Domino tilings of generalised Aztec triangles and determinant evaluations

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Universität Wien

Overview

Overview

- ENUMERATION OF DOMINO TILINGS OF GENERALISED AZTEC TRIANGLES
  (joint work with *Sylvie Corteel* and *Frederick Huang*)
- DETERMINANT EVALUATIONS
  (joint work with *Christoph Koutschan* and *Michael Schlosser*)

Di Francesco's determinant for 20V configurations

In 2021, in the context of counting certain configurations in the 20-vertex model, Di Francesco came up with the following conjecture:

Conjecture

For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(2^i\binom{i+2j+1}{2j+1}+\binom{-i+2j+1}{2j+1}\right)\\ =2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{(4i+2)!}{(n+2i+1)!}.$$

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More precisely, Di Francesco observed (and showed) that the number of domino tilings of certain regions that he called AZTEC TRIANGLES is the same as the number of these 20-vertex configurations. Furthermore, he proved that the number of domino tilings is given by one half of the above determinant.

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Di Francesco's determinant, plus a generalisation

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$$=2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1},$$

where $(\alpha)_m := \alpha(\alpha+1)\cdots(\alpha+m-1)$ *for* $m\ge 1$, *and* $(\alpha)_0 := 1$.

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$$=2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i)!}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x+4i+3)_{n-2i-1}$$
$$\times\prod_{i=0}^{\lfloor (n-2)/2\rfloor}(x-2i+3n-1)_{n-2i-2},$$

where $(\alpha)_m := \alpha(\alpha+1)\cdots(\alpha+m-1)$ *for* $m\ge 1$, *and* $(\alpha)_0 := 1$.

Di Francesco's determinant, plus a generalisation

Another CK (Christoph Koutschan) proved Di Francesco's determinant evaluation using Zeilberger's holonomic Ansatz (and heavy computer calculations). However, today's computers seem to be not powerful enough to prove the conjectures for the "x-determinants."

An independent (?) development

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SYLVIE CORTEEL: *I want to count super symplectic tableaux of triangular shape. The computer says that there is a nice product formula for them.*

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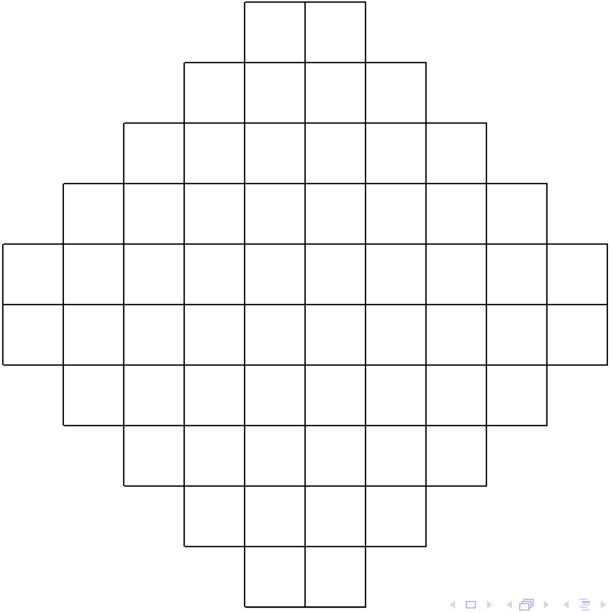
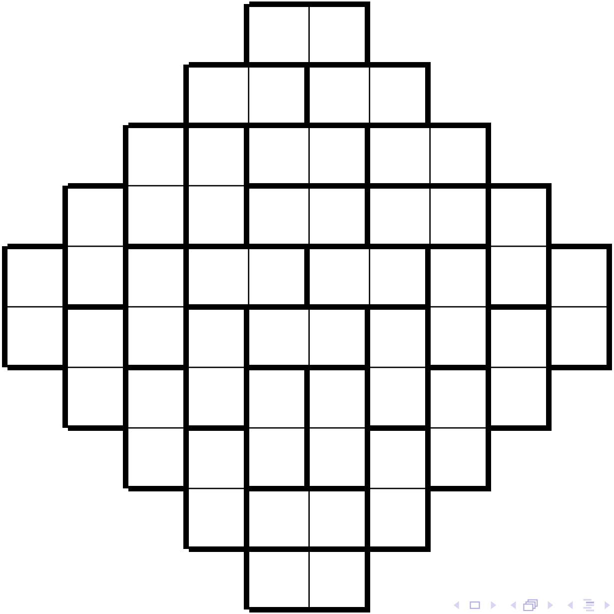
The Aztec diamond



Domino tilings of the Aztec diamond



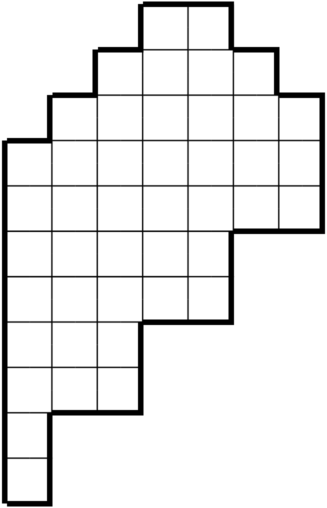
The Aztec diamond theorem

Theorem (Elkies, Kuperberg, Larsen, Propp 1992)

The number of domino tilings of the Aztec diamond of size n is

$$2^{\binom{n+1}{2}}.$$

The Aztec triangle of Di Francesco



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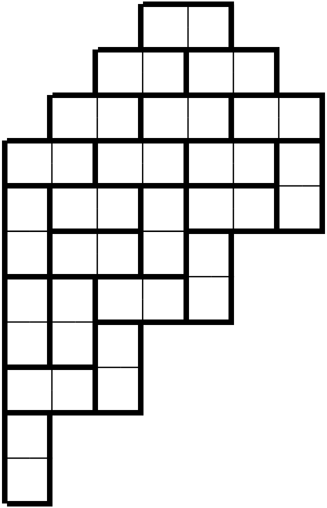

The Aztec triangle of Di Francesco

Theorem (Di Francesco + Koutschan)

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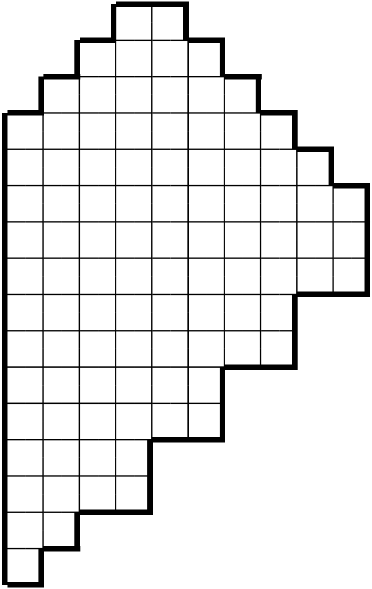
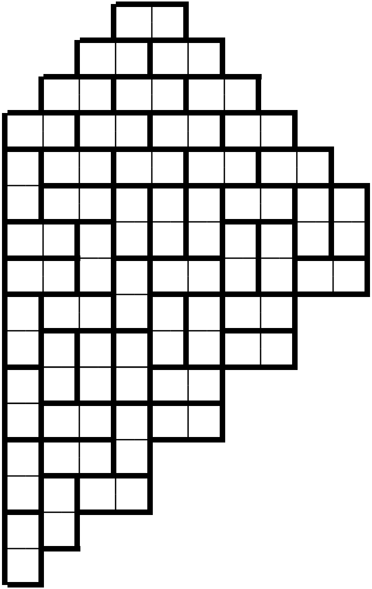


(Sylvie Corteel and Frederick Huang)

A generalised Aztec triangle



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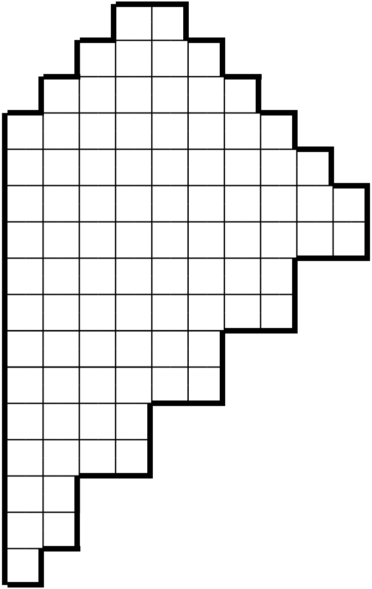

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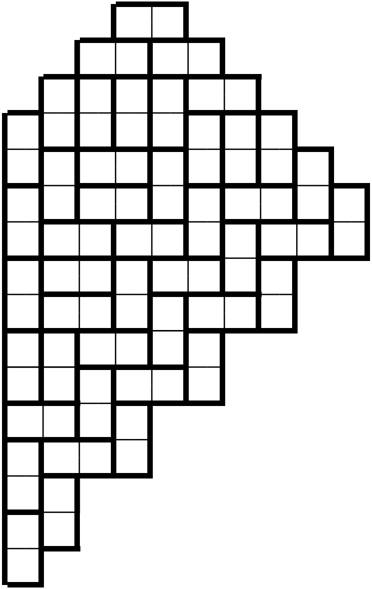

(Sylvie Corteel and Frederick Huang)

Enumeration of generalised Aztec triangles

Conjecture (Corteel, Huang)

The number of domino tilings of the (n,k)-Aztec triangle of type I is

$$\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Big/\prod_{i=1}^{k-1}(2i+1)^{k-i}\,.$$

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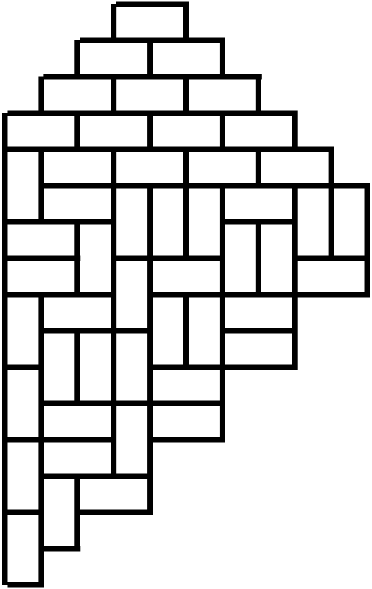
Conjecture (Corteel, Huang)

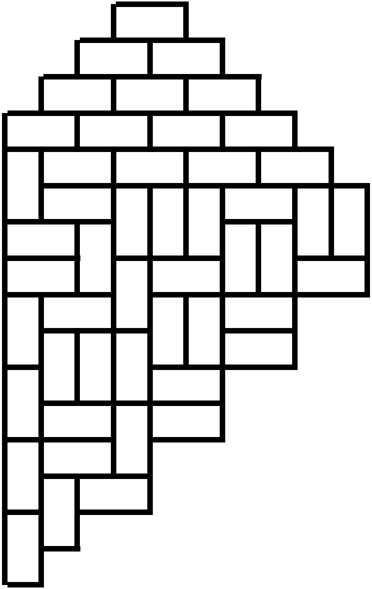
The number of domino tilings of the (n,k)-Aztec triangle of type II is

$$\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$

Generalised Aztec triangles and non-intersecting paths



Generalised Aztec triangles and NILPs



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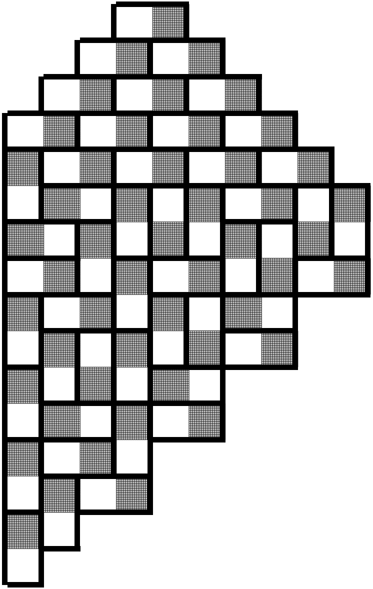

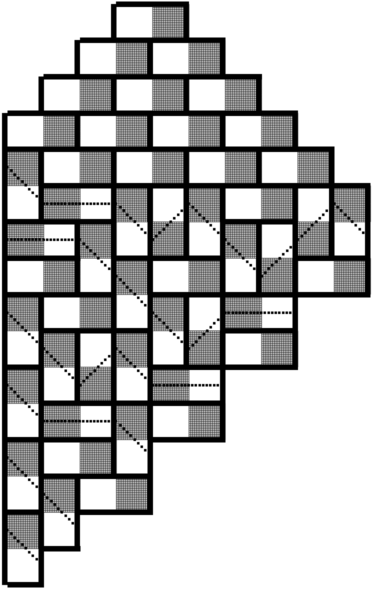

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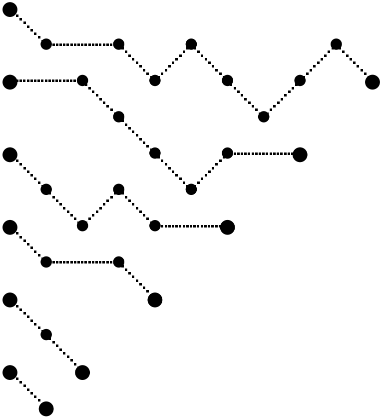

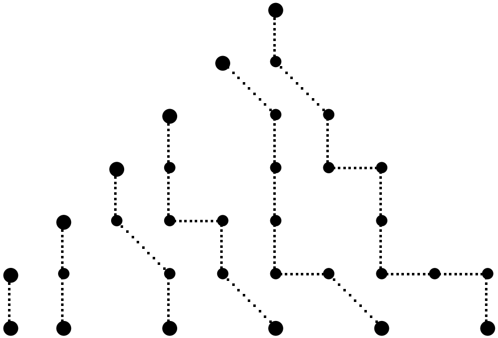

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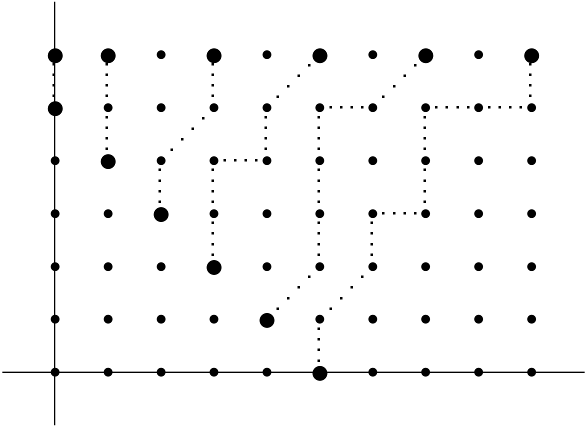

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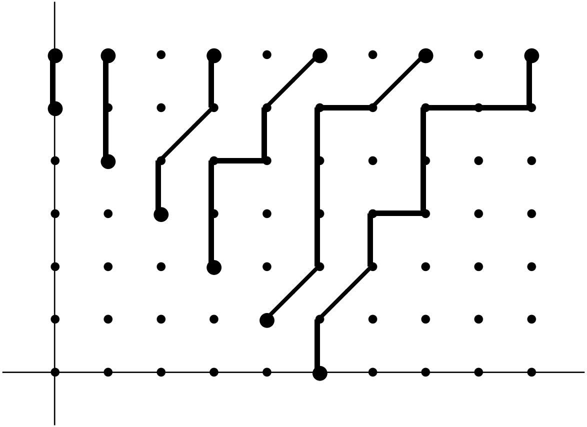

Generalised Aztec triangles and NILPs

Hence, by the Lindström–Gessel–Viennot Theorem for NILPs: *The number of domino tilings of the (n, k)-Aztec triangle of type I equals* $\det D_1(k; n)$, *where*

$$D_1(k; n) = (D(2j - i, i + n - k - 1))_{1 \le i,j \le k},$$

with $D(m, n)$ a Delannoy number, i.e., the number of paths from $(0, 0)$ to (m, n) consisting of steps $(1, 0)$, $(0, 1)$, and $(1, 1)$.

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Furthermore, the number of domino tilings of the (n,k)-Aztec triangle of type II equals $\det D_2(k;n)$, *where*

$$D_2(k;n) = \left(D(2j-i, i+n-k-1) + D(2j-i-1, i+n-k-1)\right)_{1\le i,j\le k}.$$

A determinant evaluation

We need to show:

$$\det\left(D(2j-i,i+n-k-1)\right)_{1\le i,j\le k}$$
$$=\prod_{i\ge0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i},$$

and also:

$$\det\left(D(2j-i,i+n-k-1)+D(2j-i-1,i+n-k-1)\right)_{1\le i,j\le k}$$
$$=\prod_{i\ge0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$

We have

$$D(m,n) = \langle u^m v^n \rangle \frac{1}{1-u-v-uv}$$
$$= \sum_{\ell=0}^{m} \binom{m}{\ell} \binom{n}{\ell} 2^{\ell}$$
$$= \sum_{\ell=0}^{m} \binom{m+n-\ell}{m-\ell,\, n-\ell,\, \ell}.$$

In particular, $D(m,n)$ is a polynomial in n of degree m.

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Our determinant

Here is the first determinant evaluation, rewritten:

$$\det D_1(k;n) = \det\big(D(k-2i+j, n-j-1)\big)_{0\le i,j\le k-1}$$
$$= 2^{k^2}\prod_{i=1}^{k}\frac{1}{(i)_i}\prod_{s=0}^{k-2}(n-s-1)^{\min\{\lfloor(s+1+\chi(k\text{ even}))/2\rfloor,\lfloor(k-s)/2\rfloor\}}$$
$$\cdot\prod_{s=0}^{k-1}(n-s-\tfrac{1}{2})^{\min\{\lfloor(s+1+\chi(k\text{ odd}))/2\rfloor,\lfloor(k-s+1)/2\rfloor\}}$$
$$\cdot\prod_{s=0}^{k-2}(n+k-s-1)^{\min\{\lfloor(s+2)/2\rfloor,\lfloor(k-s-\chi(k\text{ odd}))/2\rfloor\}}$$
$$\cdot\prod_{s=1}^{k-2}(n+k-s-\tfrac{1}{2})^{\min\{\lfloor(s+1)/2\rfloor,\lfloor(k-s-\chi(k\text{ even}))/2\rfloor\}}.$$

Here, $\chi(\mathcal{S})=1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S})=0$ otherwise.

How to prove such a determinant evaluation?

My favourite method:
Identification of factors/Exhaustion of factors

This method only works if there is a free parameter available, and if the determinant is a polynomial in this free parameter.

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(1) First one shows that each of the claimed factors is indeed a factor of the determinant.
(2) One shows that the degree of the conjectured evaluation is at least as large as the degree of the determinant.

    At this point one has shown that the determinant is equal to the conjectured evaluation up to a multiplicative constant.
(3) Evaluation of the multiplicative constant.

The proof

Most of the steps work out quite smoothly. The most intricate is the verification that the last factor,

$$\prod_{s=1}^{k-2}\left(n+k-s-\tfrac{1}{2}\right)^{\min\{\lfloor (s+1)/2\rfloor,\lfloor (k-s-\chi(k\text{ even}))/2\rfloor\}},$$

divides the determinant.

The proof

Among others, one has to show that

$$\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-2i}\frac{(4i-4s+1)\,(-1)^{s-1}(1-i)_{s-1}\,(\frac{1}{2})_s\,(\frac{1}{2})_{i-s}}{(2i-4s+1)\,i!\,(s-1)!}$$

$$\cdot\binom{k-2i}{\ell}\binom{-k+2s-\frac{3}{2}}{\ell}2^\ell$$

$$+\sum_{r=1}^{s}\sum_{\ell=0}^{k-2r-2s+2}(-1)^k 2^{k-2r-2s+3-\ell}\frac{2^{4r-3}(2r-\frac{1}{2})_{s-r}}{(s-r)!}$$

$$\cdot\frac{(k-2r-2s+1)!\,(k+2r-2s-1)!}{\ell!^2\,(k-2r-2s+2-\ell)!\,(k+2r-2s-\ell)!}$$

$$\cdot(-\ell^2+2k+k^2+4r-4r^2-4s-4ks+4s^2)=0,$$

for all integers k and s with $0<s$ and $k\ge 4s-1$.

The proof

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THE SHORT VERSION: Using Christoph Koutschan's *Mathematica* package `HolonomicFunctions`, this is a routine task.

The number of domino tilings of the generalised Aztec triangle of type I

Theorem

The number of domino tilings of the (n, k)-Aztec triangle of type I is

$$\prod_{i\ge0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}\,.$$

The second determinant

Recall that there was also the determinant of

$$D_2(k;n) \\ = \big(D(2j-i,i+n-k-1)+D(2j-i-1,i+n-k-1)\big)_{1\le i,j\le k}\,.$$

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Lemma

We have

$$\det D_1(k;n+\tfrac{1}{2}) = \det D_2(k;n).$$

Proof.

In fact, we have

$$D_1(k;n+\tfrac{1}{2}) = \left(\frac{(\frac{1}{2})_{j-i}}{(j-i)!}\right)_{0\le i,j\le k-1} \cdot D_2(k;n).$$

□

The number of domino tilings of the generalised Aztec triangle of type II

Theorem

The number of domino tilings of the (n, k)-Aztec triangle of type II is

$$\prod_{i\geq 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$

The “x-determinant”

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Back to:

Conjecture

For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)$$
$$\overset{?}{=} 2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1}.$$

The relation between two determinants

Define

$$D_1(k;n) := \det\left(D(2j-i, i+n-k-1)\right)_{1\le i,j\le k},$$

or, after the shifts $i \mapsto i+1$ and $j \mapsto j+1$, equivalently

$$D_1(k;n) := \det\left(D(2j-i+1, i+n-k)\right)_{0\le i,j\le k-1},$$

Furthermore, define

$$D_3(k;x) := \det_{0\le i,j\le k-1}\left(2^i\binom{x+i+2j+1}{2j+1} + \binom{x-i+2j+1}{2j+1}\right).$$

Theorem

We have

$$D_1(k;y+k) = \tfrac{1}{2}D_3(k;2y).$$

The relation between two determinants

For the proof, we use an idea of Di Francesco: consider matrices

$$A(n) := (a_{i,j})_{0\le i,j\le n-1}$$

with the entries $a_{i,j}$ given by their bivariate generating function

$$a(u,v) = \sum_{i,j\ge 0} a_{i,j} u^i v^j .$$

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We want to transform the determinant(s) $\det A(n)$ into different forms. We do this by working with the generating function $a(u,v)$ instead of the determinant $\det A(n)$ itself.

The relation between two determinants

For the proof, we use an idea of Di Francesco: consider matrices

$$A(n) := (a_{i,j})_{0 \le i,j \le n-1}$$

with the entries $a_{i,j}$ given by their bivariate generating function

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We want to transform the determinant(s) $\det A(n)$ into different forms. We do this by working with the generating function $a(u,v)$ instead of the determinant $\det A(n)$ itself.

The following two operations on the generating function do not change the value of the corresponding determinant:

- Multiplication of $a(u,v)$ by a power series in u or by a power series in v with constant coefficient 1;
- Replacement of u a power series in u with zero constant coefficient and coefficient of u equal to 1.

The relation between two determinants

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Hence:

$\longrightarrow$ Need to compute the bivariate generating function for the entries of $D_1(k; n)$;
$\longrightarrow$ Need to compute the bivariate generating function for the entries of $D_3(k; x)$;
$\longrightarrow$ Need to transform one generating function into the other by operations of the above type.

The generating function for $D_1(k;n)$

Recall:

$$D_1(k;n) := \det\left(D(2j-i+1, i+n-k)\right)_{0\le i,j\le k-1},$$

The generating function for $D_1(k;n)$

Recall:

$$D_1(k;n) := \det\left(D(2j-i+1, i+n-k)\right)_{0\le i,j\le k-1},$$

Write $N := n-k$.

After some computation, we obtain

$$\sum_{i,j\ge 0} D(2j-i+1, N+i)u^i v^j = \frac{1}{2\sqrt{v}(1-v-4uv-u^2v+u^2v^2)} \cdot \left(\left(\frac{1+\sqrt{v}}{1-\sqrt{v}}\right)^N (1-uv+\sqrt{v}(1+u)) - \left(\frac{1-\sqrt{v}}{1+\sqrt{v}}\right)^N (1-uv-\sqrt{v}(1+u)\right).$$

The generating function for $D_3(k;n)$

Recall:

$$D_3(k;x) := \det_{0\le i,j\le k-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right).$$

After some computation, we obtain

$$\sum_{i,j\ge 0}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)u^i v^j$$
$$=\frac{1}{2\sqrt{v}}\left((1-\sqrt{v})^{-x}\frac{1-2u+\sqrt{v}}{(1-2u)^2-v}-(1+\sqrt{v})^{-x}\frac{1-2u-\sqrt{v}}{(1-2u)^2-v}\right.$$
$$\left.+(1-\sqrt{v})^{-x}\frac{u(1-u-u\sqrt{v})}{(1-u)^2-u^2v}-(1+\sqrt{v})^{-x}\frac{u(1-u+u\sqrt{v})}{(1-u)^2-u^2v}\right).$$

The relation between two determinants

Here are the two generating functions (we must set $x = 2N$):

$$\sum_{i,j\ge 0} D(2j-i+1, N+i)u^i v^j = \frac{1}{2\sqrt{v}(1-v-4uv-u^2v+u^2v^2)}$$
$$\cdot\left(\left(\frac{1+\sqrt{v}}{1-\sqrt{v}}\right)^N (1-uv+\sqrt{v}(1+u)\right.$$
$$\left.-\left(\frac{1-\sqrt{v}}{1+\sqrt{v}}\right)^N (1-uv-\sqrt{v}(1+u)\right).$$

$$\sum_{i,j\ge 0}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)u^i v^j$$
$$=\frac{1}{2\sqrt{v}}\left((1-\sqrt{v})^{-x}\frac{1-2u+\sqrt{v}}{(1-2u)^2-v}-(1+\sqrt{v})^{-x}\frac{1-2u-\sqrt{v}}{(1-2u)^2-v}\right.$$
$$\left.+(1-\sqrt{v})^{-x}\frac{u(1-u-u\sqrt{v})}{(1-u)^2-u^2v}-(1+\sqrt{v})^{-x}\frac{u(1-u+u\sqrt{v})}{(1-u)^2-u^2v}\right).$$

The relation between two determinants

Now do the following with the first generating function:

- Multiply it by $(1-v)^{-N}$;
- Do the substitution $u \mapsto \frac{u}{(1-u)(1-2u)}$;
- Multiply the result by $\frac{1-2u^2}{(1-u)(1-2u)}$.

After having done these operations, the resulting bivariate generating function agrees with the second.

The "x-determinants": proved

Theorem

For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)$$
$$=2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1}.$$

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and

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j}{2j}+\binom{x-i+2j}{2j}\right)$$
$$=2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i)!}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x+4i+3)_{n-2i-1}$$
$$\times\prod_{i=0}^{\lfloor (n-2)/2\rfloor}(x-2i+3n-1)_{n-2i-2}.$$

Recall:

Theorem

For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)$$
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Determinant evaluations: variations on a theme

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Consider the determinants

$$\det_{0\le i,j\le n-1}\left(A^{i+\beta}\binom{i+Bj+\gamma}{Bj+\alpha}+\binom{-i+Bj+\delta}{Bj+\alpha}\right).$$

Are there other instances of "nice" evaluations?

Determinant evaluations: variations on a theme; $A = B = 2$

Let

$$D_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(2^{i+\beta}\binom{i+2j+\gamma}{2j+\alpha}+\binom{-i+2j+\delta}{2j+\alpha}\right).$$

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In this notation, the previous two determinant evaluations read as follows.

Theorem

For all positive integers n, we have

$$D_{1,0,x+1,x+1}(n) = 2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}$$
$$\times\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1}.$$

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In this notation, the previous two determinant evaluations read as follows.

Theorem

For all positive integers n, we have

$$D_{0,0,x,x}(n) = 2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i)!}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x+4i+3)_{n-2i-1}\times\prod_{i=0}^{\lfloor (n-2)/2\rfloor}(x-2i+3n-1)_{n-2i-2}.$$

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In the search space

$$\left\{(\alpha,\beta,\gamma,\delta) : -6 \le \alpha,\beta \le 9 \text{ and } -9 \le \gamma,\delta \le 9\right\}$$

we looked for further "nice" evaluations.

Determinant evaluations: variations on a theme; $A = B = 2$

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Theorem

The following determinant evaluations hold for all $n \geq 1$:

$$D_{-2,0,-1,-1}(n) = -2\prod_{i=2}^{n}\frac{8(2i-3)(2i-1)\,\Gamma(4i-5)\,\Gamma\left(\frac{i+1}{2}\right)}{i\,\Gamma(3i-2)\,\Gamma\left(\frac{3i-3}{2}\right)},$$

$$D_{0,2,3,-1}(n) = \prod_{i=1}^{n}\frac{3(2i-1)\,\Gamma(4i+3)\,\Gamma\left(\frac{i+1}{2}\right)}{4(i+2)\,\Gamma(3i+1)\,\Gamma\left(\frac{3i+5}{2}\right)},$$

$$D_{1,1,0,-2}(n) = -2\prod_{i=1}^{n}\frac{(2i-1)\,\Gamma(4i-3)\,\Gamma\left(\frac{i}{2}\right)}{2\,\Gamma(3i-2)\,\Gamma\left(\frac{3i}{2}\right)},$$

Determinant evaluations: variations on a theme; $A = B = 2$

$$D_{1,1,1,-1}(n) = \prod_{i=1}^{n} \frac{\Gamma\left(4i-1\right)\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(3i\right)\Gamma\left(\frac{3i-1}{2}\right)},$$

$$D_{2,1,2,0}(n) = \prod_{i=1}^{n} \frac{\Gamma\left(4i\right)\Gamma\left(\frac{i+2}{2}\right)}{\Gamma\left(3i\right)\Gamma\left(\frac{3i+2}{2}\right)},$$

$$D_{0,1,1,-1}(n) = 3\prod_{i=2}^{n} \frac{\Gamma\left(4i\right)\Gamma\left(\frac{i-1}{2}\right)}{\Gamma\left(3i+1\right)\Gamma\left(\frac{3i-3}{2}\right)}.$$

Moreover, some related determinants can be expressed in terms of these; the following identities hold (at least) for all $n \geq 4$:

$$D_{2,1,2,0}(n) = \frac{1}{8}\,D_{1,1,-1,-3}(n+1) = \frac{1}{40}\,D_{0,1,-4,-6}(n+2)$$
$$= -\frac{1}{24576}\,D_{1,2,-4,-8}(n+2),$$

Determinant evaluations: variations on a theme; $A = B = 2$

$$\begin{aligned}
D_{1,1,1,-1}(n) &= D_{2,1,1,-1}(n) = \frac{1}{3}\, D_{0,1,-2,-4}(n+1) \\
&= -\frac{1}{32}\, D_{1,1,-2,-4}(n+1) = -\frac{1}{224}\, D_{1,2,-2,-6}(n+1) \\
&= -\frac{1}{168}\, D_{0,1,-5,-7}(n+2) = -\frac{1}{3696}\, D_{0,2,-5,-9}(n+2) \\
&= -\frac{1}{337920}\, D_{1,2,-5,-9}(n+2), \\
D_{1,1,0,-2}(n) &= \frac{1}{5}\, D_{0,1,-3,-5}(n+1) = \frac{1}{1008}\, D_{1,2,-3,-7}(n+1), \\
D_{-2,1,0,-2}(n) &= D_{0,2,3,-1}(n-1), \\
D_{2,1,1,-1}(n) &= D_{4,2,4,0}(n-1), \\
D_{1,1,-2,-4}(n) &= -\frac{16}{5}\, D_{3,2,1,-3}(n-1) = \frac{64}{3}\, D_{5,3,4,-2}(n-2) \\
&= -128\, D_{7,4,7,-1}(n-3),
\end{aligned}$$

Determinant evaluations: variations on a theme; $A = B = 2$

$$D_{1,1,-1,-3}(n) = -4\, D_{3,2,2,-2}(n-1) = 16\, D_{5,3,5,-1}(n-2),$$
$$D_{1,1,0,-2}(n) = -2\, D_{3,2,3,-1}(n-1).$$

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$$D_{1,1,0,-2}(n) = -2\,D_{3,2,3,-1}(n-1).$$

Proof.

All these determinant evaluations can be proved by means of Zeilberger's holonomic Ansatz. □

Determinant evaluations: variations on a theme; $A = B = 3$

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Let

$$E_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(3^{i+\beta}\binom{i+3j+\gamma}{3j+\alpha} + \binom{-i+3j+\delta}{3j+\alpha}\right).$$

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In the search space

$$\{(\alpha,\beta,\gamma,\delta) : -6 \le \alpha,\beta \le 6 \text{ and } -8 \le \gamma,\delta \le 8\}$$
$$\cup\{(\alpha,\beta,\gamma,\delta) : 6 \le \alpha \le 10 \text{ and } 0 \le \beta \le 10 \text{ and } -10 \le \gamma,\delta \le 10\}$$

we looked for "nice" evaluations.

Determinant evaluations: variations on a theme; $A = B = 3$

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Theorem

The following determinant evaluations hold for all $n \ge 1$:

$$E_{-3,0,-1,-1}(n) = 2\prod_{i=2}^{n}\frac{2^{i+1}(2i-1)\,\Gamma(4i-5)\,\Gamma\left(\frac{i+2}{3}\right)}{i(i+1)\,\Gamma(3i-5)\,\Gamma\left(\frac{4i-1}{3}\right)},$$

$$E_{-3,1,0,-2}(n) = -2\prod_{i=2}^{n}\frac{2^{i+1}(2i-1)\,\Gamma(4i-4)\,\Gamma\left(\frac{i}{3}\right)}{i(i+1)^2\,\Gamma(3i-5)\,\Gamma\left(\frac{4i-3}{3}\right)},$$

$$E_{0,3,5,-1}(n) = \prod_{i=1}^{n}\frac{2^{i+1}(3i-2)(3i-1)\,\Gamma(4i+4)\,\Gamma\left(\frac{i+2}{3}\right)}{(i+1)(i+2)(i+3)(i+4)\,\Gamma(3i+1)\,\Gamma\left(\frac{4i+5}{3}\right)},$$

Determinant evaluations: variations on a theme; $A = B = 3$

$$E_{0,1,1,-1}(n) = \prod_{i=1}^{n} \frac{2^{i+1}\,\Gamma\left(4i-2\right)\Gamma\left(\frac{i+2}{3}\right)}{i\,\Gamma\left(3i-2\right)\Gamma\left(\frac{4i-1}{3}\right)},$$

$$E_{1,1,2,0}(n) = \prod_{i=1}^{n} \frac{2^{i}\,\Gamma\left(4i\right)\Gamma\left(\frac{i+1}{3}\right)}{3i\,\Gamma\left(3i-1\right)\Gamma\left(\frac{4i+1}{3}\right)},$$

$$E_{3,2,3,-1}(n) = \prod_{i=1}^{n} \frac{2^{i}\,\Gamma\left(4i+1\right)\Gamma\left(\frac{i+2}{3}\right)}{\Gamma\left(3i+1\right)\Gamma\left(\frac{4i+2}{3}\right)},$$

$$E_{1,0,1,1}(n) = 2\prod_{i=1}^{n} \frac{2^{i-2}\,\Gamma\left(4i-1\right)\Gamma\left(\frac{i}{3}\right)}{3\,\Gamma\left(3i-1\right)\Gamma\left(\frac{4i}{3}\right)},$$

$$E_{2,0,2,2}(n) = 2\prod_{i=1}^{n} \frac{2^{i-3}\,\Gamma\left(4i+1\right)\Gamma\left(\frac{i+2}{3}\right)}{\Gamma\left(3i+1\right)\Gamma\left(\frac{4i+2}{3}\right)}.$$

Determinant evaluations: variations on a theme; $A = B = 3$

Moreover, some related determinants can be expressed in terms of these; the following identities hold (at least) for all $n \geq 3$:

$$
\begin{aligned}
E_{0,0,0,0}(n) &= \frac{1}{2}\, E_{0,1,-1,-3}(n) = \frac{1}{5}\, E_{0,2,-2,-6}(n),\\
E_{1,0,1,1}(n) &= -\frac{1}{84}\, E_{1,3,-2,-8}(n) = 2\, E_{4,2,4,0}(n-1)\\
&= \frac{6}{5}\, E_{4,3,3,-3}(n-1),\\
E_{2,0,2,2}(n) &= 2\, E_{5,2,5,1}(n-1) = 18\, E_{8,4,8,0}(n-2)\\
&= \frac{162}{5}\, E_{8,5,7,-3}(n-2),\\
E_{-3,2,1,-3}(n) &= E_{0,3,5,-1}(n-1),\\
E_{0,1,-1,-3}(n) &= 4\, E_{3,2,3,-1}(n-1),\\
E_{1,1,0,-2}(n) &= -2\, E_{4,2,4,0}(n-1),
\end{aligned}
$$

Determinant evaluations: variations on a theme; $A = B = 3$

$$E_{1,2,-1,-5}(n) = -12\, E_{4,3,3,-3}(n-1) = -180\, E_{7,4,7,-1}(n-2),$$
$$E_{2,1,1,-1}(n) = E_{5,2,5,1}(n-1),$$
$$E_{2,2,0,-4}(n) = \frac{15}{2}\, E_{5,3,4,-2}(n-1) = -45\, E_{8,4,8,0}(n-2),$$
$$E_{2,3,-1,-7}(n) = 36\, E_{5,4,3,-5}(n-1) = -\frac{13608}{5}\, E_{8,5,7,-3}(n-2).$$

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Proof.

All these determinant evaluations can be proved by means of Zeilberger's holonomic Ansatz. □

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$$\begin{aligned} E_{1,2,-1,-5}(n) &= -12\, E_{4,3,3,-3}(n-1) = -180\, E_{7,4,7,-1}(n-2), \\ E_{2,1,1,-1}(n) &= E_{5,2,5,1}(n-1), \\ E_{2,2,0,-4}(n) &= \frac{15}{2}\, E_{5,3,4,-2}(n-1) = -45\, E_{8,4,8,0}(n-2), \\ E_{2,3,-1,-7}(n) &= 36\, E_{5,4,3,-5}(n-1) = -\frac{13608}{5}\, E_{8,5,7,-3}(n-2). \end{aligned}$$

Proof.

All these determinant evaluations can be proved by means of Zeilberger's holonomic Ansatz. □

A closer analysis of these individual evaluations plus some further experimental computations leads us to the following conjecture.

Conjecture

Let

$$\Xi(x) := \prod_{i=2}^{x} \frac{3\,\Gamma(i)\,\Gamma(4i-3)\,\Gamma(4i-2)}{2\,\Gamma(3i-2)^2\,\Gamma(3i-1)} \text{ and } \mu_m(x) := \begin{cases} 2, & \text{if } 3 \mid (x-m), \\ 1, & \text{otherwise.} \end{cases}$$

Then, for all non-negative integers x and for all $n \ge x$, we have

$$E_{0,x,-x,-3x}(n) = 2\mu_1(x)\,\Xi(x)\,(-1)^{\lfloor \frac{x}{3} \rfloor} \prod_{i=1}^{n} \frac{2^{i-1}\,\Gamma(4i-3)\,\Gamma\left(\frac{i+1}{3}\right)}{\Gamma(3i-2)\,\Gamma\left(\frac{4i-2}{3}\right)},$$

$$E_{1,x,1-x,1-3x}(n) = 2\mu_2(x)\,\Xi(x)\,(-1)^{\lfloor \frac{x+2}{3} \rfloor} \prod_{i=1}^{n} \frac{2^{i-2}\,\Gamma(4i-1)\,\Gamma\left(\frac{i}{3}\right)}{3\,\Gamma(3i-1)\,\Gamma\left(\frac{4i}{3}\right)},$$

$$E_{2,x,2-x,2-3x}(n) = \frac{\mu_0(x)}{n}\,\Xi(x)\,(-1)^{\lfloor \frac{x+1}{3} \rfloor} \prod_{i=2}^{n} \frac{2^{i-3}\,\Gamma(4i+1)\,\Gamma\left(\frac{i-1}{3}\right)}{9\,\Gamma(3i)\,\Gamma\left(\frac{4i+2}{3}\right)}.$$

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$$E_{0,x,-x,-3x}(n) = 2\mu_1(x)\,\Xi(x)\,(-1)^{\lfloor \frac{x}{3} \rfloor} \prod_{i=1}^{n} \frac{2^{i-1}\,\Gamma(4i-3)\,\Gamma\left(\frac{i+1}{3}\right)}{\Gamma(3i-2)\,\Gamma\left(\frac{4i-2}{3}\right)},$$

$$E_{1,x,1-x,1-3x}(n) = 2\mu_2(x)\,\Xi(x)\,(-1)^{\lfloor \frac{x+2}{3} \rfloor} \prod_{i=1}^{n} \frac{2^{i-2}\,\Gamma(4i-1)\,\Gamma\left(\frac{i}{3}\right)}{3\,\Gamma(3i-1)\,\Gamma\left(\frac{4i}{3}\right)},$$

$$E_{2,x,2-x,2-3x}(n) = \frac{\mu_0(x)}{n}\,\Xi(x)\,(-1)^{\lfloor \frac{x+1}{3} \rfloor} \prod_{i=2}^{n} \frac{2^{i-3}\,\Gamma(4i+1)\,\Gamma\left(\frac{i-1}{3}\right)}{9\,\Gamma(3i)\,\Gamma\left(\frac{4i+2}{3}\right)}.$$

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

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Let

$$F_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(4^{i+\beta}\binom{i+2j+\gamma}{2j+\alpha}+\binom{-i+2j+\delta}{2j+\alpha}\right).$$

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$$\{(\alpha,\beta,\gamma,\delta) : -6 \le \alpha,\beta \le 9 \text{ and } -9 \le \gamma,\delta \le 9\}$$

we looked for "nice" evaluations.

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Theorem

The following determinant evaluations hold for all $n \ge 1$:

$$F_{1,0,1,1}(n) = 2\prod_{i=1}^{n}\frac{3^{i-1}\,\Gamma(3i-1)\,\Gamma\left(\frac{i+1}{2}\right)}{\Gamma(2i)\,\Gamma\left(\frac{3i-1}{2}\right)},$$

$$F_{1,0,2,2}(n) = 2\prod_{i=1}^{n}\frac{3^{i-1}\,\Gamma(3i)\,\Gamma\left(\frac{i}{2}\right)}{2\,\Gamma(2i)\,\Gamma\left(\frac{3i}{2}\right)},$$

$$F_{1,0,3,3}(n) = 2\prod_{i=1}^{n}\frac{3^{i}\,\Gamma(3i-1)\,\Gamma\left(\frac{i+1}{2}\right)}{\Gamma(2i)\,\Gamma\left(\frac{3i-1}{2}\right)}.$$

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

Moreover, some related determinants can be expressed in terms of these; the following identities hold (at least) for all $n \geq 4$:

$$F_{1,0,1,1}(n) = \frac{2}{3}\, F_{1,1,-1,-3}(n) = \frac{1}{21}\, F_{1,2,-3,-7}(n),$$
$$F_{1,0,2,2}(n) = -2\, F_{1,1,0,-2}(n) = \frac{2}{7}\, F_{1,2,-2,-6}(n),$$
$$F_{1,0,3,3}(n) = 2\, F_{1,1,1,-1}(n) = \frac{2}{5}\, F_{1,2,-1,-5}(n) = \frac{1}{99}\, F_{1,3,-3,-9}(n),$$
$$F_{1,1,-1,-3}(n) = -6\, F_{3,2,2,-2}(n-1) = 24\, F_{5,3,5,-1}(n-2),$$
$$F_{1,1,0,-2}(n) = -2\, F_{3,2,3,-1}(n-1).$$

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

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Proof.

All these determinant evaluations can be proved by means of Zeilberger's holonomic Ansatz. □

There are also two infinite families of evaluations.

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

There are also two infinite families of evaluations.

Theorem

Let x be an indeterminate. Then, for all integers $n \geq 1$, we have

$$\det_{0 \le i,j \le n-1} \left(4^i \binom{x+i+2j+1}{2j+1} + \binom{x-i+2j+1}{2j+1} \right) \\ = 2^{\binom{n+1}{2}+1} 3^{\binom{n}{2}} \prod_{i=1}^{n} \frac{i!}{(2i)!} \prod_{i=0}^{n-1} (x+3i+1)_{n-i}.$$

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Proof.

Even if the computer takes much longer, also this determinant evaluation can be proved by means of Zeilberger's holonomic Ansatz. □

There are also two infinite families of evaluations.

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

There are also two infinite families of evaluations.

Theorem

For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(4^i\binom{x+i+2j+3}{2j+3}+\binom{x-i+2j+3}{2j+3}\right)$$
$$=\left(2\cdot 6^{\binom{n}{2}}\prod_{i=0}^{n-1}\frac{i!}{(2i+3)!}\right)\left((x+2)(x+3)\prod_{i=0}^{n-1}(x+3i+1)_{n-i}\right)$$
$$\times\,\mathrm{Pol}_n(x),$$

where $\mathrm{Pol}_n(x)$ *is a monic polynomial in* x *of degree* $2n-2$*.*

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

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Proof.

We give a lengthy proof by identification of factors. □

What is the polynomial $\mathrm{Pol}_n(x)$?

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

What is the polynomial $\mathrm{Pol}_n(x)$?

Conjecture

The polynomial $\mathrm{Pol}_n(x)$ *in the previous theorem is given by the recurrence*

$$\begin{aligned}
&3\,\mathrm{Pol}_{n+3}(x) - 2\big(18n^2 + 9nx + 72n - 3x^2 - 3x + 49\big)\,\mathrm{Pol}_{n+2}(x)\\
&+\big(135n^4+108n^3x+810n^3-54n^2x^2+108n^2x+1395n^2-52nx^3-510nx^2\\
&-1100nx+120n-9x^4-152x^3-855x^2-1780x-1020\big)\,\mathrm{Pol}_{n+1}(x)\\
&-6(n+1)(n-x-2)(n+x+2)(3n+x+3)(3n+x+5)(3n+x+7)\,\mathrm{Pol}_n(x)\\
&= 0
\end{aligned}$$

and initial values

$$\begin{aligned}
\mathrm{Pol}_1(x) &= 1,\\
\mathrm{Pol}_2(x) &= \tfrac{1}{3}\big(3x^2 + 31x + 60\big),\\
\mathrm{Pol}_3(x) &= \tfrac{1}{9}\big(9x^4 + 234x^3 + 2061x^2 + 6956x + 7680\big).
\end{aligned}$$

Determinant evaluations: variations on a theme; $A = 2$, $B = 4$

Determinant evaluations: variations on a theme; $A=2$, $B=4$

Let

$$G_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(2^{i+\beta}\binom{i+4j+\gamma}{4j+\alpha}+\binom{-i+4j+\delta}{4j+\alpha}\right).$$

Determinant evaluations: variations on a theme; $A = 2$, $B = 4$

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In the search space

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we looked for "nice" evaluations.

Determinant evaluations: variations on a theme; $A = 2$, $B = 4$

Let

$$G_{\alpha,\beta,\gamma,\delta}(n) := \det_{0 \le i,j \le n-1} \left(2^{i+\beta} \binom{i + 4j + \gamma}{4j + \alpha} + \binom{-i + 4j + \delta}{4j + \alpha} \right).$$

Determinant evaluations: variations on a theme; $A=2$, $B=4$

Let

$$G_{\alpha,\beta,\gamma,\delta}(n) := \det_{0\le i,j\le n-1}\left(2^{i+\beta}\binom{i+4j+\gamma}{4j+\alpha}+\binom{-i+4j+\delta}{4j+\alpha}\right).$$

Conjecture

The following determinant evaluations hold for all $n\ge 1$:

$$G_{0,2,3,-1}(n)=\prod_{i=1}^{n}\frac{(2i-1)(4i-3)(4i-1)\,\Gamma(6i)\,\Gamma\left(\frac{i+3}{4}\right)}{i(i+1)(i+2)(3i-1)\,\Gamma(5i-1)\,\Gamma\left(\frac{5i+3}{4}\right)},$$

$$G_{1,3,6,0}(n)=\prod_{i=1}^{n}\frac{8(2i-1)(2i+1)^2(4i-1)(4i+1)\,\Gamma(6i+2)\,\Gamma\left(\frac{i+2}{4}\right)}{(i+1)(i+2)(i+3)(i+4)\,\Gamma(5i+2)\,\Gamma\left(\frac{5i+6}{4}\right)},$$

$$G_{1,1,0,-2}(n)=-4\prod_{i=1}^{n}\frac{(3i-2)\,\Gamma(6i-5)\,\Gamma\left(\frac{i}{4}\right)}{8\,\Gamma(5i-4)\,\Gamma\left(\frac{5i}{4}\right)},$$

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

$$G_{3,0,3,3}(n) = 2 \prod_{i=1}^{n} \frac{\Gamma\left(6i-1\right)\Gamma\left(\frac{i+3}{4}\right)}{\Gamma\left(5i\right)\Gamma\left(\frac{5i-1}{4}\right)},$$

$$G_{2,1,2,0}(n) = \prod_{i=1}^{n} \frac{\Gamma\left(6i-1\right)\Gamma\left(\frac{i+2}{4}\right)}{2(2i-1)\Gamma\left(5i-1\right)\Gamma\left(\frac{5i-2}{4}\right)}.$$

Moreover, the following identities are conjectured to hold for all $n \geq 3$:

$$\begin{aligned} G_{3,0,3,3}(n) &= \frac{2}{3}\, G_{0,1,-2,-4}(n+1) = -\frac{1}{672}\, G_{1,3,-2,-8}(n+1) \\ &= \frac{1}{63}\, G_{5,4,3,-5}(n) = \frac{4}{1002001}\, G_{6,6,3,-9}(n) \\ &= -\frac{8}{5}\, G_{9,5,8,-2}(n-1), \end{aligned}$$

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

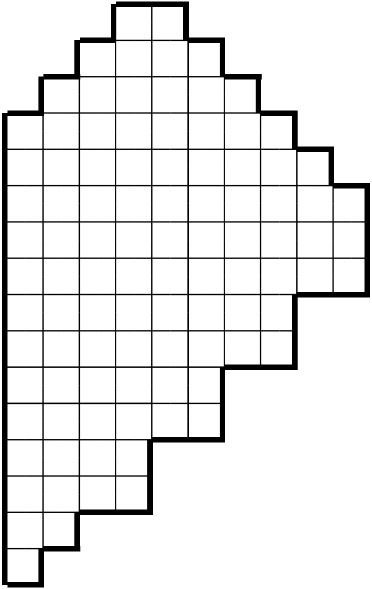
$$G_{1,1,0,-2}(n) = -\frac{1}{49}\, G_{2,3,0,-6}(n) = -\frac{2}{7}\, G_{6,4,5,-3}(n-1)$$
$$= -\frac{4}{5577}\, G_{7,6,5,-7}(n-1),$$
$$G_{2,1,2,0}(n) = 2\, G_{7,4,7,-1}(n-1).$$

Determinant evaluations: variations on a theme; $A = 4$, $B = 2$

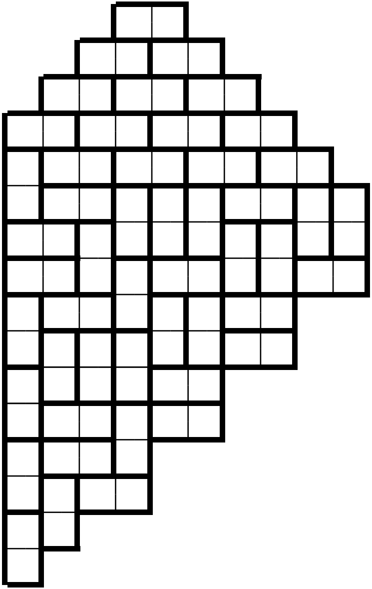
$$
\begin{aligned}
G_{1,1,0,-2}(n) &= -\frac{1}{49}\, G_{2,3,0,-6}(n) = -\frac{2}{7}\, G_{6,4,5,-3}(n-1) \\
&= -\frac{4}{5577}\, G_{7,6,5,-7}(n-1), \\
G_{2,1,2,0}(n) &= 2\, G_{7,4,7,-1}(n-1).
\end{aligned}
$$

In principle, Zeilberger's holonomic Ansatz should work for all of these. However, our computer capacities are not big enough such that the corresponding calculations can be carried out.

A generalised Aztec triangle



A generalised Aztec triangle



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