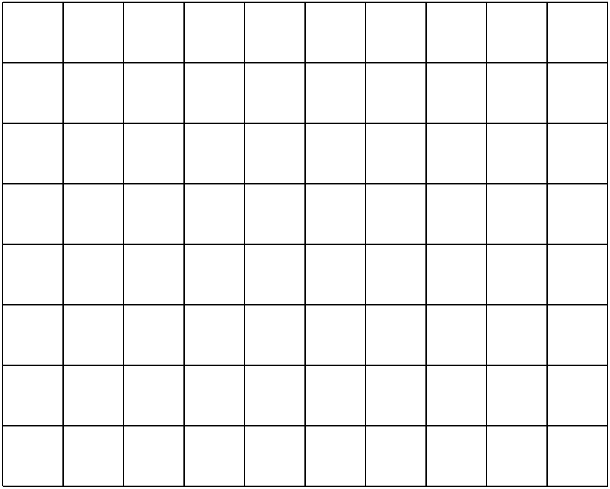
Boundary dents, the arctic circle and the arctic ellipse

Mihai Ciucu and Christian Krattenthaler

Indiana University; Universität Wien

Domino Tilings of a Rectangle



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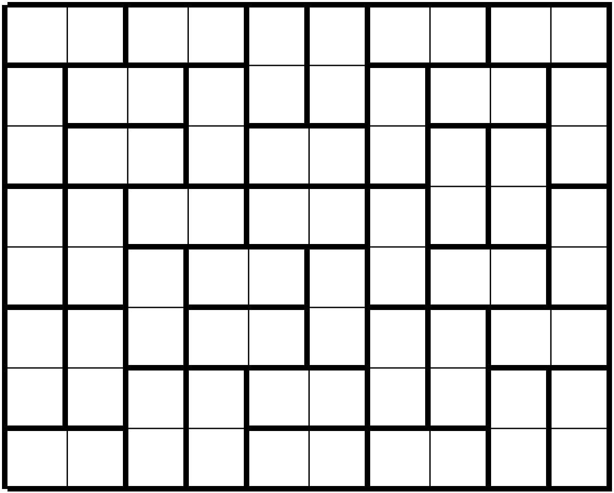

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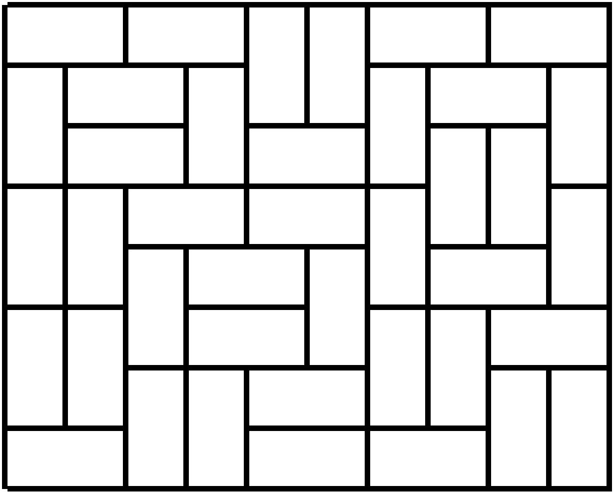


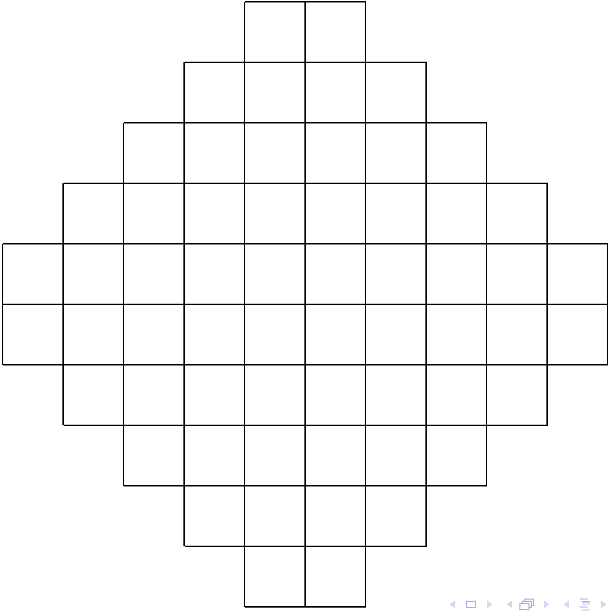
Domino Tilings of a Rectangle

Theorem (Kasteleyn 1961, Temperley and Fisher 1961)

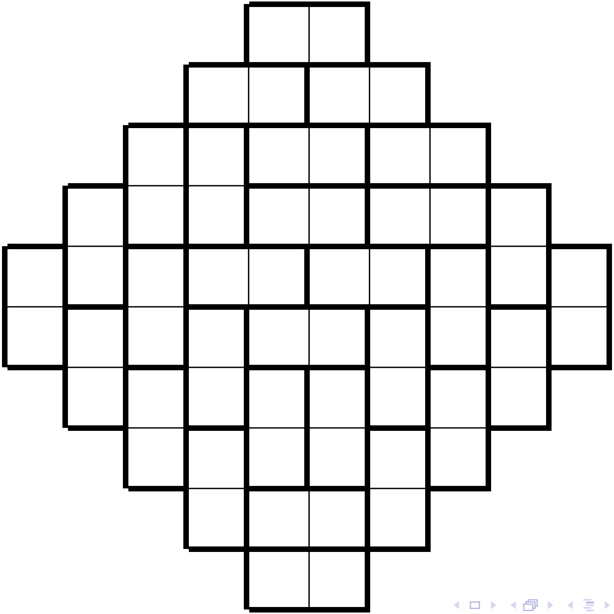
Let m and n be positive integers, with m being even. The number of domino tilings of an $m \times n$-rectangle is given by

$$2^{mn/2} \prod_{i=1}^{m/2} \prod_{j=1}^{n} \left(\cos^2 \frac{\pi i}{m+1} + \cos^2 \frac{\pi j}{n+1} \right)^{1/2}.$$

The Aztec Diamond



Domino Tilings of the Aztec Diamond



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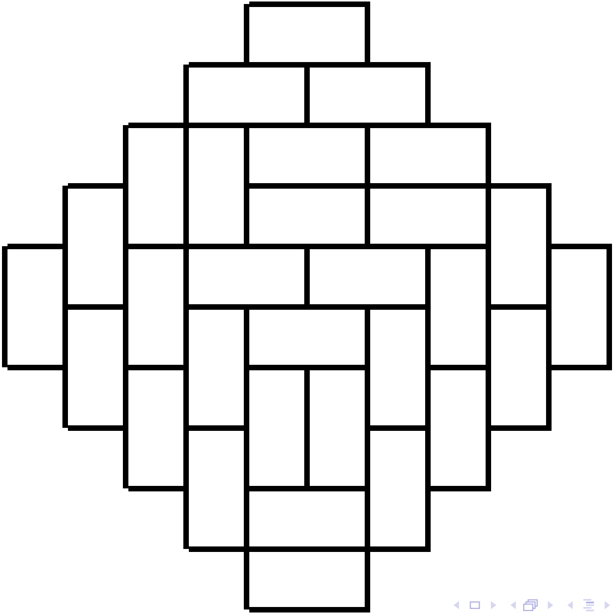

The Aztec diamond Theorem

Theorem (Elkies, Kuperberg, Larsen, Propp 1992)

The number of domino tilings of the Aztec diamond of size n is

$$2^{\binom{n+1}{2}}.$$

The Arctic Circle Theorem (Jockusch, Propp, Shor 1995)

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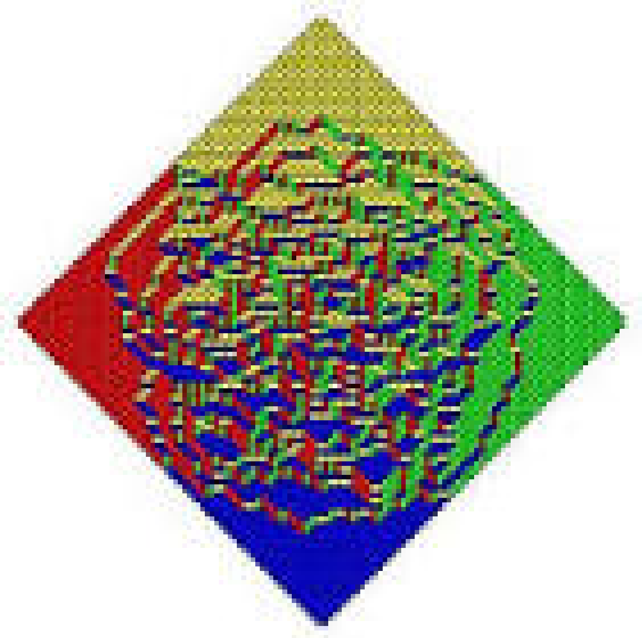

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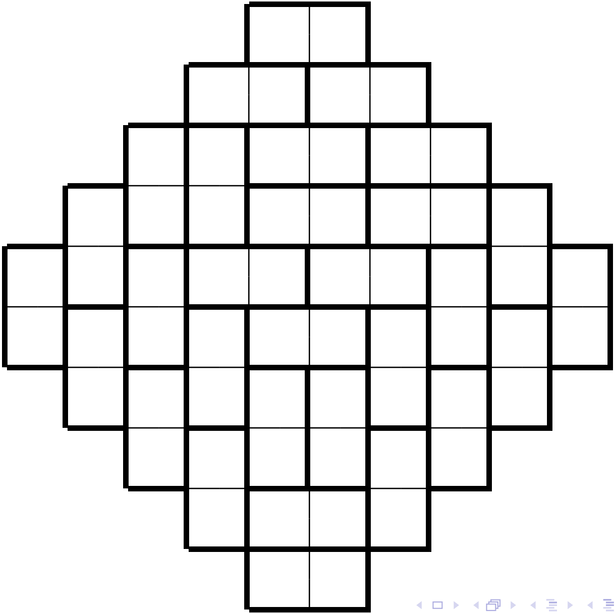

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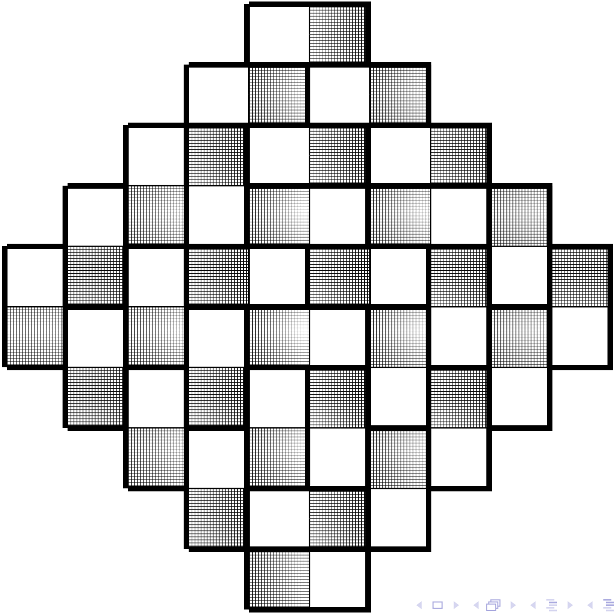

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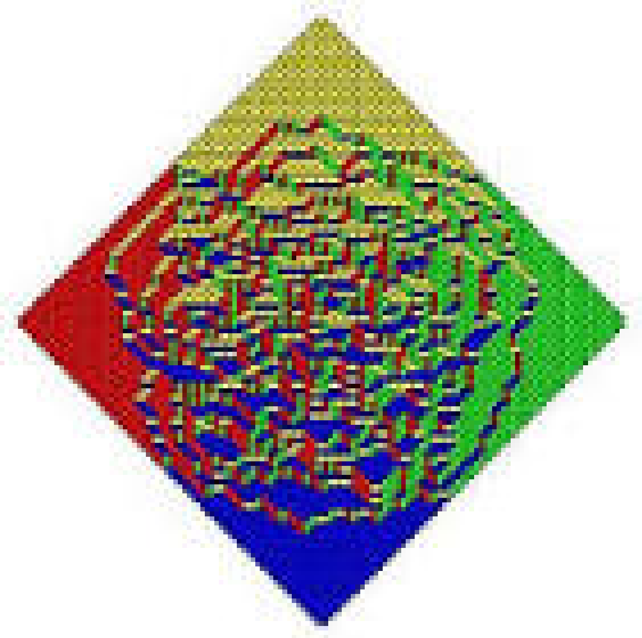

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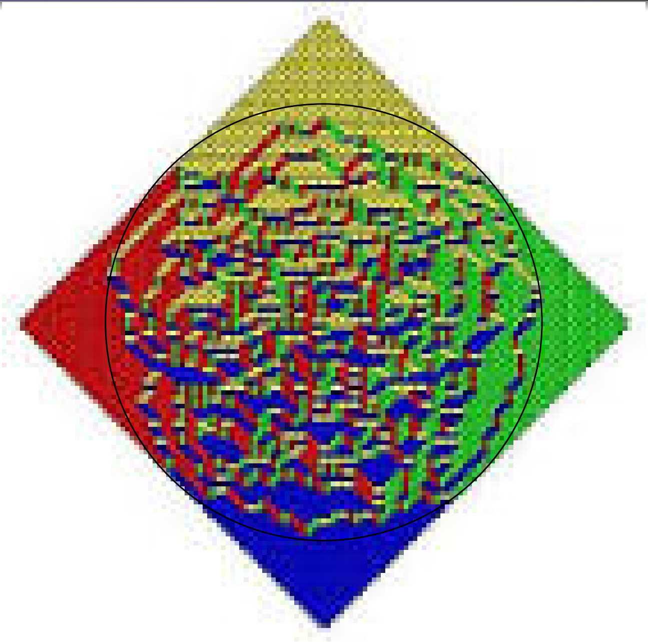
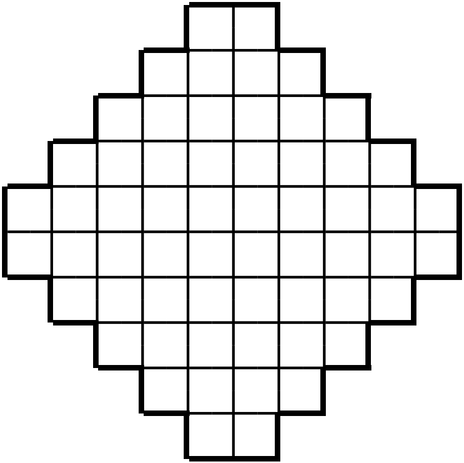


A question of Jim Propp

Is it possible to increase the number of tilings by poking holes?

A question of Jim Propp



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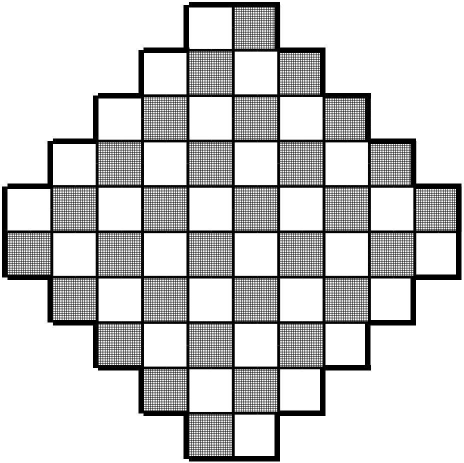

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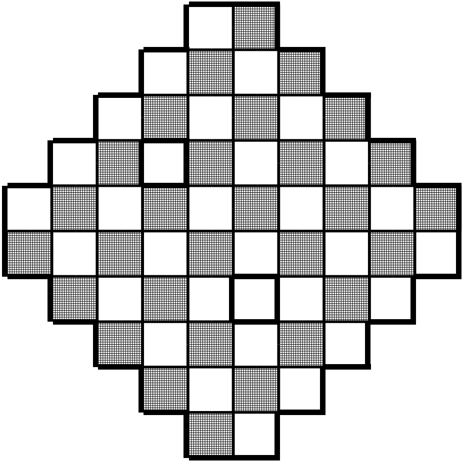
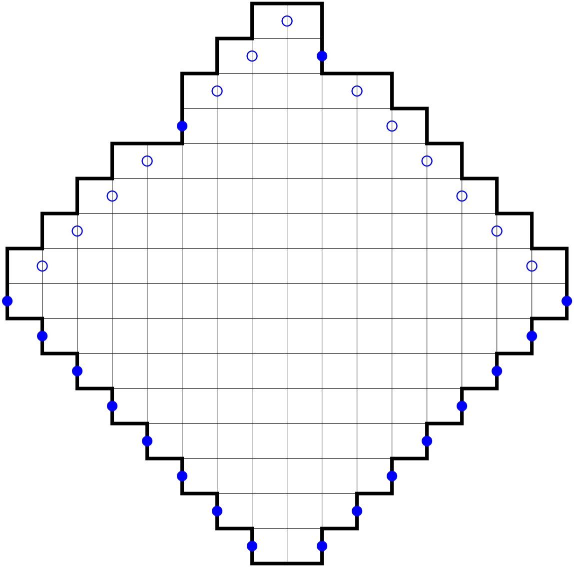


A dented Aztec diamond



Some notation

We write AD_n for the Aztec diamond of size n.

We let $AD_n^{i,j}$ denote the region obtained from AD_n by removing the ith unit square on its northwestern boundary and the jth unit square on its northeastern boundary (both counted from top to bottom).

For a region R we write $\mathrm{M}(R)$ for the number of domino tilings of R.

Main results

Theorem

For $1 \le i, j \le n$, we have:

(a).

$$\frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)} = \sum_{k=0}^{n-1} \binom{k}{i-1} \binom{k}{j-1} \frac{1}{2^{k+1}}.$$

(b).

$$\lim_{n \to \infty} \frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)} = \sum_{k=0}^{\infty} \binom{k}{i-1} \binom{k}{j-1} \frac{1}{2^{k+1}}.$$

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What is $\displaystyle\sum_{k=0}^{\infty} \binom{k}{i-1}\binom{k}{j-1}\frac{1}{2^{k+1}}$?

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We compute the bivariate generating function:

$$\begin{aligned}
\sum_{i,j\ge 0} x^i y^j \sum_{k=0}^{\infty} \binom{k}{i}\binom{k}{j}\frac{1}{2^{k+1}} &= \sum_{k=0}^{\infty} \sum_{i,j\ge 0} x^i y^j \binom{k}{i}\binom{k}{j}\frac{1}{2^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} (1+x)^k (1+y)^k \\
&= \frac{1}{2}\cdot \frac{1}{1-\frac{(1+x)(1+y)}{2}} \\
&= \frac{1}{1-x-y-xy}.
\end{aligned}$$

Main results

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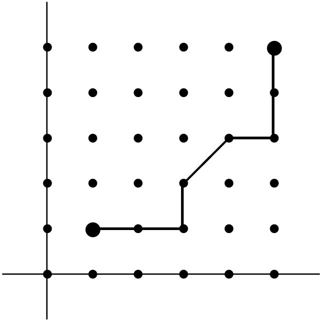
$$= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} (1+x)^k (1+y)^k$$

$$= \frac{1}{2} \cdot \frac{1}{1 - \frac{(1+x)(1+y)}{2}}$$

$$= \frac{1}{1-x-y-xy}.$$

This is the generating function of Delannoy numbers!

Delannoy paths are paths on $\mathbb{Z}^2$ using only steps $(1,0)$, $(0,1)$ or $(1,1)$.



Main results

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(b).

$$\lim_{n\to\infty} \frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)} = D(i-1, j-1),$$

where $D(k,l)$ is the Delannoy number, defined to be the number of paths on $\mathbb{Z}^2$ from $(0,0)$ to (k,l) using only steps $(1,0)$, $(0,1)$ or $(1,1)$.

Main results

Theorem

Let C be the circle inscribed in the unit square. Then as $n, i, j \to \infty$ so that $i/n \to a$ and $j/n \to b$, where $0 < a, b < 1$, we have

$$\lim_{n\to\infty} \frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)D(i-1,j-1)} = \begin{cases} 1, & \text{if the segment } [(a,0),(0,b)] \text{ is outside } C, \\ 0, & \text{if the segment } [(a,0),(0,b)] \text{ crosses } C. \end{cases}$$

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Remark. The condition "the segment $[(a,0),(0,b)]$ is outside C" is equivalent to

$$(1-a)(1-b) > \frac{1}{2}.$$

Main results

Theorem

As $n, i, j \to \infty$ so that $i/n \to a$, $j/n \to b$, we have

$$\frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)} \sim \begin{cases} \dfrac{(a+b+\sqrt{a^2+b^2})\left(\frac{(a+\sqrt{a^2+b^2})^b(b+\sqrt{a^2+b^2})^a}{a^a b^b}\right)^n}{2\sqrt{2\pi n}\sqrt{ab\sqrt{a^2+b^2}}}, & \text{if } (1-a)(1-b) > 1/2, \\[2ex] \dfrac{\sqrt{(1-a)(1-b)}\left(2a^a b^b (1-a)^{1-a}(1-b)^{1-b}\right)^{-n}}{4\pi n\sqrt{ab}\left(\frac{1}{2}-(1-a)(1-b)\right)}, & \text{if } (1-a)(1-b) < 1/2. \end{cases}$$

Main results

Corollary

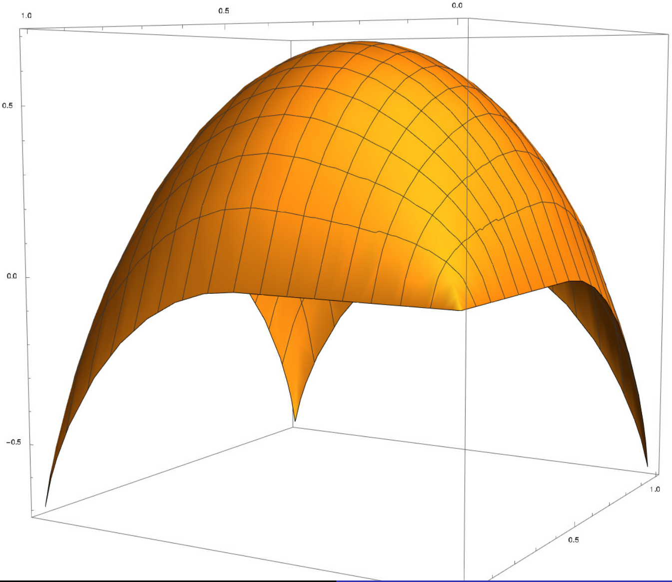
As $n, i, j \to \infty$ so that $i/n \to a$, $j/n \to b$, we have

$$\frac{1}{n} \log \frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)} \to f(a,b),$$

where the function $f(a,b)$ is defined by

$$f(a,b) = \begin{cases} \log \left(\frac{a}{b} + \sqrt{1 + \frac{a^2}{b^2}} \right)^b \left(\frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}} \right)^a, & (1-a)(1-b) > \frac{1}{2}, \\ -\log 2a^a b^b (1-a)^{1-a} (1-b)^{1-b}, & (1-a)(1-b) < \frac{1}{2}. \end{cases}$$

The function $f(a,b)$



The simplest proof of the Aztec diamond theorem

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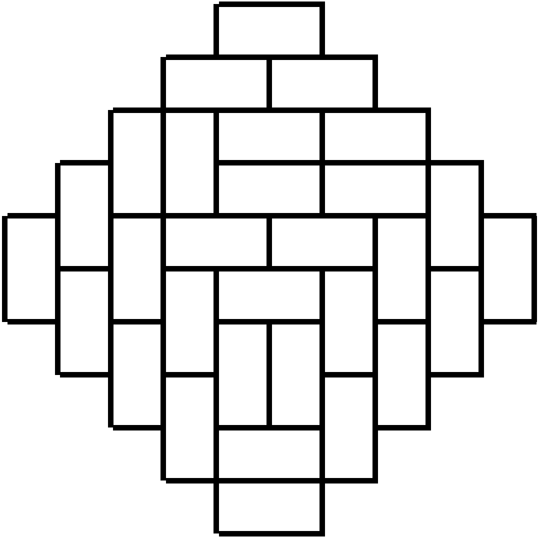

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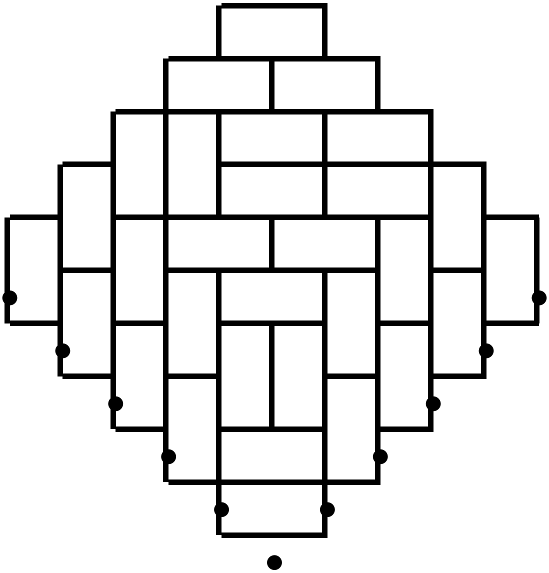

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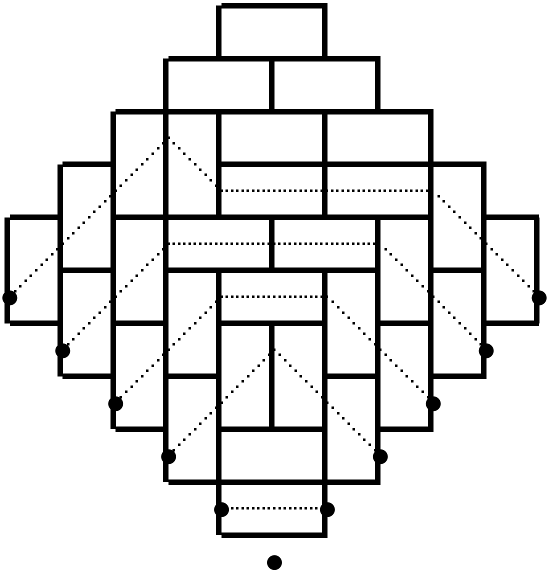

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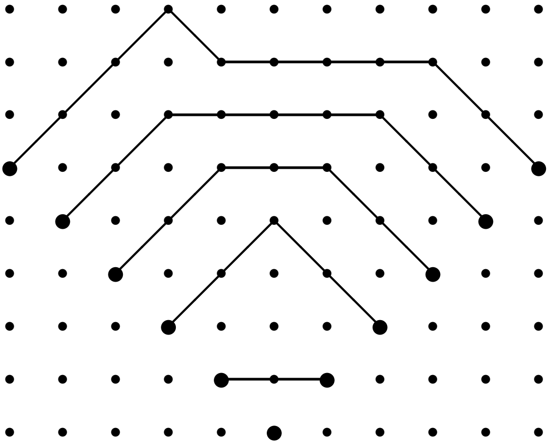

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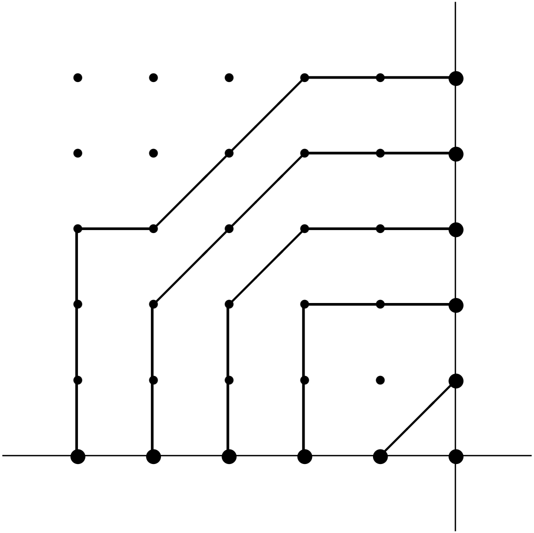

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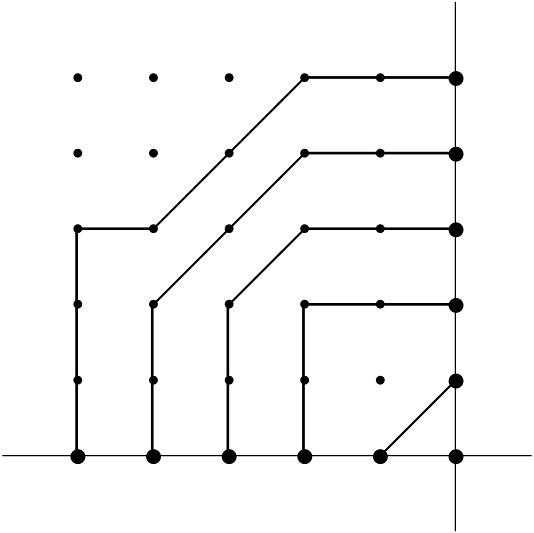

These are Delannoy paths!

Theorem (Karlin–McGregor, Lindström, Gessel–Viennot, Fisher, John–Sachs, Gronau–Just–Schade–Scheffler–Wojciechowski)

Let G be an acyclic, directed graph, and let $A_1, A_2, \dots, A_n$ and $E_1, E_2, \dots, E_n$ be vertices in the graph with the property that, for $i < j$ and $k < l$, any (directed) path from A_i to E_l intersects with any path from A_j to E_k. Then the number of families $(P_1, P_2, \dots, P_n)$ of non-intersecting (directed) paths, where the i-th path P_i runs from A_i to E_i, $i = 1, 2, \dots, n$, is given by

$$\det_{1\le i,j\le n}\left(|\mathcal{P}(A_j \to E_i)|\right),$$

where $\mathcal{P}(A \to E)$ denotes the set of paths from A to E.

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The simplest proof of the Aztec diamond theorem

By the Lindström–Gessel–Viennot theorem on non-intersecting lattice paths, we obtain

$$\mathrm{M}(AD_n) = \det \big(D(i,j)\big)_{0 \le i,j \le n},$$

where $D(i,j)$ denotes the number of Delannoy paths from $(0,0)$ to (i,j).

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Claim. We have

$$\left(D(i,j)\right)_{0\le i,j\le n} = L_n \begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 4 & & \\ & & & \ddots & \\ & & & & 2^n \end{pmatrix} L_n^t,$$

with $L_n = \left(\binom{i}{j}\right)_{0\le i,j\le n}$.

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The claim is equivalent to $D(i,j) = \sum_{k \ge 0} \binom{i}{k} \binom{j}{k} 2^k$.

The simplest proof of the Aztec diamond theorem

Theorem (Elkies, Kuperberg, Larsen, Propp 1992)

The number of domino tilings of the Aztec diamond of size n is

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Proof of the first main result

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For $1 \le i, j \le n$, we have:

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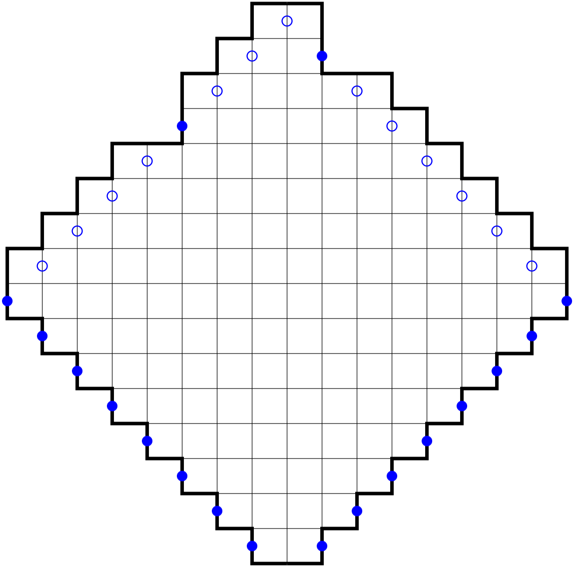
Proof of the first main result



Proof of the first main result

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$$\det \left(D(r,s)\right)_{0\le r,s\le n,\, r\ne i,\, s\ne j}\ ,$$

where the ith row and jth column is omitted.

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However, this is (up to sign) the (j,i)-entry of the inverse matrix multiplied by the determinant of the complete matrix, that is, $2^{\binom{n+1}{2}}$.

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However, this is (up to sign) the (j,i)-entry of the inverse matrix multiplied by the determinant of the complete matrix, that is, $2^{\binom{n+1}{2}}$.

The inverse matrix is easily computed from the factorisation

$$\left(D(i,j)\right)_{0\le i,j\le n} = L_n \begin{pmatrix} 1 &&&& \\ & 2 &&& \\ && 4 && \\ &&& \ddots & \\ &&&& 2^n \end{pmatrix} L_n^t,$$

with $L_n = \left(\binom{i}{j}\right)_{0\le i,j\le n}$.

Proof of the first main result

If everything is put together, we obtain:

Theorem

For $1 \le i, j \le n$, we have:

(a).

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Let C be the circle inscribed in the unit square. Then as $n, i, j \to \infty$ so that $i/n \to a$ and $j/n \to b$, where $0 < a, b < 1$, we have

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Proof of the second main result

Theorem

As $n, i, j \to \infty$ so that $i/n \to a$, $j/n \to b$, we have

$$\frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)}$$

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We must asymptotically estimate the sum

$$\sum_{k=0}^{n-1} \binom{k}{i-1} \binom{k}{j-1} \frac{1}{2^{k+1}}$$

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Let

$$\begin{aligned} F(a, b; n, k) &:= \binom{k}{an} \binom{k}{bn} 2^{-k-1} \\ &= \frac{\Gamma^2(k+1)}{\Gamma(an+1)\,\Gamma(k-an+1)\,\Gamma(bn+1)\,\Gamma(k-bn+1)} 2^{-k-1}. \end{aligned}$$

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Let

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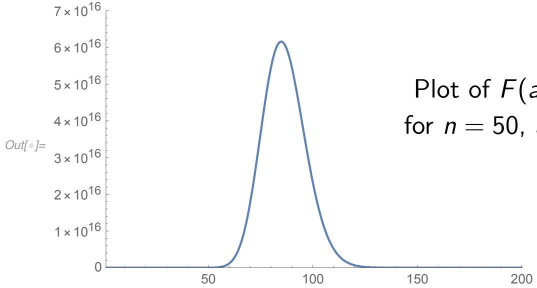

Plot of $F(a,b;n,k)$
for $n=50,\ a=b=\frac{1}{2}$

Proof of the second main result

Let

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$$\sum_{k=0}^{n-1}F(a,b;n,k),$$

as $n\to\infty$.

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Let

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We have $\psi(x)\sim\log x$ as $x\to\infty$. Hence, in a first approximation,

$$2\log(tn)-\log(tn-an)-\log(tn-bn)-\log 2=0,$$

or, equivalently,

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The solutions are $t = a + b \pm \sqrt{a^2 + b^2}$, the maximum point is $t = a + b + \sqrt{a^2 + b^2}$.

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We recall that $k = nt$.

The important question is:

Is $t_0 = a + b + \sqrt{a^2 + b^2}$ smaller than 1 or not?

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The important question is:

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It turns out that

$$a + b + \sqrt{a^2 + b^2} < 1$$

is equivalent to

$$(1 - a)(1 - b) > \frac{1}{2}.$$

Case 1. $(1-a)(1-b) > \frac{1}{2}$. By Stirling's approximation for the gamma function, we get

$$F(a,b;n,nt_0+l) = \frac{1}{2\pi\sqrt{2ab}\,n} \times \exp\left(d(a,b)n + e(a,b)\frac{l}{n} - f(a,b)\frac{l^2}{n} + O\left(\frac{l^3}{n^2}\right)\right),$$

where

$$d(a,b) = -a\log a - b\log b + b\log\left(a+\sqrt{a^2+b^2}\right) + a\log\left(b+\sqrt{a^2+b^2}\right),$$

$$e(a,b) = -\frac{\sqrt{a^2+b^2}}{\left(a+b+\sqrt{a^2+b^2}\right)^2},$$

$$f(a,b) = \frac{\sqrt{a^2+b^2}}{\left(\sqrt{a^2+b^2}+a+b\right)^2}.$$

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We use this approximation in the range $|l| < n^{3/5}$. It is not difficult to see that this range provides tthe main contribution to the sum $\sum_l F(a,b;n,nt_0+l)$, and that the other ranges are asymptotically negligible. In other words, we obtain

$$\sum_{k=0}^{n-1} F(a,b;n,k) \sim \frac{e^{d(a,b)n}}{2\pi\sqrt{2ab}\,n} \sum_{|l|<n^{3/5}} \exp\left(-f(a,b)\frac{l^2}{n}\right).$$

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$$\checkmark$$

Case 2. $(1-a)(1-b) < \frac{1}{2}$. We have

$$\sum_{k=0}^{n-1} \binom{k}{x}\binom{k}{y} 2^{-k-1} = \sum_{k=0}^{n-1} F(a,b;n,k) = \sum_{l=0}^{n-1} F(a,b;n,n-1-l)$$

$$= F(a,b;n,n-1) \sum_{l=0}^{n-1} \frac{F(a,b;n,n-1-l)}{F(a,b;n,n-1)}.$$

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$$\sum_{k=0}^{n-1}\binom{k}{x}\binom{k}{y}2^{-k-1}=\sum_{k=0}^{n-1}F(a,b;n,k)=\sum_{l=0}^{n-1}F(a,b;n,n-1-l)$$
$$=F(a,b;n,n-1)\sum_{l=0}^{n-1}\frac{F(a,b;n,n-1-l)}{F(a,b;n,n-1)}.$$

Using again Stirling's approximation, we get

$$\frac{F(a,b;n,n-1-l)}{F(a,b,n,n-1)}=\exp\left(l\cdot\log\left(2(1-a)(1-b)\right)\right.$$
$$\left.+\left(\frac{l}{n}+\frac{l^2}{n}\right)\frac{2ab-a-b}{2(1-a)(1-b)}+O\left(\frac{l^3}{n^2}\right)\right)$$

and

$$F(a,b,n,n-1)=\frac{\sqrt{(1-a)(1-b)}}{2\pi\sqrt{ab}\,n}$$
$$\times\left(2a^ab^b(1-a)^{1-a}(1-b)^{1-b}\right)^{-n}\left(1+O\left(\frac{1}{n}\right)\right).$$

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Conditional proof of the Arctic Circle Theorem

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We assume that an arctic curve for domino tilings of Aztec diamonds exists.

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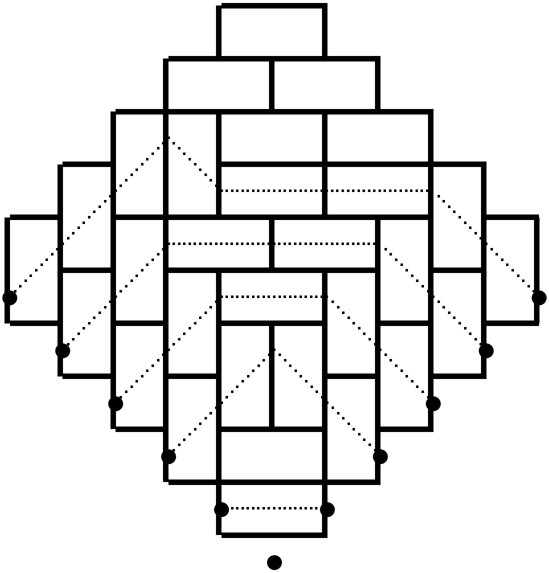

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Conditional proof of the Arctic Circle Theorem

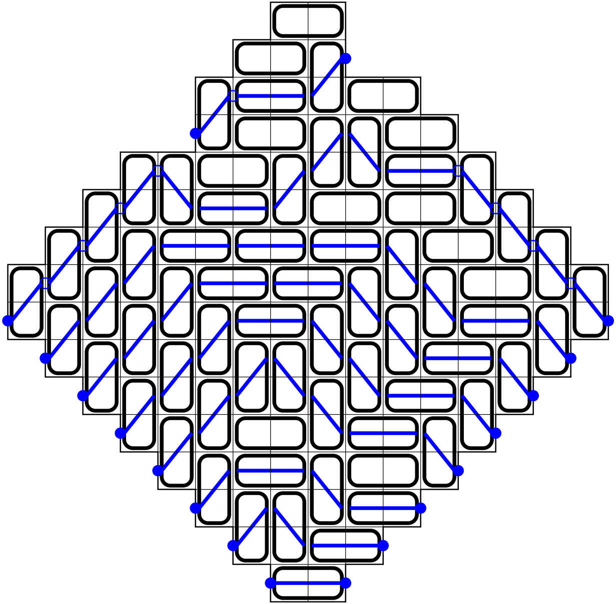
We assume that an arctic curve for domino tilings of Aztec diamonds exists.
More precisely, we assume that there exists a convex arc $\mathcal{C}$ tangent to the two bottom sides of the square such that:

(i) For any open set U containing $\mathcal{C}$ and the region above, the family of n non-intersecting Delannoy paths corresponding to a tiling of AD_n is contained in U with probability approaching 1, as $n \to \infty$

(ii) For any open set V above $\mathcal{C}$, with probability approaching 1 this family of n non-intersecting Delannoy paths is *not* contained in V.

(iii) A Delannoy path from a given a point to a another far away lying point with probability approaching 1 "looks like the line segment" connecting the two points.

Conditional proof of the Arctic Circle Theorem



Epilogue

We have completely analogous results for rhombus tilings of hexagons.