A transformation formula for elliptic hypergeometric series: three applications

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The star of this talk

$$\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\,\theta(aq^{k_i+k_j};p)^2$$

$$\times\prod_{i=1}^r \frac{\theta(aq^{2k_i};p)(a,b,c,d,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{\theta(a;p)(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{r-1-m}/\lambda,aq^{1+m};q,p)_{k_i}}$$

$$=\prod_{i=1}^r \frac{(b,c,d,ef/a;q,p)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q,p)_{i-1}}$$

$$\times\prod_{i=1}^r \frac{(aq;q,p)_m\,(aq/ef;q,p)_{m+1-r}\,(\lambda q/e,\lambda q/f;q,p)_{m-i+1}}{(\lambda q;q,p)_m\,(\lambda q/ef;q,p)_{m+1-r}\,(aq/e,aq/f;q,p)_{m-i+1}}$$

$$\times\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\,\theta(\lambda q^{k_i+k_j};p)^2$$

$$\times\prod_{i=1}^r \frac{\theta(\lambda q^{2k_i};p)\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{\theta(\lambda;p)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{r-1-m}/a,\lambda q^{1+m};q,p)_{k_i}},$$

where $\lambda=a^2q^{2-r}/bcd$.

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Given a complex number p with $|p| < 1$, we define

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Out of this, we build "*shifted factorials*":

$$(a;q,p)_m := \theta(a;p)\,\theta(aq;p)\cdots\theta(aq^{m-1};p),$$

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We also employ the short notation

$$(a_1,a_2,\ldots,a_k;q,p)_m = (a_1;q,p)_m\,(a_2;q,p)_m\cdots(a_k;q,p)_m.$$

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$$\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\,\theta(aq^{k_i+k_j};p)^2$$

$$\times\prod_{i=1}^r \frac{\theta(aq^{2k_i};p)(a,b,c,d,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{\theta(a;p)(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{r-1-m}/\lambda,aq^{1+m};q,p)_{k_i}}$$

$$=\prod_{i=1}^r \frac{(b,c,d,ef/a;q,p)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q,p)_{i-1}}$$

$$\times\prod_{i=1}^r \frac{(aq;q,p)_m\,(aq/ef;q,p)_{m+1-r}\,(\lambda q/e,\lambda q/f;q,p)_{m-i+1}}{(\lambda q;q,p)_m\,(\lambda q/ef;q,p)_{m+1-r}\,(aq/e,aq/f;q,p)_{m-i+1}}$$

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$$\times\prod_{i=1}^r \frac{\theta(\lambda q^{2k_i};p)\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{\theta(\lambda;p)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{r-1-m}/a,\lambda q^{1+m};q,p)_{k_i}},$$

where $\lambda = a^2q^{2-r}/bcd$.

The star of this talk: q-case

$$\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2\,(1-aq^{k_i+k_j})^2$$
$$\times\prod_{i=1}^r \frac{(1-aq^{2k_i})(a,b,c,d,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q)_{k_i}}{(1-a)(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{r-1-m}/\lambda,aq^{1+m};q)_{k_i}}$$
$$=\prod_{i=1}^r \frac{(b,c,d,ef/a;q)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q)_{i-1}}$$
$$\times\prod_{i=1}^r \frac{(aq;q)_m\,(aq/ef;q)_{m+1-r}\,(\lambda q/e,\lambda q/f;q)_{m-i+1}}{(\lambda q;q)_m\,(\lambda q/ef;q)_{m+1-r}\,(aq/e,aq/f;q)_{m-i+1}}$$
$$\times\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2\,(1-\lambda q^{k_i+k_j})^2$$
$$\times\prod_{i=1}^r \frac{(1-\lambda q^{2k_i})\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q)_{k_i}}{(1-\lambda)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{r-1-m}/a,\lambda q^{1+m};q)_{k_i}},$$

where $\lambda = a^2q^{2-r}/bcd$.

The star of this talk: q-case, $r = 1$

$$\sum_{k=0}^{m} q^k \frac{(1-aq^{2k})(a,b,c,d,e,f,\lambda aq^{1+m}/ef,q^{-m};q)_k}{(1-a)(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{-m}/\lambda,aq^{1+m};q)_k}$$
$$= \frac{(aq,aq/ef,\lambda q/e,\lambda q/f;q)_m}{(\lambda q,\lambda q/ef,aq/e,aq/f;q)_m}$$
$$\times \sum_{k=0}^{m} q^k \frac{(1-\lambda q^{2k})\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{1+m}/ef,q^{-m};q)_k}{(1-\lambda)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{-m}/a,\lambda q^{1+m};q)_k},$$

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$$\times \sum_{k=0}^{m} q^k \frac{(1-\lambda q^{2k})\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{1+m}/ef,q^{-m};q)_k}{(1-\lambda)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{-m}/a,\lambda q^{1+m};q)_k},$$

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This is Bailey's very-well-poised ${}_{10}\phi_9$-transformation formula!

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This identity was discovered conjecturally by Ole Warnaar in 2000, and later proved independently by Rains and by Coskun and Gustafson.

3 applications

1. Enumeration of standard tableaux of skew shape

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2. Discrete analogues of Macdonald–Mehta integrals

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2. Discrete analogues of Macdonald–Mehta integrals
3. Best polynomial approximation

The first application: Counting standard Young tableaux

(joint work with Michael Schlosser)

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John Stembridge (25 May 2011):

`My student Elizabeth DeWitt has found a closed formula for the number of standard Young tableaux of skew shape, where the outer shape is a staircase and the inner shape a rectangle. Have you seen this before?`

The first application: Counting standard Young tableaux

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be two n-tuples of non-negative integers which are in non-increasing order and satisfy $\lambda_i \geq \mu_i$ for all i.
A standard Young tableau of skew shape λ/μ is an arrangement of the numbers $1, 2, \dots, \sum_{i=1}^{n}(\lambda_i - \mu_i)$ of the form

$$\begin{matrix} & & & \pi_{1,\mu_1+1} & \cdots\cdots\cdots & \pi_{1,\lambda_1} \\ & \pi_{2,\mu_2+1} & \cdots & \pi_{2,\mu_1+1} & \cdots\cdots \quad \pi_{2,\lambda_2} & \\ \iddots & & & \vdots & \iddots & \\ \pi_{n,\mu_n+1} & \cdots\cdots\cdots\cdots & & \pi_{n,\lambda_n} & & \end{matrix}$$

such that numbers along rows and columns are increasing.

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A standard Young tableau of shape $(6,5,4,3,2,1)/(3,3,0,0,0,0)$:

$$\begin{array}{cccccc} & & & 2 & 5 & 13 \\ & & & 3 & 9 & \\ 1 & 4 & 8 & 12 & & \\ 6 & 11 & 15 & & & \\ 7 & 14 & & & & \\ 10 & & & & & \end{array}$$

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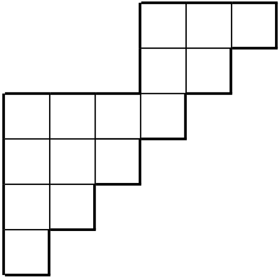


We shall do something more general here:

(1) We shall enumerate all standard Young tableaux of a skew shape, where the outer shape is a (possibly incomplete) staircase and the inner shape is a rectangle.

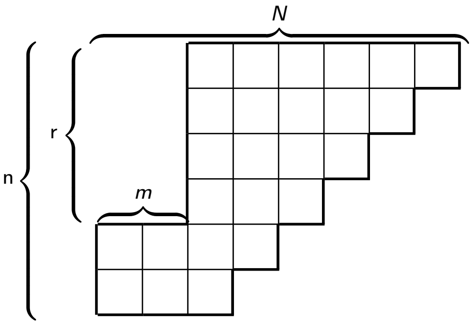
The first application: Counting standard Young tableaux

We shall do something more general here:

(1) We shall enumerate all standard Young tableaux of a skew shape, where the outer shape is a (possibly incomplete) staircase and the inner shape is a rectangle.

(2) We shall consider a q-analogue.

Our goal: Let N, n, m, r be non-negative integers. Consider all standard Young tableaux of shape $(N, N-1, \dots, N-n+1)/(m^r)$, where (m^r) stands for $(m, m, \dots, m, 0, \dots, 0)$ with r components m).



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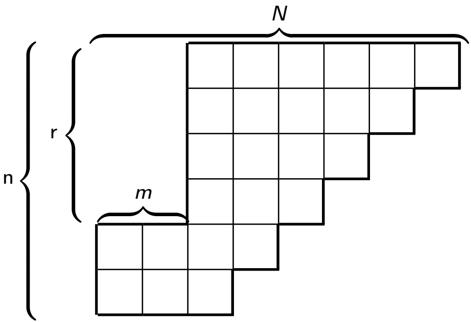

Compute $\sum_T q^{\operatorname{maj} T}$, where T ranges over all these standard Young tableaux.

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Compute $\sum_T q^{\operatorname{maj} T}$, where T ranges over all these standard Young tableaux.

The *major index* $\operatorname{maj} T$ of T is the sum of all i such that $i+1$ appears in a lower row than i.

$$\begin{array}{cccccc} & & & 2 & 5 & 13 \\ & & & 3 & 9 & \\ 1 & 4 & 8 & 12 & & \\ 6 & 11 & 15 & & & \\ 7 & 14 & & & & \\ 10 & & & & & \end{array}$$

We have $\operatorname{maj}(\,.\,) = 2+3+5+6+9+13 = 38$.

The first application: Counting standard Young tableaux

Folklore Formula (MacMahon, Stanley)

The generating function $\sum_T q^{\operatorname{maj} T}$*, where* T *ranges over all standard Young tableaux of shape* λ/μ *equals*

$$\left[\sum_{i=1}^{n}(\lambda_i-\mu_i)\right]_q! \cdot \det_{1\le i,j\le n}\left(\frac{1}{[\lambda_i-i-\mu_j+j]_q!}\right),$$

where $[m]_q! := [m]_q\,[m-1]_q\cdots[1]_q$ *with* $[\alpha]_q = 1+q+q^2+\cdots+q^{\alpha-1} = \frac{1-q^\alpha}{1-q}$.

The first application: Counting standard Young tableaux

We substitute in the formula:

$$\left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q! \det_{1\le i,j\le n}\left(\begin{cases}\dfrac{1}{[N+1-2i-m+j]_q!} & j\le r\\ \dfrac{1}{[N+1-2i+j]_q!} & j>r\end{cases}\right).$$

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We now do a Laplace expansion with respect to the first r columns:

$$\left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q!$$
$$\times \sum_{1\le k_1<\dots<k_r\le n}(-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i}\det_{1\le i,j\le r}\left(\frac{1}{[N+1-2k_i-m+j]_q!}\right)$$
$$\cdot \det_{\substack{1\le i\le n,\, i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{[N+1-2i+j]_q!}\right).$$

The first application: Counting standard Young tableaux

$$\left[\binom{N+1}{2} - \binom{N-n+1}{2} - mr\right]_q!$$
$$\times \sum_{1\le k_1<\dots<k_r\le n} (-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i} \det_{1\le i,j\le r}\left(\frac{1}{[N+1-2k_i-m+j]_q!}\right)$$
$$\cdot \det_{\substack{1\le i\le n,\, i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{[N+1-2i+j]_q!}\right).$$

$$\left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q!$$
$$\times\sum_{1\le k_1<\dots<k_r\le n}(-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i}\det_{1\le i,j\le r}\left(\frac{1}{[N+1-2k_i-m+j]_q!}\right)$$
$$\cdot\det_{\substack{1\le i\le n,\ i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{[N+1-2i+j]_q!}\right).$$

Both determinants can be evaluated by means of

$$\det_{1\le i,j\le s}\left(\frac{1}{[X_i+j]_q!}\right)$$
$$=q^{2\binom{s+1}{3}+\sum_{i=1}^s(i-1)X_i}\prod_{i=1}^s\frac{1}{[X_i+s]_q!}\prod_{1\le i<j\le s}[X_i-X_j]_q,$$

The first application: Counting standard Young tableaux

After a lot of simplification, one arrives at

$$(-1)^{\binom r2}(1+q)^{\binom n2-(n-1)r}(1-q)^{-r(r-1)}$$
$$\times\, q^{2\binom{r+1}3+2\binom{n-r+1}3+(N+1-m)\binom r2+(N+1+r)\binom{n-r}2-4\binom{n+1}3+2r\binom{n+1}2-2r^2}$$
$$\times \left[\binom{N+1}2-\binom{N-n+1}2-mr\right]_q!$$
$$\times \prod_{i=1}^n \frac{[i-1]_{q^2}!}{[N+n+1-2i]_q!}\prod_{i=1}^r \frac{[N+n-1]_q!}{[n-1]_{q^2}!\,[N-m+r-1]_q!}$$
$$\times \sum_{0\le k_1<\dots<k_r\le n-1} q^{-2\sum_{i=1}^r(2i-1)k_i}\prod_{1\le i<j\le r}(1-q^{-2(k_i-k_j)})^2$$
$$\cdot \prod_{i=1}^r \frac{(q^{N-m+r-1};q^{-2})_{k_i}\,(q^{N-m+r-2};q^{-2})_{k_i}\,(q^{2n-2};q^{-2})_{k_i}}{(q^{N+n-1};q^{-2})_{k_i}\,(q^{N+n-2};q^{-2})_{k_i}\,(q^{-2};q^{-2})_{k_i}},$$

where $[2\alpha]_q!! = [2\alpha]_q\,[2\alpha-2]_q\cdots[2]_q$, and, by convention, $k_{r+1}=n+1$.

The first application: Counting standard Young tableaux

In the elliptic transformation formula of Warnaar–Rains–Coskun/Gustafson, we let $p = 0$, $d \to aq/d$, $f \to aq/f$, and then $a \to 0$. Next we perform the substitutions $b \to q^b$, $c \to q^c$, etc.

The first application: Counting standard Young tableaux

Corollary

For all non-negative integers m, r and s, we have

$$\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2 \cdot \prod_{i=1}^r \frac{(dq^{k_i};q)_s\,(b;q)_{k_i}\,(q^{-m};q)_{k_i}}{(q;q)_{k_i}\,(f;q)_{k_i}}$$

$$= \frac{q^{\binom{r+s}{3}+\binom{r+1}{3}+s\binom{r}{2}-m\binom{r+s}{2}}}{f^{\binom{r}{2}}\,(q;q)_{r+s-1}^{s-1}} \prod_{i=1}^r \frac{(b;q)_{i-1}\,(bq^{s+r+i-m-1}/f;q)_{m-r+1}}{(q^{i-m}/f;q)_{m-i+1}}$$

$$\times \prod_{i=1}^{r+s-1} \frac{(q;q)_{i-1}\,(q;q)_m}{(q;q)_{m-i}} \prod_{i=r}^{r+s-1} \frac{(dq^{1-r}/b;q)_i}{(q;q)_{r+s-i-1}\,(d;q)_{i-r}\,(fq^{1-r-s}/b;q)_i}$$

$$\times \sum_{0\le \ell_1<\ell_2<\cdots<\ell_s\le r+s-1} q^{\sum_{i=1}^s (2i-1)\ell_i} \prod_{1\le i<j\le s} (1-q^{\ell_i-\ell_j})^2 \cdot \prod_{i=1}^s \frac{(d;q)_{\ell_i}\,(fq^{1-r-s}/b;q)_{\ell_i}\,(q^{1-r-s};q)_{\ell_i}}{(q;q)_{\ell_i}\,(dq^{1-r}/b;q)_{\ell_i}\,(q^{-m};q)_{\ell_i}}.$$

Corollary

For all non-negative integers m, r and s, we have

$$\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2 \cdot \prod_{i=1}^r \frac{(dq^{k_i};q)_s\,(b;q)_{k_i}\,(q^{-m};q)_{k_i}}{(q;q)_{k_i}\,(f;q)_{k_i}}$$

$$= \frac{q^{\binom{r+s}{3}+\binom{r+1}{3}+s\binom{r}{2}-m\binom{r+s}{2}}}{f^{\binom{r}{2}}\,(q;q)_{r+s-1}^{s-1}} \prod_{i=1}^r \frac{(b;q)_{i-1}\,(bq^{s+r+i-m-1}/f;q)_{m-r+1}}{(q^{i-m}/f;q)_{m-i+1}}$$

$$\times \prod_{i=1}^{r+s-1} \frac{(q;q)_{i-1}\,(q;q)_m}{(q;q)_{m-i}} \prod_{i=r}^{r+s-1} \frac{(dq^{1-r}/b;q)_i}{(q;q)_{r+s-i-1}\,(d;q)_{i-r}\,(fq^{1-r-s}/b;q)_i}$$

$$\times \sum_{0\le \ell_1<\ell_2<\cdots<\ell_s\le r+s-1} q^{\sum_{i=1}^s (2i-1)\ell_i} \prod_{1\le i<j\le s} (1-q^{\ell_i-\ell_j})^2 \cdot \prod_{i=1}^s \frac{(d;q)_{\ell_i}\,(fq^{1-r-s}/b;q)_{\ell_i}\,(q^{1-r-s};q)_{\ell_i}}{(q;q)_{\ell_i}\,(dq^{1-r}/b;q)_{\ell_i}\,(q^{-m};q)_{\ell_i}}.$$

The first application: Counting standard Young tableaux

Theorem

If $N-n$ is even, the generating function $\sum_T q^{\operatorname{maj}(T)}$ for standard Young tableaux T of shape $(N,N-1,\dots,N-n+1)/(m^r)$ equals

$$(-1)^{\binom{(N-n)/2}{2}+\frac{1}{2}r(N-n)}(1+q)^{\binom{n}{2}-\binom{(N-n)/2}{2}-mr}(1-q)^{-\binom{(N-n)/2}{2}-r(N-n)}$$
$$\times q^{\frac{1}{2}mr(r+m-2n)+\frac{1}{2}r(N-n)\left(\frac{1}{2}(N-3n)-m+1\right)+\binom{n+1}{3}+(N-n)\left(\binom{n}{2}+\binom{(N-n)/2}{2}\right)}$$
$$\times \frac{\left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q!}{\left[r+\frac{N-n-2}{2}\right]_{q^2}!^{(N-n)/2}\left[\frac{N+n-2}{2}\right]_{q^2}!^{(N-n)/2}}\frac{\prod_{i=1}^{(N+n)/2}[i-1]_{q^2}!}{\prod_{i=1}^{n}[N-n+2i-1]_q!}$$
$$\times \prod_{i=1}^{r}\frac{\left[\frac{N-n}{2}+i-1\right]_{q^2}![n+m-r+2i-1]_q!\left(q^{n+m-r+2i};q^2\right)_{(N-n)/2}}{[m+i-1]_{q^2}![N-m-r+2i-1]_q!}$$

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$$\times\, q^{\frac{1}{2}mr(r+m-2n)+\frac{1}{2}r(N-n)\left(\frac{1}{2}(N-3n)-m+1\right)+\binom{n+1}{3}+(N-n)\left(\binom{n}{2}+\binom{(N-n)/2}{2}\right)}$$
$$\times\, \frac{\left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q!}{\left[r+\frac{N-n-2}{2}\right]_{q^2}!^{(N-n)/2}\left[\frac{N+n-2}{2}\right]_{q^2}!^{(N-n)/2}}\,\frac{\prod_{i=1}^{(N+n)/2}[i-1]_{q^2}!}{\prod_{i=1}^{n}[N-n+2i-1]_q!}$$
$$\times\prod_{i=1}^{r}\frac{\left[\frac{N-n}{2}+i-1\right]_{q^2}!\,[n+m-r+2i-1]_q!\,\left(q^{n+m-r+2i};q^2\right)_{(N-n)/2}}{[m+i-1]_{q^2}!\,[N-m-r+2i-1]_q!}$$

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$$\times \frac{\left[\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right]_q!}{\left[r+\frac{N-n-2}{2}\right]_{q^2}!^{(N-n)/2}\left[\frac{N+n-2}{2}\right]_{q^2}!^{(N-n)/2}}\frac{\prod_{i=1}^{(N+n)/2}[i-1]_{q^2}!}{\prod_{i=1}^{n}[N-n+2i-1]_q!}$$
$$\times\prod_{i=1}^{r}\frac{\left[\frac{N-n}{2}+i-1\right]_{q^2}!\,[n+m-r+2i-1]_q!\,\left(q^{n+m-r+2i};q^2\right)_{(N-n)/2}}{[m+i-1]_{q^2}!\,[N-m-r+2i-1]_q!}$$

to be continued . . .

The first application: Counting standard Young tableaux

$$\times \sum_{0\le \ell_1<\ell_2<\dots<\ell_{(N-n)/2}\le r+\frac{N-n-2}{2}} q^{\sum_{i=1}^{(N-n)/2}(N+n-2(2i-1))l_i} \prod_{1\le i<j\le \frac{N-n}{2}} [\ell_j-\ell_i]_{q^2}^2$$

$$\cdot \prod_{i=1}^{\frac{N-n}{2}} \left(\begin{bmatrix} \frac{N-n-2}{2}+r \\ \ell_i \end{bmatrix}_{q^2} \left(q^{2-N-n};q^2\right)_{\ell_i} \left(q^{n+m-r-2i+1};q^2\right)_{r+i-\ell_i-1} \right.$$

$$\left. \cdot \frac{\left(q^{N-m-r-2i+2};q^2\right)_{r+i-\ell_i-1}}{\left(q^{N+m-r-2i+2};q^2\right)_{r+i-\ell_i-1}} \right),$$

and there is a similar statement if $N-n$ is odd.

In the case of a full staircase (i.e., $n = N$), the formula reduces to DeWitt's original result.

Corollary

The generating function $\sum_T q^{\operatorname{maj}(T)}$ for standard Young tableaux T of shape $(n, n-1, \dots, 1)/(m^r)$ equals

$$q^{\frac{1}{2}mr(r+m-2n)+\binom{n+1}{3}}(1+q)^{\binom{n}{2}-mr}\left[\binom{n+1}{2}-mr\right]_q!$$
$$\times \prod_{i=1}^{n}\frac{[i-1]_{q^2}!}{[2i-1]_q!}\prod_{i=1}^{r}\frac{[i-1]_{q^2}!\,[n+m-r+2i-1]_q!}{[m+i-1]_{q^2}!\,[n-m-r+2i-1]_q!}.$$

The "next" case ($N = n + 1$):

Corollary

The generating function $\sum_T q^{\operatorname{maj}(T)}$ *for standard Young tableaux* T *of shape* $(n+1, n, \dots, 2)/(m^r)$ *equals*

$$(1+q)^{\binom{n}{2}-(m-1)r} q^{\frac{1}{2}mr(r+m-2n+2)+r(1-n-m)+\binom{n+1}{3}+\binom{n}{2}}$$

$$\times \left[\binom{n+2}{2} - mr - 1\right]_q! \prod_{i=1}^{n} \frac{[i-1]_{q^2}!}{[2i]_q!}$$

$$\times \prod_{i=1}^{r} \frac{\left[i-1\right]_{q^2}! \, [n+m-r+2i-1]_q!}{[m+i-1]_{q^2}! \, [n-m-r+2i]_q!}$$

$$\times \sum_{\ell=0}^{r} \frac{(-1)^r q^{2nl_1}}{(1-q^2)^r} \begin{bmatrix} r \\ \ell \end{bmatrix}_{q^2}$$

$$\cdot \frac{\left(q^{-2n}; q^2\right)_\ell \left(q^{n+m-r}; q^2\right)_{r-\ell} \left(q^{n-m-r+1}; q^2\right)_{r-\ell}}{\left(q^{n+m-r+1}; q^2\right)_{r-\ell}}.$$

In general:
The generating function for standard Young tableaux of shape $(N, N-1, \dots, N-n)/(m^r)$ equals an $\lceil (N-n)/2 \rceil$-fold hypergeometric sum.

The first application: Counting standard Young tableaux

John Stembridge:

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`I think her approach is much simpler;`

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```
I think her approach is much simpler;
but I don't think it would extend to the ``next
case'' you mention.
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The second application: Discrete M–M-integrals

(joint work with Richard Brent and Ole Warnaar)

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`Together with Richard Brent, I have recently been looking at sums of the form`

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \left| \prod_{1\le i<j\le r} (k_i^\alpha - k_j^\alpha) \right|^\gamma \prod_{i=1}^r |k_i|^\delta \binom{2n}{n+k_i},$$

`which we call "discrete Mehta-type integrals".`

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which we call "discrete Mehta-type integrals".

At least, for $\alpha, \gamma \in \{1,2\}$ and small δ, we believe that these sums can be evaluated in closed form.

The Mehta integral

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$$(2\pi)^{-r/2}\int_{\mathbb{R}^r}\left|\prod_{1\le i<j\le r}(t_i-t_j)\right|^\gamma \prod_{i=1}^{r} e^{-t_i^2/2}\,dt_1\cdots dt_r$$

$$=\prod_{i=1}^{r}\frac{\Gamma(1+i\gamma/2)}{\Gamma(1+\gamma/2)}.$$

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"Discrete Mehta integrals"

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The second application: Discrete M–M-integrals

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"Discrete analogues of Macdonald–Mehta integrals"

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}}\left|\prod_{1\le i<j\le r}(k_i^{\alpha}-k_j^{\alpha})\right|^{\gamma}\prod_{i=1}^{r}|k_i|^{\delta}\binom{2n}{n+k_i}=??$$

The second application: Discrete M–M-integrals

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`Needless to tell you that the case` $\alpha = 1$, $\gamma = 2$, $\delta = 0$ `follows from specialising a rectangular Schur functions in two sets of variables.`

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Indeed,

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}}\prod_{1\le i<j\le r}(k_i-k_j)^2\prod_{i=1}^{r}\binom{2n}{n+k_i}\binom{2m}{m+k_i}$$
$$=\prod_{i=1}^{r}\binom{m+n}{i-1}^2\binom{2n}{i-1}\binom{2m}{i-1}(2m+2n-i-r+2)!\,(i-1)!^5$$

can be proved in various ways, one of which is by the use of Schur functions (and a *q-analogue* as well), as I pointed out in a paper 15 years ago.

The second application: Discrete M–M-integrals

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But, say,

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}}\prod_{1\le i<j\le r}(k_i^2-k_j^2)^2\prod_{i=1}^{r}k_i^2\binom{2n}{n+k_i}\binom{2m}{m+k_i}=??$$

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And what about a q-analogue?

The second application: Discrete M–M-integrals

Our discrete analogue of Macdonald–Mehta integrals:

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \left| \prod_{1\le i<j\le r} (k_i^\alpha - k_j^\alpha) \right|^\gamma \prod_{i=1}^{r} |k_i|^\delta \binom{2n}{n+k_i}.$$

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We found (and proved) closed form evaluations in the following cases:

α	γ	δ	G
1	1	0	A_{r-1}
1	2	$0,1$	A_{r-1}, –
2	1	$0,1,2$	D_r, B_r, –
2	2	$0,1,2,3$	D_r, –, B_r, –

The second application: Discrete M–M-integrals

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α	γ	δ	G
1	1	0	A_{r-1}
1	2	0, 1	A_{r-1}, –
2	1	0, 1, 2	D_r, B_r, –
2	2	0, 1, 2, 3	D_r, –, B_r, –

We also (eventually) found q-analogues in most cases.

The second application: Discrete M–M-integrals

I shall concentrate in this talk on the discrete analogues of Macdonald–Mehta integrals

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \left| \prod_{1\le i<j\le r} (k_i^\alpha - k_j^\alpha) \right|^\gamma \prod_{i=1}^r |k_i|^\delta \binom{2n}{n+k_i}.$$

with $\gamma = 2$.

How to approach:

Theorem

For all non-negative integers or half-integers m and n and a positive integer r, we have

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\le i<j\le r} (k_j^2 - k_i^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i} \binom{2m}{m+k_i}$$
$$= r!\, 2^{(m+n+1)r-3\binom{r+1}{2}} \prod_{i=1}^{r} \frac{(2n)!}{(2n-2i+1)!} \frac{(2m)!}{(2m-2i+1)!} \cdot \frac{(2i-1)!\,(2m+2n-2i-2r+1)!!}{(m+n-i+1)!}.$$

The second application: Discrete M–M-integrals

How to evaluate

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (k_i^2-k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i}\binom{2m}{m+k_i} = ??$$

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The above sum is equivalent to

$$2^r\,r!\sum_{0\le k_1<\dots<k_r}\prod_{1\le i<j\le r}(k_i^2-k_j^2)^2\prod_{i=1}^r k_i^2\binom{2n}{n+k_i}\binom{2m}{m+k_i}=??$$

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How can one generate

$$\prod_{1\le i<j\le r}(k_i^2-k_j^2)^2\prod_{i=1}^r k_i^2\ ?$$

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Our first idea: By non-intersecting lattice paths!

The second application: Discrete M–M-integrals

We shall be concerned with paths in the integer lattice consisting of up-steps $(1, 1)$ and down-steps $(1, -1)$.

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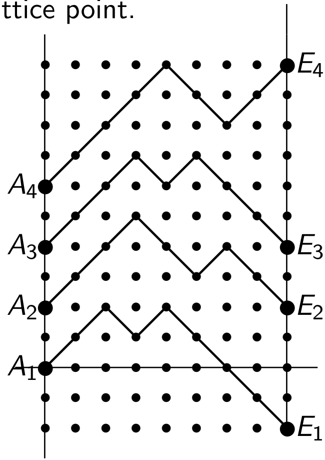
A family of paths is called *non-intersecting* if no two paths in the family meet in a lattice point.



The second application: Discrete M–M-integrals

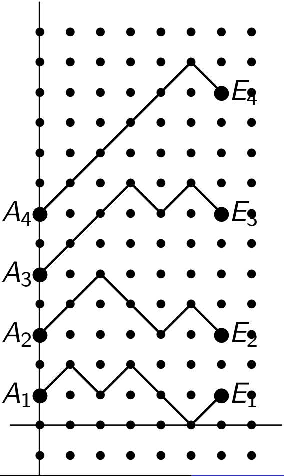
Theorem (Karlin–McGregor, Lindström, Gessel–Viennot, Fisher, John–Sachs, Gronau–Just–Schade–Scheffler–Wojciechowski)

Let G be an acyclic, directed graph, and let $A_1, A_2, \dots, A_r$ and $E_1, E_2, \dots, E_r$ be vertices in the graph with the property that, for $i < j$ and $k < l$, any (directed) path from A_i to E_l intersects with any path from A_j to E_k. Then the number of families $(P_1, P_2, \dots, P_r)$ of non-intersecting (directed) paths, where the i-th path P_i runs from A_i to E_i, $i = 1, 2, \dots, r$, is given by

$$\det_{1 \le i,j \le r} \left(|\mathcal{P}(A_j \to E_i)| \right),$$

where $\mathcal{P}(A \to E)$ denotes the set of paths from A to E.

Let $A_i = (0, 2i - 1)$ and $E_i = (n, k_i - 1)$, $i = 1, 2, \ldots, r$, with $k_i \equiv n \pmod 2$. Here, the non-intersecting lattice paths that we consider have the the additional property that paths never run below the x-axis.



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By the K-McG,L,G-V,F,J-S,G-J-S-S-W theorem on non-intersecting lattice paths, the number of families of these non-intersecting lattice paths is again given by a determinant. The individual entries are obtained by the reflection principle:

$$\det_{1\le i,j\le r}\left(\binom{n}{j+\frac{1}{2}(n-k_i)} - \binom{n}{-j+1+\frac{1}{2}(n-k_i)}\right).$$

The second application: Discrete M–M-integrals

This determinant

$$\det_{1\le i,j\le r}\left(\binom{n}{j+\frac{1}{2}(n-k_i)}-\binom{n}{-j+1+\frac{1}{2}(n-k_i)}\right)$$

This determinant can be evaluated:

$$\det_{1\le i,j\le r}\left(\binom{n}{j+\frac{1}{2}(n-k_i)}-\binom{n}{-j+1+\frac{1}{2}(n-k_i)}\right)$$
$$=\prod_{1\le i<j\le r}\left(\tfrac{1}{2}(k_j-k_i)\right)\left(\tfrac{1}{2}(k_j+k_i-2)\right)$$
$$\times\prod_{i=1}^{r}\frac{(k_i-1)\,(n+2i-2)!}{\left(\frac{1}{2}(n-k_i)+r\right)!\,\left(\frac{1}{2}(n+k_i)+r-1\right)!}.$$

(ADC1, Theorem 30; dimension formula for irreducible representations of $Sp_{2n}(\mathbb{C})$ in disguise)

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$$=\prod_{1\le i<j\le r}\left(\tfrac{1}{2}(k_j-k_i)\right)\left(\tfrac{1}{2}(k_j+k_i-2)\right)$$

$$\times\prod_{i=1}^{r}\frac{(k_i-1)\,(n+2i-2)!}{\left(\frac{1}{2}(n-k_i)+r\right)!\,\left(\frac{1}{2}(n+k_i)+r-1\right)!}.$$

(ADC1, Theorem 30; dimension formula for irreducible representations of $Sp_{2n}(\mathbb{C})$ in disguise)

One can "smell" the type B Vandermonde product: one only needs to replace k_i by $2k_i+1$ (which you need to take if n is odd).

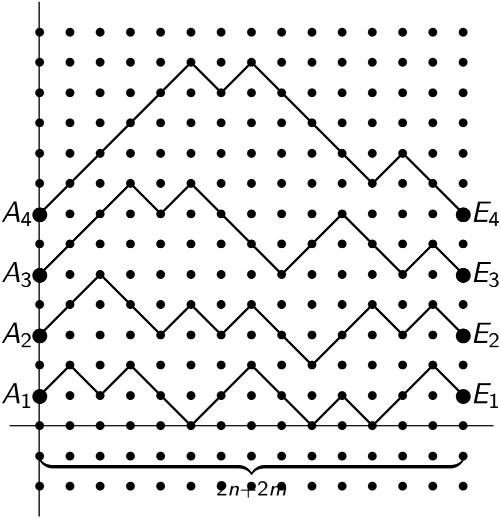
The second application: Discrete M–M-integrals

Here is the one-picture proof of

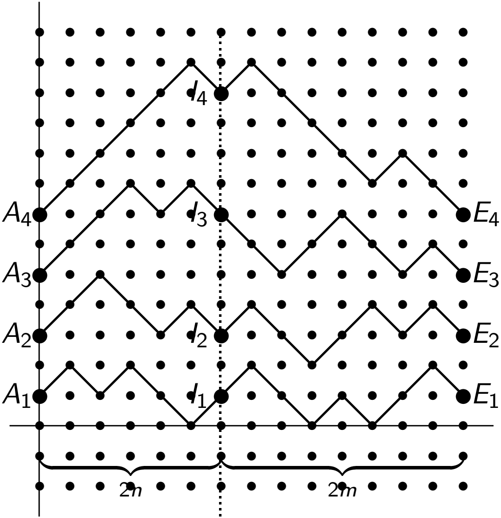
$$\sum_{0\le k_1<\cdots<k_r} \prod_{1\le i<j\le r} (k_i^2-k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i}\binom{2m}{m+k_i}$$

$$= 2^{(m+n)r-3\binom{r+1}{2}} \prod_{i=1}^{r} \frac{(2n)!}{(2n-2i+1)!}\,\frac{(2m)!}{(2m-2i+1)!} \cdot \frac{(2i-1)!\,(2m+2n-2i-2r+1)!!}{(m+n-i+1)!}.$$

The second application: Discrete M–M-integrals



The second application: Discrete M–M-integrals



The second application: Discrete M–M-integrals

This did prove that identity, but we did not manage to "tweak" this approach to produce a q-analogue.

Hence:

Hence:

Our second idea: brute force!

The second application: Discrete M–M-integrals

Hence:

Our second idea: brute force! Here applied to:

Theorem

For all non-negative integers m and n and a positive integer r, we have

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\le i<j\le r} (k_i-k_j)^2 \prod_{i=1}^{r} |k_i| \binom{2n}{n+k_i}\binom{2m}{m+k_i}$$

$$= r! \prod_{i=1}^{\lceil r/2\rceil} \left(\frac{\Gamma^2(i)\,\Gamma(2n+1)}{\Gamma(n-i+2)\,\Gamma(n-i+1)} \cdot \frac{\Gamma(2m+1)\,\Gamma(m+n-i-\lceil r/2\rceil+2)}{\Gamma(m-i+2)\,\Gamma(m-i+1)\,\Gamma(m+n-i+2)} \right)$$

$$\times \prod_{i=1}^{\lfloor r/2\rfloor} \frac{\Gamma(i)\,\Gamma(i+1)\,\Gamma(2n+1)\,\Gamma(2m+1)\,\Gamma(m+n-i-\lfloor r/2\rfloor+1)}{\Gamma^2(n-i+1)\,\Gamma^2(m-i+1)\,\Gamma(m+n-i+2)}.$$

The second application: Discrete M–M-integrals

How to evaluate

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\le i<j\le r} (k_i - k_j)^2 \prod_{i=1}^{r} |k_i| \binom{2n}{n+k_i} \binom{2m}{m+k_i} \text{ ?}$$

The second application: Discrete M–M-integrals

How to evaluate

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\le i<j\le r} (k_i - k_j)^2 \prod_{i=1}^{r} |k_i| \binom{2n}{n+k_i} \binom{2m}{m+k_i} \ ?$$

Write

$$\prod_{1\le i<j\le r} (k_i - k_j)$$

$$= \det_{1\le i,j\le r} \begin{pmatrix} 1 & k_i & (n^2 - k_i^2) & k_i(n^2 - k_i^2) & (n^2 - k_i^2)((n-1)^2 - k_i^2) & \dots \end{pmatrix},$$

In other words,

$$\prod_{1\le i<j\le r}(k_i-k_j)=\pm\det M(N),$$

where $M(N)=\big(M_{i,j}(N)\big)_{1\le i,j\le r}$ is the $r\times r$ matrix defined by

$$M_{i,j}(N)=(-1)^{2\lfloor (j-1)/2\rfloor}\,k_i^{\chi(j\text{ even})}\,(-N-k_i)_{\lfloor (j-1)/2\rfloor}\,(-N+k_i)_{\lfloor (j-1)/2\rfloor},$$

Here, $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise, and the Pochhammer symbol $(\alpha)_m$ is defined by
$(\alpha)_m:=\alpha(\alpha+1)\cdots(\alpha+m-1)$ for $m\ge 1$, and $(\alpha)_0:=1$.

The second application: Discrete M–M-integrals

So,

$$\prod_{1\le i<j\le r}(k_i-k_j)^2=\det M(n)\cdot\det M(m).$$

Thus, our sum becomes

$$\sum_{\sigma,\tau\in S_r}\operatorname{sgn}\sigma\tau\prod_{i=1}^{r}\Bigg(\sum_{k_i=-\infty}^{\infty}\Bigg(|k_i|\,k_i^{\chi(\sigma(i)\text{ even})+\chi(\tau(i)\text{ even})}$$
$$\times\frac{(2n)!}{(n+k_i-\lfloor(\sigma(i)-1)/2\rfloor)!\,(n-k_i-\lfloor(\sigma(i)-1)/2\rfloor)!}$$
$$\times\frac{(2m)!}{(m+k_i-\lfloor(\tau(i)-1)/2\rfloor)!\,(m-k_i-\lfloor(\tau(i)-1)/2\rfloor)!}\Bigg)\Bigg).$$

So,

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$$\times\frac{(2n)!}{(n+k_i-\lfloor(\sigma(i)-1)/2\rfloor)!\,(n-k_i-\lfloor(\sigma(i)-1)/2\rfloor)!}$$

$$\times\frac{(2m)!}{(m+k_i-\lfloor(\tau(i)-1)/2\rfloor)!\,(m-k_i-\lfloor(\tau(i)-1)/2\rfloor)!}\bigg)\Bigg).$$

The point here is that the inner sum is a *single* sum, which can be evaluated.

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The second application: Discrete M–M-integrals

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CASE 1. *$\sigma(i)$ and $\tau(i)$ are both odd.* The sum can be evaluated by means of the hypergeometric summation formula (*"Dixon's summation"*)

$$ {}_3F_2\left[\begin{matrix} a, b, -N \\ 1+a-b, 1+a+N \end{matrix}; 1\right] = \frac{(1+a)_N\,(1+\frac{a}{2}-b)_N}{(1+\frac{a}{2})_N\,(1+a-b)_N}, $$

where N is a non-negative integer.

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CASE 2. *$\sigma(i)$ and $\tau(i)$ have different parity.* In this case, the sum vanishes for trivial reasons.

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where N is a non-negative integer.

CASE 2. *$\sigma(i)$ and $\tau(i)$ have different parity.* In this case, the sum vanishes for trivial reasons.

CASE 3. *$\sigma(i)$ and $\tau(i)$ are both even.* Here, Dixon's summation applies again after the application of a contiguous relation.

After substituting all this, and also simplifying the sum over the two summations over σ and τ, one obtains the determinant of a checkerboard matrix

$$r!\det_{1\le i,j\le r}(A_{i,j}),$$

with

$$A_{k,l}=\begin{cases}\frac{1}{(m+n-K-L)}\cdot\frac{(2n)!}{(n-K)!\,(n-K-1)!}\cdot\frac{(2m)!}{(m-L)!\,(m-L-1)!}, & \text{if } k,l \text{ odd},\\ \frac{1}{(m+n-K-L-1)(m+n-K-L)}\cdot\frac{(2n)!}{(n-K-1)!^2}\cdot\frac{(2m)!}{(m-L-1)!^2}, & \text{if } k,l \text{ even},\\ 0, & \text{otherwise},\end{cases}$$

where $K=\lfloor (k-1)/2\rfloor$ and $L=\lfloor (l-1)/2\rfloor$.

The second application: Discrete M–M-integrals

Rows and columns of a checkerboard matrix can be reordered simultaneously, so that it becomes a block matrix, and therefore its determinant factors into the product of two determinants:

$$\det_{1\le i,j\le r}(A_{i,j}) = \det_{1\le i,j\le \lceil r/2\rceil}(A_{2i-1,2j-1}) \cdot \det_{1\le i,j\le \lfloor r/2\rfloor}(A_{2i,2j}).$$

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Aside from some factors, the first determinant is

$$\det_{1\le i,j\le \lceil r/2\rceil}\left(\frac{1}{m+n-i-j+2}\right),$$

while the second is

$$\det_{1\le i,j\le \lfloor r/2\rfloor}\left(\frac{1}{(m+n-i-j+1)(m+n-i-j+2)}\right).$$

The second application: Discrete M–M-integrals

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$$\det_{1\le i,j\le \lfloor r/2\rfloor}\left(\frac{1}{(m+n-i-j+1)(m+n-i-j+2)}\right).$$

Both are easy to evaluate.

This gives the claimed theorem:

Theorem

For all non-negative integers m and n and a positive integer r, we have

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\le i<j\le r} (k_i-k_j)^2 \prod_{i=1}^{r} |k_i| \binom{2n}{n+k_i}\binom{2m}{m+k_i}$$
$$= r! \prod_{i=1}^{\lceil r/2\rceil} \left(\frac{\Gamma^2(i)\,\Gamma(2n+1)}{\Gamma(n-i+2)\,\Gamma(n-i+1)} \cdot \frac{\Gamma(2m+1)\,\Gamma(m+n-i-\lceil r/2\rceil+2)}{\Gamma(m-i+2)\,\Gamma(m-i+1)\,\Gamma(m+n-i+2)} \right)$$
$$\times \prod_{i=1}^{\lfloor r/2\rfloor} \frac{\Gamma(i)\,\Gamma(i+1)\,\Gamma(2n+1)\,\Gamma(2m+1)\,\Gamma(m+n-i-\lfloor r/2\rfloor+1)}{\Gamma^2(n-i+1)\,\Gamma^2(m-i+1)\,\Gamma(m+n-i+2)}.$$

The method also works in the previous case, and in further cases.

The second application: Discrete M–M-integrals

The method also works in the previous case, and in further cases.

Alas, we failed another time to “put q in”.

Let us go back to, say,

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}}\prod_{1\le i<j\le r}(k_i^2-k_j^2)^2\prod_{i=1}^{r}k_i^2\binom{2n}{n+k_i}\binom{2m}{m+k_i}=??$$

Let us go back to, say,

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (k_i^2-k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i}\binom{2m}{m+k_i} = ??$$

How would a q-analogue look like? Wouldn't it contain

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2(1-q^{k_i+k_j})^2 \times \text{stuff ?}$$

The second application: Discrete M–M-integrals

How did our gigantic transformation formula (with $p = 0$) look like?

How did our gigantic transformation formula (with $p = 0$) look like?

$$\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2\,(1-aq^{k_i+k_j})^2$$

$$\times\prod_{i=1}^{r} \frac{(1-aq^{2k_i})(a,b,c,d,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q)_{k_i}}{(1-a)(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{r-1-m}/\lambda,aq^{1+m};q)_{k_i}}$$

$$=\prod_{i=1}^{r} \frac{(b,c,d,ef/a;q)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q)_{i-1}}$$

$$\times\prod_{i=1}^{r} \frac{(aq;q)_m\,(aq/ef;q)_{m+1-r}\,(\lambda q/e,\lambda q/f;q)_{m-i+1}}{(\lambda q;q)_m\,(\lambda q/ef;q)_{m+1-r}\,(aq/e,aq/f;q)_{m-i+1}}$$

$$\times\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} (1-q^{k_i-k_j})^2\,(1-\lambda q^{k_i+k_j})^2$$

$$\times\prod_{i=1}^{r} \frac{(1-\lambda q^{2k_i})\,(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{2-r+m}/ef,q^{-m};q)_{k_i}}{(1-\lambda)\,(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{r-1-m}/a,\lambda q^{1+m};q)_{k_i}},$$

So,

The second application: Discrete M–M-integrals

So, if one chooses $a = q^2$, $d = q^{1-n}$, $e = q^{1-m}$, and $f = q^2$ in this transformation formula and one gets:

Theorem

For all non-negative integers m and n and a positive integer r, we have

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\le i<j\le r} [k_j - k_i]_q^2 \, [k_i + k_j]_q^2 \cdot \prod_{i=1}^{r} q^{k_i^2 - 2ik_i} \left| [k_i]_{q^2} \, [k_i]_q^2 \right| \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}_q \begin{bmatrix} 2m \\ m+k_i \end{bmatrix}_q$$

$$= r! \left(\frac{2}{[2]_q} \right)^r q^{-2\binom{r+1}{3} - \binom{r+1}{2}} \prod_{i=1}^{r} \left(\frac{\Gamma_q(2n+1)}{\Gamma_q^2(n-i+1)} \cdot \frac{\Gamma_q(2m+1)}{\Gamma_q^2(m-i+1)} \times \frac{\Gamma_q(i)\,\Gamma_q(i+1)\,\Gamma_q(m+n-i-r+1)}{\Gamma_q(m+n-i+2)} \right).$$

However: we still don't know a q-analogue of:

Theorem

For all non-negative integers m and n and a positive integer r, we have

$$\sum_{k_1,\dots,k_r=-n}^{n} \prod_{1\le i<j\le r} (k_i-k_j)^2 \prod_{i=1}^{r} |k_i| \binom{2n}{n+k_i}\binom{2m}{m+k_i}$$

$$= r! \prod_{i=1}^{\lceil r/2\rceil} \left(\frac{\Gamma^2(i)\,\Gamma(2n+1)}{\Gamma(n-i+2)\,\Gamma(n-i+1)} \cdot \frac{\Gamma(2m+1)\,\Gamma(m+n-i-\lceil r/2\rceil+2)}{\Gamma(m-i+2)\,\Gamma(m-i+1)\,\Gamma(m+n-i+2)} \right)$$

$$\times \prod_{i=1}^{\lfloor r/2\rfloor} \frac{\Gamma(i)\,\Gamma(i+1)\,\Gamma(2n+1)\,\Gamma(2m+1)\,\Gamma(m+n-i-\lfloor r/2\rfloor+1)}{\Gamma^2(n-i+1)\,\Gamma^2(m-i+1)\,\Gamma(m+n-i+2)}.$$

The third application: Best polynomial approximation

(joint work with Han Feng and Yuan Xu)

The third application: Best polynomial approximation

(joint work with Han Feng and Yuan Xu)

Yuan Xu (26 August 2017):

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In work in approximation theory, I encountered a
certain determinant (see the attachment).  On the
basis of computer experiments, I believe that this
determinant can be evaluated in closed form.  Have
you seen it before?
```

The determinant

Let

$$f(s_1,s_2,r,i,j) := \binom{r}{j-i} \frac{(s_1+i)_{j-i}}{(s_1+s_2+i+j-1)_{j-i}\,(s_1+s_2+r+2i)_{j-i}}.$$

Form the matrix

$$M(r) := \begin{pmatrix} f(s_1,s_2,r,i,j) & \text{for } 0 \le i < r \\ (-1)^{j-i-r} f(s_2,s_1,r,i-r,j) & \text{for } r \le i < 2r \end{pmatrix}_{0 \le i,j \le 2r-1}.$$

Then $\det M(r)$ seems to be "nice".

The determinant

For example, the matrix $M(2)$ is

$$\begin{pmatrix} 1 & \frac{2s_1}{(S)(S+2)} & \frac{s_1(s_1+1)}{(S+1)(S+2)^2(S+3)} & 0 \\ 0 & 1 & \frac{2(s_1+1)}{(S+2)(S+4)} & \frac{(s_1+1)(s_1+2)}{(S+3)(S+4)^2(S+5)} \\ 1 & -\frac{2s_2}{(S)(S+2)} & \frac{s_2(s_2+1)}{(S+1)(S+2)^2(S+3)} & 0 \\ 0 & 1 & -\frac{2(s_2+1)}{(S+2)(S+4)} & \frac{(s_2+1)(s_2+2)}{(S+3)(S+4)^2(S+5)} \end{pmatrix},$$

with $S = s_1 + s_2$.

A generalised determinant

Let

$$f(s_1,s_2,r,i,j) := \binom{r}{j-i} \frac{(s_1+i)_{j-i}}{(s_1+s_2+i+j-1)_{j-i}\,(s_1+s_2+r+2i)_{j-i}}.$$

Form the matrix

$$M(r_1,r_2) := \begin{pmatrix} f(s_1,s_2,r_1,i,j) \qquad \text{for } 0\le i<r_2 \\ (-1)^{j-i-r_2} f(s_2,s_1,r_2,i-r_2,j) \\ \qquad \text{for } r_2\le i<r_1+r_2 \end{pmatrix}_{0\le i,j\le r_1+r_2-1}.$$

Then $\det M(r_1,r_2)$ seems to be "nice".

The third application: Best polynomial approximation

Where does this come from?

The third application: Best polynomial approximation

Where does this come from?

Consider the triangle

$$\triangle := \{(x,y) : x \ge 0,\ y \ge 0,\ x+y \le 1\}.$$

Define the Jacobi-type weight function

$$\varpi_{\alpha,\beta,\gamma}(x,y) := x^\alpha y^\beta (1-x-y)^\gamma, \quad \alpha,\beta,\gamma > -1.$$

Define

$$E_n(f)_{\alpha,\beta,\gamma} = E_n(f)_{L^2(\varpi_{\alpha,\beta,\gamma})} := \inf_p \|f-p\|_{L^2(\varpi_{\alpha,\beta,\gamma})},$$

where the minimum is over all polynomials in two variables of degree at most n.

The main theorem

Theorem

Let $\alpha, \beta, \gamma > -1$, and let r be a positive integer. For $f \in W_2^r(\varpi_{\alpha,\beta,\gamma})$, we have

$$E_n(f)_{\alpha,\beta,\gamma} \le \frac{c}{n^r}\Big[E_{n-r}(\partial_1^r f)_{\alpha+r,\beta,\gamma+r} + E_{n-r}(\partial_2^r f)_{\alpha,\beta+r,\gamma+r} + E_{n-r}(\partial_3^r f)_{\alpha+r,\beta+r,\gamma}\Big]$$

for $n \ge 3r$, where c is a constant independent of n and f. Here, $W_2^r(\varpi_{\alpha,\beta,\gamma})$ is a certain Sobolev space.

The third application: Best polynomial approximation

Which are the main ingredients?

The third application: Best polynomial approximation

Which are the main ingredients?

(1) The polynomials

$$J_{k,n}^{\alpha,\beta,\gamma}(x,y) := (x+y)^k J_k^{\alpha,\beta}\left(\frac{y-x}{x+y}\right) J_{n-k}^{2k+\alpha+\beta+1,\gamma}(1-2x-2y),$$

$$0 \le k \le n,$$

are orthogonal for the 2-dimensional Jacobi-type weight $\varpi_{\alpha,\beta,\gamma}$ on the triangle $\triangle$, where

$$J_n^{\alpha,\beta}(t) = \frac{1}{(n+\alpha+\beta+1)_n} P_n^{(\alpha,\beta)}(t),$$

with $P_n^{(\alpha,\beta)}$ the usual Jacobi polynomials.

Which are the main ingredients?

Which are the main ingredients?

(2) The following determinant evaluation:

Theorem

With $f(s_1,s_2,r,i,j)$ as defined before and

$$M(r) := \begin{pmatrix} f(s_1,s_2,r,i,j) & \text{for } 0 \le i < r \\ (-1)^{j-i-r} f(s_2,s_1,r,i-r,j) & \text{for } r \le i < 2r \end{pmatrix}_{0 \le i,j \le 2r-1}.$$

the determinant of $M(r)$ equals

$$(-1)^r \prod_{j=1}^{r} \frac{1}{(s_1+s_2+2r+j-2)_r}.$$

Let

$$f(s_1,s_2,r,i,j) := \binom{r}{j-i} \frac{(s_1+i)_{j-i}}{(s_1+s_2+i+j-1)_{j-i}\,(s_1+s_2+r+2i)_{j-i}}$$

and

$$M(r_1,r_2) := \begin{pmatrix} f(s_1,s_2,r_1,i,j) \qquad \text{for } 0 \le i < r_2 \\ (-1)^{j-i-r_2} f(s_2,s_1,r_2,i-r_2,j) \\ \qquad \text{for } r_2 \le i < r_1+r_2 \end{pmatrix}_{0 \le i,j \le r_1+r_2-1}.$$

Let

$$f(s_1,s_2,r,i,j) := \binom{r}{j-i} \frac{(s_1+i)_{j-i}}{(s_1+s_2+i+j-1)_{j-i}\,(s_1+s_2+r+2i)_{j-i}}$$

and

$$M(r_1,r_2) := \begin{pmatrix} f(s_1,s_2,r_1,i,j) & \text{for } 0 \le i < r_2 \\ (-1)^{j-i-r_2} f(s_2,s_1,r_2,i-r_2,j) & \\ & \text{for } r_2 \le i < r_1+r_2 \end{pmatrix}_{0 \le i,j \le r_1+r_2-1}.$$

How to calculate the determinant of $M(r_1,r_2)$?

Let

$$f(s_1, s_2, r, i, j) := \binom{r}{j-i} \frac{(s_1 + i)_{j-i}}{(s_1 + s_2 + i + j - 1)_{j-i} (s_1 + s_2 + r + 2i)_{j-i}}$$

and

$$M(r_1, r_2) := \begin{pmatrix} f(s_1, s_2, r_1, i, j) & \text{for } 0 \le i < r_2 \\ (-1)^{j-i-r_2} f(s_2, s_1, r_2, i - r_2, j) & \\ & \text{for } r_2 \le i < r_1 + r_2 \end{pmatrix}_{0 \le i,j \le r_1 + r_2 - 1}.$$

How to calculate the determinant of $M(r_1, r_2)$?

Laplace expansion again!

Laplace expansion

Laplace expansion

Write M for $M(r_1, r_2)$ for short.

Then

$$\det M = \sum_{0 \le k_0 < \cdots < k_{r_2-1} \le r_1+r_2-1} (-1)^{\binom{r_2}{2}+\sum_{i=0}^{r_2-1} k_i} \det M_{0,\dots,r_2-1}^{k_0,\dots,k_{r_2-1}} \cdot \det M_{r_2,\dots,r_1+r_2-1}^{l_0,\dots,l_{r_1-1}},$$

where $M_{b_1,\dots,b_r}^{a_1,\dots,a_r}$ denotes the submatrix of M consisting of rows $a_1, \dots, a_r$ and columns $b_1, \dots, b_r$, and $\{l_0, \dots, l_{r_1-1}\}$ is the complement of $\{k_0, \dots, k_{r_2-1}\}$ in $\{0, 1, \dots, r_1 + r_2 - 1\}$.

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Also here, it turns out that it is not difficult to evaluate the minors which appear in this sum.

The third application: Best polynomial approximation

After a lot of simplification, one arrives at

$$(-1)^{r_1 r_2} \prod_{i=0}^{r_1+r_2-1} \frac{(s_2)_i\,(s_1+s_2+i-2)!\,(i+s_1+s_2-1)_i}{(s_1+s_2+2i-2)!\,(r_1+r_2-i-1)!\,(s_1+s_2+r_1+r_2+i-2)!}$$

$$\times \prod_{i=0}^{r_2-1} \frac{(s_1+s_2+r_1+2i-2)!\,(s_1+s_2+r_1+2i-1)!\,(r_1+i)!}{(s_1)_i\,(s_1+s_2+r_1+i-2)!\,(r_1+r_2-1)!\,(s_1+s_2)_{r_1+r_2-1}}$$

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$$\times \sum_{0\le k_0<\cdots<k_{r_2-1}\le r_1+r_2-1} (-1)^{\sum_{i=0}^{r_2-1} k_i} \prod_{0\le i<j\le r_2-1} (k_j-k_i)^2(k_i+k_j+s_1+s_2-1)^2$$

$$\cdot \prod_{i=0}^{r_2-1} \frac{(s_1+s_2-1+2k_i)}{(s_1+s_2-1)} \cdot \frac{(s_1+s_2-1)_{k_i}\,(s_1)_{k_i}\,(-r_1-r_2+1)_{k_i}}{k_i!\,(s_2)_{k_i}\,(s_1+s_2+r_1+r_2-1)_{k_i}}.$$

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Now apply the $p=0$, $q\to 1$ case of the transformation formula.

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4. Is there a q-analogue of this one discrete Macdonald–Mehta integral?