The poset of bipartitions

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Theorem (MacMahon)

The inversion number and major index statistics are equidistributed over each rearrangement class $R(a_1, a_2, \ldots, a_k)$.

$R(a_1, a_2, \ldots, a_k)$ is the set of all words consisting of a_i letters i, $1 \le i \le k$.

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Theorem (Foata–Zeilberger)

The statistics $\mathbf{maj}_U(w)$ *and* $\mathbf{inv}_U(w)$ *are equidistributed over each rearrangement class* $R(a_1, a_2, \ldots, a_k)$ *if an only if* U *is a* bipartitional *relation.*

The definition of a bipartitional relation

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A relation $U \subseteq X \times X$ on a finite set X is a *bipartitional relation*, if both U and $(X \times X) \setminus U$ are transitive.

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Definition (Foata-Zeilberger)

Let $(B_1, B_2, \ldots, B_k)$ be an ordered partition of X, and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in \{0, 1\}^k$. The bipartitional relation U represented as $U(B_1^{\varepsilon_1}, B_2^{\varepsilon_2}, \ldots, B_k^{\varepsilon_k})$ is

$$(x, y) \in U \iff \begin{cases} x \in B_i \text{ and } y \in B_j \text{ for some } i < j, \\ \text{or} \\ x, y \in B_i \text{ for some } i \text{ satisfying } \varepsilon_i = 1. \end{cases}$$

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Example

$$U = \{(3,1),(3,2),(1,1),(1,2),(2,1),(2,2)\}$$

is represented as

$$U(\{3\}^0, \{1,2\}^1), \quad \text{or as} \quad U(\{3\}, \{\underline{1}, \underline{2}\}).$$

We denote the set of bipartitional relations on X by $\mathrm{Bip}(X)$.

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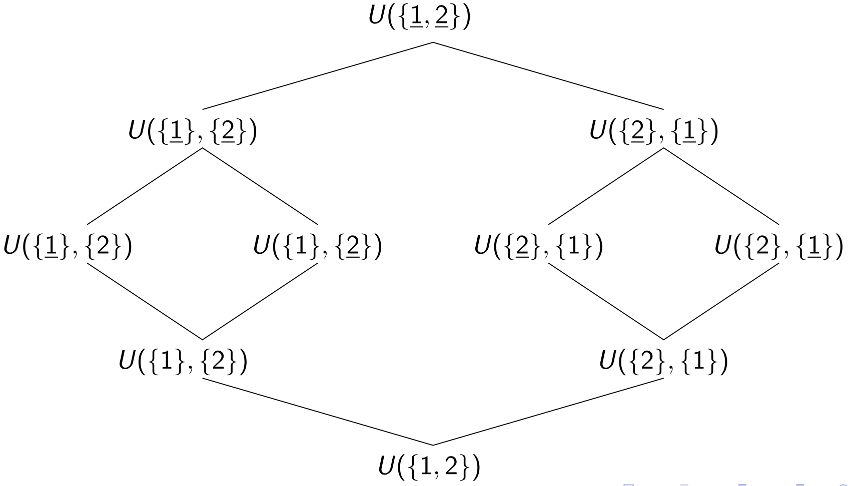

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- It is self-dual under complementation: $X \times X \setminus U(B_1^{\varepsilon_1}, B_2^{\varepsilon_2}, \dots, B_k^{\varepsilon_k}) = U(B_k^{1-\varepsilon_{k-1}}, B_{k-1}^{1-\varepsilon_{k-1}}, \dots, B_1^{1-\varepsilon_1})$.

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- It is a lattice. (Join $U \vee V$ is transitive closure of $U \cup V \cup U \circ V$)
- It is *not even modular*.

The *Möbius function* is an important invariant of a poset.
It is a function μ which assigns to each interval in a poset an integer.
By definition, it is the inverse with respect to convolution of the so-called *zeta function*. In simple terms, this is

$$\mu([x,x]) = 1 \quad \text{for all } x,$$

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The precise theorems will be stated at the end of this talk.

Order complex of a poset

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Definition

Let P be a partially ordered set (poset). The *order complex* $\Delta(P)$ of P is the *simplicial complex*

$$\Delta(P) = \big\{\{x_1, \ldots, x_k\} : x_1 < \cdots < x_k \text{ in } P\big\}.$$

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Theorem (P. Hall)

Let P be a partially ordered set (poset) with minimum $\hat{0}$ and maximum $\hat{1}$. We have

$$\mu([\hat{0}, \hat{1}]) = \tilde{\chi}(\Delta(P \setminus \{\hat{0}, \hat{1}\})),$$

where $\tilde{\chi}(.)$ denotes the reduced Euler characteristics.

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where $\tilde{\chi}(\,.\,)$ denotes the reduced Euler characteristics.

The *reduced Euler characteristics* $\widetilde{\chi}(\Delta)$ of a simplicial complex Δ is

$$-1+f_0-f_1+f_2-+\cdots,$$

where f_i denotes the number of faces (cells) of Δ of dimension i (i.e., containing $i+1$ elements).

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In particular: If P is a poset with minimum $\hat{0}$ and maximum $\hat{1}$, then:

- if $\Delta(P \setminus \{\hat{0}, \hat{1}\})$ is contractible, then $\mu\left([\hat{0}, \hat{1}]\right) = 0$.
- if $\Delta(P \setminus \{\hat{0}, \hat{1}\})$ is homotopy equivalent to a sphere of dimension m, then $\mu\left([\hat{0}, \hat{1}]\right) = (-1)^m$.

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- from the interval systems, one can construct an acyclic matching on the face poset of the order complex;
- properties of the interval system of a maximal chain C_i tell one which cells are the critical cells under this matching.

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Corollary

We have $\mu([\emptyset, X \times X]) = (-1)^{|X|}$.

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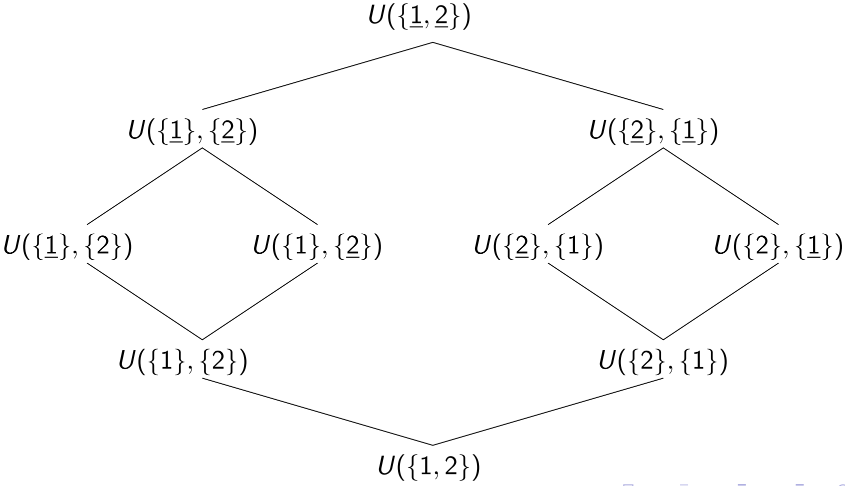
Definition

We say that an interval $[U, V] \subseteq \mathrm{Bip}(X)$ is *regular* if for every x belonging to a nonunderlined block in U and to an underlined block in V, the block containing x in U is equal to the block containing x in V. Otherwise we call $[U, V]$ *irregular*.

The case $|X| = 2$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Proposition

Every regular interval $[U, V] \subseteq \mathrm{Bip}(X)$ *is isomorphic to a direct product of Boolean lattices and lattices of the form* $\mathrm{Bip}(B)$*, where each* B *is a block in the ordered bipartition representation of* U *and of* V *such that* B *is nonunderlined in* U *and underlined in* B*.*

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Corollary

If $[U, V] \subseteq \mathrm{Bip}(\{1, 2, \ldots, n\})$ *is regular, then* $\mu([U, V]) = (-1)^{rk(U)-rk(V)}$.

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Corollary

If $[U, V] \subseteq \mathrm{Bip}(\{1, 2, \ldots, n\})$ *is not regular, then* $\mu([U, V]) = 0$*.*