Elliptic hypergeometric series and applications

Christian Krattenthaler

Universität Wien

Alin Bostan (11 April 2018):

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We would like to invite you to give a plenary talk to
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http://www.lmpt.univ-tours.fr/~raschel/COMBINEPIC.html

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C.K. (3 May 2018):

Je suis tres honnore par l'invitation.
Il sera un plaisir pour moi d'en faire partie.
Or, en materiere d'"elliptic", je n'ai rien d'en parler ; "combinatorics", normalement, ca va.

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merci beaucoup !  Pourrais-tu aussi bientôt nous
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`Je reflechis toujours ...`

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C.K. (31 March 2019):

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Je reflechis toujours ...
```

Alin Bostan (31 March 2019):

```
Si je peux me permettre une suggestion :
il y a un an, tu nous écrivais, avec beaucoup de
modestie et d'humour :
> en materiere d'"elliptic", je n'ai rien d'en parler ;
> "combinatorics", normalement, ca va.
Or, nous sommes au courant de ton papier
elliptico-combinatoire :

  "The Determinant of an Elliptic Sylvesteresque
  Matrix"

n'aurais-tu pas envie de nous parler de ça ?
```

Elliptic hypergeometric series and applications

Elliptic hypergeometric series and applications

- Hypergeometric Series

Overview

Elliptic hypergeometric series and applications

- Hypergeometric Series
- Basic Hypergeometric Series

Elliptic hypergeometric series and applications

- Hypergeometric Series
- Basic Hypergeometric Series
- Elliptic Hypergeometric Series

Elliptic hypergeometric series and applications

- Hypergeometric Series
- Basic Hypergeometric Series
- Elliptic Hypergeometric Series
- Three Applications

The "building blocks" are the *shifted factorials* (*Pochhammer symbols*)

$$(\alpha)_m = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1).$$

The (*generalised*) *hypergeometric series* is defined by

$${}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right] = \sum_{l=0}^{\infty} \frac{(a_1)_l\cdots(a_r)_l}{l!\,(b_1)_l\cdots(b_s)_l} z^l \ .$$

Hypergeometric Series

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If one chooses $a_i = -n$ for some i, *where n is a non-negative integer*, then the infinite sum becomes a *terminating* sum since

$$(-n)_l = (-n)(-n+1)\cdots(-1)\cdot 0\cdot 1\cdots(-n+l-1) = 0$$

for l large enough.

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Hypergeometric Series

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Hypergeometric series are everywhere:

$$ e^x={}_0F_0\left[\begin{matrix}-\\ -\end{matrix};x\right], $$

$$ \log x=x\,{}_2F_1\left[\begin{matrix}1,1\\ 2\end{matrix};-x\right], $$

$$ \cos x={}_0F_1\left[\begin{matrix}-\\ \frac{1}{2}\end{matrix};-\frac{x^2}{4}\right], $$

$$ J_\alpha(x)=\left(\frac{x}{2}\right)^{\alpha}{}_0F_1\left[\begin{matrix}-\\ \alpha+1\end{matrix};-\frac{x^2}{4}\right]. $$

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$$\sum_{k=0}^{n} \binom{m}{k} \binom{n}{k} = {}_2F_1\left[\begin{matrix} -m, -n \\ 1 \end{matrix}; 1\right].$$

This is a special case of the classical *Gauß hypergeometric series*

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right],$$

which converges for $|x| < 1$, and also for $x = 1$ and $\Re(c - a - b) > 0$.

Summation formulas

Summation formulas

Gauß already proved that

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, $$

provided $\Re(c-a-b) > 0$.

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provided $\Re(c-a-b) > 0$.

If one specialises $b = -n$, one obtains the *Chu–Vandermonde summation formula*:

$$ {}_2F_1\left[\begin{matrix} a, -n \\ c \end{matrix}; 1\right] = \frac{(c-a)_n}{(c)_n}, $$

where n is a non-negative integer.

Summation formulas

Summation formulas

The *Pfaff–Saalschütz summation formula*:

$$ {}_3F_2\left[\begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix}; 1\right] = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}, $$

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provided n is a non-negative integer.

Dougall's summation formula

$$ {}_7F_6\left[\begin{matrix} a, a/2+1, b, c, d, \\ a/2, 1+a-b, 1+a-c, 1+a-d, \end{matrix}\right. \qquad \left.\begin{matrix} 1+2a-b-c-d+n, -n \\ -a+b+c+d-n, a+1+n \end{matrix}; 1\right] $$

$$ = \frac{(1+a)_n\,(1+a-b-c)_n\,(1+a-b-d)_n\,(1+a-c-d)_n}{(1+a-b)_n\,(1+a-c)_n\,(1+a-d)_n\,(1+a-b-c-d)_n}, $$

provided n is a non-negative integer.

Transformation formulas

Transformation formulas

Euler's transformation formula:

$$ {}_2F_1\left[\begin{matrix}a,b\\c\end{matrix};z\right]=(1-z)^{c-a-b}{}_2F_1\left[\begin{matrix}c-a,c-b\\c\end{matrix};z\right], $$

provided $|z|<1$.

Kummer's transformation formula:

$$ {}_3F_2\left[\begin{matrix}a,b,c\\d,e\end{matrix};1\right]=\frac{\Gamma(e)\,\Gamma(d+e-a-b-c)}{\Gamma(e-a)\,\Gamma(d+e-b-c)}{}_3F_2\left[\begin{matrix}a,d-b,d-c\\d,d+e-b-c\end{matrix};1\right], $$

provided both series converge.

Whipple's transformation formula:

$$ {}_4F_3\left[\begin{matrix}a,b,c,-n\\e,f,1+a+b+c-e-f-n\end{matrix};1\right]=\frac{(e-a)_n\,(f-a)_n}{(e)_n\,(f)_n} $$

$$ \times{}_4F_3\left[\begin{matrix}-n,a,1+a+c-e-f-n,1+a+b-e-f-n\\1+a+b+c-e-f-n,1+a-e-n,1+a-f-n\end{matrix};1\right] $$

where n is a non-negative integer.

Transformation formulas

A transformation between two ${}_9F_8$-series:

$$
\begin{aligned}
&{}_9F_8\left[\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, f, \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f, \end{matrix}\right.\\
&\qquad\qquad\left.\begin{matrix} \lambda+a+1+m-e-f, -m \\ e+f-m-\lambda, a+1+m \end{matrix}; 1\right]\\
&= \frac{(1+a)_m\,(1+a-e-f)_m\,(1+\lambda-e)_m\,(1+\lambda-f)_m}{(1+\lambda)_m\,(1+\lambda-e-f)_m\,(1+a-e)_m\,(1+a-f)_m}\\
&\times {}_9F_8\left[\begin{matrix} \lambda, \frac{\lambda}{2}+1, \lambda+b-a, \lambda+c-a, \lambda+d-a, e, f, \\ \frac{\lambda}{2}, 1+a-b, 1+a-c, 1+a-d, 1+\lambda-e, 1+\lambda-f, \end{matrix}\right.\\
&\qquad\qquad\left.\begin{matrix} \lambda+a+1+m-e-f, -m \\ e+f-m-a, \lambda+1+m \end{matrix}; 1\right],
\end{aligned}
$$

where $\lambda = 2a+1-b-c-d$ and m is a non-negative integer.

Interruption

Interruption

Aren't these series extremely clumsy, random?

Who can remember these formulas?

Aren't these too many?

Interruption

Aren't these series extremely clumsy, random?

Interruption

Aren't these series extremely clumsy, random?

No!

Interruption

Aren't these series extremely clumsy, random?

No!

Let us look at the Pfaff–Saalschütz summation:

$$ {}_3F_2\left[\begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix}; 1\right] = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}. $$

This hypergeometric series is *terminating* and *balanced*, which by definition means that

(sum of upper parameters) + 1 = (sum of lower parameters).

Interruption

Let us look at Dougall's summation:

$$
{}_7F_6\left[\begin{matrix} a, a/2+1, b, c, d, \\ a/2, 1+a-b, 1+a-c, 1+a-d, \end{matrix}\right.
$$

$$
\left.\begin{matrix} 1+2a-b-c-d+n, -n \\ -a+b+c+d-n, a+1+n \end{matrix}; 1\right]
$$

$$
= \frac{(1+a)_n\,(1+a-b-c)_n\,(1+a-b-d)_n\,(1+a-c-d)_n}{(1+a-b)_n\,(1+a-c)_n\,(1+a-d)_n\,(1+a-b-c-d)_n}.
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Let us look at Dougall's summation:

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{}_7F_6\left[\begin{matrix} a, a/2+1, b, c, d, \\ a/2, 1+a-b, 1+a-c, 1+a-d, \end{matrix}\right.
\left.\begin{matrix} 1+2a-b-c-d+n, -n \\ -a+b+c+d-n, a+1+n \end{matrix}; 1\right]
$$

$$
= \frac{(1+a)_n\,(1+a-b-c)_n\,(1+a-b-d)_n\,(1+a-c-d)_n}{(1+a-b)_n\,(1+a-c)_n\,(1+a-d)_n\,(1+a-b-c-d)_n}.
$$

This series is *terminating* and *very-well-poised*, which by definition means that there is a *leading* (*upper*) *parameter* a, and the other upper/lower parameters form pairs all of which sum to $a+1$, one of these pairs being $(\frac{a}{2}+1, \frac{a}{2})$. The "effect" of the latter pair in the series is the "very-well-poised" factor

$$
\frac{(\frac{a}{2}+1)_l}{(\frac{a}{2})_l} = \frac{a+2l}{a}.
$$

Interruption

The ${}_9F_8$-transformation formula

$$
\begin{aligned}
&{}_9F_8\left[\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, f, \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f, \end{matrix}\right. \\
&\qquad\qquad\left.\begin{matrix} \lambda+a+1+m-e-f, -m \\ e+f-m-\lambda, a+1+m \end{matrix}; 1\right] \\
&= \frac{(1+a)_m\,(1+a-e-f)_m\,(1+\lambda-e)_m\,(1+\lambda-f)_m}{(1+\lambda)_m\,(1+\lambda-e-f)_m\,(1+a-e)_m\,(1+a-f)_m} \\
&\times {}_9F_8\left[\begin{matrix} \lambda, \frac{\lambda}{2}+1, \lambda+b-a, \lambda+c-a, \lambda+d-a, e, f, \\ \frac{\lambda}{2}, 1+a-b, 1+a-c, 1+a-d, 1+\lambda-e, 1+\lambda-f, \end{matrix}\right. \\
&\qquad\qquad\qquad\qquad\left.\begin{matrix} \lambda+a+1+m-e-f, -m \\ e+f-m-a, \lambda+1+m \end{matrix}; 1\right]
\end{aligned}
$$

is a transformation formula between two *terminating*, *very-well-poised* ${}_9F_8$-series.

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There is even more “order/structure”:

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The ${}_9F_8$-transformation formula implies all the other summation formulas we have seen.

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The ${}_9F_8$-transformation formula implies all the other summation formulas we have seen.

Namely, we have:

$$
\begin{aligned}
&{}_9F_8\text{-transformation formula} \Rightarrow \text{Dougall}\\
&\qquad\qquad \Rightarrow \text{Pfaff–Saalschütz} \Rightarrow \text{Gauß} \Rightarrow \text{Chu–Vandermonde}.
\end{aligned}
$$

Interruption

In the ${}_9F_8$-transformation formula

$$
{}_9F_8\left[\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, f, \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f, \end{matrix}\right.
\left.\begin{matrix} \lambda+a+1+m-e-f, -m \\ e+f-m-\lambda, a+1+m \end{matrix}; 1\right]
$$

$$
= \frac{(1+a)_m\,(1+a-e-f)_m\,(1+\lambda-e)_m\,(1+\lambda-f)_m}{(1+\lambda)_m\,(1+\lambda-e-f)_m\,(1+a-e)_m\,(1+a-f)_m}
$$

$$
\times\, {}_9F_8\left[\begin{matrix} \lambda, \frac{\lambda}{2}+1, \lambda+b-a, \lambda+c-a, \lambda+d-a, e, f, \\ \frac{\lambda}{2}, 1+a-b, 1+a-c, 1+a-d, 1+\lambda-e, 1+\lambda-f, \end{matrix}\right.
\left.\begin{matrix} \lambda+a+1+m-e-f, -m \\ e+f-m-a, \lambda+1+m \end{matrix}; 1\right],
$$

where $\lambda = 2a+1-b-c-d$, make the substitution $b \to a+1-c$. This implies $\lambda = a-d$. Thus, the right-hand series "collapses" to just one term, the left-hand series reduces to a ${}_7F_6$-series, and one obtains Dougall's summation.

Interruption

In Dougall's formula

$$
{}_7F_6\left[\begin{matrix} a, a/2+1, b, c, d, \\ a/2, 1+a-b, 1+a-c, 1+a-d, \end{matrix}\right.
\left.\begin{matrix} 1+2a-b-c-d+n, -n \\ -a+b+c+d-n, a+1+n \end{matrix}; 1\right]
$$
$$
= \frac{(1+a)_n\,(1+a-b-c)_n\,(1+a-b-d)_n\,(1+a-c-d)_n}{(1+a-b)_n\,(1+a-c)_n\,(1+a-d)_n\,(1+a-b-c-d)_n}.
$$

make the substitutions $a \to a-n-1$, $d = -m$ with m a non-negative integer, and then let $n \to \infty$;

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In Dougall's formula

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{}_7F_6\left[\begin{matrix} a, a/2+1, b, c, d, \\ a/2, 1+a-b, 1+a-c, 1+a-d, \end{matrix}\right.
$$
$$
\left.\begin{matrix} 1+2a-b-c-d+n, -n \\ -a+b+c+d-n, a+1+n \end{matrix}; 1\right]
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$$
= \frac{(1+a)_n\,(1+a-b-c)_n\,(1+a-b-d)_n\,(1+a-c-d)_n}{(1+a-b)_n\,(1+a-c)_n\,(1+a-d)_n\,(1+a-b-c-d)_n}.
$$

make the substitutions $a \to a-n-1$, $d = -m$ with m a non-negative integer, and then let $n \to \infty$; the result is

$$
{}_3F_2\left[\begin{matrix} b, c, -m \\ a, 1+b+c-a-m \end{matrix}; 1\right] = \frac{(a-b)_m\,(a-c)_m}{(a)_m\,(a-b-c)_m}.
$$

This is the Pfaff–Saalschütz summation!

Interruption

In the Pfaff–Saalschütz summation

$$ {}_3F_2\left[\begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix}; 1\right] = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n} $$

let $n \to \infty$.

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In the Pfaff–Saalschütz summation

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let $n \to \infty$. This yields

$$ {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. $$

This is Gauß' summation formula!

Interruption

In the Pfaff–Saalschütz summation

$$ {}_3F_2\left[\begin{matrix}a, b, -n\\ c, 1+a+b-c-n\end{matrix}; 1\right] = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n} $$

let $n \to \infty$. This yields

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This is Gauß' summation formula!

We have already seen that by specialising $b = -n$, where n is a non-negative integer, we get the Chu–Vandermonde summation formula.

Interruption

Who can remember these formulas?

Interruption

Who can remember these formulas?

Nobody.

Interruption

Who can remember these formulas?

Nobody.

Aren't these too many?

Interruption

Who can remember these formulas?

Nobody.

Aren't these too many?

No!

Interruption

Who can remember these formulas?

Nobody.

Aren't these too many?

No!
An extensive table of summation and transformation formulas (encompassing the tables in the standard book by Lucy Slater) can be found in the documentation for my *Mathematica* package HYP at

`https://www.mat.univie.ac.at/~kratt/hyp_hypq/hyp.html`

Moreover, HYP allows you to use these formulas in order to manipulate hypergeometric series.

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$$\sum_{j=0}^{q} \frac{2^j}{j!\,(2j-1)!!}(-q)_j\,(q-m)_j = \frac{\binom{2m}{2q}}{\binom{m}{q}}.$$

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The authors say that they were not able to find the identity in the literature.
Obviously so! It is absolutely necessary to write the binomial sum in hypergeometric notation:

$${}_2F_1\left[\begin{matrix}-q, q-m\\ 1/2\end{matrix};1\right].$$

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We recognise this series as a special instances of a ${}_2F_1[1]$-series that can be evaluated by means of the Chu–Vandermonde summation formula

$$ {}_2F_1\left[\begin{matrix} a, -n \\ c \end{matrix}; 1\right] = \frac{(c-a)_n}{(c)_n}, $$

where n is a non-negative integer.

Multidimensional hypergeometric series

Motivated by work on Wigner and Racah coefficients, L. C. Biedenharn and J. D. Louck initiated a theory of *multi-dimensional hypergeometric series* which they called $U(n)$ *hypergeometric series*.

Basic Hypergeometric Series

The "building blocks" are the *q-shifted factorials*

$$(\alpha; q)_m = (1-\alpha)(1-\alpha q)(1-\alpha q^2)\cdots(1-\alpha q^{m-1}).$$

The *basic hypergeometric series* is defined by

$$
{}_r\phi_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z\right]
= \sum_{l=0}^{\infty} \frac{(a_1;q)_l \cdots (a_r;q)_l}{(q;q)_l (b_1;q)_l \cdots (b_s;q)_l} \left((-1)^l q^{\binom{l}{2}}\right)^{s-r+1} z^l \, .
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If one chooses $a_i = q^{-n}$ for some i, *where n is a non-negative integer*, then the infinite sum becomes a *terminating* sum since

$$(q^{-n})_l = (1-q^{-n})\cdots(1-q^{-1})\cdot(1-q^0)\cdot(1-q)\cdots(1-q^{-n+l-1}) = 0$$

for l large enough.

Basic Hypergeometric Series

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$$

If one makes the substitution $a_i \to q^{a_i}$, $b_i \to q^{b_i}$, $z \to z/(1-q)^{r-s-1}$, then the above basic hypergeometric series reduces to the (ordinary) hypergeometric series

$$
{}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right].
$$

Summation formulas

Summation formulas

The q-analogue of Gauß' summation formula is Heine's identity

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right] = \frac{(c/a;q)_\infty\,(c/b;q)_\infty}{(c;q)_\infty\,(c/ab;q)_\infty}. $$

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If one specialises $b = q^{-n}$, one obtains the *q-Chu–Vandermonde summation formula*:

$$ {}_2\phi_1\left[\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, \frac{cq^n}{a}\right] = \frac{(c/a;q)_n}{(c;q)_n}, $$

where n is a non-negative integer.

Summation formulas

Summation formulas

The *q-Pfaff–Saalschütz summation formula*:

$$ {}_3\phi_2\left[\begin{matrix} a, b, q^{-n} \\ c, abq^{1-n}/c \end{matrix}; q, q\right] = \frac{(c/a;q)_n (c/b;q)_n}{(c;q)_n (c/ab;q)_n}, $$

provided n is a non-negative integer.

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provided n is a non-negative integer.

This hypergeometric series is *terminating* and *balanced*, which by definition means that

(product of upper parameters)$\times q =$ (product of lower parameters).

Summation formulas

Jackson's summation formula (the q-analogue of Dougall's formula):

$$
{}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2q^{1+n}/bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcd/aq^n, aq^{1+n} \end{matrix}; q, q\right]
= \frac{(aq;q)_n(aq/bc;q)_n(aq/bd;q)_n(aq/cd;q)_n}{(aq/b;q)_n(aq/c;q)_n(aq/d;q)_n(aq/bcd;q)_n},
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provided n is a non-negative integer.

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= \frac{(aq;q)_n(aq/bc;q)_n(aq/bd;q)_n(aq/cd;q)_n}{(aq/b;q)_n(aq/c;q)_n(aq/d;q)_n(aq/bcd;q)_n},
$$

provided n is a non-negative integer.

This series is *terminating* and *very-well-poised*, which by definition means that there is a *leading* (*upper*) *parameter* a, and the other upper/lower parameters form pairs whose product is always aq, two of these pairs being $(q\sqrt{a}, \sqrt{a})$ and $(-q\sqrt{a}, -\sqrt{a})$.

Summation formulas

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$$\frac{(q\sqrt{a};q)_l\,(-q\sqrt{a};q)_l}{(\sqrt{a};q)_l\,(-\sqrt{a};q)_l} = \frac{1-aq^{2l}}{1-a}.$$

Transformation formulas

The q-analogue of the ${}_9F_8$-transformation formula:

$$
{}_{10}\phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f, \lambda aq^{1+m}/ef, q^{-m} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-m}/\lambda, aq^{1+m} \end{matrix}; q, q\right]
$$
$$
= \frac{(aq;q)_m\,(aq/ef;q)_m\,(\lambda q/e;q)_m\,(\lambda q/f;q)_m}{(\lambda q;q)_m\,(\lambda q/ef;q)_m\,(aq/e;q)_m\,(aq/f;q)_m}
$$
$$
\times {}_{10}\phi_9\left[\begin{matrix} \lambda, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{1+m}/ef, q^{-m} \\ \sqrt{\lambda}, -\sqrt{\lambda}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-m}/a, \lambda q^{1+m} \end{matrix}; q, q\right],
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where $\lambda = a^2q/bcd$ and m is a non-negative integer.

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$$

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$$

where $\lambda = a^2q/bcd$ and m is a non-negative integer.

It is a transformation formula between terminating, very-well-poised hypergeometric series.

Summation and transformation formulas

Also here we have:

${}_{10}\phi_9$-transformation formula $\Rightarrow$ Jackson

$\Rightarrow$ q-Pfaff–Saalschütz $\Rightarrow$ q-Gauß $\Rightarrow$ q-Chu–Vandermonde.

Summation and transformation formulas

An extensive table of summation and transformation formulas (encompassing the tables in the standard book by George Gasper and Mizan Rahman) can be found in the documentation for my *Mathematica* package HYPQ at

`https://www.mat.univie.ac.at/~kratt/hyp_hypq/hyp.html`

Moreover, HYPQ allows you to use these formulas in order to manipulate hypergeometric series.

Multidimensional basic hypergeometric series

Stephen Milne, Bob Gustafson, and followers extended the work of Biedenharn and Louck to develop a theory of *multi-dimensional basic hypergeometric series associated with root systems*.

Given a complex number p with $|p| < 1$, we define

$$\theta(x;p) := \prod_{j=0}^{\infty}(1-p^j x)(1-p^{j+1}/x).$$

Note: $\theta(x;0) = 1-x$.

Elliptic Hypergeometric Series

Given a complex number p with $|p| < 1$, we define

$$\theta(x;p) := \prod_{j=0}^{\infty}(1 - p^j x)(1 - p^{j+1}/x).$$

Note: $\theta(x;0) = 1 - x$.

The building blocks are "*elliptic shifted factorials*":

$$(a;q,p)_m := \theta(a;p)\,\theta(aq;p)\cdots\theta(aq^{m-1};p),$$

Note: $(a;q,0)_m = (1-a)(1-aq)\cdots(1-aq^{m-1}) = (a;q)_m$.

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We also employ the short notation

$$(a_1,a_2,\dots,a_k;q,p)_m = (a_1;q,p)_m\,(a_2;q,p)_m\cdots(a_k;q,p)_m.$$

A transformation formula

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In work on solutions to the Yang–Baxter equation, Frenkel and Turaev found the *elliptic hypergeometric transformation formula*

$$\sum_{k=0}^{\infty} q^k \frac{\theta(aq^{2k};p)(a,b,c,d,e,f,\lambda aq^{1+m}/ef,q^{-m};q,p)_k}{\theta(a;p)(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{-m}/\lambda,aq^{1+m};q,p)_k}$$

$$= \frac{(aq,aq/ef,\lambda q/e,\lambda q/f;q,p)_m}{(\lambda q,\lambda q/ef,aq/e,aq/f;q,p)_m}$$

$$\times \sum_{k=0}^{\infty} q^k \frac{\theta(\lambda q^{2k};p)\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{1+m}/ef,q^{-m};q,p)_k}{\theta(\lambda;p)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{-m}/a,\lambda q^{1+m};q,p)_k}$$

where $\lambda = a^2q/bcd$ and m is a non-negative integer.

It generalises the earlier ${}_{10}\phi_9$-transformation formula.

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$$= \frac{(aq,aq/ef,\lambda q/e,\lambda q/f;q,p)_m}{(\lambda q,\lambda q/ef,aq/e,aq/f;q,p)_m}$$
$$\times \sum_{k=0}^{\infty} q^k \frac{\theta(\lambda q^{2k};p)\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{1+m}/ef,q^{-m};q,p)_k}{\theta(\lambda;p)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{-m}/a,\lambda q^{1+m};q,p)_k}$$

where $\lambda = a^2q/bcd$ and m is a non-negative integer.

It generalises the earlier ${}_{10}\phi_9$-transformation formula.

By setting $b = aq/c$ the right-hand sum "collapses" to the term for $k = 0$.

A summation formula

This yields *Frenkel and Turaev's summation formula*

$$\sum_{k=0}^{\infty} q^k \frac{\theta(aq^{2k};p)(a,d,e,f,a^2q^{1+m}/def,q^{-m};q,p)_k}{\theta(a;p)(q,aq/d,aq/e,aq/f,defq^{-m}/a,aq^{1+m};q,p)_k} \\ = \frac{(aq,aq/ef,aq/de,aq/df;q,p)_m}{(aq/d,aq/def,aq/e,aq/f;q,p)_m},$$

provided m is a non-negative integer.

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$$= \frac{(aq,aq/ef,aq/de,aq/df;q,p)_m}{(aq/d,aq/def,aq/e,aq/f;q,p)_m},$$

provided m is a non-negative integer.

It generalises Jackson's ${}_8\phi_7$-summation formula.

However: Since one cannot take limits, respectively infinite elliptic hypergeometric series are not well-defined since they diverge, there are no elliptic analogues of the (q-)Pfaff–Saalschütz summation or the (q-)Gauß summation!

Nonetheless, Ole Warnaar, Eric Rains, Hjalmar Rosengren, Michael Schlosser, Vyacheslav Spiridonov and followers developed the results of Frenkel and Turaev into a rich theory, and in particular so a theory of *multi-dimensional elliptic hypergeometric series associated with root systems*. This multi-dimensional theory contains much of the multi-dimensional (basic) hypergeometric theory as special cases.

A Multidimensional Elliptic Hypergeometric Series

$$\sum_{0\le k_1<k_2<\dots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\,\theta(aq^{k_i+k_j};p)^2$$

$$\times\prod_{i=1}^r \frac{\theta(aq^{2k_i};p)(a,b,c,d,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{\theta(a;p)(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{r-1-m}/\lambda,aq^{1+m};q,p)_{k_i}}$$

$$=\prod_{i=1}^r \frac{(b,c,d,ef/a;q,p)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q,p)_{i-1}}$$

$$\times\prod_{i=1}^r \frac{(aq;q,p)_m\,(aq/ef;q,p)_{m+1-r}\,(\lambda q/e,\lambda q/f;q,p)_{m-i+1}}{(\lambda q;q,p)_m\,(\lambda q/ef;q,p)_{m+1-r}\,(aq/e,aq/f;q,p)_{m-i+1}}$$

$$\times\sum_{0\le k_1<k_2<\dots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\,\theta(\lambda q^{k_i+k_j};p)^2$$

$$\times\prod_{i=1}^r \frac{\theta(\lambda q^{2k_i};p)\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{\theta(\lambda;p)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{r-1-m}/a,\lambda q^{1+m};q,p)_{k_i}},$$

where $\lambda=a^2q^{2-r}/bcd$.

The one-dimensional special case ($r = 1$) of this transformation formula is

$$\sum_{k=0}^{m} q^k \frac{\theta(aq^{2k};p)\,(a,b,c,d,e,f,\lambda aq^{1+m}/ef,q^{-m};q,p)_k}{\theta(a;p)\,(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{-m}/\lambda,aq^{1+m};q,p)_k}$$
$$= \frac{(aq,aq/ef,\lambda q/e,\lambda q/f;q,p)_m}{(\lambda q,\lambda q/ef,aq/e,aq/f;q,p)_m}$$
$$\times \sum_{k=0}^{m} q^k \frac{\theta(\lambda q^{2k};p)\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{1+m}/ef,q^{-m};q,p)_k}{\theta(\lambda;p)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{-m}/a,\lambda q^{1+m};q,p)_k}$$

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where $\lambda = a^2q^{2-r}/bcd$.

This is Frenkel and Turaev's transformation formula!

$$\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\,\theta(aq^{k_i+k_j};p)^2$$

$$\times\prod_{i=1}^r \frac{\theta(aq^{2k_i};p)(a,b,c,d,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{\theta(a;p)(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{r-1-m}/\lambda,aq^{1+m};q,p)_{k_i}}$$

$$=\prod_{i=1}^r \frac{(b,c,d,ef/a;q,p)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q,p)_{i-1}}$$

$$\times\prod_{i=1}^r \frac{(aq;q,p)_m\,(aq/ef;q,p)_{m+1-r}\,(\lambda q/e,\lambda q/f;q,p)_{m-i+1}}{(\lambda q;q,p)_m\,(\lambda q/ef;q,p)_{m+1-r}\,(aq/e,aq/f;q,p)_{m-i+1}}$$

$$\times\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\,\theta(\lambda q^{k_i+k_j};p)^2$$

$$\times\prod_{i=1}^r \frac{\theta(\lambda q^{2k_i};p)\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{\theta(\lambda;p)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{r-1-m}/a,\lambda q^{1+m};q,p)_{k_i}},$$

where $\lambda = a^2q^{2-r}/bcd$.

$$\langle \text{big expression} \rangle = \langle \text{another big expression} \rangle$$

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This identity was discovered conjecturally by Ole Warnaar in 2000, and later proved independently by Rains and by Coskun and Gustafson.

It was new even when specialised to the q-case, or to the (ordinary) hypergeometric case.

3 applications

1. Enumeration of standard tableaux of skew shape

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2. Discrete analogues of Macdonald–Mehta integrals

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1. Enumeration of standard tableaux of skew shape
2. Discrete analogues of Macdonald–Mehta integrals
3. Best polynomial approximation

The first application: Counting standard Young tableaux

(joint work with MICHAEL SCHLOSSER)

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(joint work with Michael Schlosser)

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be two n-tuples of non-negative integers which are in non-increasing order and satisfy $\lambda_i \ge \mu_i$ for all i.
A standard Young tableau of skew shape λ/μ is an arrangement of the numbers $1, 2, \dots, \sum_{i=1}^n (\lambda_i - \mu_i)$ of the form

$$\begin{array}{ccccccc}
 & & & \pi_{1,\mu_1+1} & \cdots\cdots\cdots & & \pi_{1,\lambda_1} \\
 & \pi_{2,\mu_2+1} & \cdots & \pi_{2,\mu_1+1} & \cdots\cdots & \pi_{2,\lambda_2} & \\
 \iddots & & & \vdots & \iddots & & \\
\pi_{n,\mu_n+1} & \cdots\cdots\cdots\cdots\cdots & & \pi_{n,\lambda_n} & & &
\end{array}$$

such that numbers along rows and columns are increasing.

The first application: Counting standard Young tableaux

A standard Young tableau of skew shape λ/μ is an arrangement of the numbers $1, 2, \dots, \sum_{i=1}^{n}(\lambda_i - \mu_i)$ of the form

$$\begin{array}{ccccccccc}
 & & & & \pi_{1,\mu_1+1} & & \cdots\cdots\cdot & & \pi_{1,\lambda_1} \\
 & & \pi_{2,\mu_2+1} & \cdots & \pi_{2,\mu_1+1} & & \cdots\cdot & \pi_{2,\lambda_2} & \\
 & \iddots & & & \vdots & & \iddots & & \\
\pi_{n,\mu_n+1} & & \cdots\cdots\cdots\cdots & & & \pi_{n,\lambda_n} & & &
\end{array}$$

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 & \iddots & & \vdots & \iddots & & \\
\pi_{n,\mu_n+1} & & \cdots\cdots\cdots\cdots\cdots & & \pi_{n,\lambda_n} & &
\end{array}$$

such that numbers along rows and columns are increasing.

A standard Young tableau of shape $(6,5,4,3,2,1)/(3,3,0,0,0,0)$:

$$\begin{array}{cccccc}
 & & & 2 & 5 & 13 \\
 & & & 3 & 9 & \\
1 & 4 & 8 & 12 & & \\
6 & 11 & 15 & & & \\
7 & 14 & & & & \\
10 & & & & &
\end{array}$$

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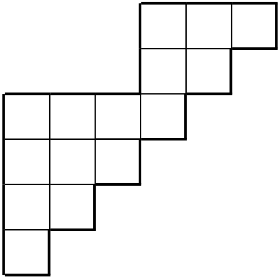
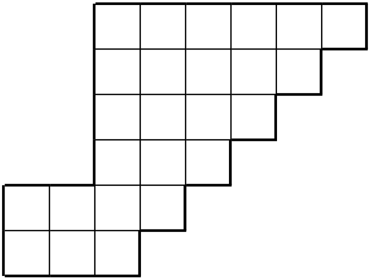
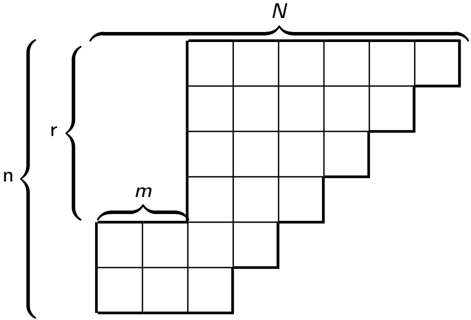


We shall do something more general than DeWitt here: we shall enumerate all standard Young tableaux of a skew shape, where the outer shape is a (possibly incomplete) staircase and the inner shape is a rectangle.



Our goal: Let N, n, m, r be non-negative integers. Compute the number of all standard Young tableaux of shape $(N, N-1, \dots, N-n+1)/(m^r)$, where (m^r) stands for $(m, m, \dots, m, 0, \dots, 0)$ with r components m).



Aitken's Formula

The number of all standard Young tableaux of shape λ/μ equals

$$\left(\sum_{i=1}^{n}(\lambda_i-\mu_i)\right)!\cdot\det_{1\le i,j\le n}\left(\frac{1}{(\lambda_i-i-\mu_j+j)!}\right).$$

We substitute in Aitken's formula:

$$\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!\det_{1\le i,j\le n}\left(\begin{cases}\dfrac{1}{(N+1-2i-m+j)!} & j\le r\\ \dfrac{1}{(N+1-2i+j)!} & j>r\end{cases}\right).$$

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We now do a Laplace expansion with respect to the first r columns:

$$\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!$$
$$\times\sum_{1\le k_1<\dots<k_r\le n}(-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i}\det_{1\le i,j\le r}\left(\frac{1}{(N+1-2k_i-m+j)!}\right)$$
$$\cdot\det_{\substack{1\le i\le n,\ i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{(N+1-2i+j)!}\right).$$

The first application: Counting standard Young tableaux

$$\left(\binom{N+1}{2} - \binom{N-n+1}{2} - mr\right)!$$
$$\times \sum_{1\le k_1<\dots<k_r\le n} (-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i} \det_{1\le i,j\le r}\left(\frac{1}{(N+1-2k_i-m+j)!}\right)$$
$$\cdot \det_{\substack{1\le i\le n,\, i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{(N+1-2i+j)!}\right).$$

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$$\cdot \det_{\substack{1\le i\le n,\, i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{(N+1-2i+j)!}\right).$$

Both determinants can be evaluated by means of

$$\det_{1\le i,j\le s}\left(\frac{1}{(X_i+j)!}\right)=\prod_{i=1}^{s}\frac{1}{(X_i+s)!}\prod_{1\le i<j\le s}(X_i-X_j),$$

After a lot of simplification, one arrives at

$$(-1)^{\binom{r}{2}} 2^{\binom{r}{2}+\binom{n-r}{2}} \left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!$$
$$\times \prod_{i=1}^{n} \frac{(i-1)!}{(N+n+1-2i)!} \prod_{i=1}^{r} \frac{(N+n-1)!}{(n-1)!\,(N-m+r-1)!}$$
$$\times \sum_{0\le k_1<\dots<k_r\le n-1} \prod_{1\le i<j\le r} (k_j-k_i)^2$$
$$\cdot \prod_{i=1}^{r} \frac{\left(-\frac{N-m+r-1}{2}\right)_{k_i} \left(-\frac{N-m+r-2}{2}\right)_{k_i} (-n+1)_{k_i}}{\left(-\frac{N+n-1}{2}\right)_{k_i} \left(-\frac{N+n-2}{2}\right)_{k_i} k_i!}.$$

After a lot of simplification, one arrives at

$$(-1)^{\binom{r}{2}} 2^{\binom{r}{2}+\binom{n-r}{2}} \left(\binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!$$
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Michael Schlosser immediately saw that our multi-dimensional transformation formula can be applied!

In the elliptic multi-dimensional transformation formula, we let $p = 0$, $d \to aq/d$, $f \to aq/f$, and then $a \to 0$. Next we perform the substitutions $b \to q^b$, $c \to q^c$, etc., we divide both sides of the identity obtained so far by $(1-q)^{\binom{r}{2}}$, and we let $q \to 1$.

Corollary

For all non-negative integers m, r and s, we have

$$\sum_{0\le k_1<k_2<\dots<k_r\le m}\prod_{1\le i<j\le r}(k_i-k_j)^2\prod_{i=1}^{r}\frac{(d+k_i)_s\,(b)_{k_i}\,(-m)_{k_i}}{k_i!\,(f)_{k_i}}$$

$$=\frac{(-1)^{\binom r2}}{(r+s-1)!^{s-1}}\prod_{i=1}^{r}\frac{(b)_{i-1}\,(-f+b+s+2r-i-m)_{m-r+1}}{(-f-m+i)_{m-i+1}}$$

$$\times\prod_{i=1}^{r+s-1}\frac{(i-1)!\,m!}{(m-i)!}\prod_{i=r}^{r+s-1}\frac{(d-b+1-r)_i}{(r+s-i-1)!\,(d)_{i-r}\,(f-b-s+1-r)_i}$$

$$\times\sum_{0\le \ell_1<\ell_2<\dots<\ell_s\le r+s-1}\prod_{1\le i<j\le s}(\ell_i-\ell_j)^2$$

$$\times\prod_{i=1}^{s}\frac{(d)_{\ell_i}\,(f-b-s+1-r)_{\ell_i}\,(-r-s+1)_{\ell_i}}{\ell_i!\,(d-b+1-r)_{\ell_i}\,(-m)_{\ell_i}}.$$

Corollary

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$$\sum_{0\le k_1<k_2<\cdots<k_r\le m}\prod_{1\le i<j\le r}(k_i-k_j)^2\prod_{i=1}^{r}\frac{(d+k_i)_s\,(b)_{k_i}\,(-m)_{k_i}}{k_i!\,(f)_{k_i}}$$

$$=\frac{(-1)^{\binom{r}{2}}}{(r+s-1)!^{s-1}}\prod_{i=1}^{r}\frac{(b)_{i-1}\,(-f+b+s+2r-i-m)_{m-r+1}}{(-f-m+i)_{m-i+1}}$$

$$\times\prod_{i=1}^{r+s-1}\frac{(i-1)!\,m!}{(m-i)!}\prod_{i=r}^{r+s-1}\frac{(d-b+1-r)_i}{(r+s-i-1)!\,(d)_{i-r}\,(f-b-s+1-r)_i}$$

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The first application: Counting standard Young tableaux

Theorem

If $N-n$ is even, the number of standard Young tableaux of shape $(N,N-1,\dots,N-n+1)/(m^r)$ equals

$$(-1)^{\binom{(N-n)/2}{2}+\frac{1}{2}r(N-n)}2^{\binom{n}{2}+(N-n-m)r}\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!$$

$$\times\frac{1}{\left(r+\frac{N-n-2}{2}\right)!^{(N-n)/2}\left(\frac{N+n-2}{2}\right)!^{(N-n)/2}}\frac{\prod_{i=1}^{(N+n)/2}(i-1)!}{\prod_{i=1}^{n}(N-n+2i-1)!}$$

$$\times\prod_{i=1}^{r}\frac{\left(\frac{N-n}{2}+i-1\right)!\,(n+m-r+2i-1)!\,\left(\frac{n+m-r}{2}+i\right)_{(N-n)/2}}{(m+i-1)!\,(N-m-r+2i-1)!}$$

$$\times\sum_{0\le\ell_1<\ell_2<\dots<\ell_{(N-n)/2}\le r+\frac{N-n-2}{2}}(-1)^{\sum_{i=1}^{(N-n)/2}\ell_i}\left(\prod_{1\le i<j\le\frac{N-n}{2}}(\ell_i-\ell_j)^2\right)$$

$$\cdot\prod_{i=1}^{\frac{N-n}{2}}\binom{\frac{N-n-2}{2}+r}{\ell_i}\frac{\left(\frac{N+n}{2}-\ell_i\right)_{\ell_i}\left(\frac{n+m-r+1}{2}-i\right)_{r+i-\ell_i-1}\left(\frac{N-m-r+2}{2}-i\right)_{r+i-\ell_i-1}}{\left(\frac{N+m-r+2}{2}-i\right)_{r+i-\ell_i-1}},$$

and there is a similar statement if $N-n$ is odd.

In the case of a full staircase (i.e., $n = N$), the formula reduces to DeWitt's original result.

Corollary

The number of standard Young tableaux of shape $(n, n-1, \dots, 1)/(m^r)$ *equals*

$$2^{\binom{n}{2}-rm}\left(\binom{n+1}{2}-mr\right)!\prod_{i=1}^{n}\frac{(i-1)!}{(2i-1)!} \times\prod_{i=1}^{r}\frac{(n+m-r+2i-1)!\,(i-1)!}{(m+i-1)!\,(n-m-r+2i-1)!},$$

The "next" case:

Corollary

The number of standard Young tableaux of shape $(n+1, n, \dots, 2)/(m^r)$ *equals*

$$2^{\binom{n}{2}-(m-1)r}\left(\binom{n+2}{2}-mr-1\right)!\prod_{i=1}^{n}\frac{(i-1)!}{(2i)!}$$
$$\times\prod_{i=1}^{r}\frac{(n+m-r+2i-1)!\,(i-1)!}{(m+i-1)!\,(n-m-r+2i)!}$$
$$\times\sum_{\ell=0}^{r}(-1)^{r-\ell}\binom{r}{\ell}\frac{(n-\ell+1)_\ell\left(\frac{n+m-r}{2}\right)_{r-\ell}\left(\frac{n-m-r+1}{2}\right)_{r-\ell}}{\left(\frac{n+m-r+1}{2}\right)_{r-\ell}}.$$

In general:
The number of standard Young tableaux of shape $(N, N-1, \dots, N-n)/(m^r)$ equals an $\lceil (N-n)/2 \rceil$-fold hypergeometric sum.

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The second application: Discrete M–M-integrals

(joint work with Richard Brent and Ole Warnaar)

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Ole Warnaar (15 May 2015):

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$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \Bigg| \prod_{1\le i<j\le r} (k_i^\alpha - k_j^\alpha) \Bigg|^\gamma \prod_{i=1}^r |k_i|^\delta \binom{2n}{n+k_i},$$

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`which we call "discrete Mehta-type integrals".`

`At least, for` $\alpha,\gamma\in\{1,2\}$ `and small` δ`, we believe that these sums can be evaluated in closed form.`

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$$(2\pi)^{-r/2}\int_{\mathbb{R}^r}\left|\prod_{1\le i<j\le r}(t_i-t_j)\right|^{\gamma}\prod_{i=1}^{r}e^{-t_i^2/2}\,dt_1\cdots dt_r
=\prod_{i=1}^{r}\frac{\Gamma(1+i\gamma/2)}{\Gamma(1+\gamma/2)}.$$

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“Discrete analogues of Macdonald–Mehta integrals”

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$$= \prod_{i=1}^{r} \binom{m+n}{i-1}^2 \binom{2n}{i-1}\binom{2m}{i-1} (2m+2n-i-r+2)!\,(i-1)!^5$$

can be proved in various ways, one of which is by the use of Schur functions (and a *q-analogue* as well), as I pointed out in a paper 15 years ago.

The second application: Discrete M–M-integrals

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And what about a q-analogue?

Our discrete analogue of Macdonald–Mehta integrals:

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \left| \prod_{1\le i<j\le r} (k_i^\alpha - k_j^\alpha) \right|^\gamma \prod_{i=1}^r |k_i|^\delta \binom{2n}{n+k_i}.$$

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We found (and proved) closed form evaluations in the following cases:

α	γ	δ
1	1	0
1	2	0, 1
2	1	0, 1, 2
2	2	0, 1, 2, 3

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α	γ	δ
1	1	0
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We also (eventually) found q-analogues in most cases.

We struggled longest with the discrete analogues of Macdonald–Mehta integrals with $\gamma = 2$:

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \left| \prod_{1\le i<j\le r} (k_i^\alpha - k_j^\alpha) \right|^2 \prod_{i=1}^{r} |k_i|^\delta \binom{2n}{n+k_i}.$$

The second application: Discrete M–M-integrals

For example, we observed that

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (k_i^2-k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i} \binom{2m}{m+k_i}$$

$$= \sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (k_i-k_j)^2(k_i+k_j)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i} \binom{2m}{m+k_i}$$

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evaluates "nicely".
How would a q-analogue look like? Wouldn't it contain

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2(1-q^{k_i+k_j})^2 \times \text{stuff ?}$$

The second application: Discrete M–M-integrals

How did our gigantic transformation formula (with $p = 0$) look like?

The second application: Discrete M–M-integrals

How did our gigantic transformation formula (with $p=0$) look like?

$$\sum_{0\le k_1<k_2<\dots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2\,(1-aq^{k_i+k_j})^2$$

$$\times\prod_{i=1}^r \frac{(1-aq^{2k_i})(a,b,c,d,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q)_{k_i}}{(1-a)(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{r-1-m}/\lambda,aq^{1+m};q)_{k_i}}$$

$$=\prod_{i=1}^r \frac{(b,c,d,ef/a;q)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q)_{i-1}}$$

$$\times\prod_{i=1}^r \frac{(aq;q)_m\,(aq/ef;q)_{m+1-r}\,(\lambda q/e,\lambda q/f;q)_{m-i+1}}{(\lambda q;q)_m\,(\lambda q/ef;q)_{m+1-r}\,(aq/e,aq/f;q)_{m-i+1}}$$

$$\times\sum_{0\le k_1<k_2<\dots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2\,(1-\lambda q^{k_i+k_j})^2$$

$$\times\prod_{i=1}^r \frac{(1-\lambda q^{2k_i})\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q)_{k_i}}{(1-\lambda)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{r-1-m}/a,\lambda q^{1+m};q)_{k_i}},$$

So,

So, if one chooses $a = q^2$, $d = q^{1-n}$, $e = q^{1-m}$, and $f = q^2$, and then lets $q \to 1$ in this transformation formula, and one gets:

Theorem

For all non-negative integers m and n and a positive integer r, we have

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\le i<j\le r} (k_i^2 - k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i}\binom{2m}{m+k_i}$$
$$= r!\,2^{-r} \prod_{i=1}^{r} \left(\frac{\Gamma_{q^{1/2}}(2i)\,\Gamma(2n+1)\,\Gamma(2m+1)}{\Gamma(m+n-i+2)\,\Gamma(m+n-i-r+2)} \right.$$
$$\left. \times \frac{\Gamma(2m+2n-2i-2r+3)}{\Gamma(2n-2i+2)\,\Gamma(2m-2i+2)} \right).$$

And, if we don't let $q \to 1$, we get a q-analogue for free:

Theorem

For all non-negative integers m and n and a positive integer r, we have

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\le i<j\le r} [k_j-k_i]_q^2\,[k_i+k_j]_q^2 \times \prod_{i=1}^{r} q^{k_i^2-(2i-\frac{1}{2})k_i} \left|[k_i]_{q^2}\,[k_i]_q\right| \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}_q \begin{bmatrix} 2m \\ m+k_i \end{bmatrix}_q$$

$$= r!\left(\frac{2}{[2]_q}\right)^r [2]_{q^{1/2}}^{-r}\, q^{-2\binom{r+1}{3}-\frac{1}{2}\binom{r+1}{2}} \times \prod_{i=1}^{r} \left(\frac{\Gamma_{q^{1/2}}(2i)\,\Gamma_q(2n+1)\,\Gamma_q(2m+1)}{\Gamma_q(m+n-i+2)\,\Gamma_q(m+n-i-r+2)} \times \frac{\Gamma_{q^{1/2}}(2m+2n-2i-2r+3)}{\Gamma_{q^{1/2}}(2n-2i+2)\,\Gamma_{q^{1/2}}(2m-2i+2)} \right).$$

Similar specialisations worked for the other cases with $\gamma = 2$:

α	γ	δ
1	2	$0, 1$
2	2	$0, 1, 2, 3$

except for $\alpha = \delta = 1$. For that case, we have a (lengthy) ad hoc argument, but we don't know a q-analogue.

The third application: Best polynomial approximation

(joint work with Han Feng, Yuan Xu, and Gaurav Bhatnagar)

The third application: Best polynomial approximation

(joint work with Han Feng, Yuan Xu, and Gaurav Bhatnagar)

Yuan Xu (26 August 2017):

`In work in approximation theory, I encountered a certain determinant (see the attachment). On the basis of computer experiments, I believe that this determinant can be evaluated in closed form. Have you seen it before?`

The determinant

Let

$$f(s_1,s_2,r,i,j) := \binom{r}{j-i}\frac{(s_1+i)_{j-i}}{(s_1+s_2+i+j-1)_{j-i}\,(s_1+s_2+r+2i)_{j-i}}.$$

Form the matrix

$$M(r) := \begin{pmatrix} f(s_1,s_2,r,i,j) & \text{for } 0\le i<r \\ (-1)^{j-i-r} f(s_2,s_1,r,i-r,j) & \text{for } r\le i<2r \end{pmatrix}_{0\le i,j\le 2r-1}.$$

Then $\det M(r)$ seems to be "nice".

The determinant

For example, the matrix $M(2)$ is

$$\begin{pmatrix} 1 & \frac{2s_1}{(S)(S+2)} & \frac{s_1(s_1+1)}{(S+1)(S+2)^2(S+3)} & 0 \\ 0 & 1 & \frac{2(s_1+1)}{(S+2)(S+4)} & \frac{(s_1+1)(s_1+2)}{(S+3)(S+4)^2(S+5)} \\ 1 & -\frac{2s_2}{(S)(S+2)} & \frac{s_2(s_2+1)}{(S+1)(S+2)^2(S+3)} & 0 \\ 0 & 1 & -\frac{2(s_2+1)}{(S+2)(S+4)} & \frac{(s_2+1)(s_2+2)}{(S+3)(S+4)^2(S+5)} \end{pmatrix},$$

with $S = s_1 + s_2$.

A generalised determinant

Let

$$f(s_1,s_2,r,i,j) := \binom{r}{j-i} \frac{(s_1+i)_{j-i}}{(s_1+s_2+i+j-1)_{j-i}\,(s_1+s_2+r+2i)_{j-i}}.$$

Form the matrix

$$M(r_1,r_2) := \begin{pmatrix} f(s_1,s_2,r_1,i,j) \qquad \text{for } 0 \le i < r_2 \\ (-1)^{j-i-r_2} f(s_2,s_1,r_2,i-r_2,j) \\ \qquad\qquad \text{for } r_2 \le i < r_1+r_2 \end{pmatrix}_{0 \le i,j \le r_1+r_2-1}.$$

Then $\det M(r_1,r_2)$ seems to be "nice".

Where does this come from?

Where does this come from?

Consider the triangle

$$\triangle := \{(x,y) : x \geq 0,\ y \geq 0,\ x+y \leq 1\}.$$

Define the Jacobi-type weight function

$$\varpi_{\alpha,\beta,\gamma}(x,y) := x^\alpha y^\beta (1-x-y)^\gamma, \quad \alpha,\beta,\gamma > -1.$$

Define

$$E_n(f)_{\alpha,\beta,\gamma} = E_n(f)_{L^2(\varpi_{\alpha,\beta,\gamma})} := \inf_p \|f-p\|_{L^2(\varpi_{\alpha,\beta,\gamma})},$$

where the minimum is over all polynomials in two variables of degree at most n.

The main theorem

Theorem

Let $\alpha, \beta, \gamma > -1$, and let r be a positive integer. For $f \in W_2^r(\varpi_{\alpha,\beta,\gamma})$, we have

$$E_n(f)_{\alpha,\beta,\gamma} \le \frac{c}{n^r}\Big[E_{n-r}(\partial_1^r f)_{\alpha+r,\beta,\gamma+r} + E_{n-r}(\partial_2^r f)_{\alpha,\beta+r,\gamma+r} \\ + E_{n-r}(\partial_3^r f)_{\alpha+r,\beta+r,\gamma}\Big]$$

for $n \ge 3r$, where c is a constant independent of n and f. Here, $W_2^r(\varpi_{\alpha,\beta,\gamma})$ is a certain Sobolev space.

The third application: Best polynomial approximation

Which are the main ingredients?

Which are the main ingredients?

(1) The polynomials

$$J_{k,n}^{\alpha,\beta,\gamma}(x,y) := (x+y)^k J_k^{\alpha,\beta}\left(\frac{y-x}{x+y}\right) J_{n-k}^{2k+\alpha+\beta+1,\gamma}(1-2x-2y),$$
$$0 \le k \le n,$$

are orthogonal for the 2-dimensional Jacobi-type weight $\varpi_{\alpha,\beta,\gamma}$ on the triangle $\triangle$, where

$$J_n^{\alpha,\beta}(t) = \frac{1}{(n+\alpha+\beta+1)_n} P_n^{(\alpha,\beta)}(t),$$

with $P_n^{(\alpha,\beta)}$ the usual Jacobi polynomials.

Which are the main ingredients?

Which are the main ingredients?

(2) The following determinant evaluation:

Theorem

With $f(s_1,s_2,r,i,j)$ as defined before and

$$M(r) := \begin{pmatrix} f(s_1,s_2,r,i,j) & \text{for } 0 \le i < r \\ (-1)^{j-i-r} f(s_2,s_1,r,i-r,j) & \text{for } r \le i < 2r \end{pmatrix}_{0 \le i,j \le 2r-1}.$$

the determinant of $M(r)$ equals

$$(-1)^r \prod_{j=1}^{r} \frac{1}{(s_1+s_2+2r+j-2)_r}.$$

The third application: Best polynomial approximation

Let

$$f(s_1, s_2, r, i, j) := \binom{r}{j-i} \frac{(s_1+i)_{j-i}}{(s_1+s_2+i+j-1)_{j-i}\,(s_1+s_2+r+2i)_{j-i}}$$

and

$$M(r_1, r_2) := \begin{pmatrix} f(s_1, s_2, r_1, i, j) & \text{for } 0 \le i < r_2 \\ (-1)^{j-i-r_2} f(s_2, s_1, r_2, i-r_2, j) & \\ & \text{for } r_2 \le i < r_1 + r_2 \end{pmatrix}_{0 \le i,j \le r_1+r_2-1}.$$

The third application: Best polynomial approximation

Let

$$f(s_1,s_2,r,i,j) := \binom{r}{j-i} \frac{(s_1+i)_{j-i}}{(s_1+s_2+i+j-1)_{j-i}\,(s_1+s_2+r+2i)_{j-i}}$$

and

$$M(r_1,r_2) := \begin{pmatrix} f(s_1,s_2,r_1,i,j) \qquad \text{for } 0\le i<r_2 \\ (-1)^{j-i-r_2} f(s_2,s_1,r_2,i-r_2,j) \\ \qquad\qquad \text{for } r_2\le i<r_1+r_2 \end{pmatrix}_{0\le i,j\le r_1+r_2-1}.$$

How to calculate the determinant of $M(r_1,r_2)$?

The third application: Best polynomial approximation

Let

$$f(s_1, s_2, r, i, j) := \binom{r}{j-i} \frac{(s_1+i)_{j-i}}{(s_1+s_2+i+j-1)_{j-i}\,(s_1+s_2+r+2i)_{j-i}}$$

and

$$M(r_1, r_2) := \begin{pmatrix} f(s_1, s_2, r_1, i, j) & \text{for } 0 \le i < r_2 \\ (-1)^{j-i-r_2} f(s_2, s_1, r_2, i-r_2, j) & \\ & \text{for } r_2 \le i < r_1 + r_2 \end{pmatrix}_{0 \le i,j \le r_1+r_2-1}.$$

How to calculate the determinant of $M(r_1, r_2)$?

Laplace expansion again!

Laplace expansion

Laplace expansion

Write M for $M(r_1, r_2)$ for short.

Then

$$\det M = \sum_{0 \le k_0 < \dots < k_{r_2-1} \le r_1+r_2-1} (-1)^{\binom{r_2}{2}+\sum_{i=0}^{r_2-1} k_i} \det M_{0,\dots,r_2-1}^{k_0,\dots,k_{r_2-1}} \cdot \det M_{r_2,\dots,r_1+r_2-1}^{l_0,\dots,l_{r_1-1}},$$

where $M_{b_1,\dots,b_r}^{a_1,\dots,a_r}$ denotes the submatrix of M consisting of rows $a_1, \dots, a_r$ and columns $b_1, \dots, b_r$, and $\{l_0, \dots, l_{r_1-1}\}$ is the complement of $\{k_0, \dots, k_{r_2-1}\}$ in $\{0, 1, \dots, r_1 + r_2 - 1\}$.

Laplace expansion

Write M for $M(r_1, r_2)$ for short.

Then

$$\det M = \sum_{0 \le k_0 < \cdots < k_{r_2-1} \le r_1+r_2-1} (-1)^{\binom{r_2}{2}+\sum_{i=0}^{r_2-1} k_i} \det M_{0,\dots,r_2-1}^{k_0,\dots,k_{r_2-1}} \cdot \det M_{r_2,\dots,r_1+r_2-1}^{l_0,\dots,l_{r_1-1}},$$

where $M_{b_1,\dots,b_r}^{a_1,\dots,a_r}$ denotes the submatrix of M consisting of rows $a_1, \dots, a_r$ and columns $b_1, \dots, b_r$, and $\{l_0, \dots, l_{r_1-1}\}$ is the complement of $\{k_0, \dots, k_{r_2-1}\}$ in $\{0, 1, \dots, r_1 + r_2 - 1\}$.

Also here, it turns out that it is not difficult to evaluate the minors which appear in this sum.

The third application: Best polynomial approximation

After a lot of simplification, one arrives at

$$(-1)^{r_1 r_2} \prod_{i=0}^{r_1+r_2-1} \frac{(s_2)_i\,(s_1+s_2+i-2)!\,(i+s_1+s_2-1)_i}{(s_1+s_2+2i-2)!\,(r_1+r_2-i-1)!\,(s_1+s_2+r_1+r_2+i-2)!}$$

$$\times \prod_{i=0}^{r_2-1} \frac{(s_1+s_2+r_1+2i-2)!\,(s_1+s_2+r_1+2i-1)!\,(r_1+i)!}{(s_1)_i\,(s_1+s_2+r_1+i-2)!\,(r_1+r_2-1)!\,(s_1+s_2)_{r_1+r_2-1}}$$

$$\times \prod_{i=0}^{r_1-1} \frac{(s_1+s_2+r_2+2i-2)!\,(s_1+s_2+r_2+2i-1)!\,(r_2+i)!}{(s_2)_i\,(s_1+s_2+r_2+i-2)!}$$

$$\times \sum_{0\le k_0<\cdots<k_{r_2-1}\le r_1+r_2-1} (-1)^{\sum_{i=0}^{r_2-1} k_i} \prod_{0\le i<j\le r_2-1} (k_j-k_i)^2(k_i+k_j+s_1+s_2-1)^2$$

$$\cdot \prod_{i=0}^{r_2-1} \frac{(s_1+s_2-1+2k_i)}{(s_1+s_2-1)} \cdot \frac{(s_1+s_2-1)_{k_i}\,(s_1)_{k_i}\,(-r_1-r_2+1)_{k_i}}{k_i!\,(s_2)_{k_i}\,(s_1+s_2+r_1+r_2-1)_{k_i}}.$$

After a lot of simplification, one arrives at

$$(-1)^{r_1 r_2} \prod_{i=0}^{r_1+r_2-1} \frac{(s_2)_i\,(s_1+s_2+i-2)!\,(i+s_1+s_2-1)_i}{(s_1+s_2+2i-2)!\,(r_1+r_2-i-1)!\,(s_1+s_2+r_1+r_2+i-2)!}$$

$$\times \prod_{i=0}^{r_2-1} \frac{(s_1+s_2+r_1+2i-2)!\,(s_1+s_2+r_1+2i-1)!\,(r_1+i)!}{(s_1)_i\,(s_1+s_2+r_1+i-2)!\,(r_1+r_2-1)!\,(s_1+s_2)_{r_1+r_2-1}}$$

$$\times \prod_{i=0}^{r_1-1} \frac{(s_1+s_2+r_2+2i-2)!\,(s_1+s_2+r_2+2i-1)!\,(r_2+i)!}{(s_2)_i\,(s_1+s_2+r_2+i-2)!}$$

$$\times \sum_{0\le k_0<\cdots<k_{r_2-1}\le r_1+r_2-1} (-1)^{\sum_{i=0}^{r_2-1} k_i} \prod_{0\le i<j\le r_2-1} (k_j-k_i)^2(k_i+k_j+s_1+s_2-1)^2$$

$$\cdot \prod_{i=0}^{r_2-1} \frac{(s_1+s_2-1+2k_i)}{(s_1+s_2-1)} \cdot \frac{(s_1+s_2-1)_{k_i}\,(s_1)_{k_i}\,(-r_1-r_2+1)_{k_i}}{k_i!\,(s_2)_{k_i}\,(s_1+s_2+r_1+r_2-1)_{k_i}}.$$

Now apply the $p=0$, $q\to 1$ case of the transformation formula.

The third application: Best polynomial approximation

An elliptic generalisation:

Theorem

We consider the $(r_1+r_2)\times(r_1+r_2)$ *matrix* $F=(f'_{ij})$*, where*

$$f'_{ij}=\begin{cases} f_{ij}(s_1,s_2,t_1,t_2,r_1,r_2), & \text{for } 0\le i\le r_2-1,\\ f_{i-r_2,j}(s_2,s_1,s_1s_2/t_1,s_1s_2/t_2,r_2,r_1), & \text{for } r_2\le i\le r_1+r_2-1,\end{cases}$$

with

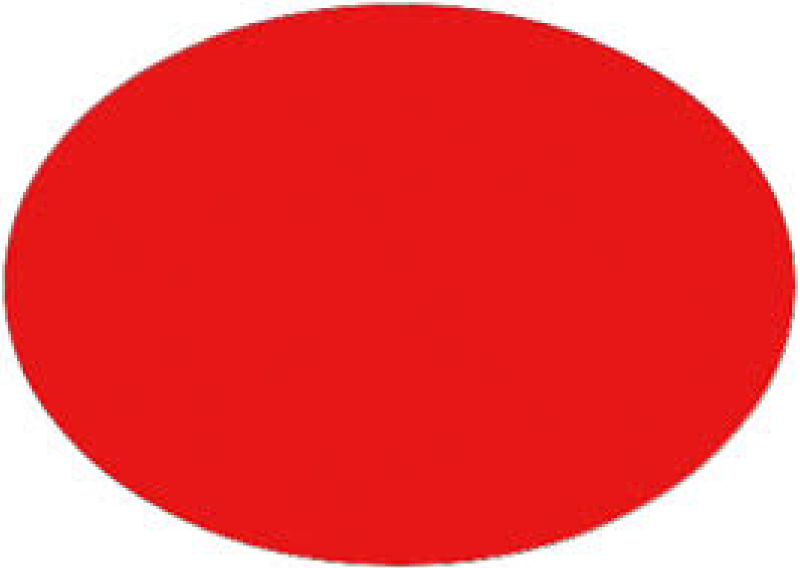
$$f_{ij}=f_{ij}(s_1,s_2,t_1,t_2,r_1,r_2):=q^{\binom{j-i}{2}+r_2(j-i)}\frac{(q;q,p)_{r_1}}{(q;q,p)_{r_1-j+i}} \times\frac{\left(s_1q^i,t_1q^i,t_2q^i,s_1s_2^2q^{r_1-r_2+i}/t_1t_2;q,p\right)_{j-i}}{\left(q,s_1s_2q^{i+j-1},s_1s_2q^{r_1+2i};q,p\right)_{j-i}}.$$

The third application: Best polynomial approximation

Then

$$\det F = (-1)^{r_1 r_2} \left(\frac{t_1 t_2}{s_2}\right)^{r_1 r_2} q^{\frac{1}{2} r_1 r_2 (r_1 + 4 r_2 - 3)} \\ \times \prod_{j=1}^{r_1} \frac{\left(s_2 q^{-r_2+j}/t_1, s_2 q^{-r_2+j}/t_2, s_1 s_2 q^{-r_2+j}/t_1 t_2; q, p\right)_{r_2}}{\left(s_1 s_2 q^{r_1+r_2+j-2}; q, p\right)_{r_2}}.$$

Elliptic?



Transient Transcendence in Transylvania?