1}\left(\sum_{a\in\mathbb{N}}(G(a,i+1)) - \sum_{a\in\mathbb{N}}(G(a,i))\right)\right) =$
$\lim\limits_{b\to\infty}\left(\sum_{i=-b}^{k-1}\left(\sum_{a\in\mathbb{N}}(G(a,i+1) - G(a,i))\right)\right) =$
$\lim\limits_{b\to\infty}\left(\sum_{i=-b}^{k-1}\left(\lim\limits_{a\to\infty}\left(\sum_{j=0}^{a}(G(j,i+1) - G(j,i))\right)\right)\right) =$
$\lim\limits_{b\to\infty}\left(\sum_{i=-b}^{k-1}\left(\lim\limits_{a\to\infty}\left(\sum_{j=0}^{a}(F(j+1,i) - F(j,i))\right)\right)\right) =$
$\lim\limits_{b\to\infty}\left(\sum_{i=-b}^{k-1}\left(\lim\limits_{a\to\infty}(F(a+1,i) - F(0,i))\right)\right) =$
$\lim\limits_{b\to\infty}\left(\sum_{i=-b}^{k-1}\left(\lim\limits_{a\to\infty}(F(a,i)) - F(0,i)\right)\right) =$
$\sum_{i=-\infty}^{k-1}\left(\lim\limits_{a\to\infty}(F(a,i)) - F(0,i)\right)$. □

Proposition 3.4.2. There exist dual WZ identities

Let $F(n,k), G(n,k), F_1(n,k), G_1(n,k), F_2(n,k), G_3(n,k)$ be expressions such that:

1. $F(n,k) = P(n,k) \cdot \dfrac{\prod_{i=0}^{m}(\Gamma(\alpha_i \cdot n + \beta_i \cdot k + \gamma_i))}{\prod_{i=0}^{r}(\Gamma(\delta_i \cdot n + \varepsilon_i \cdot k + \varphi_i))} \cdot x^k$, being:
   - $P(u,v) \in K[u,v]$,
   - n, k integer parameters,
   - $m, r \in \mathbb{N}$,
   - $\alpha_i, \beta_i, \delta_j, \varepsilon_j \in \mathbb{Z}$, for all $i \in \{0,...,m\}$ and $j \in \{0,...,r\}$,
   - $\gamma_i \in K$ or γ_i a parameter taking values in K, for all $i \in \{0,...,m\}$,
   - $\varphi_i \in K$ or φ_i a parameter taking values in K, for all $j \in \{0,...,r\}$,
   - $x \in K$ or x a parameter taking values in K,
2. $(F(n,k), G(n,k))$ is a WZ pair,
3. for all $i \in \{1,2\}$, $F_i(n,k) = L_i(n,k) \cdot F(n,k)$ and $G_i(n,k) = L_i(n,k) \cdot G(n,k)$, being:
   - $L_1(n,k) = \dfrac{(-1)^{\alpha_I \cdot n + \beta_I \cdot k}}{\Gamma(\alpha_I \cdot n + \beta_I \cdot k + \gamma_I) \cdot \Gamma(1 - \alpha_I \cdot n - \beta_I \cdot k - \gamma_I)}$, for some $I \in \{0,...,m\}$,
   - $L_2(n,k) = \dfrac{\Gamma(\delta_J \cdot n + \varepsilon_J \cdot k + \varphi_J) \cdot \Gamma(1 - \delta_J \cdot n - \varepsilon_J \cdot k - \varphi_J)}{(-1)^{\delta_J \cdot n + \varepsilon_J \cdot k}}$, for some $J \in \{0,...,r\}$.

Then $(F_1(n,k), G_1(n,k))$ and $(F_2(n,k), G_2(n,k))$ are WZ pairs (cf. subsection "Dual identities" of Section 7.3 of [Petkovšek et al.]).

For example, it can be easily checked that, considering $F(n,k) = \binom{n}{k}^2 \cdot \binom{2\cdot n}{n}^{-1}$ and $G(n,k) = \frac{(2\cdot k-3-3\cdot n)\cdot k^2}{2\cdot(2\cdot n+1)\cdot(n-k+1)^2} \cdot F(n,k)$, $(F(n,k), G(n,k))$ is a WZ pair; and note that $F(n,k) = \frac{n!^4}{(n-k)!^2\cdot k!^2\cdot(2\cdot n)!} = \frac{\Gamma(1\cdot n+0\cdot k+1)^4}{\Gamma(1\cdot n+(-1)\cdot k+1)^2\cdot\Gamma(0\cdot n+1\cdot k+1)^2\cdot\Gamma(2\cdot n+0\cdot k+1)}$. Considering now $L(n,k) = \frac{\Gamma(0\cdot n+1\cdot k+1)^2\cdot\Gamma(1-0\cdot n-1\cdot k-1)^2}{((-1)^{0\cdot n+1\cdot k})^2} \cdot \frac{\Gamma(2\cdot n+0\cdot k+1)\cdot\Gamma(1-2\cdot n-0\cdot k-1)}{(-1)^{2\cdot n+0\cdot k}} \cdot \frac{((-1)^{1\cdot n+0\cdot k})^4}{\Gamma(1\cdot n+0\cdot k+1)^4\cdot\Gamma(1-1\cdot n-0\cdot k-1)^4} = \frac{k!^2\cdot(-k-1)!^2\cdot(2\cdot n)!\cdot(-2\cdot n-1)!}{n!^4\cdot(-n-1)!^4}$ and applying Proposition 3.4.2 seven times,

$(L(n,k) \cdot F(n,k), L(n,k) \cdot G(n,k))$, i.e. $\left(\frac{(-1-k)!^2 \cdot (-1-2\cdot n)!}{(-1-n)!^4 \cdot (n-k)!^2}, \frac{(3\cdot n+3-2\cdot k)\cdot(-k)!^2\cdot(-2-2\cdot n)!}{2\cdot(n+1-k)!^2\cdot(-1-n)!^4}\right)$, is another WZ pair.

Proposition 3.4.1 does not induce an involution on the combinatorial identitities (i.e. the companion WZ identity of a companion WZ identity is not always the original identity), but Proposition 3.4.2 does. However, WZ dualization is not always commuting with specialization (i.e. a WZ dual of a particular case of a given combinatorial identity does not always coincide with a particular case of a WZ dual of such identity).

Proposition 3.4.3. There exist Zeilberger's definite-made-indefinite identities

Let $n_0, r \in \mathbb{N}$ such that $n_0 < r$ and $F(n,k)$ a doubly hypergeometric term which is well-defined in $\{n_0, ..., r\}^2$ and which has a WZ mate $G(n,k)$ which is well-defined in $\{n_0, ..., r-1\} \times \{n_0, ..., r\}$ and such that $G(i, n_0) = 0$, for all $i \in \{n_0, ..., r-1\}$. Then $\sum_{b=n_0}^{r}(F(r,b)) = F(n_0, n_0) + \sum_{i=n_0}^{r-1}(G(i, i+1) + F(i+1, i+1))$.

Proof $\sum_{i=n_0}^{r-1}(G(i,i+1) + F(i+1,i+1)) =$
$\sum_{i=n_0}^{r-1}(G(i,i+1) - G(i,n_0) + F(i+1,i+1)) =$
$\sum_{i=n_0}^{r-1}(\sum_{b=n_0}^{i}(G(i,b+1) - G(i,b)) + F(i+1,i+1)) =$
$\sum_{i=n_0}^{r-1}(\sum_{b=n_0}^{i}(F(i+1,b) - F(i,b)) + F(i+1,i+1)) =$
$\sum_{i=n_0}^{r-1}(\sum_{b=n_0}^{i+1}(F(i+1,b)) - \sum_{b=n_0}^{i}(F(i,b))) =$
$\sum_{b=n_0}^{r}(F(r,b)) - F(n_0,n_0)$. □

Proposition 3.4.3 can be utilized for asymptotics and speeding up table making, since the summands depend only on a single variable (cf. Example 7.3.6 of [Petkovšek et al.]).

4 HYPERGEOMETRIC SERIES

From the concept of hypergeometric sequence, the so-called hypergeometric series will be now defined. This class of series is of central importance, since its generality reaches the majority of the functions used in mathematics. Some famous identities will be proven, with the help of the WZ method; and an example of how they can be used to prove combinatorial identities will be shown.

During this chapter, let K be a field of characteristic zero such that $K \subseteq \mathbb{C}$.

4.1 POCHHAMMER SYMBOLS

Hypergeometric series can be expressed in terms of the so-called Pochhammer symbols. In this section will be explained how.

Definition 4.1.1. Let s be a series. Then s is said to be **hypergeometric** if there exist an expression $f(k)$ involving k as integer parameter (it can involve more parameters) and such that the following conditions hold:

- when all the parameters of $f(k)$ take concrete values, the result lies in K,
- $\frac{f(k+1)}{f(k)}$ is a rational expression in k,
- $s = \sum_{j=0}^{\infty}(f(j))$.

For example, $\sum_{j=0}^{\infty}\left(\binom{2\cdot j}{j}\cdot x^j\right)$ is hypergeometric, since given $a \in K(x)^{\mathbb{N}}$ such that $a(j) = \binom{2\cdot j}{j}\cdot x^j$, for all $j \in \mathbb{N}$, $a \in \text{hyp}(K(x))$.

Definition 4.1.2. Let $f \in K^K$ and $n \in \mathbb{Z}$. Then f is said to be the n^{th} **Pochhammer symbol**, or the n^{th} **rising factorial**, if $f(\alpha) = \prod_{k=0}^{n-1}(\alpha+k)$, for all $\alpha \in K$, fact which is denoted by $f(\alpha) = (\alpha)_n$, for all $\alpha \in K$.

For example, $(-1)_3 = \prod_{k=0}^{2}(k-1) = (-1)\cdot 0 \cdot 1 = 0$.

Proposition 4.1.3. Some properties of the Pochhammer symbol
Let $\alpha \in K$ and $n \in \mathbb{Z}$. Then the following conditions hold:

1. if $n \notin \mathbb{N}^+$, then $(\alpha)_n = 1$,
2. if $n \in \mathbb{N}$, then $(1)_n = n!$,
3. if $n \in \mathbb{N}$ and $\alpha \notin \mathbb{Z}\backslash\mathbb{N}^+$, then $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$.

Proof
1. Recall that, by convention, given $f \in K^{\mathbb{N}}$, $j \in \mathbb{N}$ and $i \in \mathbb{Z}$ such that $i < j$, $\prod_{k=j}^{i}(f(k)) = 1$.
2. Immediate.
3. Apply iteratively the fact that $\Gamma(\beta+1) = \beta\cdot\Gamma(\beta)$, for all $\beta \in K\backslash(\mathbb{Z}\backslash\mathbb{N}^+)$. □

From now on, given $p, q \in \mathbb{N}^+$ and expressions $\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q, t$ such that, when all their parameters take concrete values, the result lies in K:

- $\displaystyle\sum_{n\in\mathbb{N}}\left(\frac{(\alpha_1)_n\cdot ...\cdot(\alpha_p)_n\cdot t^n}{(\beta_1)_n\cdot ...\cdot(\beta_q)_n\cdot n!}\right)$ will be denoted by ${}_pF_q\left[\begin{matrix}\alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_q\end{matrix}\ ;\ t\right]$,
- $\displaystyle\sum_{n\in\mathbb{N}}\left(\frac{(\alpha_1)_n\cdot ...\cdot(\alpha_p)_n\cdot t^n}{n!}\right)$ will be denoted by ${}_pF_0\left[\begin{matrix}\alpha_1 & \cdots & \alpha_p \\ & - & \end{matrix}\ ;\ t\right]$,

- $\sum_{n\in\mathbb{N}}\left(\frac{t^n}{(\beta_1)_n\cdot\ldots\cdot(\beta_q)_n\cdot n!}\right)$ will be denoted by ${}_0F_q\left[\begin{matrix} & - & \\ \beta_1 & \cdots & \beta_q\end{matrix};t\right]$,
- $\sum_{n\in\mathbb{N}}\left(\frac{t^n}{n!}\right)$ (i.e. $\exp(t)$) will be denoted by ${}_0F_0\left[\begin{matrix}-\\-\end{matrix};t\right]$.

For example, $\sum_{n\in\mathbb{N}}\left(\frac{(1/2)_n\cdot(4\cdot x)^n}{n!}\right)={}_1F_0\left[\begin{matrix}1/2\\-\end{matrix};4\cdot x\right]$.

Proposition 4.1.4. Recognition of hypergeometric series
Let s be a series such that $\mathrm{term}_0(s)=1$, $p,q\in\mathbb{N}^+$ and $\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q,t$ expressions such that, when all their parameters take concrete values, the result lies in K. Then the following conditions hold:

1. if $\frac{\mathrm{term}_{n+1}(s)}{\mathrm{term}_n(s)}=\frac{(n+\alpha_1)\cdot\ldots\cdot(n+\alpha_p)}{(n+\beta_1)\cdot\ldots\cdot(n+\beta_q)\cdot(n+1)}\cdot t$, for all $n\in\mathbb{N}$, then $s={}_pF_q\left[\begin{matrix}\alpha_1&\cdots&\alpha_p\\\beta_1&\cdots&\beta_q\end{matrix};t\right]$,
2. if $\frac{\mathrm{term}_{n+1}(s)}{\mathrm{term}_n(s)}=\frac{(n+\alpha_1)\cdot\ldots\cdot(n+\alpha_p)}{n+1}\cdot t$, for all $n\in\mathbb{N}$, then $s={}_pF_0\left[\begin{matrix}\alpha_1&\cdots&\alpha_p\\&-&\end{matrix};t\right]$,
3. if $\frac{\mathrm{term}_{n+1}(s)}{\mathrm{term}_n(s)}=\frac{1}{(n+\beta_1)\cdot\ldots\cdot(n+\beta_q)\cdot(n+1)}\cdot t$, for all $n\in\mathbb{N}$, then $s={}_0F_q\left[\begin{matrix}&-&\\\beta_1&\cdots&\beta_q\end{matrix};t\right]$,
4. if $\frac{\mathrm{term}_{n+1}(s)}{\mathrm{term}_n(s)}=\frac{1}{n+1}\cdot t$, for all $n\in\mathbb{N}$, then $s={}_0F_0\left[\begin{matrix}-\\-\end{matrix};t\right]$.

Proof (Of case 1, the rest are analogous) $\frac{\mathrm{term}_{n+1}(s)}{\mathrm{term}_n(s)}=\frac{(n+\alpha_1)\cdot\ldots\cdot(n+\alpha_p)}{(n+\beta_1)\cdot\ldots\cdot(n+\beta_q)\cdot(n+1)}\cdot t=$

$$\frac{\frac{\prod_{k=0}^{n}(\alpha_1+k)\cdot\ldots\cdot\prod_{k=0}^{n}(\alpha_p+k)}{\prod_{k=0}^{n}(\beta_1+k)\cdot\ldots\cdot\prod_{k=0}^{n}(\beta_q+k)\cdot(n+1)!}\cdot t^{n+1}}{\frac{\prod_{k=0}^{n-1}(\alpha_1+k)\cdot\ldots\cdot\prod_{k=0}^{n-1}(\alpha_p+k)}{\prod_{k=0}^{n-1}(\beta_1+k)\cdot\ldots\cdot\prod_{k=0}^{n-1}(\beta_q+k)\cdot n!}\cdot t^n}=\frac{\frac{(\alpha_1)_{n+1}\cdot\ldots\cdot(\alpha_p)_{n+1}}{(\beta_1)_{n+1}\cdot\ldots\cdot(\beta_q)_{n+1}\cdot(n+1)!}\cdot t^{n+1}}{\frac{(\alpha_1)_n\cdot\ldots\cdot(\alpha_p)_n}{(\beta_1)_n\cdot\ldots\cdot(\beta_q)_n\cdot n!}\cdot t^n},$$

for all $n\in\mathbb{N}$. Applying that $\mathrm{term}_0(s)=1$ and Proposition 4.1.3, $s(t)=\sum_{n\in\mathbb{N}}\left(\frac{(\alpha_1)_n\cdot\ldots\cdot(\alpha_p)_n\cdot t^n}{(\beta_1)_n\cdot\ldots\cdot(\beta_q)_n\cdot n!}\right)$.

□

For example, $\sum_{n\in\mathbb{N}}\left(\binom{2\cdot n}{n}\cdot x^n\right)={}_1F_0\left[\begin{matrix}1/2\\-\end{matrix};4\cdot x\right]$, since $\binom{2\cdot 0}{0}\cdot x^0=1$ and $\frac{\binom{2\cdot(n+1)}{n+1}\cdot x^{n+1}}{\binom{2\cdot n}{n}\cdot x^n}=\frac{n+1/2}{n+1}\cdot 4\cdot x$, for all $n\in\mathbb{N}$.

Many of the usual series are hypergeometric, or at least can be easily expressed in terms of hypergeometric series; eg., given an indeterminate x over K and $p\in\mathbb{N}$, $J_p(x)=\sum_{n\in\mathbb{N}}\left(\frac{(-1)^n\cdot(x/2)^{2\cdot n+p}}{(n+p)!\cdot n!}\right)=\frac{(x/2)^p}{p!}\cdot\sum_{n\in\mathbb{N}}\left(\frac{(-(x/2)^2)^n}{(p+1)_n\cdot n!}\right)=\frac{(x/2)^p}{p!}\cdot{}_0F_1\left[\begin{matrix}-\\p+1\end{matrix};-\left(\frac{x}{2}\right)^2\right]$ (recall that $J_p(x)$ is the so-called Bessel function of order p).

Recall that a series is said to be terminating if it has only finitely many nonzero terms.

Proposition 4.1.5. Criterion about termination of hypergeometric series
Let $p, q \in \mathbb{N}^+$ and $\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_q, t$ expressions such that, when all their parameters take concrete values, the result lies in K. Then ${}_pF_q\left[\begin{matrix}\alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_q\end{matrix}\ ;\ t\right]$ and ${}_pF_0\left[\begin{matrix}\alpha_1 & \cdots & \alpha_p \\ & - & \end{matrix}\ ;\ t\right]$ are terminating if, and only if, $\alpha_j \in \mathbb{Z}\backslash\mathbb{N}^+$, for some $j \in \{1, ..., p\}$.

Proof $\alpha_j \in \mathbb{Z}\backslash\mathbb{N}^+$, for some $j \in \{1, ..., p\}$ $\Leftrightarrow$
there exist $j \in \{1, ..., p\}$ and $m \in \mathbb{N}$ such that $\prod_{k=0}^{n-1}(\alpha_j + k) = 0$, for all $n \in \mathbb{N}$ such that $m \leqslant n$ $\Leftrightarrow$
there exist $j \in \{1, ..., p\}$ and $m \in \mathbb{N}$ such that $(\alpha_j)_n = 0$, for all $n \in \mathbb{N}$ such that $m \leqslant n$ $\Leftrightarrow$
$\dfrac{(\alpha_1)_n \cdot ... \cdot (\alpha_p)_n \cdot t^n}{(\beta_1)_n \cdot ... \cdot (\beta_q)_n \cdot n!} = 0$, for all $n \in \mathbb{N}$ such that $m \leqslant n$, for some $m \in \mathbb{N}$ $\Leftrightarrow$
${}_pF_q\left[\begin{matrix}\alpha_1 & \cdots & \alpha_p \\ \beta_1 & \cdots & \beta_q\end{matrix}\ ;\ t\right]$ is terminating.
For the other case, the reasoning is analogous. □

4.2 SOME HYPERGEOMETRIC IDENTITIES

In this section, some of the most famous hypergeometric identities will be shown.

Proposition 4.2.1. Dougall's Identity
Let $n \in \mathbb{N}$ and $\alpha, \beta, \gamma, \delta \in K$ such that $\alpha/2, \alpha-\beta+1, \alpha-\gamma+1, \alpha-\delta+1, \beta+\gamma+\delta-\alpha-n, 1+\alpha+n, \alpha-\beta-\gamma-\delta+1 \in K\backslash(\mathbb{Z}\backslash\mathbb{N}^+)$. Then

$$ {}_7F_6\left[\begin{matrix}\alpha,\ 1+\alpha/2,\ \beta,\ \gamma,\ \delta,\ n+2\cdot\alpha-\beta-\gamma-\delta+1,\ -n \\ \alpha/2,\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha-\delta+1,\ \beta+\gamma+\delta-\alpha-n,\ 1+\alpha+n\end{matrix}\ ;\ 1\right] = $$
$$ \frac{(\alpha+1)_n \cdot (\alpha-\gamma-\delta+1)_n \cdot (\alpha-\beta-\delta+1)_n \cdot (\alpha-\beta-\gamma+1)_n}{(\alpha-\beta-\gamma-\delta+1)_n \cdot (\alpha-\beta+1)_n \cdot (\alpha-\gamma+1)_n \cdot (\alpha-\delta+1)_n}. $$

Proof Let

$$ F(m,k) = \frac{(\alpha)_k \cdot (1+\alpha/2)_k \cdot (\beta)_k \cdot (\gamma)_k \cdot (\delta)_k \cdot (m+2\cdot\alpha-\beta-\gamma-\delta+1)_k \cdot (-m)_k}{(\alpha/2)_k \cdot (\alpha-\beta+1)_k \cdot (\alpha-\gamma+1)_k \cdot (\alpha-\delta+1)_k \cdot (\beta+\gamma+\delta-\alpha-m)_k \cdot (1+\alpha+m)_k \cdot k!} \cdot $$
$$ \frac{(\alpha-\beta-\gamma-\delta+1)_m \cdot (\alpha-\beta+1)_m \cdot (\alpha-\gamma+1)_m \cdot (\alpha-\delta+1)_m}{(\alpha+1)_m \cdot (\alpha-\gamma-\delta+1)_m \cdot (\alpha-\beta-\delta+1)_m \cdot (\alpha-\beta-\gamma+1)_m}. $$

$F(m,k)$ is a doubly hypergeometric term, so the WZ method is applicable to it. Applying then the WZ method to $F(m,k)$, it is obtained the output $G(m,k)$, being $G(m,k) = R(m,k) \cdot F(m,k)$, being $R(m,k) = \dfrac{(\alpha-\gamma+k)\cdot(\alpha-\beta+k)\cdot(\delta-\alpha-k)\cdot(\alpha-\beta-\gamma-\delta-k+m+1)}{(\alpha+2\cdot k)\cdot(2\cdot\alpha-\beta-\gamma-\delta+1+m)\cdot(\alpha-\gamma-\delta+1+m)\cdot(\alpha-\beta-\delta+1+m)} \cdot$
$\dfrac{(2\cdot\alpha-\beta-\gamma-\delta+2\cdot m+2)\cdot k}{(\alpha-\beta-\gamma+m-1)\cdot(1-k+m)}$.

$R(m,0) = 0$, so $G(m,0) = 0$. [0]

$F(m+1,k) - F(m,k) = G(m,k+1) - G(m,k)$, so $\sum_{j\in\mathbb{N}}(F(m+1,j) - F(m,j)) = \sum_{j\in\mathbb{N}}(G(m,j+1) - G(m,j))$; i.e. $\sum_{j\in\mathbb{N}}(F(m+1,j) - F(m,j)) = \lim\limits_{j\to\infty}(G(m,j+1) - G(m,0))$. Hence, $\sum_{j\in\mathbb{N}}(F(m+1,j)) - \sum_{j\in\mathbb{N}}(F(m,j)) = \lim\limits_{j\to\infty}(G(m,j+1)) - G(m,0) \overset{[0]}{=} \lim\limits_{j\to\infty}(G(m,j+1))$. [1]

$(-i)_j = 0$, for all $i, j \in \mathbb{N}$ such that $i < j$, so $\lim\limits_{j\to\infty}(G(i,j+1)) = 0$, for all $i \in \mathbb{N}$. Then, applying [1], $\sum_{j\in\mathbb{N}}(F(i+1,j)) - \sum_{j\in\mathbb{N}}(F(i,j)) = 0$, for all $i \in \mathbb{N}$; which yields the existence of $\varepsilon \in K$ such that $\sum_{j\in\mathbb{N}}(F(i,j)) = \varepsilon$, for all $i \in \mathbb{N}$. [2]

Applying Proposition 4.1.3, $(\varphi)_0 = 1$, for all $\varphi \in K$; and it is clear that $(-0)_j$, for all $j \in \mathbb{N}^+$. Thus, $\sum_{j\in\mathbb{N}}(F(0,j)) = 1$. Applying [2], $\sum_{j\in\mathbb{N}}(F(n,j)) = 1$; i.e.

$$
{}_7F_6\left[\begin{matrix} \alpha,\ 1+\alpha/2,\ \beta,\ \gamma,\ \delta,\ n+2\cdot\alpha-\beta-\gamma-\delta+1,\ -n \\ \alpha/2,\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha-\delta+1,\ \beta+\gamma+\delta-\alpha-n,\ 1+\alpha+n \end{matrix};1\right] =
$$

$$
\frac{(\alpha+1)_n\cdot(\alpha-\gamma-\delta+1)_n\cdot(\alpha-\beta-\delta+1)_n\cdot(\alpha-\beta-\gamma+1)_n}{(\alpha-\beta-\gamma-\delta+1)_n\cdot(\alpha-\beta+1)_n\cdot(\alpha-\gamma+1)_n\cdot(\alpha-\delta+1)_n}.
$$

□

Proposition 4.2.2. Pfaff - Saalschütz Identity
Let $n \in \mathbb{N}$ and $\alpha,\beta,\gamma \in K$ such that $\gamma, 1+\alpha+\beta-\gamma-n, \gamma-\alpha-\beta \in K\backslash(\mathbb{Z}\backslash\mathbb{N}^+)$. Then

$$
{}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ -n \\ \gamma, 1+\alpha+\beta-\gamma-n \end{matrix};1\right] = \frac{(\gamma-\beta)_n\cdot(\gamma-\alpha)_n}{(\gamma-\alpha-\beta)_n\cdot(\gamma)_n}.
$$

Proof ${}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ -n \\ \gamma,\ 1+\alpha+\beta-\gamma-n \end{matrix};1\right] =$

$$
\lim_{t\to\infty}\left({}_7F_6\left[\begin{matrix} t,\ 1+t/2,\ \alpha,\ \beta,\ 1+t-\gamma,\ n+t-\alpha-\beta+\gamma,\ -n \\ t/2,\ 1+t-\alpha,\ 1+t-\beta,\ \gamma,\ 1+\alpha+\beta-\gamma-n,\ 1+t+n \end{matrix};1\right]\right) \overset{4.2.1}{=}
$$

$$
\lim_{t\to\infty}\left(\frac{(t+1)_n\cdot(\gamma-\beta)_n\cdot(\gamma-\alpha)_n\cdot(1+t-\alpha-\beta)_n}{(\gamma-\alpha-\beta)_n\cdot(1+t-\alpha)_n\cdot(1+t-\beta)_n\cdot(\gamma)_n}\right) = \frac{(\gamma-\beta)_n\cdot(\gamma-\alpha)_n}{(\gamma-\alpha-\beta)_n\cdot(\gamma)_n}.
$$

□

Proposition 4.2.3. Dixon's Theorem
Let $\alpha,\beta,\gamma \in K$ such that $\alpha/2, \alpha+1-\gamma, \alpha+1-\beta, \gamma-\alpha/2+\beta, \alpha+1, \alpha+1-\beta-\gamma, \alpha/2+1-\beta, \alpha/2+1-\gamma, \alpha/2+1-\beta-\gamma \in K\backslash(\mathbb{Z}\backslash\mathbb{N}^+)$ and $\mathrm{Rl}(1+\alpha/2-\beta-\gamma) > 0$. Then

$$
{}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ \gamma \\ \alpha+1-\beta,\ \alpha+1-\gamma \end{matrix};1\right] = \frac{(\alpha/2)!\cdot(\alpha-\beta)!\cdot(\alpha-\gamma)!\cdot(\alpha/2-\beta-\gamma)!}{\alpha!\cdot(\alpha/2-\beta)!\cdot(\alpha/2-\gamma)!\cdot(\alpha-\beta-\gamma)!}.
$$

Proof ${}_3F_2\left[\begin{matrix} \alpha,\ \gamma,\ \beta \\ \alpha+1-\gamma,\ \alpha+1-\beta \end{matrix};1\right] = {}_5F_4\left[\begin{matrix} \alpha,\ 1+\alpha/2,\ \alpha/2,\ \gamma,\ \beta \\ \alpha/2,\ 1+\alpha/2,\ \alpha+1-\gamma,\ \alpha+1-\beta \end{matrix};1\right] =$

$$
\lim_{t\to\infty}\left({}_7F_6\left[\begin{matrix} \alpha,\ 1+\alpha/2,\ \alpha/2,\ \gamma,\ \beta,\ t+3\cdot\alpha/2+1-\beta-\gamma,\ -t \\ \alpha/2,\ 1+\alpha/2,\ \alpha+1-\gamma,\ \alpha+1-\beta,\ \gamma-\alpha/2+\beta-t,\ 1+\alpha+t \end{matrix};1\right]\right) \overset{4.2.1}{=}
$$

$$
\lim_{t\to\infty}\left(\frac{(\alpha+1)_t\cdot(\alpha+1-\beta-\gamma)_t\cdot(\alpha/2+1-\beta)_t\cdot(\alpha/2+1-\gamma)_t}{(\alpha/2+1-\beta-\gamma)_t\cdot(1+\alpha/2)_t\cdot(1+\alpha-\gamma)_t\cdot(\alpha+1-\beta)_t}\right) \overset{4.1.3}{=}
$$

$$
\lim_{t\to\infty}\left(\frac{\frac{\Gamma(\alpha+1+t)}{\Gamma(\alpha+1)}\cdot\frac{\Gamma(\alpha+1-\beta-\gamma+t)}{\Gamma(\alpha+1-\beta-\gamma)}\cdot\frac{\Gamma(\alpha/2+1-\beta+t)}{\Gamma(\alpha/2+1-\beta)}\cdot\frac{\Gamma(\alpha/2+1-\gamma+t)}{\Gamma(\alpha/2+1-\gamma)}}{\frac{\Gamma(\alpha/2+1-\beta-\gamma+t)}{\Gamma(\alpha/2+1-\beta-\gamma)}\cdot\frac{\Gamma(1+\alpha/2+t)}{\Gamma(1+\alpha/2)}\cdot\frac{\Gamma(1+\alpha-\gamma+t)}{\Gamma(1+\alpha-\gamma)}\cdot\frac{\Gamma(\alpha+1-\beta+t)}{\Gamma(\alpha+1-\beta)}}\right) =
$$

$$
\frac{\Gamma(\alpha/2+1-\beta-\gamma)\cdot\Gamma(1+\alpha/2)\cdot\Gamma(1+\alpha-\gamma)\cdot\Gamma(\alpha+1-\beta)}{\Gamma(\alpha+1)\cdot\Gamma(\alpha+1-\beta-\gamma)\cdot\Gamma(\alpha/2+1-\beta)\cdot\Gamma(\alpha/2+1-\gamma)} =
$$

$$
\frac{(\alpha/2)!\cdot(\alpha-\beta)!\cdot(\alpha-\gamma)!\cdot(\alpha/2-\beta-\gamma)!}{\alpha!\cdot(\alpha/2-\beta)!\cdot(\alpha/2-\gamma)!\cdot(\alpha-\beta-\gamma)!}.
$$

□

Proposition 4.2.4. Kummer's Theorem
Let $\alpha,\beta \in K$ such that $\alpha/2, \alpha+1-\beta, -\alpha/2+\beta, \alpha+1, \alpha/2+1-\beta \in K\backslash(\mathbb{Z}\backslash\mathbb{N}^+)$ and $\mathrm{Rl}(\beta) < 1$. Then

$$
{}_2F_1\left[\begin{matrix} \alpha,\ \beta \\ \alpha+1-\beta \end{matrix};-1\right] = \frac{(\alpha/2)!\cdot(\alpha-\beta)!}{\alpha!\cdot(\alpha/2-\beta)!}.
$$

Proof ${}_2F_1\left[\begin{matrix} \alpha,\ \beta \\ \alpha+1-\beta \end{matrix};-1\right] = \lim_{t\to\infty}\left({}_3F_2\left[\begin{matrix} \alpha,\ -t,\ \beta \\ \alpha+1+t,\ \alpha+1-\beta \end{matrix};1\right]\right) \overset{4.2.3}{=}$

$$\lim_{t\to\infty}\left(\frac{(\alpha/2)!\cdot(\alpha-\beta)!\cdot(\alpha+t)!\cdot(\alpha/2-\beta+t)!}{\alpha!\cdot(\alpha/2-\beta)!\cdot(\alpha/2+t)!\cdot(\alpha-\beta+t)!}\right)=\frac{(\alpha/2)!\cdot(\alpha-\beta)!}{\alpha!\cdot(\alpha/2-\beta)!}.$$ □

Proposition 4.2.5. Gauß's Identity
Let $\alpha,\beta,\gamma\in K$ such that $\gamma, 1+\alpha+\beta-\gamma, \gamma-\alpha-\beta, \gamma-\beta, \gamma-\alpha \in K\backslash(\mathbb{Z}\backslash\mathbb{N}^+)$ and $[\beta\in\mathbb{Z}\backslash\mathbb{N}^+$ or $\mathrm{Rl}(\gamma-\alpha-\beta)>0]$. Then

$${}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};1\right]=\frac{\Gamma(\gamma-\alpha-\beta)\cdot\Gamma(\gamma)}{\Gamma(\gamma-\beta)\cdot\Gamma(\gamma-\alpha)}.$$

Proof $${}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};1\right]=\lim_{t\to\infty}\left({}_3F_2\left[\begin{matrix}\alpha,\ \beta,\ -t\\ \gamma, 1+\alpha+\beta-\gamma-t\end{matrix};1\right]\right)\overset{4.2.2}{=}\lim_{t\to\infty}\left(\frac{(\gamma-\beta)_t\cdot(\gamma-\alpha)_t}{(\gamma-\alpha-\beta)_t\cdot(\gamma)_t}\right)=\frac{\Gamma(\gamma-\alpha-\beta)\cdot\Gamma(\gamma)}{\Gamma(\gamma-\beta)\cdot\Gamma(\gamma-\alpha)}.$$ □

Proposition 4.2.6. Chu - Vandermonde Identity
Let $n\in\mathbb{N}$ and $\beta,\gamma\in K$ such that $\gamma, 1-n+\beta-\gamma, \gamma+n-\beta, \gamma-\beta\in K\backslash(\mathbb{Z}\backslash\mathbb{N}^+)$. Then

$${}_2F_1\left[\begin{matrix}-n,\ \beta\\ \gamma\end{matrix};1\right]=\frac{(\gamma-\beta)_n}{(\gamma)_n}.$$

Proof $${}_2F_1\left[\begin{matrix}-n,\ \beta\\ \gamma\end{matrix};1\right]\overset{4.2.5}{=}\frac{\Gamma(\gamma+n-\beta)\cdot\Gamma(\gamma)}{\Gamma(\gamma-\beta)\cdot\Gamma(\gamma+n)}=\frac{(\gamma-\beta)_n}{(\gamma)_n}.$$ □

The previous proofs have been chosen in order to show the connections between the results, but all of them could have been proven by using directly the WZ method (cf. Section 7.2 of [Petkovšek et al.]).

Many other hypergeometric identities can be found in Section 3.5, Section 7.3 and Section 7.4 of [Petkovšek et al.], and in [Gessel].

4.3 AN EXAMPLE

The following example gives an idea of how to prove combinatorial identities by identifying, if possible, the underlying hypergeometric series.

The goal is to prove $\sum_{k\in\mathbb{N}}\left((-1)^k\cdot\binom{2\cdot n}{k}\cdot\binom{2\cdot k}{k}\cdot\binom{4\cdot n-2\cdot k}{2\cdot n-k}\right)=\binom{2\cdot n}{n}^2$, for all $n\in\mathbb{N}$.

$$\sum_{k\in\mathbb{N}}\left((-1)^k\cdot\binom{2\cdot n}{k}\cdot\binom{2\cdot k}{k}\cdot\binom{4\cdot n-2\cdot k}{2\cdot n-k}\right)=$$
$$(-1)^0\cdot\binom{2\cdot n}{0}\cdot\binom{2\cdot 0}{0}\cdot\binom{4\cdot n-2\cdot 0}{2\cdot n-0}\cdot\sum_{k\in\mathbb{N}}\left(\frac{(-2\cdot n)_k\cdot(-2\cdot n)_k\cdot(1/2)_k}{(1)_k\cdot(1/2-2\cdot n)_k\cdot k!}\right)=$$
$$\binom{4\cdot n}{2\cdot n}\cdot{}_3F_2\left[\begin{matrix}-2\cdot n,\ -2\cdot n,\ 1/2\\ 1,\ 1/2-2\cdot n\end{matrix};1\right]=\binom{4\cdot n}{2\cdot n}\cdot\lim_{\alpha\to-2\cdot n}\left(\lim_{\gamma\to1/2}\left({}_3F_2\left[\begin{matrix}\alpha,\ \alpha,\ \gamma\\ 1,\ \alpha+1-\gamma\end{matrix};1\right]\right)\right)\overset{4.2.3}{=}$$
$$\binom{4\cdot n}{2\cdot n}\cdot\lim_{\alpha\to-2\cdot n}\left(\lim_{\gamma\to1/2}\left(\frac{(\alpha/2)!\cdot(\alpha-\gamma)!\cdot(-\alpha/2-\gamma)!}{\alpha!\cdot(-\alpha/2)!\cdot(\alpha/2-\gamma)!\cdot(-\gamma)!}\right)\right)=$$
$$\binom{4\cdot n}{2\cdot n}\cdot\lim_{\alpha\to-2\cdot n}\left(\lim_{\gamma\to1/2}\left(\frac{\Gamma(1+\alpha/2)\cdot\Gamma(1+\alpha-\gamma)\cdot\Gamma(1-\alpha/2-\gamma)}{\Gamma(1+\alpha)\cdot\Gamma(1-\alpha/2)\cdot\Gamma(1+\alpha/2-\gamma)\cdot\Gamma(1-\gamma)}\right)\right)=$$
$$\binom{4\cdot n}{2\cdot n}\cdot\lim_{\alpha\to-2\cdot n}\left(\lim_{\gamma\to1/2}\left(\frac{\frac{\pi\cdot\pi\cdot\Gamma(1-\alpha/2-\gamma)}{\Gamma(-\alpha/2)\cdot\sin(\pi\cdot(1+\alpha/2))\cdot\Gamma(\gamma-\alpha)\cdot\sin(\pi\cdot(1+\alpha-\gamma))}}{\frac{\pi\cdot\Gamma(1-\alpha/2)\cdot\pi\cdot\Gamma(1-\gamma)}{\Gamma(-\alpha)\cdot\sin(\pi\cdot(1+\alpha))\cdot\Gamma(\gamma-\alpha/2)\cdot\sin(\pi\cdot(1-\gamma+\alpha/2))}}\right)\right)=$$

$$\binom{4\cdot n}{2\cdot n}\cdot\lim_{\alpha\to-2\cdot n}\left(\lim_{\gamma\to1/2}\left(\frac{\sin(\pi\cdot(1+\alpha))\cdot\sin(\pi\cdot(1-\gamma+\alpha/2)}{\sin(\pi\cdot(1+\alpha/2))\cdot\sin(\pi\cdot(1+\alpha-\gamma))}\cdot\right.\right.$$
$$\left.\left.\frac{\Gamma(-\alpha)\cdot\Gamma(\gamma-\alpha/2)\cdot\Gamma(1-\alpha/2-\gamma)}{\Gamma(-\alpha/2)\cdot\Gamma(1-\alpha/2)\cdot\Gamma(\gamma-\alpha)\cdot\Gamma(1-\gamma)}\right)\right)=$$
$$\binom{4\cdot n}{2\cdot n}\cdot\lim_{\alpha\to-2\cdot n}\left(\frac{\sin(\pi\cdot(1+\alpha))\cdot\sin(\pi\cdot((\alpha+1)/2)}{\sin(\pi\cdot(1+\alpha/2))\cdot\sin(\pi\cdot(\alpha+1/2))}\cdot\frac{\Gamma(-\alpha)\cdot\Gamma((1-\alpha)/2)^2}{\Gamma(-\alpha/2)\cdot\Gamma(1-\alpha/2)\cdot\Gamma(1/2-\alpha)\cdot\Gamma(1/2)}\right)=$$
$$\binom{4\cdot n}{2\cdot n}\cdot2\cdot\frac{\Gamma(2\cdot n)\cdot\Gamma(n+1/2)^2}{\Gamma(n)\cdot\Gamma(n+1)\cdot\Gamma(2\cdot n+1/2)\cdot\Gamma(1/2)}=\binom{4\cdot n}{2\cdot n}\cdot2\cdot\frac{(2\cdot n-1)!\cdot(n-1/2)!^2}{(n-1)!\cdot n!\cdot(2\cdot n-1/2)!\cdot(-1/2)!}=$$
$$\binom{4\cdot n}{2\cdot n}\cdot\binom{2\cdot n}{n}\cdot\frac{(n-1/2)!^2}{(2\cdot n-1/2)!\cdot(-1/2)!}=\binom{4\cdot n}{2\cdot n}\cdot\binom{2\cdot n}{n}\cdot\frac{((1/2)_n\cdot(-1/2)!)^2}{(1/2)_{2\cdot n}\cdot(-1/2)!^2}=$$
$$\binom{4\cdot n}{2\cdot n}\cdot\binom{2\cdot n}{n}\cdot\frac{((n-1/2)\cdot(n-3/2)\cdot\ldots\cdot1/2)^2}{(2\cdot n-1/2)\cdot(2\cdot n-3/2)\cdot\ldots\cdot1/2}=$$
$$\binom{4\cdot n}{2\cdot n}\cdot\binom{2\cdot n}{n}\cdot\frac{\left(\frac{(2\cdot n-1)\cdot(2\cdot n-3)\cdot\ldots\cdot1}{2^n}\right)^2}{\frac{(2\cdot(2\cdot n)-1)\cdot(2\cdot(2\cdot n)-3)\cdot\ldots\cdot1}{2^{2\cdot n}}}=\binom{4\cdot n}{2\cdot n}\cdot\binom{2\cdot n}{n}\cdot\frac{\left(\frac{(2\cdot n)!}{4^n\cdot n!}\right)^2}{\frac{(2\cdot(2\cdot n))!}{4^{2\cdot n}\cdot(2\cdot n)!}}=$$
$$\binom{4\cdot n}{2\cdot n}\cdot\binom{2\cdot n}{n}\cdot\frac{((2\cdot n)!)^3}{(4\cdot n)!\cdot n!^2}=\binom{2\cdot n}{n}^2\text{, for all } n\in\mathbb{N}^+. \qquad [0]$$

Note that Euler's reflection theorem (i.e. $\Gamma(a)\cdot\Gamma(1-a)\cdot\sin(\pi\cdot a)=\pi$, for all $a\in\mathbb{C}\backslash(\mathbb{Z}\backslash\mathbb{N}^+)$) has been applied here.

Now, as $\sum_{k\in\mathbb{N}}\left((-1)^k\cdot\binom{2\cdot0}{k}\cdot\binom{2\cdot k}{k}\cdot\binom{4\cdot0-2\cdot k}{2\cdot0-k}\right)=$

$1\cdot\binom{0}{0}\cdot\binom{0}{0}\cdot\binom{0}{0}+\sum_{k\in\mathbb{N}^+}\left((-1)^k\cdot\binom{0}{k}\cdot\binom{2\cdot k}{k}\cdot\binom{-2\cdot k}{-k}\right)=$

$1+\sum_{k\in\mathbb{N}^+}\left((-1)^k\cdot0\cdot\binom{2\cdot k}{k}\cdot0\right)=1=\binom{0}{0}=\binom{2\cdot0}{0}^2$, for all $n\in\mathbb{N}$,

[0] yields that $\sum_{k\in\mathbb{N}}\left((-1)^k\cdot\binom{2\cdot n}{k}\cdot\binom{2\cdot k}{k}\cdot\binom{4\cdot n-2\cdot k}{2\cdot n-k}\right)=\binom{2\cdot n}{n}^2$, for all $n\in\mathbb{N}$.

5 DIFFERENCE RINGS

So far, the work has been focused on solving difference equations directly, viz. working on the structure $K^{\mathbb{N}}$, being K a field of characteristic zero. This chapter, however, will present techniques of resolution of equations over certain structures, called ring $R\Pi\Sigma^*$-extensions, which have the interesting property that such equations can be interpreted as difference equations.

One of the interests of this theory lies in the fact that the class of linear difference equations which are represented now is bigger than the one in which the coefficients are polynomial, extending then the theory of the previous chapters. For example, at the end of the chapter it will be sketched how to solve the linear difference $\mathbb{Q}(\iota)$-equation given by $y(n+1) + \frac{\iota^n}{n+1} \cdot y(n) = 0$ (recall that ι denotes the imaginary unit).

The central notion in this chapter, as the title claims, is that of difference ring.

Definition 5.0.1. Let P be an ordered pair. Then P is said to be a **difference ring** (resp. **difference field**) if $P = (A, \sigma)$, for some commutative unital ring (resp. field) A of characteristic zero and some $\sigma \in \mathrm{Aut}_{\mathbf{CRing}}(A)$ (resp. $\mathrm{Aut}_{\mathbf{Field}}(A)$).

For example, $(\mathbb{Q}(\iota), \mathrm{id}_{\mathbb{Q}(\iota)})$ is a difference field.

So, during this chapter, let (A, σ) be a difference ring and (F, τ) a difference field.

Proposition 5.0.2. Relation between difference rings and difference fields
The following conditions hold:

1. if A is a field, then (A, σ) is a difference field,
2. (F, τ) is a difference ring.

Proof
1. **Field** is a full subcategory of **CRing**, so $\mathrm{Aut}_{\mathbf{CRing}}(A) = \mathrm{Aut}_{\mathbf{Field}}(A)$. Then $\sigma \in \mathrm{Aut}_{\mathbf{Field}}(A)$; and consequently (A, σ) is a difference field.
2. Immediate. □

5.1 PERIOD, ORDER, CONSTANTS AND SEMICONSTANTS

The first step prior to sketch how to solve equations over ring $R\Pi\Sigma^*$-extensions is to formalize the concepts of period, order and semiconstants.

Definition 5.1.1. Let $o \in \mathbb{N}^A$. Then o is said to be the **order function** of A if $o(a) = \begin{cases} 0 & \text{if } a^n \neq 1, \text{ for all } n \in \mathbb{N}^+ \\ \min(\{n \in \mathbb{N}^+ \mid a^n = 1\}) & \text{otherwise} \end{cases}$, for all $a \in A$, fact which is denoted by $o = \mathrm{ord}$.

Definition 5.1.2. Let $p \in \mathbb{N}^{(A^*)}$. Then p is said to be the **period function** of A if $p(a) = \begin{cases} 0 & \text{if } \sigma^n(a) \neq a, \text{ for all } n \in \mathbb{N}^+ \\ \min(\{n \in \mathbb{N}^+ \mid \sigma^n(a) = a\}) & \text{otherwise} \end{cases}$, for all $a \in A^*$, fact which is denoted by $p = \mathrm{per}$.

Definition 5.1.3. Let $a \in A$. Then a is said to be a **constant** if $\sigma(a) = a$.

From now on, $\{a \in A \mid a \text{ is a constant}\}$ will be denoted by $\mathrm{const}(A, \sigma)$.

For example, in the difference ring $(\mathbb{Q}(\iota), \mathrm{id}_{\mathbb{Q}(\iota)})$, the following conditions hold:
- $\mathrm{ord}(\iota) = \min(\{n \in \mathbb{N}^+ \mid \iota^n = 1\}) = 4$,
- $\mathrm{per}(\iota) = \min(\{n \in \mathbb{N}^+ \mid \mathrm{id}^n_{\mathbb{Q}(\iota)}(\iota) = \iota\}) = 1$,
- $\mathrm{const}(\mathbb{Q}(\iota), \mathrm{id}_{\mathbb{Q}(\iota)}) = \mathbb{Q}(\iota)$.

Proposition 5.1.4. Constants form a substructure containing the rational numbers
The following conditions hold:
1. $\mathbb{Q} \lesssim_{\mathbf{CRing}} \mathrm{const}(A, \sigma) \leqslant_{\mathbf{CRing}} A$,
2. $\mathbb{Q} \lesssim_{\mathbf{Field}} \mathrm{const}(F, \tau) \leqslant_{\mathbf{Field}} F$.

Proof It is well-known from algebra that any commutative unital ring of characteristic zero is a **CRing**-extension of (some copy of) $\mathbb{Q}$; and that any automorphism over such a ring fixes every element of (such a copy of) $\mathbb{Q}$. [0]
From [0] follows that $\mathbb{Q} \lesssim_{\mathbf{CRing}} A$ and $\sigma(c) = c$, for all $c \in \mathbb{Q}$. [1]
Let $a, b \in \mathrm{const}(A, \sigma)$. It is clear that $\sigma(a) = a$ and $\sigma(b) = b$, so $\sigma(a + b) = \sigma(a) + \sigma(b) = a + b$ and $\sigma(a \cdot b) = \sigma(a) \cdot \sigma(b) = a \cdot b$. Thus, $\sigma(a - b) = \sigma(a + (-1) \cdot b) = a + (-1) \cdot b = a - b$. Hence, $a \cdot b, a - b \in \mathrm{const}(A, \sigma)$. As [1] yields that $\sigma(1) = 1$, $1 \in \mathrm{const}(A, \sigma)$ too; so necessarily $\mathrm{const}(A, \sigma) \leqslant_{\mathbf{CRing}} A$. Applying [0], $\mathbb{Q} \lesssim_{\mathbf{CRing}} \mathrm{const}(A, \sigma) \leqslant_{\mathbf{CRing}} A$.
The fact that $\mathbb{Q} \lesssim_{\mathbf{Field}} \mathrm{const}(F, \tau) \leqslant_{\mathbf{Field}} F$ can be proven similarly. □

Definition 5.1.5. Let $a \in A$ and $G \leqslant_{\mathbf{Grp}} A^*$ (recall that A^*, i.e. the set of units of A, forms a multiplicative group). Then a is said to be a **semiconstant** over G if $\sigma(a) = u \cdot a$, for some $u \in G$.

From now on, given $G \leqslant_{\mathbf{Grp}} A^*$, $\{a \in A \mid a \text{ is a semiconstant over } G\}$ will be denoted by $\mathrm{sconst}_G(A, \sigma)$.

Definition 5.1.6. (A, σ) is said to be **constant-stable** if $\mathrm{const}(A, \sigma)$ is a field and $\mathrm{const}(A, \sigma^k) = \mathrm{const}(A, \sigma)$, for all $k \in \mathbb{N}^+$.

Definition 5.1.7. (A, σ) is said to be **strong constant-stable** if (A, σ) is constant-stable and $\{a \in A \mid a \text{ is a root of unity}\} \subseteq \mathrm{const}(A, \sigma)$.

For example, $(\mathbb{Q}(\iota), \mathrm{id}_{\mathbb{Q}(\iota)})$ is a strong constant-stable difference ring; indeed, $\{a \in \mathbb{Q}(\iota) \mid a \text{ is a root of unity}\} = \{\pm 1, \pm \iota\} \subseteq \mathbb{Q}(\iota) = \mathrm{const}(\mathbb{Q}(\iota), \mathrm{id}_{\mathbb{Q}(\iota)}) = \mathrm{const}(\mathbb{Q}(\iota), \mathrm{id}^k_{\mathbb{Q}(\iota)})$, for all $k \in \mathbb{N}^+$.

5.2 EXTENSIONS

Having defined the previous concepts, it is now possible to formalize what a ring $R\Pi\Sigma^*$-extension is, showing also characterizations of concepts that will be very useful in the sequel.

Definition 5.2.1. Let (C, ρ) be a difference ring (resp. difference field). Then (C, ρ) is said to be a ring (resp. field) **extension** of (A, σ) (resp. (F, τ)) if $A \leqslant_{\mathbf{CRing}} C$ and $\rho|_A = \sigma$ (resp. $F \leqslant_{\mathbf{Field}} C$ and $\rho|_F = \tau$), fact which is denoted by $(A, \sigma) \leqslant_R (C, \rho)$ (resp. $(F, \tau) \leqslant_F (C, \rho)$).

For example, given an indeterminate k over $\mathbb{Q}(\iota)$ and $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k) = k + 1$, $(\mathbb{Q}(\iota)(k), \tau_0)$ is a field extension of $(\mathbb{Q}(\iota), \mathrm{id}_{\mathbb{Q}(\iota)})$.

Lemma 5.2.2. Let $a \in A^*$, $b \in A$ and t an indeterminate over A. Then there exists a unique difference ring $(\tilde{A}, \tilde{\sigma})$ such that $(A, \sigma) \leqslant_R (\tilde{A}, \tilde{\sigma})$, $\tilde{A} = A[t]$ and $\tilde{\sigma}(t) = a \cdot t + b$ (cf. Lemma 2.1.(1) of [Schneider I]).

Definition 5.2.3. Let $(\tilde{A}, \tilde{\sigma})$ be a difference ring such that $(A, \sigma) \leqslant_R (\tilde{A}, \tilde{\sigma})$, $a \in A^*$, $b \in A$ and t an indeterminate over A. Then $(\tilde{A}, \tilde{\sigma})$ is said to be the **unimonomial ring extension of polynomial function type (u.r.e.p)** of (A, σ) with respect to a, b and t if $\tilde{A} = A[t]$ and $\tilde{\sigma}(t) = a \cdot t + b$.

For example, given an indeterminate k over $\mathbb{Q}(\iota)$, $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k) = k + 1$, an indeterminate x over $\mathbb{Q}(\iota)(k)$ and $\tilde{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}(\mathbb{Q}(\iota)(k)[x])$ such that $\tilde{\tau}|_{\mathbb{Q}(\iota)(k)} = \tau_0$ and $\tilde{\tau}(x) = \iota \cdot x$, $(\mathbb{Q}(\iota)(k)[x], \tilde{\tau})$ is the u.r.e.p of $(\mathbb{Q}(\iota)(k), \tau_0)$ with respect to ι, 0 and x.

Lemma 5.2.4. Let $a \in A^*$ and t an indeterminate over A. Then there exists a unique difference ring $(\tilde{A}, \tilde{\sigma})$ such that $(A, \sigma) \leqslant_R (\tilde{A}, \tilde{\sigma})$, $\tilde{A} = A\left[t, \frac{1}{t}\right]$ and $\tilde{\sigma}(t) = a \cdot t$ (cf. Lemma 2.1.(2) of [Schneider I]).

Definition 5.2.5. Let $(\tilde{A}, \tilde{\sigma})$ be a difference ring such that $(A, \sigma) \leqslant_R (\tilde{A}, \tilde{\sigma})$, $a \in A^*$ and t an indeterminate over A. Then $(\tilde{A}, \tilde{\sigma})$ is said to be the **unimonomial ring extension of Laurent polynomial function type (u.r.e.l)** of (A, σ) with respect to a and t if $\tilde{A} = A\left[t, \frac{1}{t}\right]$ and $\tilde{\sigma}(t) = a \cdot t$.

For example, given an indeterminate k over $\mathbb{Q}(\iota)$, $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k) = k + 1$, an indeterminate x over $\mathbb{Q}(\iota)(k)$, $\tilde{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}(\mathbb{Q}(\iota)(k)[x])$ such that $\tilde{\tau}|_{\mathbb{Q}(\iota)(k)} = \tau_0$ and $\tilde{\tau}(x) = \iota \cdot x$, an indeterminate t over $\mathbb{Q}(\iota)(k)[x]$ and $\hat{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right]\right)$ such that $\hat{\tau}|_{\mathbb{Q}(\iota)(k)[x]} = \tilde{\tau}$ and $\hat{\tau}(t) = x \cdot k \cdot t$, $\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right], \hat{\tau}\right)$ is the u.r.e.l of $(\mathbb{Q}(\iota)(k)[x], \tilde{\tau})$ with respect to $x \cdot k$ and t.

Proposition 5.2.6. The u.r.e.l extends the u.r.e.p
Let $a \in A^*$, t an indeterminate over A, $(\tilde{A}, \tilde{\sigma})$ the u.r.e.p of (A, σ) with respect to a, 0 and t and $(\hat{A}, \hat{\sigma})$ the u.r.e.l of (A, σ) with respect to a and t. Then $(\tilde{A}, \tilde{\sigma}) \leqslant_R (\hat{A}, \hat{\sigma})$ (cf. Lemma 2.1.(2) of [Schneider I]).

Lemma 5.2.7. Let $f \in F^*$, $g \in F$ and t an indeterminate over F. Then there exists a unique difference field $(\tilde{F}, \tilde{\tau})$ such that $(F, \tau) \leqslant_F (\tilde{F}, \tilde{\tau})$, $\tilde{F} = F(t)$ and $\tilde{\tau}(t) = f \cdot t + g$ (cf. Lemma 2.1.(3) of [Schneider I]).

Definition 5.2.8. Let $(\tilde{F}, \tilde{\tau})$ be a difference field such that $(F, \tau) \leqslant_F (\tilde{F}, \tilde{\tau})$, $f \in F^*$, $g \in F$ and t an indeterminate over F. Then $(\tilde{F}, \tilde{\tau})$ is said to be the **unimonomial ring extension of rational function type (u.f.e.r)** of (F, τ) with respect to f, g and t if $\tilde{F} = F(t)$ and $\tilde{\tau}(t) = f \cdot t + g$.

For example, given an indeterminate k over $\mathbb{Q}(\iota)$ and $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k) = k + 1$, $(\mathbb{Q}(\iota)(k), \tau_0)$ is the u.f.e.r of $(\mathbb{Q}(\iota), \mathrm{id}_{\mathbb{Q}(\iota)})$ with respect to 1, 1 and k.

Proposition 5.2.9. The u.f.e.r extends the u.r.e.p and the u.r.e.l
Let $f \in F^*$, $g \in F$, t an indeterminate over F, (F_1, τ_1) the u.r.e.p of (F, τ) with respect to f, g and t, (F_2, τ_2) the u.r.e.l of (F, τ) with respect to f and t and (F_3, τ_3) the u.f.e.r of (F, τ) with respect to f, g and t. Then $(F_1, \tau_1) \leqslant_R (F_3, \tau_3)$ and, if $g = 0$, then $(F_1, \tau_1) \leqslant_R (F_2, \tau_2) \leqslant_R (F_3, \tau_3)$ (cf. Lemma 2.1.(3) of [Schneider I]).

Definition 5.2.10. Let t be an indeterminate over A (resp. F) and (C,ρ) an u.r.e.p (resp. u.f.e.r) of (A,σ) (resp. (F,τ)) with respect to t. Then (C,ρ) is said to be a **ring** (resp. **field**) **Σ^*-extension** of (A,σ) (resp. (F,τ)) if $\rho(t)-t \in A^*$ and $\mathrm{const}(C,\rho)=\mathrm{const}(A,\sigma)$ (resp. $\rho(t)-t\in F^*$ and $\mathrm{const}(C,\rho)=\mathrm{const}(F,\tau)$).

For example, given an indeterminate k over $\mathbb{Q}(\iota)$ and $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k)=k+1$, $(\mathbb{Q}(\iota)(k),\tau_0)$ is the field Σ^*-extension of $(\mathbb{Q}(\iota),\mathrm{id}_{\mathbb{Q}(\iota)})$ with respect to 1, 1 and k, since $\tau_0(k)-k=1\in\mathbb{Q}(\iota)\backslash\{0\}$ and $\mathrm{const}(\mathbb{Q}(\iota)(k),\tau_0)=\mathrm{const}(\mathbb{Q}(\iota),\mathrm{id}_{\mathbb{Q}(\iota)})=\mathbb{Q}(\iota)$.

Lemma 5.2.11. Let $G \leqslant_{\mathbf{Grp}} A^*$, $b\in A$, t an indeterminate over A and $(\tilde{A},\tilde{\sigma})$ the u.r.e.p of (A,σ) with respect to 1, b and t. If $\mathrm{sconst}_G(A,\sigma)\backslash\{0\}\leqslant_{\mathbf{Grp}} A^*$, then the following conditions are equivalent:

1. $\tilde{\sigma}(f)=u\cdot f$, for some $f\in\tilde{A}\backslash A$ and $u\in G$,
2. $\sigma(g)=g+b$, for some $g\in A$,
3. $\mathrm{const}(A,\sigma)\subsetneqq\mathrm{const}(\tilde{A},\tilde{\sigma})$

(cf. Lemma 3.8 of [Schneider I]).

Proposition 5.2.12. Characterization of ring Σ^*-extension

Let $b\in A$ and $(\tilde{A},\tilde{\sigma})$ the u.r.e.p of (A,σ) with respect to 1 and b; and assume that $\mathrm{const}(A,\sigma)$ is a field. Then $(\tilde{A},\tilde{\sigma})$ is a ring Σ^*-extension of (A,σ) if, and only if, $\sigma(a)\neq a+b$, for all $a\in A$.

Proof Let $G=\{1\}$. Then $\mathrm{sconst}_G(A,\sigma)=\{c\in A \mid \sigma(c)=c\}=\mathrm{const}(A,\sigma)$, so, by hypothesis, $\mathrm{sconst}_G(A,\sigma)$ is a field. Hence $\mathrm{sconst}_G(A,\sigma)\backslash\{0\}\leqslant_{\mathbf{Grp}} A^*$. Thus, applying Lemma 5.2.11, there exists $g\in A$ such that $\sigma(g)=g+b$ if, and only if, $\mathrm{const}(A,\sigma)\subsetneqq\mathrm{const}(\tilde{A},\tilde{\sigma})$. And, as $(\tilde{A},\tilde{\sigma})$ is the u.r.e.p of (A,σ), there exists $g\in A$ such that $\sigma(g)=g+b$ if, and only if, $(\tilde{A},\tilde{\sigma})$ is not a ring Σ^*-extension of (A,σ). I.e. $\sigma(g)\neq g+b$, for all $g\in A$, if, and only if, $(\tilde{A},\tilde{\sigma})$ is a ring Σ^*-extension of (A,σ). □

Definition 5.2.13. Let t be an indeterminate over A (resp. F) and (C,ρ) an u.r.e.l (resp. u.f.e.r) of (A,σ) (resp. (F,τ)) with respect to t. Then (C,ρ) is said to be a **ring** (resp. **field**) **Π-extension** of (A,σ) (resp. (F,τ)) if $\frac{\rho(t)}{t}\in A^*$ and $\mathrm{const}(C,\rho)=\mathrm{const}(A,\sigma)$ (resp. $\frac{\rho(t)}{t}\in F^*$ and $\mathrm{const}(C,\rho)=\mathrm{const}(F,\tau)$).

An example of this notion will be shown after characterizing it.

Lemma 5.2.14. Let t be an indeterminate over A, $(\tilde{A},\tilde{\sigma})$ an u.r.e.l of (A,σ) with respect to t, $u\in A$, $n\in\mathbb{N}$, $g_{-n},\dots,g_n\in A$ and $g=\sum_{i=-n}^{n}(g_i\cdot t^i)$. If $\tilde{\sigma}(g)=u\cdot g$, then $\sigma(g_i)=\frac{u\cdot g_i\cdot t^i}{\tilde{\sigma}(t^i)}$, for all $i\in\{-n,\dots,n\}$ (cf. Lemma 3.15 of [Schneider I]).

Proposition 5.2.15. Characterization of ring Π-extension

Let $a\in A^*$ and $(\tilde{A},\tilde{\sigma})$ the u.r.e.l of (A,σ) with respect to a. Then $(\tilde{A},\tilde{\sigma})$ is a ring Π-extension of (A,σ) if, and only if, $\sigma(g)\neq a^m\cdot g$, for all $g\in A\backslash\{0\}$ and $m\in\mathbb{Z}\backslash\{0\}$.

Proof Let t be the indeterminate over A such that $(\tilde{A},\tilde{\sigma})$ is the u.r.e.l of (A,σ) with respect to a and t.

$\Rightarrow$) (Contrapositive argument) Let $m\in\mathbb{Z}\backslash\{0\}$ and $g\in A\backslash\{0\}$ such that $\sigma(g)=a^m\cdot g$. Then $\sigma(g)=a^m\cdot g$ and $\tilde{\sigma}(t^m)=(\tilde{\sigma}(t))^m=(a\cdot t)^m=a^m\cdot t^m$, so $\sigma\left(\frac{g}{t^m}\right)=\frac{g}{t^m}$ and then $\frac{g}{t^m}\in\mathrm{const}(\tilde{A},\tilde{\sigma})$. Obviously $\frac{g}{t^m}\notin A$. In particular, $\frac{g}{t^m}\notin\mathrm{const}(A,\sigma)$, so $\frac{g}{t^m}\in\mathrm{const}(\tilde{A},\tilde{\sigma})$ yields $\mathrm{const}(\tilde{A},\tilde{\sigma})\neq\mathrm{const}(A,\sigma)$ and thus $(\tilde{A},\tilde{\sigma})$ is not a ring Π-extension of (A,σ).

$\Leftarrow$) (Reductio ad absurdum) Let $n\in\mathbb{N}$, $g_{-n},\dots,g_n\in A$ such that $\tilde{\sigma}(\sum_{i=-n}^{n}(g_i\cdot t^i))=\sum_{i=-n}^{n}(g_i\cdot t^i)\in\tilde{A}\backslash A$ and $g=\sum_{i=-n}^{n}(g_i\cdot t^i)$. $\sigma(g)=g\in\tilde{A}\backslash A$ yields $g_m\neq 0$, for some $m\in\{-n,\dots,n\}$, so, by Lemma 5.2.14, $\sigma(g_m)=\frac{g_m\cdot t^m}{\tilde{\sigma}(t^m)}$, i.e. $\sigma(g_m)=a^{-m}\cdot g_m$. Contradiction with the hypotheses. □

For example, given an indeterminate k over $\mathbb{Q}(\iota)$, $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k) = k+1$, an indeterminate x over $\mathbb{Q}(\iota)(k)$, $\tilde{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}(\mathbb{Q}(\iota)(k)[x])$ such that $\tilde{\tau}|_{\mathbb{Q}(\iota)(k)} = \tau_0$ and $\tilde{\tau}(x) = \iota \cdot x$, an indeterminate t over $\mathbb{Q}(\iota)(k)[x]$ and $\hat{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right]\right)$ such that $\hat{\tau}|_{\mathbb{Q}(\iota)(k)[x]} = \tilde{\tau}$ and $\hat{\tau}(t) = x \cdot k \cdot t$, applying Proposition 5.2.15 to the fact that $\mathrm{ord}(x \cdot k) = 0$ and $\tilde{\tau}(g) \neq (x \cdot k)^m \cdot g$, for all $g \in \mathbb{Q}(\iota)(k)[x]\backslash\{0\}$ and $m \in \mathbb{Z}\backslash\{0\}$, $\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right], \hat{\tau}\right)$ is the ring Π-extension of $(\mathbb{Q}(\iota)(k)[x], \tilde{\tau})$ with respect to $x \cdot k$ and t.

Lemma 5.2.16. Let $\alpha \in A^*$ such that $\alpha^\lambda = 1$, for some $\lambda \in \mathbb{N}$ such that $1 < \lambda$, and t an indeterminate over A such that $t^\lambda = 1$. Then there exists the u.r.e.p of (A, σ) with respect to α, 0 and t (cf. Lemma 2.6 of [Schneider I] and the paragraph preceding it).

Definition 5.2.17. Let t be an indeterminate over A, $\alpha \in A^*$ such that $\alpha^\lambda = 1$, for some $\lambda \in \mathbb{N}$ such that $1 < \lambda$, and $(\tilde{A}, \tilde{\sigma})$ the u.r.e.p of (A, σ) with respect to α, 0 and t. Then $(\tilde{A}, \tilde{\sigma})$ is said to be an **algebraic ring extension** of order λ of (A, σ) with respect to α and t if $t^\lambda = 1$.

Definition 5.2.18. Let $\lambda \in \mathbb{N}$ such that $1 < \lambda$ and $(\tilde{A}, \tilde{\sigma})$ an algebraic ring extension of order λ of (A, σ). Then $(\tilde{A}, \tilde{\sigma})$ is said to be a **root-of-unity ring extension** of order λ of (A, σ), or a **ring R-extension** of order λ of (A, σ), if $\mathrm{const}(\tilde{A}, \tilde{\sigma}) = \mathrm{const}(A, \sigma)$.

An example of these notions will be shown after characterizing one of them.

Proposition 5.2.19. Characterization of ring R-extension
Let $\alpha \in A^*$, $\lambda \in \mathbb{N}$ such that $1 < \lambda$ and $(\tilde{A}, \tilde{\sigma})$ an algebraic ring extension of order λ of (A, σ) with respect to α. Then $(\tilde{A}, \tilde{\sigma})$ is a ring R-extension of (A, σ) if, and only if, $\sigma(g) \neq \alpha^m \cdot g$, for all $g \in A\backslash\{0\}$ and $m \in \{1, ..., \lambda - 1\}$.

Proof Let t be the indeterminate over A such that $(\tilde{A}, \tilde{\sigma})$ is an algebraic ring extension of order λ of (A, σ) with respect to α and t.

$\Rightarrow$) (Contrapositive argument) Given $m \in \{1, ..., \lambda - 1\}$ and $g \in A\backslash\{0\}$ such that $\sigma(g) = \alpha^m \cdot g$, $\tilde{\sigma}(t^m) = (\tilde{\sigma}(t))^m = (\alpha \cdot t)^m = \alpha^m \cdot t^m$, so $\tilde{\sigma}(g \cdot t^{\lambda - m}) = g \cdot t^{\lambda - m}$ and then $g \cdot t^{\lambda - m} \in \mathrm{const}(\tilde{A}, \tilde{\sigma})$. Obviously $g \cdot t^{\lambda - m} \notin A$. In particular, $g \cdot t^{\lambda - m} \notin \mathrm{const}(A, \sigma)$, so $g \cdot t^{\lambda - m} \in \mathrm{const}(\tilde{A}, \tilde{\sigma})$ yields that $\mathrm{const}(\tilde{A}, \tilde{\sigma}) \neq \mathrm{const}(A, \sigma)$ and thus $(\tilde{A}, \tilde{\sigma})$ is not a ring R-extension of (A, σ).

$\Leftarrow$) (Reductio ad absurdum) Let $g_0, ..., g_{\lambda-1} \in A$ such that $\tilde{\sigma}(\sum_{i=0}^{\lambda-1}(g_i \cdot t^i)) = \sum_{i=0}^{\lambda-1}(g_i \cdot t^i) \in \tilde{A}\backslash A$ and $g = \sum_{i=0}^{\lambda-1}(g_i \cdot t^i) \in \tilde{A}\backslash A$. $\sigma(g) = g$ yields $g_r \neq 0$, for some $r \in \{1, ..., \lambda - 1\}$, so, by coefficient comparison, $\tilde{\sigma}(g_r) = \alpha^{\lambda - r} \cdot g_r$. But $(\tilde{A}, \tilde{\sigma})$ is a ring R-extension of (A, σ), $1 \in A\backslash\{0\}$ and $m := \mathrm{ord}(\alpha) < \lambda$, so $\sigma(1) = 1 = \alpha^m \cdot 1$. Contradiction with the hypotheses. □

For example, given an indeterminate k over $\mathbb{Q}(\iota)$, $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k) = k+1$, an indeterminate x over $\mathbb{Q}(\iota)(k)$ such that $x^4 = 1$ and $\tilde{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}(\mathbb{Q}(\iota)(k)[x])$ such that $\tilde{\tau}|_{\mathbb{Q}(\iota)(k)} = \tau_0$ and $\tilde{\tau}(x) = \iota \cdot x$, applying Proposition 5.2.19 to the fact that $\mathrm{ord}(\iota) = 4$ and $\tau(g) \neq \iota^m \cdot g$, for all $g \in \mathbb{Q}(\iota)(k)\backslash\{0\}$ and $m \in \{1, 2, 3\}$, $(\mathbb{Q}(\iota)(k)[x], \tilde{\tau})$ is the ring R-extension of order 4 of $(\mathbb{Q}(\iota)(k), \tau)$ with respect to ι and x.

Proposition 5.2.20. Characterization of the semiconstants of the ring R-extensions
Let $G \leqslant_{\mathbf{Grp}} A^*$, $\lambda \in \mathbb{N}^+$, t an indeterminate over A and $(\tilde{A}, \tilde{\sigma})$ a ring R-extension of order λ of (A, σ) with respect to t. If $\mathrm{sconst}_G(A, \sigma)\backslash\{0\} \leqslant_{\mathbf{Grp}} A^*$ and $\frac{\tilde{\sigma}(t)}{t} \in G$, then $\mathrm{sconst}_G(\tilde{A}, \tilde{\sigma})\backslash\{0\} \leqslant_{\mathbf{Grp}} \tilde{A}^*$ and $\mathrm{sconst}_G(\tilde{A}, \tilde{\sigma}) = \{h \cdot t^m \mid [h \in \mathrm{sconst}_G(A, \sigma) \ \wedge \ m \in \{0, ..., \lambda - 1\}]\}$ (cf. Proposition 3.23 of [Schneider I]).

From now on, given a commutative unital ring R of characteristic zero and an indeterminate t over R, $R\langle t \rangle$ will denote any element of $\left\{R[t], R\left[t, \frac{1}{t}\right]\right\}$.

Definition 5.2.21. Let t be an indeterminate over A and $\tilde{\sigma} \in \mathrm{Aut}_{\mathbf{CRing}}(A\langle t\rangle)$. Then the following conditions hold:

- $(A\langle t\rangle, \tilde{\sigma})$ is said to be a **ring RΠ-extension** of (A, σ) with respect to t if $(A\langle t\rangle, \tilde{\sigma})$ is a ring R-extension or a ring Π-extension of (A, σ) with respect to t,
- $(A\langle t\rangle, \tilde{\sigma})$ is said to be a **ring RΣ*-extension** of (A, σ) with respect to t if $(A\langle t\rangle, \tilde{\sigma})$ is a ring R-extension or a ring Σ*-extension of (A, σ) with respect to t,
- $(A\langle t\rangle, \tilde{\sigma})$ is said to be a **ring ΠΣ*-extension** of (A, σ) with respect to t if $(A\langle t\rangle, \tilde{\sigma})$ is a ring Π-extension or a ring Σ*-extension of (A, σ) with respect to t,
- $(A\langle t\rangle, \tilde{\sigma})$ is said to be a **ring RΠΣ*-extension** of (A, σ) with respect to t if $(A\langle t\rangle, \tilde{\sigma})$ is a ring R-extension, a ring Π-extension or a ring Σ*-extension of (A, σ) with respect to t.

Definition 5.2.22. Let t be an indeterminate over F and $\tilde{\tau} \in \mathrm{Aut}_{\mathbf{Field}}(F(t))$. Then $(F(t), \tilde{\tau})$ is said to be a **field ΠΣ*- extension** of (F, τ) with respect to t if $(F(t), \tilde{\tau})$ is a field Π-extension or a field Σ*-extension of (F, τ) with respect to t.

Definition 5.2.23. Let $r \in \mathbb{N}$ and $t_0, ..., t_r$ indeterminates over A. Then $A\langle t_0\rangle ... \langle t_r\rangle$ is said to be a **nested ring RΠΣ*-** (resp. **RΠ-, RΣ*-, ΠΣ*-, R-, Π-, Σ*-) extension** of A if $A\langle t_0\rangle$ is a ring RΠΣ*- (resp. RΠ-, RΣ*-, ΠΣ*-, R-, Π-, Σ*-) extension of A, ... and $A\langle t_0\rangle ... \langle t_r\rangle$ is a ring RΠΣ*- (resp. RΠ-, RΣ*-, ΠΣ*-, R-, Π-, Σ*-) extension of $A\langle t_0\rangle ... \langle t_{r-1}\rangle$ (equivalently, it can be said, losing formality (since index -1 does not exist) but gaining brievity, "if $A\langle t_0\rangle ... \langle t_i\rangle$ is a ring RΠΣ*- (resp. RΠ-, RΣ*-, ΠΣ*-, R-, Π-, Σ*-) extension of $A\langle t_0\rangle ... \langle t_{i-1}\rangle$, for all $i \in \{0, ..., r\}$").

Note that here (A, σ) has been identified with A, but this is slightly informal. This omission of the automorphism will be frequently made during this chapter.

For example, given an indeterminate k over $\mathbb{Q}(\iota)$, $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k) = k + 1$, an indeterminate x over $\mathbb{Q}(\iota)(k)$ such that $x^4 = 1$, $\tilde{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}(\mathbb{Q}(\iota)(k)[x])$ such that $\tilde{\tau}|_{\mathbb{Q}(\iota)(k)} = \tau_0$ and $\tilde{\tau}(x) = \iota \cdot x$, an indeterminate t over $\mathbb{Q}(\iota)(k)[x]$ and $\hat{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right]\right)$ such that $\hat{\tau}|_{\mathbb{Q}(\iota)(k)[x]} = \tilde{\tau}$ and $\hat{\tau}(t) = x \cdot k \cdot t$, the following conditions hold:

- $\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right], \hat{\tau}\right)$ is a nested ring Π-extension of $(\mathbb{Q}(\iota)(k)[x], \tilde{\tau})$,
- $\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right], \hat{\tau}\right)$ is a nested ring RΠ-extension of $(\mathbb{Q}(\iota)(k), \tau_0)$,
- $\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right], \hat{\tau}\right)$ is a nested ring RΠΣ*-extension of $(\mathbb{Q}(\iota), \mathrm{id}_{\mathbb{Q}(\iota)})$.

From now on, given $G \leqslant_{\mathbf{Grp}} A^*$, $r \in \mathbb{N}$, $t_0, ..., t_r$ indeterminates over A and $E = A\langle t_0\rangle ... \langle t_r\rangle$, $\{g \cdot t_0^{m_0} \cdot ... \cdot t_r^{m_r} \mid [g \in G \ \wedge \ \forall\, i \in \{0, ..., r\}, \ [m_i \in \mathbb{Z} \ \wedge$
$[A\langle t_0\rangle ... \langle t_i\rangle$ is a ring Σ*-extension of $A\langle t_0\rangle ... \langle t_{i-1}\rangle \ \Rightarrow \ m_i = 0]]]\}$ will be denoted by G_A^E.

Definition 5.2.24. Let $G \leqslant_{\mathbf{Grp}} A^*$, $r \in \mathbb{N}$, $t_0, ..., t_r$ indeterminates over A, $\tilde{\sigma} \in \mathrm{Aut}_{\mathbf{CRing}}(A\langle t_0\rangle ... \langle t_r\rangle)$ such that $(A\langle t_0\rangle ... \langle t_r\rangle, \tilde{\sigma})$ is a nested ring RΠΣ*- (resp. RΠ-, RΣ*-, ΠΣ*-, R-, Π-, Σ*-) extension of (A, σ) and $E = A\langle t_0\rangle ... \langle t_r\rangle$. Then $(A\langle t_0\rangle ... \langle t_r\rangle, \tilde{\sigma})$ is said to be G-**simple** if $\frac{\tilde{\sigma}(t_i)}{t_i} \in G_A^E$, for all $i \in \{0, ..., r\}$ such that $A\langle t_0\rangle ... \langle t_i\rangle$ is a ring RΠ-extension of $A\langle t_0\rangle ... \langle t_{i-1}\rangle$.

For example, given an indeterminate k over $\mathbb{Q}(\iota)$, $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k) = k + 1$, an indeterminate x over $\mathbb{Q}(\iota)(k)$ such that $x^4 = 1$, $\tilde{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}(\mathbb{Q}(\iota)(k)[x])$ such that $\tilde{\tau}|_{\mathbb{Q}(\iota)(k)} = \tau_0$ and $\tilde{\tau}(x) = \iota \cdot x$, an indeterminate t over $\mathbb{Q}(\iota)(k)[x]$, $\hat{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right]\right)$ such that $\hat{\tau}|_{\mathbb{Q}(\iota)(k)[x]} = \tilde{\tau}$ and $\hat{\tau}(t) = x \cdot k \cdot t$ and $G \leqslant_{\mathbf{Grp}} \mathbb{Q}(\iota)\backslash\{0\}$, $\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right], \hat{\tau}\right)$ is a not a G-simple nested ring RΠΣ*-extension of $(\mathbb{Q}(\iota), \mathrm{id}_{\mathbb{Q}(\iota)})$, since $\dfrac{\hat{\tau}(t)}{t} = x \cdot k \notin G_{\mathbb{Q}(\iota)}^{\mathbb{Q}(\iota)(k)[x][t, \frac{1}{t}]} = \{g \cdot k^0 \cdot x^{m_1} \cdot t^{m_2} \,||[g \in G \ \wedge \ m_1, m_2 \in \mathbb{Z}]\}$.

Proposition 5.2.25. Characterization of the semiconstants of the nested ring $\Pi\Sigma^*$ -extensions
Let $r \in \mathbb{N}$, $t_0, ..., t_r$ indeterminates over A, $\tilde{A} = A\langle t_0\rangle...\langle t_r\rangle$, $\tilde{\sigma} \in \mathrm{Aut}_{\mathbf{CRing}}(\tilde{A})$ such that $(\tilde{A}, \tilde{\sigma})$ is a nested ring $\Pi\Sigma^*$-extension of (A, σ) and $G \leqslant_{\mathbf{Grp}} A^*$. If $\mathrm{sconst}_G(A, \sigma)\backslash\{0\} \leqslant_{\mathbf{Grp}} A^*$, $\tilde{G} := G_A^{\tilde{A}}$ and $(\tilde{A}, \tilde{\sigma})$ is G-simple, then $\mathrm{sconst}_{\tilde{G}}(\tilde{A}, \tilde{\sigma})\backslash\{0\} \leqslant_{\mathbf{Grp}} \tilde{A}^*$ and $\mathrm{sconst}_{\tilde{G}}(\tilde{A}, \tilde{\sigma}) = \{h \cdot t_0^{m_0} \cdot ... \cdot t_r^{m_r} \mid [h \in \mathrm{sconst}_G(A, \sigma) \ \wedge \ \forall\, i \in \{0, ..., r\}, \ [m_i \in \mathbb{Z} \ \wedge \ [A\langle t_0\rangle...\langle t_i\rangle$ is a ring Σ^*-extension of $A\langle t_0\rangle...\langle t_{i-1}\rangle \ \Rightarrow \ m_i = 0]]]\}$.

Proof (Outline, induction on r)
Case 0 Immediate.
Case r Induction Hypothesis (I.H.).
Case $r+1$ Let $\hat{G} = G_A^{\tilde{A}\langle t_{r+1}\rangle}$.

Case a): $\tilde{A}\langle t_{r+1}\rangle$ is a $\hat{G}$-simple ring Σ^*-extension of $\tilde{A}$. In this case, $G_A^{\tilde{A}\langle t_{r+1}\rangle} = G_A^{\tilde{A}}$, i.e. $\hat{G} = \tilde{G}$; so it can be proven that $\mathrm{sconst}_{\hat{G}}(\tilde{A}\langle t_{r+1}\rangle) = \mathrm{sconst}_{\tilde{G}}(\tilde{A}\langle t_{r+1}\rangle) = \mathrm{sconst}_{\tilde{G}}(\tilde{A})$ and $\mathrm{sconst}_{\hat{G}}(\tilde{A}\langle t_{r+1}\rangle)\backslash\{0\} \leqslant_{\mathbf{Grp}} \tilde{A}^* \leqslant_{\mathbf{Grp}} \tilde{A}\langle t_{r+1}\rangle^*$ (cf. Theorem 3.12 of [Schneider I]).

Case b): $\tilde{A}\langle t_{r+1}\rangle$ is a $\hat{G}$-simple ring Π-extension of $\tilde{A}$. In this case, it can be proven that $\mathrm{sconst}_{\hat{G}}(\tilde{A}\langle t_{r+1}\rangle) = \{h \cdot t_{r+1}^m \mid [m \in \mathbb{Z} \ \wedge \ h \in \mathrm{sconst}_{\tilde{G}}(\tilde{A}, \tilde{\sigma})]\}$ (cf. Theorem 3.20 of [Schneider I]). Applying I.H., $\mathrm{sconst}_{\hat{G}}(\tilde{A}\langle t_{r+1}\rangle) = \{h \cdot t_0^{m_0} \cdot ... \cdot t_{r+1}^{m_{r+1}} \mid [h \in \mathrm{sconst}_G(A, \sigma) \ \wedge \ \forall\, i \in \{0, ..., r+1\}, \ [m_i \in \mathbb{Z} \ \wedge \ [A\langle t_0\rangle...\langle t_i\rangle$ is a ring Σ^*-extension of $A\langle t_0\rangle...\langle t_{i-1}\rangle \ \Rightarrow \ m_i = 0]]]\}$ and then $\mathrm{sconst}_{\hat{G}}(\tilde{A}\langle t_{r+1}\rangle)\backslash\{0\} \leqslant_{\mathbf{Grp}} \tilde{A}\langle t_{r+1}\rangle^*$.

□

Definition 5.2.26. Let $r \in \mathbb{N}$ and $t_0, ..., t_r$ indeterminates over F. Then $F(t_0)...(t_r)$ is said to be a **nested field $\Pi\Sigma^*$- (resp. Π-, Σ^*-) extension** of F if $F(t_0)...(t_i)$ is a field $\Pi\Sigma^*$- (resp. Π-, Σ^*-) extension of $F(t_0)...(t_{i-1})$, for all $i \in \{0, ..., r\}$.

Definition 5.2.27. Let $r \in \mathbb{N}$ and $t_0, ..., t_r$ indeterminates over F. Then $F(t_0)...(t_r)$ is said to be a **$\Pi\Sigma^*$- (resp. Π-, Σ^*-) field** if $F(t_0)...(t_r)$ is a nested field $\Pi\Sigma^*$- (resp. Π-, Σ^*-) extension of F and $\mathrm{const}(F(t_0)...(t_r)) = \mathrm{const}(F)$.

For example, given an indeterminate k over $\mathbb{Q}(\iota)$ and $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k) = k+1$, $(\mathbb{Q}(\iota)(k), \tau_0)$ is a $\Pi\Sigma^*$-field. In order to avoid confusions, it is remarked that this example represents a case of Definition 5.2.27 in which $r = 0$ and $F = \mathbb{Q}(\iota)$, not in which $r = 1$ and $F = \mathbb{Q}$; recall that ι is not an indeterminate over $\mathbb{Q}$ (i.e. ι is not a transcendental element over $\mathbb{Q}$, since $\iota^2 + 1 = 0$; cf. page 550 of [Hungerford]).

5.3 THE MAIN PROBLEMS

The objective of this section is to make precise how resolution of equations over ring $R\Pi\Sigma^*$-extensions actually gets reflected into resolution of difference equations.

For the considerations from this section and the following one making sense, it is necessary to assume also that (A, σ) and (F, τ) are **computable**, i.e. there exist algorithms that can perform the automorphism and the standard operations, including zero recognition and deciding constructively if an element is invertible. Note that, in that case, it is also possible to decide if an element is a constant.

The problem "Given a difference ring (C, ρ) and $f \in C$, decide constructively if there exists $g \in C$ such that $\rho(g) - g = f$." is called **Telescoping Problem (T)**.

T is the first point in which this chapter connects with the previous ones: the automorphism must be understood as representing the shift operator, viz. given a difference ring (C, ρ) and $f \in C$, if f represents a sequence y, then to find $g \in C$ such that $\rho(g) - g = f$ means to solve the difference equation given by $x(n+1) - x(n) = y(n)$.

A generalization of T is the so-called **Parameterized Telescoping Problem (PT)**, viz. "Given a difference ring (C, ρ), $r \in \mathbb{N}$ and $f_0, \dots, f_r \in C$, decide constructively if there exist $c_0, \dots, c_r \in \mathrm{const}(C, \rho)$ and $g \in C$ such that $\rho(g) - g = c_0 \cdot f_0 + \dots + c_r \cdot f_r$ and $c_i \neq 0$, for some $i \in \{0, \dots, r\}$.".

Given an indeterminate k over F and $\tilde{\tau} \in \mathrm{Aut}_{\mathbf{Field}}(F(k))$ such that $(F(k), \tilde{\tau})$ is a $\Pi\Sigma^*$-field, Karr's algorithm and its enhanced versions, Schneider's algorithms, solve PT for $(F(k), \tilde{\tau})$ (cf. [Karr I], [Karr II], [Schneider II] and [Schneider III]).

Therefore, it is clear that such algorithms allow one to deal with indefinite summation, but note that with definite summation too, since the previous problem covers Zeilberger's creative telescoping. Indeed, given indeterminates k and n over F, $\tilde{\tau} \in \mathrm{Aut}_{\mathbf{Field}}(F(k)(n))$ such that $(F(k)(n), \tilde{\tau})$ is a difference field, $r \in \mathbb{N}$ and $H(n,k) \in F(k)(n)$, one is interested in deciding constructively if there exist $G(n,k) \in F(k)(n)$ and $a_0(n), \dots, a_r(n) \in \mathrm{const}(F(k)(n), \tilde{\tau})$ such that $\tilde{\tau}(G(n,k)) - G(n,k) = \sum_{i=0}^{r}(a_i(n) \cdot H(n+i,k))$, i.e. such that $G(n,k+1) - G(n,k) = \sum_{i=0}^{r}(a_i(n) \cdot H(n+i,k))$ (the class of doubly hypergeometric terms can be formalized within the class of considered bivariate functions $H(n,k)$, cf. [Ocansey & Schneider]).

Note also that Karr's algorithm can be considered as the discrete counterpart of Risch's integration algorithm (cf. [Risch] and [Bronstein]).

Another interesting problem is to decide when a ring $R\Pi\Sigma^*$-extension exists.

Note that, given $b \in A$, in order to decide if (A, σ) has a ring Σ^*-extension with respect to 1 and b, it suffices to solve T for (A, σ) and b, and then apply Proposition 5.2.12.

For deciding if (A, σ) has a ring $R\Pi$-extension with respect to one of its elements, one can use the following algorithm.

Input: (A, σ) and $a \in A$.
1. If $a \notin A^*$, then return "There exists no ring $R\Pi$-extension of (A, σ) with respect to a" and STOP.
2. Compute $\mathrm{ord}(a)$, by solving the so-called **Order Problem (O)** for A^* and a, viz. "Given a group G and $g \in G$, find the order of g."; and let $\lambda = \mathrm{ord}(a)$.

At this point, it is important to remark that, if $\lambda = 0$ (resp. $\lambda > 0$), then only the construction of a ring Π- (resp. R-) extension might be possible (cf. Subsection 2.2.1 of [Schneider I] for details).

3. Solve the so-called **Multiplicative Telescoping Problem (MT)** for (A, σ) and a, viz. "Given a difference ring (C, ρ) and $c \in C^*$, let $\eta = \mathrm{ord}(c)$ and decide if there exists $g \in C \setminus \{0\}$ such that the following conditions hold: • if $\eta = 0$, then $\rho(g) = c^m \cdot g$, for some $m \in \mathbb{Z} \setminus \{0\}$, • if $\eta > 0$, then $\rho(g) = c^m \cdot g$, for some $m \in \{1, \dots, \lambda - 1\}$.": if such g and m exist, return "There exists no ring $R\Pi$-extension of (A, σ) with respect to a" and STOP; otherwise, return "There exists a ring $R\Pi$-extension of (A, σ) with respect to a" and STOP.

Note that Proposition 5.2.15 and Proposition 5.2.19 have been applied here.

Output: "There exists a ring $R\Pi$-extension of (A, σ) with respect to a" if this is the case; "There exists no ring $R\Pi$-extension of (A, σ) with respect to a" otherwise.

From now on, given $W \subseteq A$, $n \in \mathbb{N}^+$ and $\vec{f} \in (A^*)^n$, $\{(m_1, ..., m_n) \in \mathbb{Z}^n \mid \exists\, g \in W\backslash\{0\}$ such that $\sigma(g) = \vec{f}(0)^{m_1} \cdot ... \cdot \vec{f}(n-1)^{m_n} \cdot g\}$ will be denoted by $M(\vec{f}, W)$.

Lemma 5.3.1. Let $G \leqslant_{\mathbf{Grp}} A^*$, $n \in \mathbb{N}^+$ and $\vec{f} \in G^n$. If $\text{sconst}_G(A, \sigma)\backslash\{0\} \leqslant_{\mathbf{Grp}} A^*$, then the following conditions hold:

1. $M(\vec{f}, A) = M(\vec{f}, \text{sconst}_G(A, \sigma))$,
2. $M(\vec{f}, A) \leqslant_{\mathbb{Z}\textbf{-Mod}} \mathbb{Z}^n$,
3. there exists a $\mathbb{Z}$-basis for $M(\vec{f}, A)$ whose rank is nongreater than n

(cf. Lemma 2.16 of [Schneider I]).

Lemma 5.3.2. Let $G \leqslant_{\mathbf{Grp}} A^*$, $n \in \mathbb{N}^+$, $\vec{f} \in G^n$ and t an indeterminate over A. If there exists $(\tilde{A}, \tilde{\sigma})$ ring R-extension of (A, σ) with respect to t, $\text{sconst}_G(A, \sigma)\backslash\{0\} \leqslant_{\mathbf{Grp}} A^*$ and $\frac{\tilde{\sigma}(t)}{t} \in G$, then $M(\vec{f}, A) = \left\{(\vec{m}(0), ..., \vec{m}(n-1)) \mid \vec{m} \in M\left(\left(\vec{f}(0), ..., \vec{f}(n-1), \frac{t}{\tilde{\sigma}(t)}\right), A\right)\right\}$ (cf. Lemma 6.6 of [Schneider I]).

The problem "Given a difference ring (C, ρ), $G \leqslant_{\mathbf{Grp}} C^*$ such that $\text{sconst}_G(C, \rho)\backslash\{0\} \leqslant_{\mathbf{Grp}} C^*$, $n \in \mathbb{N}^+$ and $\vec{f} \in G^n$, compute a $\mathbb{Z}$-basis for $M(\vec{f}, C)$.", is called **Parameterized Multiplicative Telescoping Problem (PMT)**.

Note that, given $a \in A$, $\lambda = \text{ord}(a)$ and $G \leqslant_{\mathbf{Grp}} A^*$ such that $a \in G$ and $\text{sconst}_G(A, \sigma)\backslash\{0\} \leqslant_{\mathbf{Grp}} A^*$, MT can be reduced to PMT. Indeed, to solve PMT for (A, σ), G, 1 and (a) yields to find a $\mathbb{Z}$-basis B of $M((a), A)$ of rank nongreater than 1, so Lemma 5.3.1 can be applied: if B is empty, then $\sigma(g) \neq a^m \cdot g$, for all $g \in A\backslash\{0\}$ and $m \in \mathbb{Z}\backslash\{0\}$; otherwise B has rank 1, so $m \cdot \mathbb{Z} \cong_{\mathbb{Z}\textbf{-Mod}} M((a), A)$, for some $m \in \mathbb{N}^+$, and hence $m = \min(\{r \in \mathbb{N}^+ \mid \exists\, g \in A\backslash\{0\}$ such that $\sigma(g) = a^r \cdot g\})$.

From now on, whenever $\text{const}(A, \sigma)$ is a field, given a nonempty set W of A, $u \in A\backslash\{0\}$, $n \in \mathbb{N}^+$, $\vec{f} \in A^n$ and $K = \text{const}(A, \sigma)$, $\{(c_1, ..., c_n, g) \in K^n \times W \mid \sigma(g) - u \cdot g = \sum_{i=1}^{n}(c_i \cdot \vec{f}(i-1))\}$ will be denoted by $V(u, \vec{f}, (W, \sigma))$.

Lemma 5.3.3. Assume that $\text{const}(A, \sigma)$ is a field, and let $K = \text{const}(A, \sigma)$, $G \leqslant_{\mathbf{Grp}} A^*$, $n \in \mathbb{N}^+$, $\vec{f} \in A^n$ and $u \in G$. Then A is a K-vector space and, for all $W \leqslant_{\mathbf{Vect}_K} A$, if $\text{sconst}_G(A, \sigma)\backslash\{0\} \leqslant_{\mathbf{Grp}} A^*$, then $V(u, \vec{f}, (W, \sigma)) \leqslant_{\mathbf{Vect}_K} K^n \times W$ and $\dim_K(V(u, \vec{f}, (W, \sigma))) \leqslant n + 1$ (cf. Lemma 2.17 of [Schneider I]).

The problem "Given a difference ring (C, ρ) such that $\text{const}(C, \rho)$ is a field, $G \leqslant_{\mathbf{Grp}} C^*$ such that $\text{sconst}_G(C, \rho)\backslash\{0\} \leqslant_{\mathbf{Grp}} C^*$, $u \in G$, $n \in \mathbb{N}^+$, $\vec{f} \in G^n$ and $K = \text{const}(C, \rho)$, compute a K-basis for $V(u, \vec{f}, (C, \rho))$.", is called **Parameterized First Order Linear Difference Equations Problem (PFLDE)**.

Note that, under the assumption that $\text{const}(A, \sigma)$ is a field, and given $K = \text{const}(A, \sigma)$, $G \leqslant_{\mathbf{Grp}} A^*$ such that $\text{sconst}_G(A, \sigma)\backslash\{0\} \leqslant_{\mathbf{Grp}} A^*$, $n \in \mathbb{N}^+$ and $f \in G$, PT can be reduced to PFLDE. Indeed, solving PFLDE for (A, σ), G, 1, n, f and K, immediately yields to solve PT.

5.4 SOME RESOLUTIONS OF PMT AND PFLDE

As it has been evidenced in the previous section, solving PMT and PFLDE is a key to solve the rest of the problems. Therefore, this section will focus on sketching some ways of solving PMT and PFLDE.

5.4.1 For difference rings

First of all, two observations are precised, viz.:

- About the resolution of PMT: given $G \leqslant_{\mathbf{Grp}} A^*$ and a G-simple nested ring $\Pi\Sigma^*$-extension $(\tilde{A}, \tilde{\sigma})$ of (A, σ), if $\text{sconst}_G(A, \sigma)\backslash\{0\} \leqslant_{\mathbf{Grp}} A^*$ and $\tilde{G} := G_A^{\tilde{A}}$, then $\text{sconst}_{\tilde{G}}(\tilde{A}, \tilde{\sigma})\backslash\{0\} \leqslant_{\mathbf{Grp}} \tilde{A}^*$ (recall

Proposition 5.2.25) and, if in addition PMT is solvable for (A, σ) and G, then PMT is solvable for $(\tilde{A}, \tilde{\sigma})$ and $\tilde{G}$,

- About the resolution of PFLDE: given $G \leqslant_{\mathbf{Grp}} A^*$ and a nested ring $R\Pi\Sigma^*$-extension $(\tilde{A}, \tilde{\sigma})$ of (A, σ), if $\mathrm{sconst}_G(A, \sigma)\backslash\{0\} \leqslant_{\mathbf{Grp}} A^*$, then the following conditions hold:
  - if $(\tilde{A}, \tilde{\sigma})$ is a G-simple nested ring Σ^*-extension of (A, σ) and PFLDE is solvable for (A, σ) and G, then PFLDE is solvable for $(\tilde{A}, \tilde{\sigma})$ and $G_A^{\tilde{A}}$,
  - if $(\tilde{A}, \tilde{\sigma})$ is a G-simple nested ring Π-extension of (A, σ) and PMT and PFLDE are solvable for (A, σ) and G, then PFLDE is solvable for $(\tilde{A}, \tilde{\sigma})$ and $G_A^{\tilde{A}}$,
  - if $(\tilde{A}, \tilde{\sigma})$ is a ring R-extension of (A, σ) with respect to an indeterminate t over A, $\frac{\tilde{\sigma}(t)}{t} \in G$ and PFLDE is solvable for (A, σ) and G, then PFLDE is solvable for $(\tilde{A}, \tilde{\sigma})$ and $G_A^{\tilde{A}}$ (but note that $\mathrm{ord}(t)$ needs to be known beforehand, either as input or by solving O for G)

(cf. Theorem 6.1, Theorem 7.1 and Proposition 7.9 of [Schneider I]).

If there exist $G \leqslant_{\mathbf{Grp}} A^*$ such that $\mathrm{sconst}_G(A, \sigma)\backslash\{0\} \leqslant_{\mathbf{Grp}} A^*$, a difference ring $(\hat{A}, \hat{\sigma})$, $u, v, w \in \mathbb{N}$ such that $u \leqslant v \leqslant w$ and $t_0, \ldots, t_w$ indeterminates over A such that the following conditions hold:

- $\hat{A} = A\langle t_0\rangle\ldots\langle t_w\rangle$,
- $A\langle t_0\rangle\ldots\langle t_i\rangle$ is a ring Π-extension of $A\langle t_0\rangle\ldots\langle t_{i-1}\rangle$, for all $i \in \{0, \ldots, u\}$,
- for all $i \in \{u+1, \ldots, v\}$, $A\langle t_0\rangle\ldots\langle t_i\rangle$ is a ring R-extension of $A\langle t_0\rangle\ldots\langle t_{i-1}\rangle$ and $\frac{\hat{\sigma}(t_i)}{t_i} \in A^*$,
- $A\langle t_0\rangle\ldots\langle t_i\rangle$ is a ring Σ^*-extension of $A\langle t_0\rangle\ldots\langle t_{i-1}\rangle$, for all $i \in \{v+1, \ldots, w\}$,
- $(\hat{A}, \hat{\sigma})$ is G-simple,

then, calling $\tilde{G} = G_A^{\tilde{A}}$, the following conditions hold:

1. if PMT is solvable for (A, σ) and G, then it is solvable for $(\tilde{A}, \tilde{\sigma})$ and $\tilde{G}$,
2. if O is solvable for G and PMT and PFLDE are solvable for (A, σ) and G, then PFLDE is solvable for $(\tilde{A}, \tilde{\sigma})$ and $\tilde{G}$.

Indeed:

1. If PMT is solvable for (A, σ) and G, then, by the observation about the resolution of PMT from the beginning of this subsection, $\mathrm{sconst}_{\tilde{G}}(A\langle t_0\rangle\ldots\langle t_u\rangle)\backslash\{0\} \leqslant_{\mathbf{Grp}} A\langle t_0\rangle\ldots\langle t_u\rangle^*$ and PMT is solvable for $A\langle t_0\rangle\ldots\langle t_u\rangle$ and $\tilde{G}$, so applying iteratively Lemma 5.3.2 and Proposition 5.2.20, it can be proven that $\mathrm{sconst}_{\tilde{G}}(A\langle t_0\rangle\ldots\langle t_v\rangle)\backslash\{0\} \leqslant_{\mathbf{Grp}} A\langle t_0\rangle\ldots\langle t_v\rangle^*$ and PMT is solvable for $A\langle t_0\rangle\ldots\langle t_v\rangle$ and $\tilde{G}$. Consequently, again by the observation, PMT is solvable for $(\tilde{A}, \tilde{\sigma})$ and $\tilde{G}$.
2. If PFLDE is solvable for (A, σ) and G, then, by iterative application of the observation about the resolution of PFLDE from the beginning of this subsection and Proposition 5.2.25, $\mathrm{sconst}_{G_A^{A\langle t_0\rangle\ldots\langle t_u\rangle}}(A\langle t_0\rangle\ldots\langle t_u\rangle)\backslash\{0\} \leqslant_{\mathbf{Grp}} A\langle t_0\rangle\ldots\langle t_u\rangle^*$ and PFLDE is solvable for $A\langle t_0\rangle\ldots\langle t_u\rangle$ and $G_A^{A\langle t_0\rangle\ldots\langle t_u\rangle}$, so applying iteratively the fact that O is solvable for G, the observation and Proposition 5.2.20, $\mathrm{sconst}_{\tilde{G}}(A\langle t_0\rangle\ldots\langle t_v\rangle)\backslash\{0\} \leqslant_{\mathbf{Grp}} A\langle t_0\rangle\ldots\langle t_v\rangle^*$ and PFLDE is solvable for $A\langle t_0\rangle\ldots\langle t_v\rangle$ and $G_A^{A\langle t_0\rangle\ldots\langle t_v\rangle}$. Consequently, by iterative application of the observation and Proposition 5.2.25, PFLDE is solvable for $(\tilde{A}, \tilde{\sigma})$ and $\tilde{G}$.

5.4.2 For difference fields

Lemma 5.4.1. Let $(\tilde{A}, \tilde{\sigma})$ be a nested ring $R\Pi\Sigma^*$-extension of (A, σ), $(\hat{A}, \hat{\sigma})$ a nested ring $R\Pi\Sigma^*$-extension of $(\tilde{A}, \tilde{\sigma})$ and $G \leqslant_{\mathbf{Grp}} A^*$. Then $(G_A^{\tilde{A}})_{\tilde{A}}^{\hat{A}} = G_A^{\hat{A}}$.

<u>Proof</u>

Let $r, s \in \mathbb{N}$ such that $\tilde{A} = A\langle t_0\rangle\ldots\langle t_r\rangle$ and $\hat{A} = A\langle t_0\rangle\ldots\langle t_{r+s+1}\rangle$.

Then $(G_A^{\tilde{A}})_{\tilde{A}}^{\hat{A}} = (G_A^{A\langle t_0\rangle\ldots\langle t_r\rangle})_{A\langle t_0\rangle\ldots\langle t_r\rangle}^{A\langle t_0\rangle\ldots\langle t_{r+s+1}\rangle} =$

$\{\tilde{g} \cdot t_{r+1}^{m_{r+1}} \cdot \ldots \cdot t_{r+s+1}^{m_{r+s+1}} \mid \big[\tilde{g} \in G_A^{A\langle t_0\rangle\ldots\langle t_r\rangle} \wedge \forall\, i \in \{r+1, \ldots, r+s+1\}, [m_i \in \mathbb{Z} \wedge$

$[(A\langle t_0\rangle\ldots\langle t_r\rangle)\langle t_{r+1}\rangle\ldots\langle t_i\rangle$ is a ring Σ^*-extension of $(A\langle t_0\rangle\ldots\langle t_r\rangle)\langle t_{r+1}\rangle\ldots\langle t_{i-1}\rangle \Rightarrow m_i = 0]]]\}$

$\{g \cdot t_0^{m_0} \cdot \ldots \cdot t_{r+s+1}^{m_{r+s+1}} \mid [g \in G \wedge \forall\, i \in \{0, \ldots, r+s+1\}, [m_i \in \mathbb{Z} \wedge$

$[A\langle t_0\rangle\ldots\langle t_i\rangle$ is a ring Σ^*-extension of $A\langle t_0\rangle\ldots\langle t_{i-1}\rangle \Rightarrow m_i = 0]]]\} =$

$G_A^{A\langle t_0\rangle\ldots\langle t_{r+s+1}\rangle} = G_A^{\hat{A}}$. □

Lemma 5.4.2. Let $(E, \tilde{\tau})$ be a nested ring R-extension of (F, τ) and $G = (F^*)_F^E$. Then $\mathrm{sconst}_G(E, \tilde{\tau})\backslash\{0\} \leqslant_{\mathbf{Grp}} E^*$ (cf. Corollary 4.3 of [Schneider I]).

Given $(E, \tilde{\tau})$ an F^*-simple nested ring $R\Pi\Sigma^*$-extension of (F, τ), if O is solvable for $\mathrm{const}(F, \tau)^*$, then the following conditions hold:

1. O is solvable for $(F^*)_F^E$,
2. if (F, τ) is strong constant-stable and PMT is solvable for (F, τ) and F^*, then PMT is solvable for $(E, \tilde{\tau})$ and $(F^*)_F^E$,
3. if (F, τ) is strong constant-stable, PMT is solvable for (F, τ) and F^* and PFLDE is solvable for (F, τ^k) and F^*, for all $k \in \mathbb{N}^+$, then PFLDE is solvable for $(E, \tilde{\tau})$ and $(F^*)_F^E$.

Indeed, since $\{u \in F \mid u \text{ is a root of unity}\} \subseteq \mathrm{const}(F, \tau)$ and $\mathrm{const}(F, \tau)$ is a field, it can be proven that there exists a difference ring $(\hat{A}, \hat{\tau})$, $u, v, w \in \mathbb{N}$ such that $u \leqslant v \leqslant w$ and $t_0, \ldots, t_w$ indeterminates over F such that the following conditions hold:

- $\hat{A} = F\langle t_0\rangle\ldots\langle t_w\rangle$,
- there exists a **CRing**-isomorphism ϕ from E to $\hat{A}$,
- $[\tilde{\tau}(a) = b \;\Leftrightarrow\; \hat{\tau}(\phi(a)) = \phi(b)]$, for all $a, b \in E$,
- for all $i \in \{0, \ldots, u\}$, $F\langle t_0\rangle\ldots\langle t_i\rangle$ is a ring Π-extension of $F\langle t_0\rangle\ldots\langle t_{i-1}\rangle$ and $\frac{\hat{\tau}(t_i)}{t_i} = u_i \cdot t_0^{z_0} \cdot \ldots \cdot t_{i-1}^{z_{i-1}}$, for some $z_0, \ldots, z_{i-1} \in \mathbb{N}$ and $u_i \in F$ such that u_i is a root of unity,
- for all $i \in \{u+1, \ldots, v\}$, $F\langle t_0\rangle\ldots\langle t_i\rangle$ is a ring R-extension of $F\langle t_0\rangle\ldots\langle t_{i-1}\rangle$ and $\frac{\hat{\tau}(t_i)}{t_i} = u_i \cdot t_0^{z_0} \cdot \ldots \cdot t_{i-1}^{z_{i-1}}$, for some $z_0, \ldots, z_{i-1} \in \mathbb{Z}$ and $u_i \in F$ such that u_i is a root of unity,
- for all $i \in \{v+1, \ldots, w\}$, $F\langle t_0\rangle\ldots\langle t_i\rangle$ is a ring Σ^*-extension of $F\langle t_0\rangle\ldots\langle t_{i-1}\rangle$ and $\hat{\tau}(t_i) - t_i \in F\langle t_0\rangle\ldots\langle t_{i-1}\rangle$,
- for all $i \in \{u+1, \ldots, w\}$ and $f \in (F^*)_F^E \cap (F\langle t_0\rangle\ldots\langle t_i\rangle\backslash F\langle t_0\rangle\ldots\langle t_u\rangle)$, $\mathrm{ord}(f) = 0$,
- $(E, \tilde{\tau})$ is a G-simple nested ring $\Pi\Sigma^*$-extension of $(\hat{A}, \hat{\tau})$, being $G = (F^*)_F^{\hat{A}}$,
- $(\hat{A}, \hat{\tau})$ is an F^*-simple nested ring R-extension of (F, τ)

(cf. Lemma 4.10 of [Schneider I]).

1. Let $f \in (F^*)_F^{F\langle t_0\rangle\ldots\langle t_u\rangle}$. If $f \in F\langle t_0\rangle\ldots\langle t_j\rangle\backslash F\langle t_0\rangle\ldots\langle t_u\rangle$, for some $j \in \{u+1, \ldots, w\}$, then necessarily $\mathrm{ord}(f) = 0$; and otherwise $u \cdot t_0^{m_0} \cdot \ldots \cdot t_{s-1}^{m_{s-1}} = f$, for some $u \in F^*$, $s \in \{0, \ldots, u\}$ and $m_0, \ldots, m_{s-1} \in \mathbb{N}$, so in this case $\mathrm{ord}(f)$ can be computed (cf. Corollary 5.6.(5) of [Schneider I]).

2. Since O is solvable for $\mathrm{const}(F, \tau)^*$, then PMT can be solved for $(\hat{A}, \hat{\tau})$ and G (cf. Proof 6.15 of [Schneider I]). In particular, applying Lemma 5.4.2, $\mathrm{sconst}_G(\hat{A}, \hat{\tau})\backslash\{0\} \leqslant_{\mathbf{Grp}} \hat{A}^*$, so, by the observation about the resolution of PMT from the beginning of the previous subsection, PMT is solvable for $(E, \tilde{\tau})$ and $G_{\hat{A}}^E$. And, applying Lemma 5.4.1, $G_{\hat{A}}^E = (F^*)_F^E$. Therefore, PMT is solvable for $(E, \tilde{\tau})$ and $(F^*)_F^E$.

3. Let $H = (F^*)_F^E$. If (F, τ) is strong constant-stable and $(E, \tilde{\tau})$ is a simple ring $R\Pi\Sigma^*$-extension of (F, σ), then it can be proven that, for all $i \in \{u+1, \ldots, v\}$, $\mathrm{per}(t_i)$ and $\mathrm{ord}(t_i)$ are computable, and $\mathrm{per}(t_i) > 0$ (cf. Corollary 5.6.(3) and Corollary 5.6.(4) of [Schneider I]); and therefore also that $\mathrm{sconst}_H(E, \tilde{\tau})\backslash\{0\} \leqslant_{\mathbf{Grp}} E^*$ (cf. Theorem 2.24 of [Schneider I]). Thus PFLDE admits $(E, \tilde{\tau})$ and H. Since F^* is closed under τ, $\mathrm{sconst}_{F^*}(F, \tau^l)\backslash\{0\} = F^*$, for all $l \in \mathbb{N}^+$. In addition, Lemma 5.4.2 yields that $\mathrm{sconst}_H(\hat{A}, \hat{\tau})\backslash\{0\} \leqslant_{\mathbf{Grp}} \hat{A}^*$, so in particular $\mathrm{sconst}_{F^*}(\hat{A}, \hat{\tau})\backslash\{0\} \leqslant_{\mathbf{Grp}} \hat{A}^*$ and hence it can be proven that PFLDE is solvable for $(\hat{A}, \tilde{\tau})$ and G (cf. Proposition 7.12 of [Schneider I]). Since PMT is solvable for (F, τ) and F^* and PFLDE is solvable for (F, τ^k) and F^*, for all $k \in \mathbb{N}^+$, by 2. PMT is solvable for $(E, \tilde{\tau})$ and H. So applying iteratively the observation about the resolution from PFLDE from the beginning of the previous subsection, PFLDE is solvable for $(E, \tilde{\tau})$ and $G_{\hat{A}}^E$. Finally, applying Lemma 5.4.1, $G_{\hat{A}}^E = H$. Therefore, PFLDE is solvable for $(E, \tilde{\tau})$ and H.

For example, given an indeterminate k over $\mathbb{Q}(\iota)$, $\tau_0 \in \mathrm{Aut}_{\mathbf{Field}}(\mathbb{Q}(\iota)(k))$ such that $\tau_0|_{\mathbb{Q}(\iota)} = \mathrm{id}_{\mathbb{Q}(\iota)}$ and $\tau_0(k) = k + 1$, an indeterminate x over $\mathbb{Q}(\iota)(k)$ such that $x^4 = 1$, $\tilde{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}(\mathbb{Q}(\iota)(k)[x])$ such that $\tilde{\tau}|_{\mathbb{Q}(\iota)(k)} = \tau_0$ and $\tilde{\tau}(x) = \iota \cdot x$, an indeterminate t over $\mathbb{Q}(\iota)(k)[x]$ and $\hat{\tau} \in \mathrm{Aut}_{\mathbf{CRing}}\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right]\right)$

such that $\hat{\tau}|_{\mathbb{Q}(\iota)(k)[x]} = \tilde{\tau}$ and $\hat{\tau}(t) = x \cdot k \cdot t$, one may try to solve T for $(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right], \hat{\tau})$ and, for example, $\frac{-x}{k+1}$. Does $g \in \mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right]$ such that $\hat{\tau}(g) = \frac{-x}{k+1} \cdot g$ exist? Solving PFLDE for $\left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right], \hat{\tau}\right)$, $(\mathbb{Q}(\iota)^*)_{\mathbb{Q}(\iota)}^{\mathbb{Q}(\iota)(k)[x][t,\frac{1}{t}]}$, $\frac{-x}{k+1}$, 1, 0 and $\mathbb{Q}(\iota)$ (note that $\frac{-x}{k+1} \in (\mathbb{Q}(\iota)^*)_{\mathbb{Q}(\iota)}^{\mathbb{Q}(\iota)(k)[x][t,\frac{1}{t}]} \leqslant_{\mathbf{Grp}} \mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right]^*$), one can find the $\mathbb{Q}(\iota)$-basis $\left(\left(0, \frac{x\cdot(\iota+x^2)}{k\cdot t}\right), (1, 0)\right)$ of $V\left(\frac{-x}{k+1}, (0), \left(\mathbb{Q}(\iota)(k)[x]\left[t, \frac{1}{t}\right], \hat{\tau}\right)\right)$ (cf. Example 7.6 of [Schneider I]); which means that, considering $g = \frac{x\cdot(\iota+x^2)}{k\cdot t}$, $\hat{\tau}(g) = \frac{-x}{k+1} \cdot g$. It is easy to verify that the solution is correct: the equality $\frac{-x}{k+1} \cdot \frac{x\cdot(\iota+x^2)}{k\cdot t} = \hat{\tau}\left(\frac{x\cdot(\iota+x^2)}{k\cdot t}\right)$ is equivalent to $\frac{-x^2\cdot(\iota+x^2)}{(k+1)\cdot k\cdot t} = \frac{\hat{\tau}(x)\cdot(\iota+(\hat{\tau}(x))^2)}{\hat{\tau}(k)\cdot\hat{\tau}(t)}$, i.e. to $\frac{-x^2\cdot(\iota+x^2)}{(k+1)\cdot k\cdot t} = \frac{\iota\cdot x\cdot(\iota+(\iota\cdot x)^2)}{(k+1)\cdot x\cdot k\cdot t}$, i.e. to $x^4 = 1$, which holds.

Note that:

- $\tau_0(k) = k + 1$ induces the difference equation $s(n+1) = s(n) + 1$, whose solution is $n + c$, being $c \in \mathbb{Q}(\iota)$. So k can be interpreted as the solution n.
- $\tilde{\tau}(x) = \iota \cdot x$ induces the difference equation $s(n+1) = \iota \cdot s(n)$, whose solution is $c \cdot \iota^{n-1}$, being $c \in \mathbb{Q}(\iota)$. So x can be interpreted as the solution ι^n.
- $\hat{\tau}(t) = x \cdot k \cdot t$ induces the difference equation $s(n+1) = \iota^n \cdot n \cdot s(n)$, whose solution is $c \cdot \exp(\frac{\iota\cdot\pi\cdot(n-1)\cdot n}{4}) \cdot \Gamma(n) = c \cdot \iota^{\binom{n}{2}} \cdot (n-1)! = c \cdot \prod_{j=1}^{n-1}(j) \cdot \prod_{j=1}^{n-1}(\iota^j) = c \cdot \prod_{j=1}^{n-1}(j \cdot \iota^j)$, being $c \in \mathbb{Q}(\iota)$. So t can be interpreted as $\prod_{j=1}^{n-1}(j \cdot \iota^j)$.

Therefore, solving the previous question was actually found that the sequence given by $\dfrac{\iota^n \cdot (\iota + (-1)^n)}{n \cdot \prod_{j=1}^{n-1}(j \cdot \iota^j)}$ is a solution of the linear difference $\mathbb{Q}(\iota)$-equation given by $y(n+1) + \frac{\iota^n}{n+1} \cdot y(n) = 0$.

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