Congruence properties of Taylor coefficients of modular forms

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In a talk at a Workshop on "Computer Algebra and Combinatorics" at the Erwin Schrödinger Institute in Vienna in November 2017, Dan Romik presented his investigations on Taylor coefficients of Jacobi's theta function $\theta_3(\tau)$ at $\tau = i$.

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The setup:

Jacobi's theta function θ_3 is defined by

$$\theta_3(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \text{with } q = e^{i\pi\tau}.$$

The Taylor expansion that Romik was interested in is

$$\theta_3\left(i\frac{1-z}{1+z}\right) = \theta_3(i)(1+z)^{1/2} \sum_{n=0}^{\infty} \frac{d(n)}{(2n)!} \Phi^n z^{2n},$$

where $\Phi = \Gamma^8(1/4)/(128\pi^4)$ and $\theta_3(i) = \pi^{1/4}/\Gamma(3/4)$.

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where $\Phi = \Gamma^8(1/4)/(128\pi^4)$ and $\theta_3(i) = \pi^{1/4}/\Gamma(3/4)$.
The first few values turn out to be

$$1, 1, -1, 51, 849, -26199, 1341999, 82018251, 18703396449, \\ -993278479599, -78795859032801, 38711746282537251, \dots$$

Indeed, Romik showed that the sequence $\big(d(n)\big)_{n\geq 0}$ is a sequence of integers.

Taylor coefficients of $\theta_3(\tau)$ at $\tau = i$

At the end of his talk, based on computational data, Romik reported that the sequence $\big(d(n)\big)_{n\geq 0}$ seemed to satisfy interesting congruence properties.

Conjecture

1. *$d(n)$ eventually vanishes modulo any prime power p^e with $p \equiv 3 \pmod 4$;*
2. *$d(n)$ is eventually periodic modulo any prime power p^e with $p \equiv 1 \pmod 4$;*
3. *$d(n)$ is purely periodic modulo any 2-power 2^e.*

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- Scherer [2019] proved Item 1 for primes (i.e., for $e = 1$).
- Guerzhoy, Mertens and Rolen [2019] proved Item 2 in the more general context of Taylor coefficients of modular forms of half integer weight at complex multiplication points.
- Wakhare [2020] revisited Item 2 for primes (i.e., for $e = 1$) and proved fine results on period lengths.

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Periodicity of Taylor coefficients of modular forms

Square roots of central values of Hecke L-series

by Fernando Rodriguez Villegas and Don Zagier

§1. Introduction

In [2] numerical examples were produced suggesting that the "algebraic" part of central values of certain Hecke L-series are perfect squares. More precisely, let ψ_1 be the grossencharacter of $\mathbf{Q}(\sqrt{-7})$ defined by

$$\psi_1(\mathfrak{a}) = \left(\frac{m}{7}\right)\alpha \qquad \text{if} \quad \mathfrak{a} = (\alpha), \quad \alpha = \frac{m + n\sqrt{-7}}{2} \in \mathbf{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$$

and consider the central value $L(\psi_1^{2k-1}, k)$ of the L-series associated to an odd power of ψ_1. This value vanishes for k even by virtue of the functional equation, but for k odd one has

$$(1) \qquad L(\psi_1^{2k-1}, k) = 2\,\frac{(2\pi/\sqrt{7})^k\,\Omega^{2k-1}}{(k-1)!}\,A(k), \qquad \Omega = \frac{\Gamma(\frac{1}{7})\Gamma(\frac{2}{7})\Gamma(\frac{4}{7})}{4\pi^2},$$

with $A(1) = 1/4$, $A(3) = A(5) = 1$, $A(7) = 9$, $A(9) = 49$, $\ldots$, $A(33) = 44762286327255^2$, suggesting the conjecture

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Given a modular form $f(\tau)$ and complex multiplication point τ_0, this procedure constructs a sequence $\big(p_n(t)\big)_{n\ge 0}$ via a recurrence of the form

$$p_{n+1}(t) = a_n(t)p_n'(t) + b_n(t)p_n(t) + c_n(t)p_{n-1}(t), \quad n \ge 2,$$

where $a_n(t), b_n(t), c_n(t)$ are polynomials in t and n with integer coefficients.
The Taylor coefficients of $f(\tau)$ at $\tau = \tau_0$ are then given by

$$p_n(0), \quad n = 0, 1, 2, \dots,$$

up to some renormalisation.

Back to θ_3:

Romik's Taylor expansion at $\tau = i$:

$$\theta_3\left(i\frac{1-z}{1+z}\right) = \theta_3(i)(1+z)^{1/2}\sum_{n=0}^{\infty}\frac{d(n)}{(2n)!}\Phi^n z^{2n},$$

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where $\Phi = \Gamma^8(1/4)/(128\pi^4)$ and $\theta_3(i) = \pi^{1/4}/\Gamma(3/4)$.
The procedure of Rodríguez Villegas and Zagier yields

$$p_{n+1}(t) = (\tfrac{1}{6} - 96t^2)p_n'(t) + 16(4n+1)tp_n(t) \\ - n(n-\tfrac{1}{2})(256t^2 + \tfrac{4}{3})p_{n-1}(t),$$

with $p_{-1}(t) = 0$ and $p_0(t) = 1$, and we have $p_{2n+1}(0) = 0$ and

$$d(n) = 2^{-n}p_{2n}(0).$$

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- O'Sullivan and Risager [2013] showed that coefficients produced by such a recursive scheme are (eventually) periodic modulo any fixed integer M.

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Non-vanishing of Taylor coefficients and Poincaré series

Cormac O'Sullivan · Morten S. Risager

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Abstract We prove recursive formulas for the Taylor coefficients of cusp forms, such as Ramanujan's Delta function, at points in the upper half-plane. This allows us to show the non-vanishing of all Taylor coefficients of Delta at CM points of small discriminant as well as the non-vanishing of certain Poincaré series. At a "generic" point, all Taylor coefficients are shown to be non-zero. Some conjectures on the Taylor coefficients of Delta at CM points are stated.

for $m \geq 0$ with period 4. For $l = 7$ we have $q_{m,\mathfrak{z}_{-4}}(0)$ mod 7 equaling

$$1, 0, 2, 0, 6, 0, 1, 0, 5, 0, 0, 0, 4, 0, 0, 0, 4, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, \dots \qquad (6.2)$$

for $m \geq 0$ with all further terms $\equiv 0$ mod 7. Hence $q_{m,\mathfrak{z}_{-4}}(0)$ mod 7 has period 1 for $m \geq 21$. We next prove that (6.1), (6.2) are typical. (In what follows, by a *period* we always understand the least eventual period of a sequence.)

Theorem 6.1 *Let l be in $\mathbb{Z}_{\geq 1}$ and $\mathfrak{z} = \mathfrak{z}_D$ a CM point.*

(i) *The sequence $q_{m,\mathfrak{z}}(0)$ mod $l\mathcal{O}_K$ becomes periodic.*
(ii) *If $q_{m,\mathfrak{z}}(0)$ mod $l\mathcal{O}_K$ is periodic from $m = \alpha$ with period β then $\alpha + \beta \leq l|\mathcal{O}_K/l\mathcal{O}_K|^{2l}$.*

Proof Recall (5.16) and the map $\mathcal{O}_K[t] \to \mathcal{R}_l$ given by $p \mapsto \overline{p}$. Since $\frac{d}{dt}t^l \equiv 0$ mod l we see that $\overline{q}_n(t)$ also satisfies the recursion (1.10) which depends only on n mod l; see Remark 3. We can therefore conclude that if, for some rational integers $i_0 < j_0$,

$$\overline{q}_{i_0}(t) = \overline{q}_{j_0}(t), \qquad \overline{q}_{i_0+1}(t) = \overline{q}_{j_0+1}(t), \qquad i_0 \equiv j_0 \bmod l, \qquad (6.3)$$

then $\overline{q}_{i_0+n}(t) = \overline{q}_{j_0+n}(t)$ for all $n \in \mathbb{Z}_{\geq 0}$. But since

$$(\overline{q}_{i_0}, \overline{q}_{i_0+1}, \overline{i_0}), \ (\overline{q}_{j_0}, \overline{q}_{j_0+1}, \overline{j_0}) \in \mathcal{R}_l^2 \times (\mathbb{Z}/l\mathbb{Z}),$$

the box principle implies that (6.3) is true for some $0 \leq i_0 < j_0 \leq |\mathcal{R}_l^2 \times (\mathbb{Z}/l\mathbb{Z})| = l|\mathcal{O}_K/l\mathcal{O}_K|^{2l}$. Therefore, $q_m(0)$ mod l is periodic from at most $m = i_0$ with period dividing $j_0 - i_0$. □

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Consider the sequence $\big(p_n(t)\big)_{n\ge0}$ of polynomials $p_n(t)=\sum_{k\ge0}p_{n,k}t^k$ given by

$$p_{n+1}(t)=a_n(t)p_n'(t)+b_n(t)p_n(t)+c_n(t)p_{n-1}(t),$$

where $a_n(t),b_n(t),c_n(t)$ are given polynomials in t and n with integer coefficients.

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Claim. For considering $\big(p_n(t) \text{ modulo } M\big)_{n\ge0}$ (coefficient-wise), it suffices to consider the above recurrence modulo M for the sequence $\big(p_n^{(M)}(t)\big)_{n\ge0}$ of truncated polynomials

$$p_n^{(M)}(t)=\sum_{k=0}^{M-1}\big(p_{n,k} \bmod M\big)\cdot t^k.$$

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We consider now the map

$$\varphi : \left(n \bmod M, p_{n-1}^{(M)}(t), p_n^{(M)}(t)\right) \mapsto \left(n+1 \bmod M, p_n^{(M)}(t), p_{n+1}^{(M)}(t)\right)$$

defined via the recurrence

$$p_{n+1}^{(M)}(t) = a_n(t)\left(p_n^{(M)}(t)\right)' + b_n(t)p_n^{(M)}(t) + c_n(t)p_{n-1}^{(M)}(t) \quad \bmod (M, t^M).$$

This defines a map on a *finite* space, namely on

$$(\mathbb{Z}/M\mathbb{Z}) \times \left((\mathbb{Z}/M\mathbb{Z})[t]/(t^M)\right) \times \left((\mathbb{Z}/M\mathbb{Z})[t]/(t^M)\right).$$

More precisely, this space has M^{2M+1} elements.

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More precisely, this space has M^{2M+1} elements.

Consequently, the map φ must be (eventually) periodic, and thus also

$$\left(p_n(0) \bmod M\right) = p_n^{(M)}(0),$$

which gives our Taylor coefficients modulo M.

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On the other hand:

- Obtain only astronomic bounds on period length (namely M^{2M+1}).
- The argument cannot decide whether the Taylor coefficients eventually vanish modulo M, or when.

Taylor coefficients of $\theta_3(\tau)$ at $\tau = i$

Again back to θ_3:

Conjecture

1. *$d(n)$ eventually vanishes modulo any prime power p^e with $p \equiv 3 \pmod 4$;*
2. *$d(n)$ is eventually periodic modulo any prime power p^e with $p \equiv 1 \pmod 4$;*
3. *$d(n)$ is purely periodic modulo any 2-power 2^e.*

We have

$$d(n) = 2^{-n} p_{2n}(0),$$

where

$$p_{n+1}(t) = (\tfrac{1}{6} - 96t^2)p_n'(t) + 16(4n+1)tp_n(t) \\ - n(n - \tfrac{1}{2})(256t^2 + \tfrac{4}{3})p_{n-1}(t),$$

with $p_{-1}(t) = 0$ and $p_0(t) = 1$.

The previous argument proves periodicity of $d(n)$ modulo p^e for primes p different from 2 and 3.

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Taylor coefficients of $\theta_3(\tau)$ at $\tau = i$

Theorem

(1) *Let* p *be a prime number with* $p \equiv 3 \pmod 4$, *and let* e *be a positive integer. Then* $d(n) \equiv 0 \pmod{p^e}$ *for* $n \ge \lceil \frac{ep^2}{2} \rceil$.
(2) *Let* p *be a prime number with* $p \equiv 1 \pmod 4$, *and let* e *be a positive integer. Then the sequence* $\big(d(n)\big)_{n \ge e+1}$ *is purely periodic modulo* p^e *with (not necessarily minimal) period length* $\frac{1}{4} p^{e-1}(p-1)^2$.
(3) *Let* e *be a positive integer. The sequence* $(d(n))_{n \ge 0}$, *when taken modulo any fixed 2-power* 2^e *with* $e \ge 3$, *is purely periodic with (not necessarily minimal) period length* 2^{e-1}. *Modulo* 4, *the sequence is purely periodic with period length* 4, *the first few values of the sequence (modulo* 4*) being given by*

$$1, 1, 3, 3, 1, \dots .$$

Romik's setup

Recall the expansion

$$\theta_3\left(i\frac{1-z}{1+z}\right) = \theta_3(i)(1+z)^{1/2}\sum_{n=0}^{\infty}\frac{d(n)}{(2n)!}\Phi^n z^{2n},$$

where $\Phi = \Gamma^8(1/4)/(128\pi^4)$ and $\theta_3(i) = \pi^{1/4}/\Gamma(3/4)$.
We start by writing

$$\sigma_3(z) = \frac{1}{\sqrt{1+z}}\theta_3\left(i\frac{1-z}{1+z}\right),$$

or, equivalently,

$$\theta_3(ix) = \sqrt{\frac{2}{1+x}}\sigma_3\left(\frac{1-x}{1+x}\right).$$

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We have

$$\theta_3\left(i\frac{G(1-t)}{G(t)}\right) = \sqrt{G(t)},$$

where $G(t) = \frac{2}{\pi}K(\sqrt{t})$ with

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Substitution of $x = G(1-t)/G(t)$ in the first line yields

$$\sqrt{2}\,\sigma_3\left(\frac{G(t)-G(1-t)}{G(t)+G(1-t)}\right) = \sqrt{G(t)+G(1-t)}.$$

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Our last identity:

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Romik has shown that

$$G(t)-G(1-t)=\frac{4\,\Gamma^2(\frac{3}{4})}{\pi^{3/2}}\left(t-\tfrac{1}{2}\right){}_2F_1\left[\begin{matrix}\frac{3}{4},\frac{3}{4}\\ \frac{3}{2}\end{matrix};4\left(t-\tfrac{1}{2}\right)^2\right],$$

$$G(t)+G(1-t)=\frac{\Gamma^2(\frac{1}{4})}{\pi^{3/2}}\,{}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{1}{4}\\ \frac{1}{2}\end{matrix};4\left(t-\tfrac{1}{2}\right)^2\right].$$

Now we need to define some auxiliary sequences.

Romik's setup

We define the sequences $(u(n))_{n\geq 0}$ and $(v(n))_{n\geq 0}$ by

$$U(s) := \sum_{n\geq 0} \frac{u(n)}{(2n+1)!} s^{2n+1} = s \frac{{}_2F_1\left[\begin{matrix}\frac{3}{4}, \frac{3}{4} \\ \frac{3}{2}\end{matrix}; 4s^2\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{1}{4} \\ \frac{1}{2}\end{matrix}; 4s^2\right]}.$$

and

$$V(s) := \sum_{n\geq 0} \frac{v(n)}{2^n (2n)!} s^{2n} = {}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{1}{4} \\ \frac{1}{2}\end{matrix}; 4s^2\right]^{1/2}.$$

Romik has shown that $(u(n))_{n\geq 0}$ and $(v(n))_{n\geq 0}$ are integer sequences.

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and

$$V(s) := \sum_{n\geq 0} \frac{v(n)}{2^n(2n)!} s^{2n} = {}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{1}{4} \\ \frac{1}{2}\end{matrix}; 4s^2\right]^{1/2}.$$

Romik has shown that $(u(n))_{n\geq 0}$ and $(v(n))_{n\geq 0}$ are integer sequences.

Our last identity can then be written in the form

$$\sum_{n=0}^{\infty} \frac{d(n)}{2^n\,(2n)!} U^{2n}(s) = V(s).$$

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Romik's setup

Our last identity can then be written in the form

$$\sum_{n=0}^{\infty} \frac{d(n)}{2^n\,(2n)!}\,U^{2n}(s) = V(s).$$

Define the lower-triangular matrix $\big(R(n,k)\big)_{n,k\ge 0}$ by

$$R(n,k) := 2^{(n-k)/2}\,\frac{n!}{k!}\,\langle s^n\rangle\; U^k(s).$$

It is not difficult to see that $R(n,k)$ is always an integer.
Comparison of coefficients of $\frac{s^{2k}}{2^k(2k)!}$ yields

$$\sum_{n=0}^{k} R(2k,2n)d(n) = v(k).$$

Romik's setup

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Theorem (Romik)

The sequence $\big(d(n)\big)_{n\geq 0}$ is a sequence of integers. Moreover, the $d(n)$'s can be computed via the relation

$$d(n) = v(n) - \sum_{k=0}^{n-1} R(2n, 2k) d(k) \quad \text{and } d(0) = 1,$$

or by

$$d(n) = \sum_{k=0}^{n} R^{-1}(2n, 2k) v(k).$$

Define auxiliary sequences $(u(n))_{n\ge0}$ and $(v(n))_{n\ge0}$ and the matrix $\big(R(n,k)\big)_{n,k\ge0}$ by

$$U(s) := \sum_{n\ge0} \frac{u(n)}{(2n+1)!} s^{2n+1} = s\frac{{}_2F_1\left[\begin{matrix}\frac{3}{4},\frac{3}{4}\\ \frac{3}{2}\end{matrix};4s^2\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{1}{4}\\ \frac{1}{2}\end{matrix};4s^2\right]},$$

$$\sum_{n\ge0} \frac{v(n)}{2^n(2n)!} s^{2n} = {}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{1}{4}\\ \frac{1}{2}\end{matrix};4s^2\right]^{1/2},$$

$$R(n,k) := 2^{(n-k)/2}\frac{n!}{k!}\left\langle s^n\right\rangle U^k(s).$$

Then the sequence $(d(n))_{n\ge0}$ can be computed by

$$d(n) = v(n) - \sum_{k=0}^{n-1} R(2n,2k)d(k) \quad \text{and } d(0)=1,$$

or by

$$d(n) = \sum_{k=0}^{n} R^{-1}(2n,2k)v(k).$$

The auxiliary sequence $(u(n))_{n\geq 0}$

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An equivalent, recursive, definition is

$$u(n) = \prod_{j=1}^{n} (4j-1)^2 - \sum_{m=0}^{n-1} u(m) \binom{2n+1}{2m+1} \prod_{j=1}^{n-m} (4j-3)^2, \quad \text{with } u(0) = 1.$$

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Proposition

Given an odd prime p and a positive integer e, the number $u(n)$ is divisible by p^e for $n \geq \lfloor \frac{ep^2}{2} \rfloor$.

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$$v(n) = 2^{n-1}\prod_{j=1}^{n}(4j-3)^2 - \frac{1}{2}\sum_{m=1}^{n-1}\binom{2n}{2m} v(m)v(n-m), \quad \text{with } v(0) = 1.$$

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The matrix $(R(n,k))_{n,k\ge 0}$ is defined by

$$R(n,k) := 2^{(n-k)/2}\frac{n!}{k!}\left\langle s^n\right\rangle U^k(s),$$

where

$$U(s) := \sum_{n\ge 0}\frac{u(n)}{(2n+1)!}s^{2n+1} = s\frac{{}_2F_1\left[\begin{matrix}\frac{3}{4},\frac{3}{4}\\ \frac{3}{2}\end{matrix};4s^2\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{1}{4}\\ \frac{1}{2}\end{matrix};4s^2\right]}.$$

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Expanded out, this is

$$2^{-(n-k)}R(2n,2k) = \sum{}'\frac{(2n)!}{\prod_{i=1}^{2n} i!^{c_i}c_i!}\prod_{i=1}^{2n}u\left(\tfrac{i-1}{2}\right)^{c_i},$$

where the sum $\sum'$ is over all tuples $(c_1,c_2,\dots,c_{2n})$ of non-negative integers c_i, where $c_{2j}=0$ for all j, and

$$c_1+c_3+\cdots+c_{2n-1}=2k,$$
$$c_1+3c_3+\cdots+(2n-1)c_{2n-1}=2n.$$

The case where $p \equiv 3 \pmod 4$.

The auxiliary matrix $(R(n,k))_{n,k\ge0}$

The case where $p \equiv 3 \pmod 4$.

Proposition

Let N, K, e, f be non-negative integers with $N \ge K$, and let p be a prime number with $p \equiv 3 \pmod 4$. If $N \ge ep^2$ and $K < fp^2$, then

$$\sum_{(c_i)\in\mathcal{P}^o_{N,K}} \frac{N!}{\prod_{i=1}^N i!^{c_i}\, c_i!} \prod_{i=1}^N u\left(\tfrac{i-1}{2}\right)^{c_i}$$

is divisible by p^{e-f+1}.

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is divisible by p^{e-f+1}.

In the special case where $N = 2n$ and $K = 2k$ we conclude:

Corollary

Let n, k, e, f be non-negative integers with $n \ge k$, and let p be a prime number with $p \equiv 3 \pmod 4$. If $n \ge \lceil \frac{ep^2}{2} \rceil$ and $k < \lceil \frac{fp^2}{2} \rceil$, then $R(2n, 2k)$ is divisible by p^{e-f+1}.

Taylor coefficients of $\theta_3(\tau)$ at $\tau = i$

Taylor coefficients of $\theta_3(\tau)$ at $\tau = i$

Theorem

(1) *Let* p *be a prime number with* $p \equiv 3 \pmod 4$, *and let* e *be a positive integer. Then* $d(n) \equiv 0 \pmod{p^e}$ *for* $n \geq \lceil \frac{ep^2}{2} \rceil$.

(2) *Let* p *be a prime number with* $p \equiv 1 \pmod 4$, *and let* e *be a positive integer. Then the sequence* $\big(d(n)\big)_{n \geq e+1}$ *is purely periodic modulo* p^e *with (not necessarily minimal) period length* $\frac{1}{4}p^{e-1}(p-1)^2$.

(3) *Let* e *be a positive integer. The sequence* $(d(n))_{n\geq 0}$, *when taken modulo any fixed 2-power* 2^e *with* $e \geq 3$, *is purely periodic with (not necessarily minimal) period length* 2^{e-1}. *Modulo* 4, *the sequence is purely periodic with period length* 4, *the first few values of the sequence (modulo* 4*) being given by*

$$1, 1, 3, 3, 1, \dots .$$

Taylor coefficients of $\theta_3(\tau)$ at $\tau = i$

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Proof.

We do an induction on n. Let $n \geq \lceil \frac{ep^2}{2} \rceil$. Recall that

$$d(n) = v(n) - \sum_{k=0}^{n-1} R(2n, 2k) d(k) \quad \text{and } d(0) = 1,$$

Under the above assumption, we know that $v(n)$ is divisible by p^e. On the other hand, consider some k with $\lceil \frac{(f-1)p^2}{2} \rceil \leq k < \lceil \frac{fp^2}{2} \rceil$. We have

$$v_p\big(R(2n, 2k) d(k)\big) \geq (e - f + 1) + (f - 1) = e. \quad \square$$

Here, $v_p(\alpha) =$ maximal β such that $p^\beta \mid \alpha$.

The case where $p \equiv 1 \pmod 4$.

The inverse of the auxiliary matrix $(R(n,k))_{n,k\ge0}$

The case where $p \equiv 1 \pmod 4$.

Recall that

$$R(n,k) := 2^{(n-k)/2} \frac{n!}{k!} \left\langle s^n \right\rangle U^k(s).$$

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Recall that

$$R(n,k) := 2^{(n-k)/2} \frac{n!}{k!} \langle s^n \rangle \, U^k(s).$$

By Lagrange inversion, we obtain

$$R^{-1}(n,k) = 2^{(n-k)/2} \frac{(n-1)!}{(k-1)!} \left\langle t^{-k} \right\rangle U^{-n}(t).$$

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Expanded out, this is

$$R^{-1}(2n+k,k) = \sum_{m\ge 0}{\sum}'(-1)^m\frac{(2n+m+k-1)(2n+m+k-2)\cdots k}{n!\,(2n-1)!!}$$

$$\cdot\frac{(2n)!}{2!^{c_3}c_3!\,4!^{c_5}c_5!\cdots(2n)!^{c_{2n+1}}c_{2n+1}!}\prod_{i=1}^{2n+1}\left(\frac{u\left(\frac{i-1}{2}\right)}{i}\right)^{c_i}.$$

where the sum ${\sum}'$ is over all tuples $(0,0,c_3,0,c_5,0,\dots,0,c_{2n+1})$ of non-negative integers c_i with

$$c_3+c_5+\cdots+c_{2n+1}=m,$$

$$3c_3+5c_5+\cdots+(2n+1)c_{2n+1}=2n+m.$$

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The inverse of the auxiliary matrix $(R(n,k))_{n,k\ge 0}$

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Proposition

Let p be a prime with $p \equiv 1 \pmod 4$. Then, for all positive integers n, k, and e, we have

$$p^{\lfloor 2k/p\rfloor} R^{-1}\left(2n + p^{e-1}(p-1), 2k\right) \equiv u^{p^{e-1}}\left(\tfrac{p-1}{2}\right) \cdot (-1)^{(p-5)/4} p^{\lfloor 2k/p\rfloor} R^{-1}(2n, 2k) \pmod{p^e},$$

for $n \ge e+1$.
In particular, the sequence $\left(p^{\lfloor 2k/p\rfloor} R^{-1}(2n,2k)\right)_{n\ge e+1}$, when taken modulo any fixed p-power p^e with $e \ge 1$, is purely periodic with (not necessarily minimal) period length $\frac{1}{4}p^{e-1}(p-1)^2$.

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Theorem

(2) *Let p be a prime number with $p \equiv 1 \pmod 4$, and let e be a positive integer. Then the sequence $\big(d(n)\big)_{n\ge e+1}$ is purely periodic modulo p^e with (not necessarily minimal) period length $\frac{1}{4}p^{e-1}(p-1)^2$.*

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Proof.

Recall that

$$d(n) = \sum_{k=0}^{n} R^{-1}(2n, 2k) v(k).$$

Theorem

(2) *Let p be a prime number with $p \equiv 1 \pmod 4$, and let e be a positive integer. Then the sequence $\big(d(n)\big)_{n \ge e+1}$ is purely periodic modulo p^e with (not necessarily minimal) period length $\frac{1}{4} p^{e-1}(p-1)^2$.*

Proof.

Recall that

$$d(n) = \sum_{k=0}^{n} R^{-1}(2n, 2k) v(k).$$

We know that $v(k) \equiv 0 \pmod{p^e}$ for $k \ge \lceil \frac{ep}{2} \rceil$. Consequently, we may truncate the sum on on the right-hand side when we consider both sides modulo p^e,

$$d(n) \equiv \sum^{\lfloor ep/2 \rfloor} R^{-1}(2n, 2k) v(k) \pmod{p^e}.$$

Proof (continued).

We have

$$d(n) \equiv \sum_{k=0}^{\lfloor ep/2 \rfloor} R^{-1}(2n, 2k) v(k) \pmod{p^e}.$$

Taylor coefficients of $\theta_3(\tau)$ at $\tau = i$

Proof (continued).

We have

$$d(n) \equiv \sum_{k=0}^{\lfloor ep/2 \rfloor} R^{-1}(2n, 2k) v(k) \pmod{p^e}.$$

We know that $v(k) \equiv 0 \pmod{p^e}$ for $k \geq \lceil \frac{ep}{2} \rceil$. In other words, we have $v(k) = p^{\lfloor 2k/p \rfloor} V(k, p)$, where $V(k, p)$ is an integer. Altogether, this leads to

$$d(n) \equiv \sum_{k=1}^{\lfloor ep/2 \rfloor} R^{-1}(2n, 2k) p^{\lfloor 2k/p \rfloor} V(k, p) \pmod{p^e}, \quad \text{for } n \geq 1.$$

By the previous theorem, the sequence $\left(p^{\lfloor 2k/p \rfloor} R^{-1}(2n, 2k)\right)_{n \geq e+1}$ is purely periodic when taken modulo p^e with (not necessarily minimal) period length $\frac{1}{4} p^{e-1}(p-1)^2$.

Since, by the above congruence, the sequence $\left(d(n)\right)_{n \geq e+1}$, when taken modulo p^e, is a finite linear combination of the sequences $\left(p^{\lfloor 2k/p \rfloor} R^{-1}(2n, 2k)\right)_{n \geq e+1}$, $k = 1, 2, \ldots$, it has the same periodicity behaviour. □

The case where $p = 2$.

The case where $p=2$.

The sequence $(v(n))_{n\ge0}$ is defined by

$$\sum_{n\ge0}\frac{v(n)}{2^n(2n)!}s^{2n}={}_2F_1\left[\begin{matrix}\frac14,\frac14\\ \frac12\end{matrix};4s^2\right]^{1/2}.$$

The case where $p = 2$.

The sequence $(v(n))_{n\geq 0}$ is defined by

$$\sum_{n\geq 0} \frac{v(n)}{2^n(2n)!} s^{2n} = {}_2F_1\left[\begin{matrix}\frac{1}{4}, \frac{1}{4} \\ \frac{1}{2}\end{matrix}; 4s^2\right]^{1/2}.$$

Proposition

Let $x(j)$, $j = 0, 1, 2, \ldots$, be a sequence of integers with $x(0) = 1$, $x(1)$ and $x(2)$ odd. Then the coefficients $v_{\mathbf{x}}(n)$ in the expansion

$$\sum_{n\geq 0} \frac{v_{\mathbf{x}}(n)}{2^n\,(2n)!} t^n = \left(1 + \sum_{j\geq 1} \frac{x(j)}{(2j)!} t^{2j}\right)^{1/2}$$

are integers. Moreover, for all integers $e \geq 3$, the sequence $(v_{\mathbf{x}}(n))_{n\geq 0}$ is purely periodic modulo 2^e with (not necessarily minimal) period length 2^{e-1}.

The case where $p = 2$.

The case where $p=2$.

Recall that

$$R^{-1}(n,k)=2^{(n-k)/2}\frac{(n-1)!}{(k-1)!}\left\langle t^{-k}\right\rangle U^{-n}(t).$$

The case where $p=2$.

Recall that

$$R^{-1}(n,k) = 2^{(n-k)/2}\frac{(n-1)!}{(k-1)!}\left\langle t^{-k}\right\rangle U^{-n}(t).$$

Proposition

For fixed n and any 2-power 2^e, the sequence $\left(R^{-1}(k+2n,k)\right)_{k\ge 0}$ is purely periodic modulo 2^e with (not necessarily minimal) period length 2^e.

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(2) *Let* p *be a prime number with* $p \equiv 1 \pmod 4$, *and let* e *be a positive integer. Then the sequence* $\big(d(n)\big)_{n \ge e+1}$ *is purely periodic modulo* p^e *with (not necessarily minimal) period length* $\frac{1}{4}p^{e-1}(p-1)^2$.
(3) *Let* e *be a positive integer. The sequence* $(d(n))_{n \ge 0}$, *when taken modulo any fixed 2-power* 2^e *with* $e \ge 3$, *is purely periodic with (not necessarily minimal) period length* 2^{e-1}. *Modulo* 4, *the sequence is purely periodic with period length* 4, *the first few values of the sequence (modulo* 4*) being given by*

$$1, 1, 3, 3, 1, \dots.$$

Taylor coefficients of $\theta_3(\tau)$ at $\tau = i$

Theorem

(1) *Let p be a prime number with* $p \equiv 3 \pmod 4$, *and let e be a positive integer. Then* $d(n) \equiv 0 \pmod{p^e}$ *for* $n \geq \lceil \frac{ep^2}{2} \rceil$.
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$$1, 1, 3, 3, 1, \ldots .$$

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(3) *Let* e *be a positive integer. The sequence* $(d(n))_{n\ge0}$*, when taken modulo any fixed 2-power* 2^e *with* $e \ge 3$*, is purely periodic with (not necessarily minimal) period length* 2^{e-1}*.*

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Proof.

Recall that

$$d(n) = \sum_{k=0}^{n} R^{-1}(2n, 2k)v(k).$$

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Theorem

(3) Let e be a positive integer. The sequence $(d(n))_{n\ge 0}$, when taken modulo any fixed 2-power 2^e with $e \ge 3$, is purely periodic with (not necessarily minimal) period length 2^{e-1}.

Proof.

Recall that

$$d(n) = \sum_{k=0}^{n} R^{-1}(2n, 2k)v(k).$$

From the definition of $R^{-1}(2n, 2k)$, we see that $R^{-1}(2n, 2k) \equiv 0 \pmod{2^e}$ for $n \ge k + e$. Thus, from the above relation we obtain

$$d(n) \equiv \sum_{k=0}^{e-1} R^{-1}(2n, 2n-2k)v(n-k) \pmod{2^e}.$$

Proof (continued).

We have

$$d(n) \equiv \sum_{k=0}^{e-1} R^{-1}(2n, 2n-2k)v(n-k) \pmod{2^e}.$$

Proof (continued).

We have

$$d(n) \equiv \sum_{k=0}^{e-1} R^{-1}(2n, 2n-2k)v(n-k) \pmod{2^e}.$$

We know:

• the sequence $(v(n))_{n\ge0}$ is (purely) periodic modulo 2^e with period length 2^{e-1};
• the sequence $\big(R^{-1}(2n, 2n-2k)\big)_{n\ge0}$ is (purely) periodic modulo 2^e with period length 2^{e-1}.

This implies that each summand on the right-hand side is (purely) periodic modulo 2^e with (not necessarily minimal) period length 2^{e-1}. Since these are finitely many summands, the same must hold for $d(n)$. □

Taylor coefficients of $\theta_3(\tau)$ at $\tau = i$

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$$1, 1, 3, 3, 1, \dots.$$

What else?

Taylor coefficients of $\theta_2(\tau)$ at $\tau = i$

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Jacobi's theta function θ_2 is defined by

$$\theta_2(\tau) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}, \quad \text{with } q = e^{i\pi\tau}.$$

The Taylor expansion that we are interested in is

$$\theta_2\left(i\frac{1-z}{1+z}\right) = \theta_2(i)(1+z)^{1/2}\sum_{n=0}^{\infty}\frac{c(n)}{n!}\Psi^n z^n,$$

where $\Psi = \Gamma^4(1/4)/(16\pi^2)$ and $\theta_2(i) = \Gamma(1/4)/(2\pi)^{3/4}$.

Taylor coefficients of $\theta_2(\tau)$ at $\tau = i$

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where $\Psi = \Gamma^4(1/4)/(16\pi^2)$ and $\theta_2(i) = \Gamma(1/4)/(2\pi)^{3/4}$.
The first few values turn out to be

$$1, 1, -1, 3, 17, 9, 111, -2373, 12513, 86481, 146079, 9806643,\\ 81727857, 81072729, -22284691569, 142745006187,\\ -751645880127, 38512100339361, 305713085239359, \dots$$

One can again show that the $c(n)'s$ are always integers.

Taylor coefficients of $\theta_2(\tau)$ at $\tau = i$

Our expansion:

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One can show using similar reasoning that

$$\sum_{n=0}^{\infty}\frac{c(n)}{2^n\,n!}U^n(s) = (1+2s)^{1/4}V(s),$$

where, as before,

$$U(s) := \sum_{n\ge0}\frac{u(n)}{(2n+1)!}s^{2n+1} = s\frac{{}_2F_1\left[\begin{matrix}\frac{3}{4},\frac{3}{4}\\ \frac{3}{2}\end{matrix};4s^2\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{1}{4}\\ \frac{1}{2}\end{matrix};4s^2\right]}.$$

and

$$V(s) := \sum_{n\ge0}\frac{v(n)}{2^n(2n)!}s^{2n} = {}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{1}{4}\\ \frac{1}{2}\end{matrix};4s^2\right]^{1/2}.$$

Taylor coefficients of $\theta_2(\tau)$ at $\tau = i$

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Taylor coefficients of $\theta_3(\tau)$ at $\tau = i$

For comparison:

$$\theta_3\left(i\frac{1-z}{1+z}\right) = \theta_3(i)(1+z)^{1/2}\sum_{n=0}^{\infty}\frac{d(n)}{(2n)!}\Phi^n z^{2n},$$

where $\Phi = \Gamma^8(1/4)/(128\pi^4)$ and $\theta_3(i) = \pi^{1/4}/\Gamma(3/4)$.
We had earlier shown that

$$\sum_{n=0}^{\infty}\frac{d(n)}{2^n\,(2n)!}U^{2n}(s) = V(s).$$

where

$$U(s) := \sum_{n\ge 0}\frac{u(n)}{(2n+1)!}s^{2n+1} = s\frac{{}_2F_1\left[\begin{matrix}\frac{3}{4},\frac{3}{4}\\ \frac{3}{2}\end{matrix};4s^2\right]}{{}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{1}{4}\\ \frac{1}{2}\end{matrix};4s^2\right]}.$$

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Taylor coefficients of $\theta_2(\tau)$ at $\tau = i$

Theorem

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$$1, 1, 3, 3, 1, \dots .$$

What else?

The Eisenstein series E_4 and E_6

The Eisenstein series E_4 and E_6

The classical Eisenstein series are defined by

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n \ge 1} \frac{n^{2k-1} q^n}{1 - q^n}, \quad \text{with } q = e^{2i\pi\tau},$$

where B_{2k} is a Bernoulli number.
In particular,

$$E_4(\tau) = 1 + 240 \sum_{n \ge 1} \frac{n^3 q^n}{1 - q^n}, \; E_6(\tau) = 1 - 504 \sum_{n \ge 1} \frac{n^5 q^n}{1 - q^n}, \text{ with } q = e^{2i\pi\tau}.$$

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We have

$$E_4(\tau) = \theta_2^8(\tau) - \theta_2^4(\tau)\theta_3^4(\tau) + \theta_3^8(\tau)$$

and

$$E_6(\tau) = \frac{1}{2}\left(\theta_3^4(\tau)+\theta_2^4(\tau)\right)\left(2\theta_3^4(\tau)-\theta_2^4(\tau)\right)\left(\theta_3^4(\tau)-2\theta_2^4(\tau)\right).$$

Taylor coefficients of $E_4(\tau)$ at $\tau = i$

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The Taylor expansion that we are interested in is

$$E_4\left(i\frac{1-z}{1+z}\right) = E_4(i)(1+z)^4 \sum_{n=0}^{\infty} \frac{e_4(n)}{3 \cdot n!} \Psi^n z^n,$$

where $E_4(i) = 3\Gamma^8(1/4)/(2\pi)^6$, with Ψ the same as before.

Taylor coefficients of $E_4(\tau)$ at $\tau = i$

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where $E_4(i) = 3\Gamma^8(1/4)/(2\pi)^6$, with Ψ the same as before.
The first few values turn out to be

$$3,\ 0,\ 80,\ 0,\ 1920,\ 0,\ 184320,\ 0,\ 9338880,\ 0,\ 2194145280,\ 0,$$
$$245178040320,\ 0, 83119696773120,\ 0,\ 14017452551700480,\ 0,$$
$$9277412311805460480,\ 0, \dots$$

One can again show that the $e_4(n)'s$ are always integers.

Taylor coefficients of $E_4(\tau)$ at $\tau = i$

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One can show that

$$\sum_{n=0}^{\infty} \frac{e_4(n)}{2^n \, n!} U^n(s) = (3 + 4s^2) V^8(s).$$

Taylor coefficients of $E_4(\tau)$ at $\tau = i$

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Taylor coefficients of $E_4(\tau)$ at $\tau = i$

Theorem

(1) *Let p be a prime number with $p \equiv 3 \pmod 4$, and let e be a positive integer. Then $e_4(n) \equiv 0 \pmod{p^e}$ for $n \geq ep^2$.*
(2) *Let p be a prime number with $p \equiv 1 \pmod 4$, and let e be a positive integer. Then the sequence $\big(e_4(n)\big)_{n \geq 2e+2}$ is purely periodic modulo p^e with (not necessarily minimal) period length $\frac{1}{2}p^{e-1}(p-1)^2$.*
(3) *Given a positive integer n, the number $e_4(2n)$ is divisible by 2^{2n-1}, while $e_4(2n-1) = 0$ for $n \geq 1$.*

Taylor coefficients of $E_4(\tau)$ at $\tau = i$

Theorem

(1) *Let* p *be a prime number with* $p \equiv 3 \pmod 4$, *and let* e *be a positive integer. Then* $e_4(n) \equiv 0 \pmod {p^e}$ *for* $n \ge ep^2$.

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Taylor coefficients of $E_6(\tau)$ at $\tau = i$

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$$E_6(\tau) = 1 - 504 \sum_{n \geq 1} \frac{n^5 q^n}{1 - q^n}, \text{ with } q = e^{2i\pi\tau}.$$

The Taylor expansion that we are interested in is

$$E_6\left(i\frac{1-z}{1+z}\right) = \varepsilon_6 (1+z)^6 \sum_{n=0}^{\infty} \frac{e_6(n)}{n!} \Psi^n z^n,$$

where $\varepsilon_6 = -3\Gamma^{12}(1/4)/(2^7\pi^9)$, with Ψ the same as before.

Taylor coefficients of $E_6(\tau)$ at $\tau = i$

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$$E_6(\tau) = 1 - 504 \sum_{n \geq 1} \frac{n^5 q^n}{1 - q^n}, \text{ with } q = e^{2i\pi\tau}.$$

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where $\varepsilon_6 = -3\Gamma^{12}(1/4)/(2^7\pi^9)$, with Ψ the same as before.
The first few values turn out to be

$$0,\ 3,\ 0,\ 112,\ 0,\ 10752,\ 0,\ 903168,\ 0,\ 179601408,\ 0,\ 28339863552,$$
$$0\ ,\ \ 9094123487232, 0,\ 2243952774217728,\ 0,$$
$$1140973440312803328, 0,\ \ 403435727694166228992,\ \ldots$$

One can again show that the $e_6(n)'s$ are always integers.

Taylor coefficients of $E_6(\tau)$ at $\tau = i$

Theorem

(1) *Let p be a prime number with* $p \equiv 3 \pmod 4$, *and let e be a positive integer. Then* $e_6(n) \equiv 0 \pmod{p^e}$ *for* $n \ge ep^2$.

(2) *Let p be a prime number with* $p \equiv 1 \pmod 4$, *and let e be a positive integer. Then the sequence* $\big(e_6(n)\big)_{n \ge 2e+2}$ *is purely periodic modulo p^e with* (*not necessarily minimal*) *period length* $\frac{1}{2}p^{e-1}(p-1)^2$.

(3) *Given a positive integer n, the number $e_6(2n+1)$ is divisible by* 2^{2n}, *while* $e_6(2n) = 0$ *for* $n \ge 1$.

Taylor coefficients of even weight modular forms at $\tau = i$

Let $f(\tau)$ be a modular form of weight $2m$ which can be expressed as

$$\gamma_f\, f(\tau) = P_f(E_4(\tau), E_6(\tau)),$$

for a certain positive integer γ_f, and for a polynomial $P_f(a,b)$ in a and b with integer coefficients.

The expansion that we are interested in is

$$f\left(i\frac{1-z}{1+z}\right) = \varepsilon_m (1+z)^{2m} \sum_{n=0}^{\infty} \frac{e_f(n)}{\gamma_f \cdot n!} \Psi^n z^n,$$

where $\varepsilon_m = \Gamma^{4m}(1/4)/(2\pi)^{3m}$, with Ψ as before.

Taylor coefficients of even weight modular forms at $\tau = i$

Theorem

(1) *Let p be a prime number with $p \equiv 3 \pmod 4$, and let e be a positive integer. Then $e_f(n) \equiv 0 \pmod{p^e}$ for $n \geq ep^2$.*
(2) *Let p be a prime number with $p \equiv 1 \pmod 4$, and let e be a positive integer. Then the sequence $\big(e_f(n)\big)_{n \geq 2e+2}$ is purely periodic modulo p^e with (not necessarily minimal) period length $\frac{1}{2}p^{e-1}(p-1)^2$.*
(3) *Given a positive integer n, the number $e_f(n)$ is divisible by $2^{n-s_2(m)-\lfloor \log_2(m/3) \rfloor -1}$, and $e_f(n) = 0$ for $n \not\equiv m \pmod 2$. Here, $s_2(m)$ denotes the sum of the digits in the 2-adic representation of m.*

This covers all Eisenstein series, the modular discriminant, ...

Open problems

Open problems

Recall:

Theorem

(1) *Let p be a prime number with $p \equiv 3 \pmod 4$, and let e be a positive integer. Then $d(n) \equiv 0 \pmod{p^e}$ for $n \geq \lceil \frac{ep^2}{2} \rceil$.*

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(1) *Let p be a prime number with $p \equiv 3 \pmod 4$, and let e be a positive integer. Then $d(n) \equiv 0 \pmod{p^e}$ for $n \geq \lceil \frac{ep^2}{2} \rceil$.*

Conjecture

If $p \equiv 3 \pmod 4$, we have $d(n) \equiv 0 \pmod{p^e}$ for $n \geq \lceil \frac{ep^2}{4} \rceil$.

Open problems

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Recall:

Theorem

(2) *Let p be a prime number with $p \equiv 1 \pmod 4$, and let e be a positive integer. Then the sequence $\big(d(n)\big)_{n \ge e+1}$ is purely periodic modulo p^e with (not necessarily minimal) period length $\frac{1}{4}p^{e-1}(p-1)^2$.*

Open problems

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Conjecture

(1) *If $p \equiv 1 \pmod 4$, the sequence $\big(d(n)\big)_{n\ge 1}$, taken modulo p^e, is (eventually) periodic with (not necessarily minima l) period length $\frac{1}{8}p^{e-1}(p-1)^2$.*

(2) *If $p \equiv 1 \pmod 4$, there exists a constant $C_{p,e}$ such that:*

(i) $d\big(n + \frac{p^{e-1}(p-1)}{4}\big) \equiv C_{p,e}\, d(n) \pmod{p^e}$ *for all* $n \ge 1$;

(ii) $C_{p,e}^{(p-1)/2} \equiv 1 \pmod{p^e}$.

Open problems

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Recall:

Theorem

(3) *Let* e *be a positive integer. The sequence* $(d(n))_{n\ge0}$, *when taken modulo any fixed 2-power* 2^e *with* $e \ge 3$, *is purely periodic with (not necessarily minimal) period length* 2^{e-1}.

Open problems

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Theorem

(3) *Let e be a positive integer. The sequence $(d(n))_{n\ge0}$, when taken modulo any fixed 2-power 2^e with $e \ge 3$, is purely periodic with (not necessarily minimal) period length 2^{e-1}.*

From computer data, this seems to be the correct period length.

Open problems

Open problems

Can we handle Taylor expansions at other complex multiplication points by a similar approach?