# Some determinants of path generating functions

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We have the (surprising?) *Hankel determinant* evaluation

$$\det \begin{pmatrix} C_0 & C_1 & C_2 & \dots & C_{n-1} \\ C_1 & C_2 & C_3 & \dots & C_n \\ C_2 & C_3 & C_4 & \dots & C_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & C_n & C_{n+1} & \dots & C_{2n-2} \end{pmatrix} = 1.$$

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**Theorem**

*Let $(p_n(x))_{n\geq 0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree $n$, which is orthogonal with respect to some functional $L$, that is, $L(p_m(x)p_n(x)) = \delta_{m,n}c_n$, where the $c_n$'s are some non-zero constants and $\delta_{m,n}$ is the Kronecker delta. Let*

$$p_{n+1}(x) = (a_n + x)p_n(x) - b_n p_{n-1}(x)$$

*be the corresponding three-term recurrence which is guaranteed by Favard's theorem. Then the Hankel determinant of the moments $\mu_k = L(x^k)$ satisfies*

$$\det_{0\leq i,j\leq n-1}(\mu_{i+j}) = \mu_0^n b_1^{n-1} b_2^{n-2} \cdots b_{n-2}^2 b_{n-1}.$$

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The Catalan numbers are the moments for $U_n(x/2)$, where $U_n(x)$ denotes the $n$-th *Chebyshev polynomial* of the second kind.

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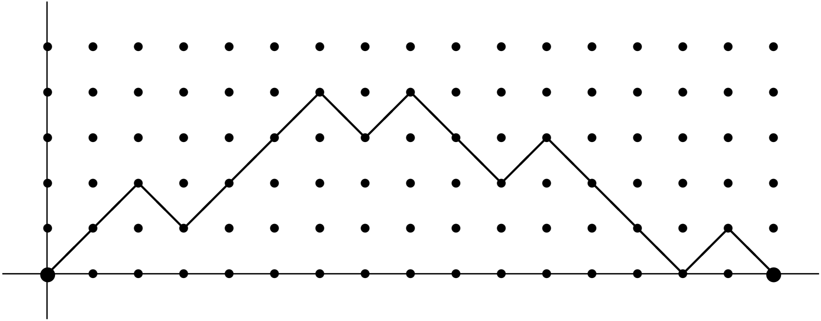
The Catalan number $C_n$ counts *Dyck paths* of length $2n$:



(A Dyck path is a lattice path from $(0,0)$ back to the $x$-axis consisting of up-steps $(1,1)$ and down-steps $(1,-1)$ never running below the $x$-axis.)

## The non-intersecting lattice path theorem

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**Theorem (Lindström, Gessel–Viennot)**

*Fix a lattice region $R$. Let $A_0, A_1, \dots, A_{n-1}$ and $E_0, E_1, \dots, E_{n-1}$ be lattice points in $R$. Then (modulo a mild technical condition) the number of all families $(P_0, P_1, \dots, P_{n-1})$ of non-intersecting paths staying in $R$, such that the $i$-th path $P_i$ runs from $A_i$ to $E_i$, $i = 0, 1, \dots, n-1$, is given by*

$$\det_{0 \le i,j \le n-1} \left( |\mathcal{P}(A_j \to E_i)| \right),$$

*where $\mathcal{P}(A \to E)$ denotes the set of paths from $A$ to $E$ staying in $R$.*

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$$\det_{0\le i,j\le n-1}\left(C_{i+j}\right)=1.$$

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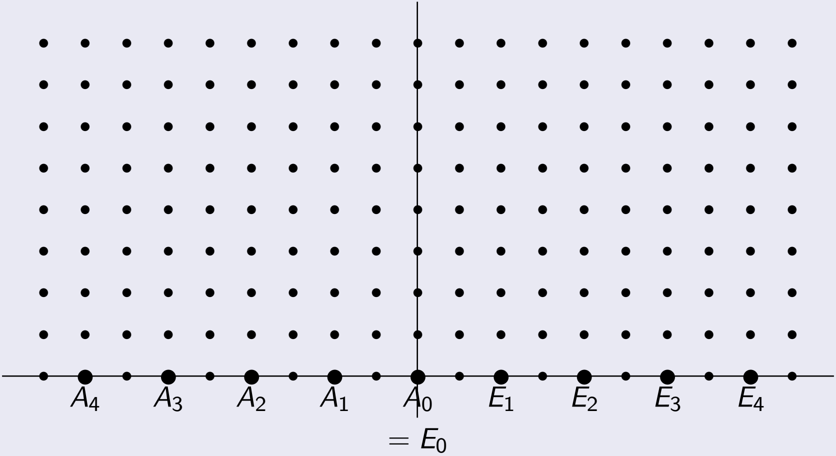

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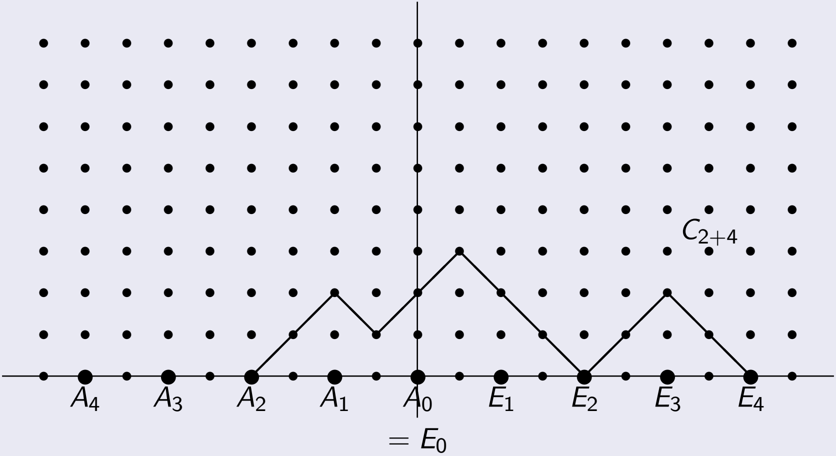

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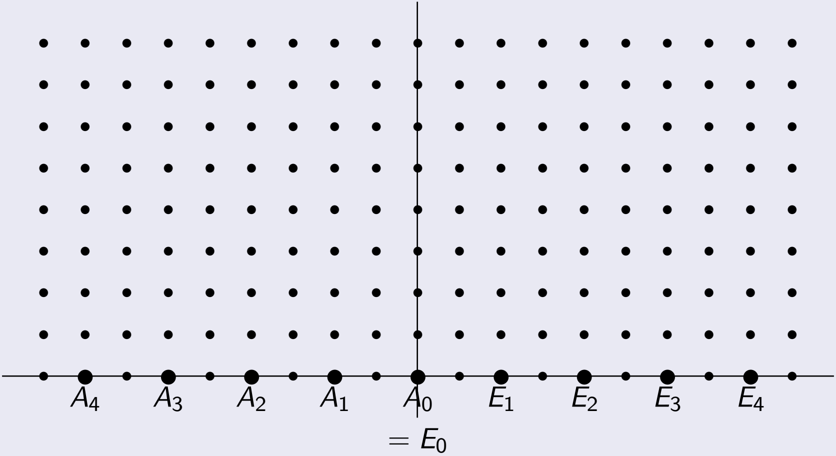

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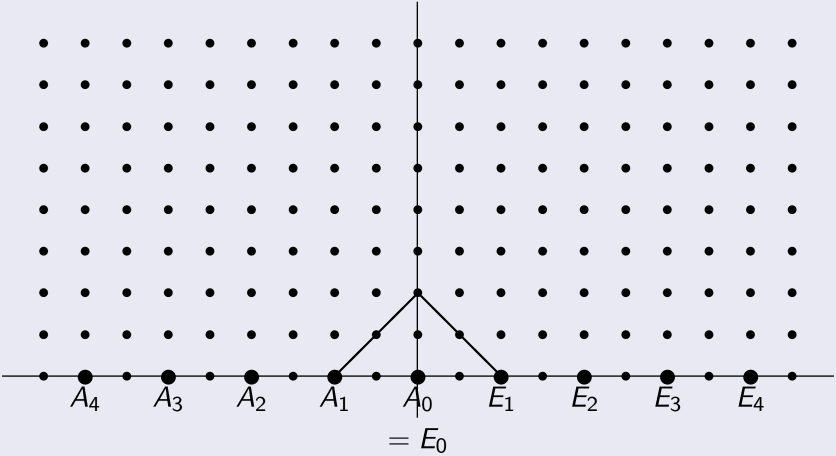

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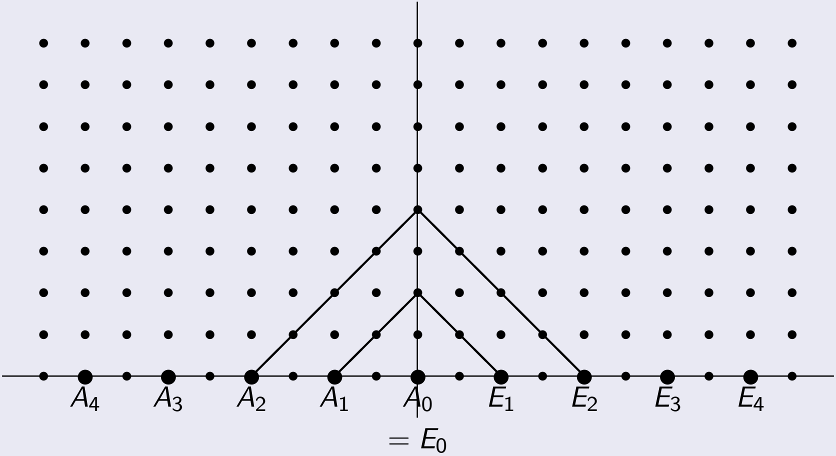

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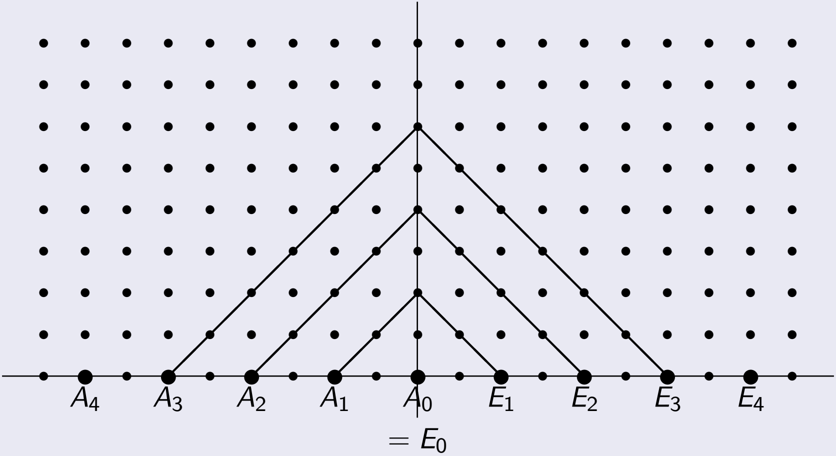

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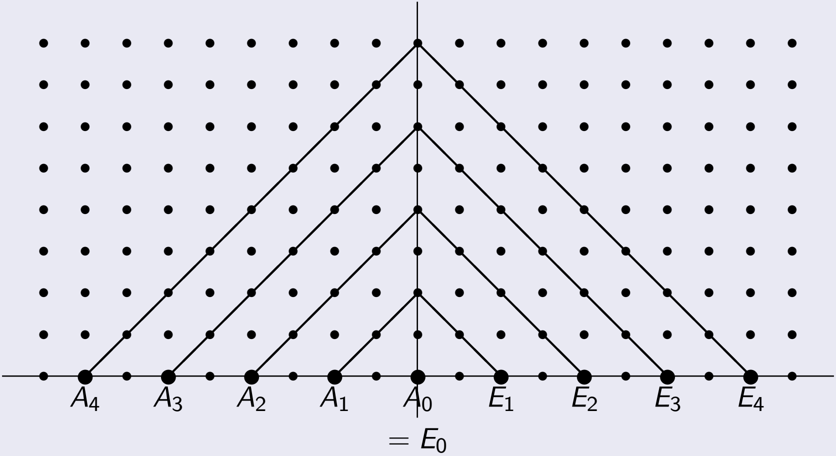

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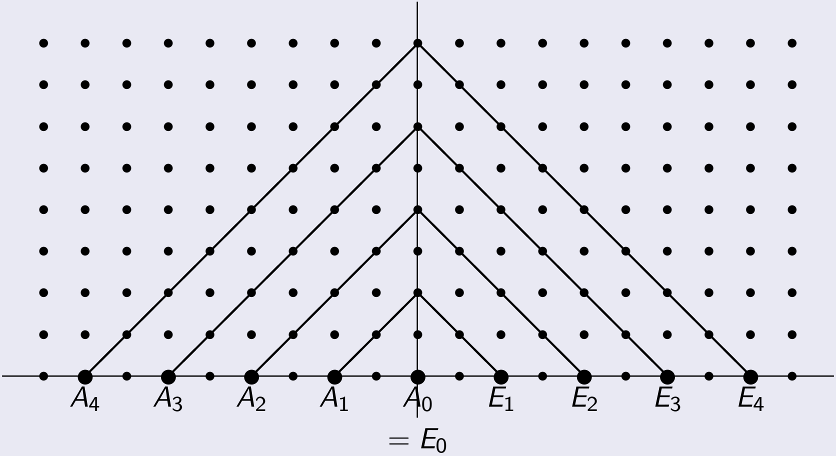

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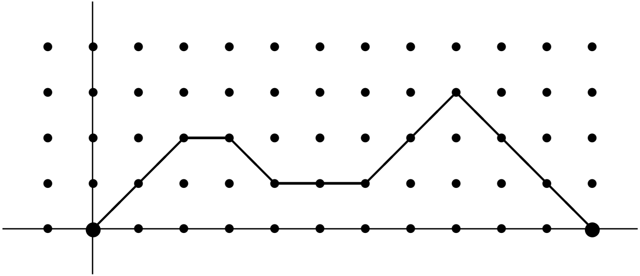

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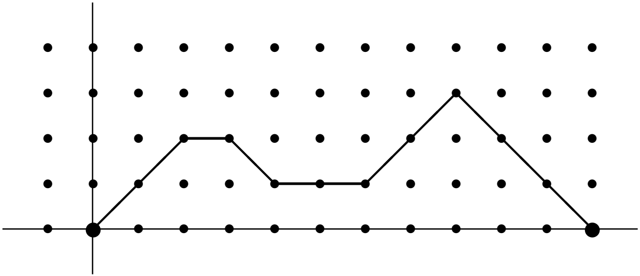

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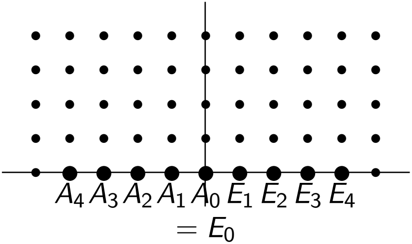

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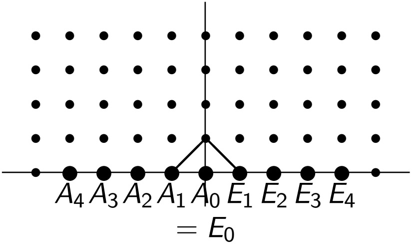

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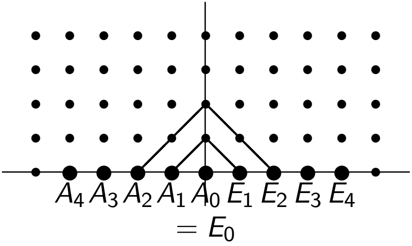

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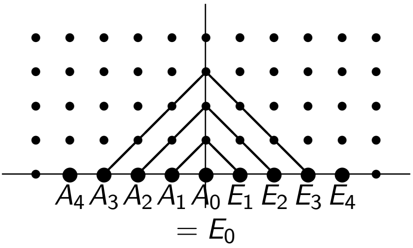

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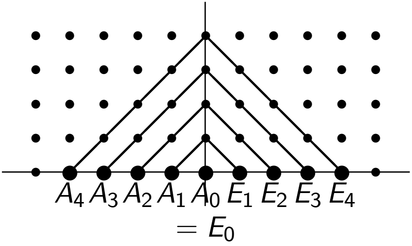

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REMARK. There is actually a closed product formula for $\det_{0\le i,j\le n-1}(C_{x_i+j})$. This is, however, not the case for the Motzkin numbers.

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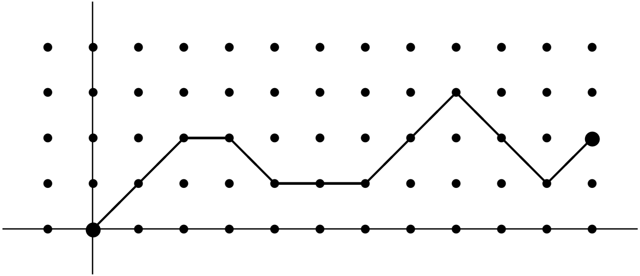

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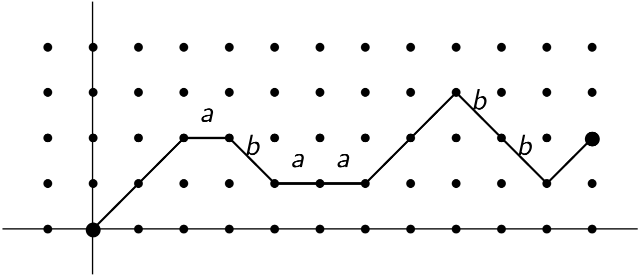

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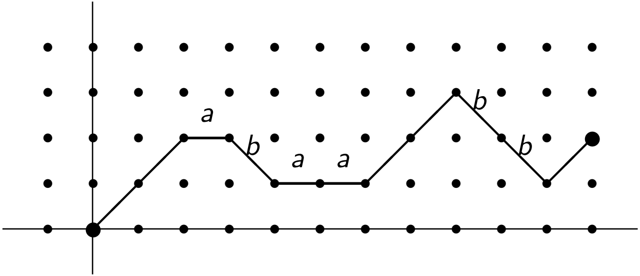


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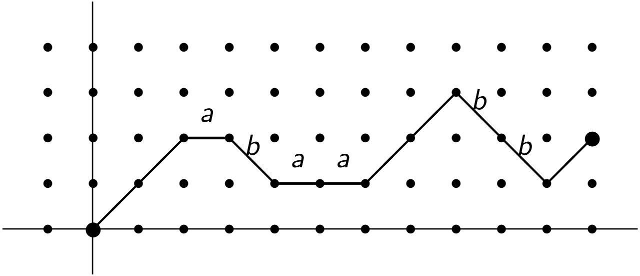


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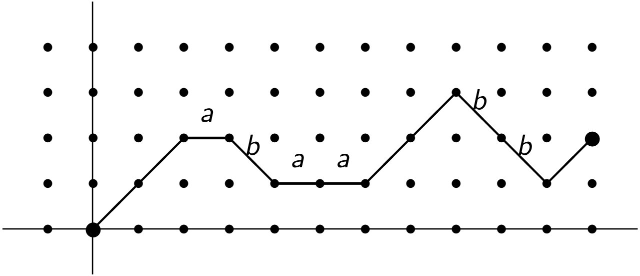


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Furthermore, let $\mathcal{P}_n^+(l, k) = \sum_P w(P)$, where $P$ runs over all Motzkin paths from $(0, l)$ to $(n, k)$.

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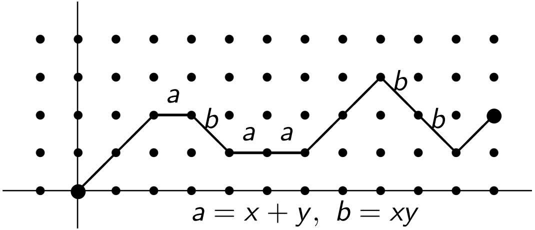
We have

$$C_n = \mathcal{P}^+(0,0)\big|_{a=0,b=1} = \mathcal{P}^+(0,0)\big|_{x=-y=\sqrt{-1}},$$
$$M_n = \mathcal{P}^+(0,0)\big|_{a=b=1} = \mathcal{P}^+(0,0)\big|_{x=y^{-1}=\omega},$$

where $\omega$ is a primitive sixth root of unity.



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In[3]:= D[n_,k_]:=Det[Table[W[i+j,0,k]-
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>   0}
In[6]:= Table[Factor[D[n,2]],{n,1,9}]

                          9  9           27  27
Out[6]= {0, 0, -1, 0, 0, x  y , 0, 0, -(x   y  )}
```

## Theorem

*For all positive integers n und k, we have*

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}^{+}_{i+j}(0,k)\right) = \begin{cases} (-1)^{n_1\binom{k+1}{2}}(xy)^{(k+1)^2\binom{n_1}{2}} & n = n_1(k+1), \\ 0 & n \not\equiv 0 \pmod{k+1}. \end{cases}$$

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## Theorem

*For all positive integers n und k, we have*

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}^{+}_{i+j+1}(0,k)\right) = \begin{cases} (-1)^{n_1\binom{k+1}{2}}(xy)^{(k+1)^2\binom{n_1}{2}}\frac{y^{(k+1)(n_1+1)}-x^{(k+1)(n_1+1)}}{y^{k+1}-x^{k+1}} & n=n_1(k+1),\\ (-1)^{n_1\binom{k+1}{2}+\binom{k}{2}}(xy)^{(k+1)^2\binom{n_1}{2}+n_1k(k+1)} \\ \qquad\times\frac{y^{(k+1)(n_1+1)}-x^{(k+1)(n_1+1)}}{y^{k+1}-x^{k+1}} & n=n_1(k+1)+k,\\ 0 & n\not\equiv 0,k \pmod{k+1}. \end{cases}$$

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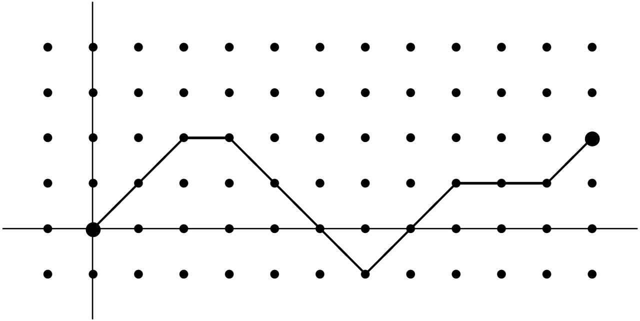
Let us consider *three-step paths*:



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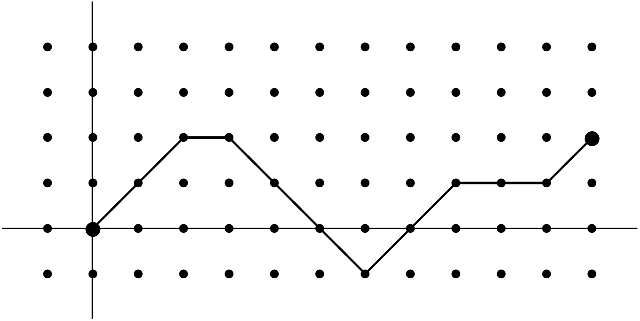

Let $\mathcal{P}_n(l,k)=\sum_P w(P)$, where $P$ runs over all three-step paths from $(0,l)$ to $(n,k)$.

## Theorem

*For all positive integers n and k, we have*

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}_{i+j}(0,k)\right) = \begin{cases} (-1)^{kn_1+\binom{k}{2}}(xy)^{k(n_1-1)(2kn_1-k+1)} & n = 2kn_1 - k + 1, \\ (-1)^{kn_1}(xy)^{kn_1(2kn_1-k-1)} & n = 2kn_1, \\ 0 & n \not\equiv 0, k+1 \pmod{2k}. \end{cases}$$

## Theorem

*For all positive integers n and integers $k \geq 2$, we have*

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}_{i+j+1}(0,k)\right)$$
$$= \begin{cases} (-1)^{k(n_1-1)-1}(xy)^{kn_1(2kn_1-k-3)+k}P_{n-k+2,k}(x,y) & \\ \qquad\qquad n = 2kn_1-1, & \\ (-1)^{kn_1+\binom{k}{2}}(xy)^{k(n_1-1)(2kn_1-k+1)}P_{n,k}(x,y) & \\ \qquad\qquad n = 2kn_1-k+1, & \\ (-1)^{kn_1+\binom{k+1}{2}}(xy)^{k(n_1-1)(2kn_1-k-1)}P_{n-k,k}(x,y) & \\ (-1)^{kn_1}(xy)^{kn_1(2kn_1-k-1)}P_{n,k}(x,y) & \\ 0 & n \not\equiv 0, k, k+1, 2k-1 \pmod{2k}, \end{cases}$$

*where*

$$P_{m,k}(x,y) = \begin{cases} \frac{x^{m+k}+(-1)^{m/k}y^{m+k}}{x^k+y^k} \\ \frac{\left(x^{k\lfloor m/k\rfloor+k}+(-1)^{\lfloor m/k\rfloor}y^{k\lfloor m/k\rfloor+k}\right)\left(x^{m-k\lfloor m/k\rfloor}+(-1)^{\lfloor m/k\rfloor}y^{m-k\lfloor m/k\rfloor}\right)}{x^k+y^k} \end{cases}$$

By specialising the variables $x$ und $y$, one can derive numerous formulae for binomial determinants, determinants of Catalan and ballot numbers, determinants of (generalised) Motzkin numbers.

By specialising the variables $x$ und $y$, one can derive numerous formulae for binomial determinants, determinants of Catalan and ballot numbers, determinants of (generalised) Motzkin numbers. For example:

## Corollary

*For all positive integers $n$ und $k$, we have*

$$\det_{0\le i,j\le n-1}\left(\binom{2i+2j+4}{i+j+k+2}\right) = \begin{cases} (-1)^{n_1 k} & n=2n_1k, \\ (-1)^{n_1k+\binom{k+2}{2}} & n=2n_1k-k-1, \\ 2(-1)^{n_1k+\binom{k+1}{2}}(n+k) & n=2n_1k-k, \\ 2(-1)^{n_1k+k}(n+1) & n=2n_1k-1, \\ 0 & n\not\equiv 0,k-1,k,2k-1 \pmod{2k}, \end{cases}$$

## Corollary

*For all positive integers n and $k \ge 2$, we have*

$$\det_{0\le i,j\le n-1}\left(\sum_{\ell\ge 0}\binom{i+j+1}{\ell,\ell+k}\right)=\begin{cases}(-1)^{kn_1/2} & n=kn_1 \text{ and } k\equiv 0 \pmod 6,\\ (-1)^{\binom{n_1+1}{2}} & n=kn_1 \text{ and } k\equiv 3 \pmod{12},\\ (-1)^{\binom{n_1}{2}} & n=kn_1 \text{ and } k\equiv 9 \pmod{12},\\ (-1)^{kn_1+\binom{k+1}{2}} & n=6kn_1-5k \text{ and } 3\nmid k,\\ 3(-1)^{k(n_1+1)+\lfloor (k+1)/6\rfloor} & n=6kn_1-5k+1 \text{ and } 3\nmid k,\\ 3(-1)^{k(n_1+1)+\lfloor k/3\rfloor} & n=6kn_1-4k-1 \text{ and } 3\nmid k,\\ 2(-1)^{kn_1+1} & n=6kn_1-4k \text{ and } 3\nmid k,\\ 2(-1)^{kn_1+\binom{k}{2}+1} & n=6kn_1-3k \text{ and } 3\nmid k,\\ 3(-1)^{k(n_1+1)+\lfloor (k+4)/6\rfloor} & n=6kn_1-3k+1 \text{ and } 3\nmid k,\\ 3(-1)^{kn_1+\lfloor k/3\rfloor+1} & n=6kn_1-2k-1 \text{ and } 3\nmid k,\\ (-1)^{k(n_1+1)} & n=6kn_1-2k \text{ and } 3\nmid k,\\ (-1)^{kn_1+\binom{k+1}{2}} & n=6kn_1-k \text{ and } 3\nmid k,\\ (-1)^{kn_1} & n=6kn_1 \text{ and } 3\nmid k,\\ 0 & \text{otherwise.}\end{cases}$$

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We show that our determinants can be transformed into equivalent, but more accessible determinants.

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This transformation is most conveniently explained by using non-intersecting paths again.

Let us consider

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}^{+}_{i+j}(0,k)\right).$$

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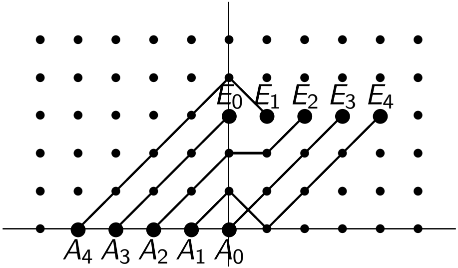
Its combinatorial interpretation in terms of non-intersecting lattice paths is (here, $n=5$, $k=3$):



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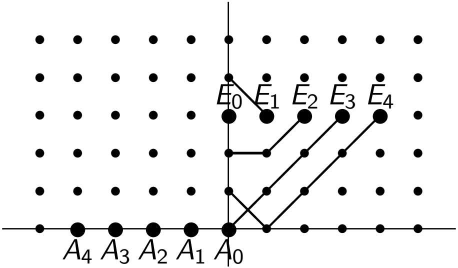

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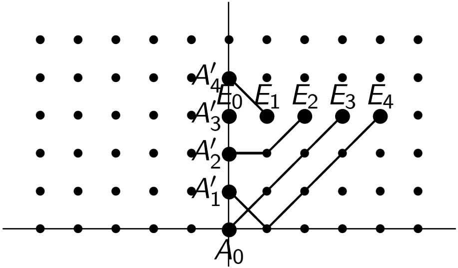

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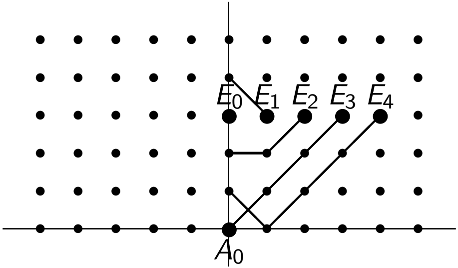


By the (generalised) non-intersecting lattice paths theorem, the weighted generating function is again a determinant:

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}^+_{j}(i,k)\right).$$

Now we make use of the *reflection principle*, which allows us to express the restricted weighted counts $\mathcal{P}^+(l,k)$ in terms of the unrestricted ones:

$$\mathcal{P}_n^+(l,k) = \mathcal{P}_n(l,k) - (xy)^{l+1}\mathcal{P}_n(-l-2,k).$$

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Hence, the determinant in our first theorem equals

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}_j^+(i,k)\right) = \det_{0\le i,j\le n-1}\left(\mathcal{P}_j(i,k) - (xy)^{i+1}\mathcal{P}_j(-i-2,k)\right),$$

and there are similar transformations for all the other determinants.

So, we should prove:

**Theorem**

*For all positive integers $n$ and non-negative integers $k$, we have*

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}_j(i,k) - \ (xy)^{i+1}\mathcal{P}_j(-i-2,k)\right)$$
$$= \begin{cases} \qquad (-1)^{n_1\binom{k+1}{2}}(xy)^{(k+1)^2\binom{n_1}{2}} & n = n_1(k+1), \\ 0 & n \not\equiv 0 \pmod{k+1}. \end{cases}$$

In fact, we can introduce another parameter:

**Theorem**

*For all positive integers n and non-negative integers k, we have*

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}_j(i,k) - (xy)^{i+1}\mathcal{P}_j(-i-2,k)\right)$$
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In fact, we can introduce another parameter:

**Theorem**

*For all positive integers n and non-negative integers k, we have*

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}_j(i,k)-q(xy)^{i+1}\mathcal{P}_j(-i-2,k)\right)$$
$$=\begin{cases} q^{k\lfloor\frac{n_1}{2}\rfloor}(-1)^{n_1\binom{k+1}{2}}(xy)^{(k+1)^2\binom{n_1}{2}} & n=n_1(k+1),\\ 0 & n\not\equiv 0 \pmod{k+1}.\end{cases}$$

And the same works for all the other determinants:

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$$\det_{0\le i,j\le n-1}\left(\mathcal{P}_{j+1}(i,k)-q(xy)^{i+1}\mathcal{P}_{j+1}(-i-2,k)\right)$$
$$=\begin{cases}(-1)^{n_1\binom{k+1}{2}}q^{k\lfloor\frac{n_1}{2}\rfloor}(xy)^{(k+1)^2\binom{n_1}{2}}\\ \qquad\times\sum_{s=0}^{n_1}q^{\min\{s,n_1-s\}}x^{s(k+1)}y^{(n_1-s)(k+1)} & \\ & n=n_1(k+1),\\ (-1)^{n_1\binom{k+1}{2}+\binom{k}{2}}q^{k\lceil\frac{n_1}{2}\rceil}(xy)^{(k+1)^2\binom{n_1}{2}+n_1k(k+1)}\\ \qquad\times\sum_{s=0}^{n_1}q^{\min\{s,n_1-s\}}x^{s(k+1)}y^{(n_1-s)(k+1)} & \\ & n=n_1(k+1)+k,\\ 0 & n\not\equiv 0,k \pmod{k+1}.\end{cases}$$

And the same works for all the other determinants:

**Theorem**

*For all positive integers n and k, we have*

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}_j(i,k)+q(xy)^i\mathcal{P}_j(-i,k)\right)$$
$$=\begin{cases}(-1)^{kn_1+\binom{k}{2}}(1+q)q^{k(n_1-1)}(xy)^{k(n_1-1)(2kn_1-k+1)} \\ \qquad\qquad n=2kn_1-k+1,\\ (-1)^{kn_1}(1+q)q^{kn_1-1}(xy)^{kn_1(2kn_1-k-1)} \\ \qquad\qquad n=2kn_1,\\ 0 \qquad\qquad n\not\equiv 0,k+1 \pmod{2k}.\end{cases}$$

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## Theorem

*For all positive integers n and k, we have*

$$\det_{0\le i,j\le n-1}\left(\mathcal{P}_{j+1}(i,k)+q(xy)^i\mathcal{P}_{j+1}(-i,k)\right)$$

$$=\begin{cases}(-1)^{k(n_1-1)-1}(1+q)q^{kn_1-2}(xy)^{kn_1(2kn_1-k-3)+k}P_{n-k+2,k}(x,y,q) & \\ \qquad n=2kn_1-1, & \\ (-1)^{kn_1+\binom{k}{2}}(1+q)q^{k(n_1-1)}(xy)^{k(n_1-1)(2kn_1-k+1)}P_{n,k}(x,y,q) & \\ \qquad n=2kn_1-k+1, & \\ (-1)^{kn_1+\binom{k+1}{2}}(1+q)q^{k(n_1-1)}(xy)^{k(n_1-1)(2kn_1-k-1)}P_{n-k,k}(x,y,q) & \\ \qquad n=2kn_1-k, & \\ (-1)^{kn_1}(1+q)q^{kn_1-1}(xy)^{kn_1(2kn_1-k-1)}P_{n,k}(x,y,q) & \\ \qquad n=2kn_1, & \\ 0 & n\not\equiv 0,k,k+1,2k-1 \pmod{2k},\end{cases}$$

*where* $P_{m,k}(x,y,q)$

$$=\begin{cases}\sum_{s=0}^{m/k}(-1)^s q^{\min\{s,\frac{m}{k}-s\}}x^{sk}y^{m-sk} & \text{if } m\equiv 0 \pmod{k},\\ \sum_{s=0}^{\lfloor m/k\rfloor}(-1)^s q^{\min\{s,\lceil m/k\rceil-s\}}\left(x^{sk}y^{m-sk}+x^{m-sk}y^{sk}\right) & \text{if } m\not\equiv 0 \pmod{k},\end{cases}$$

# How does one prove these determinant evaluations?

## How does one prove **these** determinant evaluations?

By applying (tedious) row operations, one can convert all these determinants into "saw-tooth" forms:

$$
\det\begin{pmatrix}
0&0&0&*&*&*&*&*&*&*&*&*&*&*&*&*&\\
0&0&*&*&*&*&*&*&*&*&*&*&*&*&*&*&\\
0&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&\\
*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&*&\\
0&0&0&0&0&0&0&*&*&*&*&*&*&*&*&*&\\
0&0&0&0&0&0&*&*&*&*&*&*&*&*&*&*&\\
0&0&0&0&0&*&*&*&*&*&*&*&*&*&*&*&\\
0&0&0&0&*&*&*&*&*&*&*&*&*&*&*&*&\\
0&0&0&0&0&0&0&0&0&0&0&*&*&*&*&*&\\
0&0&0&0&0&0&0&0&0&0&*&*&*&*&*&*&\\
0&0&0&0&0&0&0&0&0&*&*&*&*&*&*&*&\\
0&0&0&0&0&0&0&0&*&*&*&*&*&*&*&*&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&\\
0&0&0&0&0&0&0&0&0&0&0&0&*&*&*&*&\\
&&&&&&&&&&&&&&&&\ddots
\end{pmatrix}
$$

## How does one prove these determinant evaluations?

In particular, if the determinant should not have the "right" size, it will vanish:

$$\det\left(\begin{array}{ccccccccccccccc}
0&0&0&*&*&*&*&*&*&*&*&*&*&*&*\\
0&0&*&*&*&*&*&*&*&*&*&*&*&*&*\\
0&*&*&*&*&*&*&*&*&*&*&*&*&*&*\\
*&*&*&*&*&*&*&*&*&*&*&*&*&*&*\\
0&0&0&0&0&0&0&*&*&*&*&*&*&*&*\\
0&0&0&0&0&0&*&*&*&*&*&*&*&*&*\\
0&0&0&0&0&*&*&*&*&*&*&*&*&*&*\\
0&0&0&0&*&*&*&*&*&*&*&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&*&*&*&*&*\\
0&0&0&0&0&0&0&0&0&*&*&*&*&*&*\\
0&0&0&0&0&0&0&0&*&*&*&*&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&*\\
0&0&0&0&0&0&0&0&0&0&0&0&0&*&*
\end{array}\right)$$

## How does one prove these determinant evaluations?

If the determinant does have the “right” size, then it is simply equal to the product of the left-most entries in each row:

$$
\det \begin{pmatrix}
0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & *
\end{pmatrix}
$$

# Open Questions

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(2) Is there a connection with symplectic and orthogonal characters?

## Symplectic (and orthogonal) characters

# Symplectic (and orthogonal) characters

$$
\begin{aligned}
&sp_\lambda(x_1, x_2, \dots, x_n) \\
&= \det_{1 \le i,j \le \lambda_1} \left( e_{\lambda'_i - i + j}(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}) - e_{\lambda'_i - i - j}(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}) \right),
\end{aligned}
$$

## Symplectic (and orthogonal) characters

$$sp_\lambda(x_1, x_2, \dots, x_n)$$
$$= \det_{1\le i,j\le \lambda_1} \left(e_{\lambda'_i - i + j}(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}) - e_{\lambda'_i - i - j}(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1})\right),$$

$$sp_\lambda(x_1, x_2, \dots, x_n)$$
$$= \frac{1}{2} \det_{1\le i,j\le n} \left(h_{\lambda_i - i + j}(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}) + h_{\lambda_i - i - j + 2}(x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1})\right),$$

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$$\det_{0\le i,j\le n-1} \left(\mathcal{P}_j(i,k) - q(xy)^{i+1}\mathcal{P}_j(-i-2,k)\right)$$

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$$\det_{0\le i,j\le n-1} \left( \mathcal{P}_j(i,k) - q(xy)^{i+1} \mathcal{P}_j(-i-2,k) \right)$$

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$$\det_{0\le i,j\le n-1} \left( \mathcal{P}_j(i,k) + q(xy)^{i} \mathcal{P}_j(-i,k) \right)$$

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(3) Is it possible to put this under one roof with determinant evaluations of Eğecioğlu, Redmond and Ryavec?

# The determinants of Eğecioğlu, Redmond and Ryavec

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For example, they prove

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As specialisations of our results, we obtain determinant evaluations for

$$\det_{0\le i,j\le n-1}\left(\binom{2i+2j+4}{i+j+k+2}\right),$$

for example.