Two applications of useful functions

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'Special functions' should be more appropriately called 'useful functions'

(Pál Turán)

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The 'useful functions' of this talk will be hypergeometric series.

$$(\alpha;q)_m := (1-\alpha)(1-\alpha q)(1-\alpha q^2)\cdots(1-\alpha q^{m-1})$$

$$\begin{aligned}{}_8W_7\left(a;b,c,d,e,f;q,\frac{a^2q^2}{bcdef}\right)\\ =\frac{(aq,aq/de,aq/df,aq/ef;q)_\infty}{(aq/d,aq/e,aq/f,aq/def;q)_\infty}\\ \times {}_4\phi_3\left[\begin{matrix}aq/bc,d,e,f\\ aq/b,aq/c,def/a\end{matrix};q,q\right]\end{aligned}$$

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$\longrightarrow$ the *Mathematica* packages HYP and HYPQ

(joint work with Michael Schlosser)

The first application: Counting standard Young tableaux

(joint work with Michael Schlosser)

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ be two n-tuples of non-negative integers which are in non-increasing order and satisfy $\lambda_i \ge \mu_i$ for all i.
A standard Young tableau of skew shape λ/μ is an arrangement of the numbers $1, 2, \dots, \sum_{i=1}^n (\lambda_i - \mu_i)$ of the form

$$\begin{array}{cccccccc}
 & & & \pi_{1,\mu_1+1} & & \cdots\cdots\cdots & & \pi_{1,\lambda_1} \\
 & & \pi_{2,\mu_2+1} \quad \cdots & \pi_{2,\mu_1+1} & \cdots\cdots & & \pi_{2,\lambda_2} & \\
 & \iddots & & \vdots & & \iddots & & \\
\pi_{n,\mu_n+1} & & \cdots\cdots\cdots\cdots\cdots & & \pi_{n,\lambda_n} & & &
\end{array}$$

such that numbers along rows and columns are increasing.

The first application: Counting standard Young tableaux

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such that numbers along rows and columns are increasing.

A standard Young tableau of shape $(6, 5, 4, 3, 2, 1)/(3, 3, 0, 0, 0, 0)$:

$$\begin{matrix} & & & 2 & 5 & 13 \\ & & & 3 & 9 & \\ 1 & 4 & 8 & 12 & & \\ 6 & 11 & 15 & & & \\ 7 & 14 & & & & \\ 10 & & & & & \end{matrix}$$

John Stembridge:

`My student Elizabeth DeWitt has found a closed formula for the number of standard Young tableaux of skew shape, where the outer shape is a staircase and the inner shape a rectangle. Have you seen this before?`

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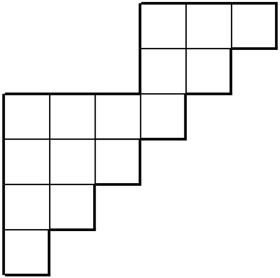
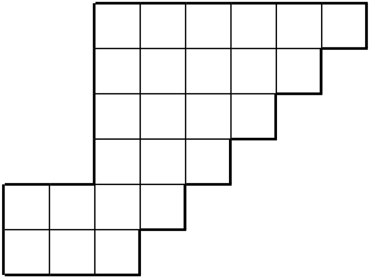
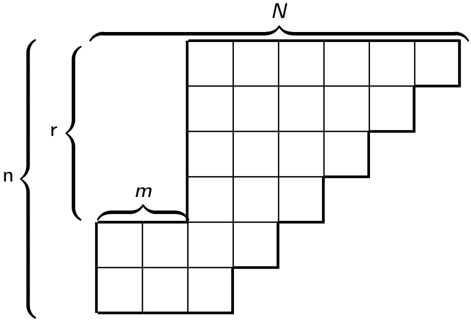


The first application: Counting standard Young tableaux

We shall do something more general than DeWitt here: we shall enumerate all standard Young tableaux of a skew shape, where the outer shape is a (possibly incomplete) staircase and the inner shape is a rectangle.



Our goal: Let N, n, m, r be non-negative integers. Compute the number of all standard Young tableaux of shape $(N, N-1, \ldots, N-n+1)/(m^r)$, where (m^r) stands for $(m, m, \ldots, m, 0, \ldots, 0)$ with r components m).



Aitken's Formula

The number of all standard Young tableaux of shape λ/μ equals

$$\left(\sum_{i=1}^{n}(\lambda_i-\mu_i)\right)!\cdot\det_{1\le i,j\le n}\left(\frac{1}{(\lambda_i-i-\mu_j+j)!}\right).$$

We substitute in Aitken's formula:

$$\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!\det_{1\le i,j\le n}\left(\begin{cases}\dfrac{1}{(N+1-2i-m+j)!} & j\le r\\ \dfrac{1}{(N+1-2i+j)!} & j>r\end{cases}\right).$$

We substitute in Aitken's formula:

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We now do a Laplace expansion with respect to the first r columns:

$$\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!$$
$$\times\sum_{1\le k_1<\dots<k_r\le n}(-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i}\det_{1\le i,j\le r}\left(\frac{1}{(N+1-2k_i-m+j)!}\right)$$
$$\cdot\det_{\substack{1\le i\le n,\ i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{(N+1-2i+j)!}\right).$$

The first application: Counting standard Young tableaux

$$\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!$$

$$\times\sum_{1\le k_1<\dots<k_r\le n}(-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i}\det_{1\le i,j\le r}\left(\frac{1}{(N+1-2k_i-m+j)!}\right)$$

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$$\left(\binom{N+1}{2} - \binom{N-n+1}{2} - mr\right)!$$
$$\times \sum_{1\le k_1<\dots<k_r\le n} (-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i} \det_{1\le i,j\le r}\left(\frac{1}{(N+1-2k_i-m+j)!}\right)$$
$$\cdot \det_{\substack{1\le i\le n,\, i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{(N+1-2i+j)!}\right).$$

Both determinants can be evaluated by means of

$$\det_{1\le i,j\le s}\left(\frac{1}{(X_i+j)!}\right) = \prod_{i=1}^{s} \frac{1}{(X_i+s)!} \prod_{1\le i<j\le s} (X_i - X_j),$$

The first application: Counting standard Young tableaux

After a lot of simplification, one arrives at

$$(-1)^{\binom{r}{2}} 2^{\binom{r}{2}+\binom{n-r}{2}} \left(\binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!$$
$$\times \prod_{i=1}^{n} \frac{(i-1)!}{(N+n+1-2i)!} \prod_{i=1}^{r} \frac{(N+n-1)!}{(n-1)!\,(N-m+r-1)!}$$
$$\times \sum_{0\le k_1<\cdots<k_r\le n-1} \prod_{1\le i<j\le r} (k_j-k_i)^2$$
$$\cdot \prod_{i=1}^{r} \frac{\left(-\frac{N-m+r-1}{2}\right)_{k_i} \left(-\frac{N-m+r-2}{2}\right)_{k_i} (-n+1)_{k_i}}{\left(-\frac{N+n-1}{2}\right)_{k_i} \left(-\frac{N+n-2}{2}\right)_{k_i} k_i!}.$$

After a lot of simplification, one arrives at

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$$\times \sum_{0\le k_1<\dots<k_r\le n-1} \prod_{1\le i<j\le r} (k_j-k_i)^2$$
$$\cdot \prod_{i=1}^{r} \frac{\left(-\frac{N-m+r-1}{2}\right)_{k_i} \left(-\frac{N-m+r-2}{2}\right)_{k_i} (-n+1)_{k_i}}{\left(-\frac{N+n-1}{2}\right)_{k_i} \left(-\frac{N+n-2}{2}\right)_{k_i} k_i!}.$$

$\longrightarrow$ multiple hypergeometric series associated to root systems!

The first application: Counting standard Young tableaux

An elliptic transformation formula (Rains, Coskun and Gustafson)

Let a, b, c, d, e, f be indeterminates, let m be a nonnegative integer, and $r \geq 1$. Then

$$\sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^r (2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\,\theta(aq^{k_i+k_j};p)^2$$
$$\times \prod_{i=1}^r \frac{\theta(aq^{2k_i};p)(a,b,c,d,e,f;q,p)_{k_i}}{\theta(a;p)(q,aq/b,aq/c,aq/d,aq/e,aq/f;q,p)_{k_i}}$$
$$\times \prod_{i=1}^r \frac{(\lambda aq^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{(efq^{r-1-m}/\lambda,aq^{1+m};q,p)_{k_i}}$$

The first application: Counting standard Young tableaux

$$= \prod_{i=1}^{r} \frac{(b, c, d, ef/a; q, p)_{i-1}}{(\lambda b/a, \lambda c/a, \lambda d/a, ef/\lambda; q, p)_{i-1}}$$

$$\times \prod_{i=1}^{r} \frac{(aq; q, p)_m \, (aq/ef; q, p)_{m+1-r} \, (\lambda q/e, \lambda q/f; q, p)_{m-i+1}}{(\lambda q; q, p)_m \, (\lambda q/ef; q, p)_{m+1-r} \, (aq/e, aq/f; q, p)_{m-i+1}}$$

$$\times \sum_{0 \le k_1 < k_2 < \cdots < k_r \le m} q^{\sum_{i=1}^{r} (2i-1)k_i} \prod_{1 \le i < j \le r} \theta(q^{k_i - k_j}; p)^2 \, \theta(\lambda q^{k_i + k_j}; p)^2$$

$$\times \prod_{i=1}^{r} \frac{\theta(\lambda q^{2k_i}; p)(\lambda, \lambda b/a, \lambda c/a, \lambda d/a, e, f; q, p)_{k_i}}{\theta(\lambda; p)(q, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f; q, p)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(\lambda a q^{2-r+m}/ef, q^{-m}; q, p)_{k_i}}{(efq^{r-1-m}/\lambda, \lambda q^{1+m}; q, p)_{k_i}},$$

where $\lambda = a^2 q^{2-r}/bcd$.

In the elliptic transformation formula, we let $p = 0$, $d \to aq/d$, $f \to aq/f$, and then $a \to 0$. Next we perform the substitutions $b \to q^b$, $c \to q^c$, etc., we divide both sides of the identity obtained so far by $(1-q)^{\binom{r}{2}}$, and we let $q \to 1$.

The first application: Counting standard Young tableaux

Corollary

For all non-negative integers m, r and s, we have

$$\sum_{0\le k_1<k_2<\cdots<k_r\le m}\ \prod_{1\le i<j\le r}(k_i-k_j)^2\prod_{i=1}^{r}\frac{(d+k_i)_s\,(b)_{k_i}\,(-m)_{k_i}}{k_i!\,(f)_{k_i}}$$

$$=\frac{(-1)^{\binom r2}}{(r+s-1)!^{s-1}}\prod_{i=1}^{r}\frac{(b)_{i-1}\,(-f+b+s+2r-i-m)_{m-r+1}}{(-f-m+i)_{m-i+1}}$$

$$\times\prod_{i=1}^{r+s-1}\frac{(i-1)!\,m!}{(m-i)!}\prod_{i=r}^{r+s-1}\frac{(d-b+1-r)_i}{(r+s-i-1)!\,(d)_{i-r}\,(f-b-s+1-r)_i}$$

$$\times\sum_{0\le \ell_1<\ell_2<\cdots<\ell_s\le r+s-1}\ \prod_{1\le i<j\le s}(\ell_i-\ell_j)^2$$

$$\times\prod_{i=1}^{s}\frac{(d)_{\ell_i}\,(f-b-s+1-r)_{\ell_i}\,(-r-s+1)_{\ell_i}}{\ell_i!\,(d-b+1-r)_{\ell_i}\,(-m)_{\ell_i}}.$$

The first application: Counting standard Young tableaux

Corollary

For all non-negative integers m, r and s, we have

$$\sum_{0\le k_1<k_2<\cdots<k_{\color{red}r}\le m}\ \prod_{1\le i<j\le r}(k_i-k_j)^2\prod_{i=1}^{r}\frac{(d+k_i)_{\color{red}s}\,(b)_{k_i}\,(-m)_{k_i}}{k_i!\,(f)_{k_i}}$$

$$=\frac{(-1)^{\binom r2}}{(r+s-1)!^{s-1}}\prod_{i=1}^{r}\frac{(b)_{i-1}\,(-f+b+s+2r-i-m)_{m-r+1}}{(-f-m+i)_{m-i+1}}$$

$$\times\prod_{i=1}^{r+s-1}\frac{(i-1)!\,m!}{(m-i)!}\prod_{i=r}^{r+s-1}\frac{(d-b+1-r)_i}{(r+s-i-1)!\,(d)_{i-r}\,(f-b-s+1-r)_i}$$

$$\times\sum_{0\le \ell_1<\ell_2<\cdots<\ell_{\color{red}s}\le r+s-1}\ \prod_{1\le i<j\le s}(\ell_i-\ell_j)^2$$

$$\times\prod_{i=1}^{s}\frac{(d)_{\ell_i}\,(f-b-s+1-r)_{\ell_i}\,(-r-s+1)_{\ell_i}}{\ell_i!\,(d-b+1-r)_{\ell_i}\,(-m)_{\ell_i}}.$$

The first application: Counting standard Young tableaux

Theorem

If $N-n$ is even, the number of standard Young tableaux of shape $(N, N-1, \dots, N-n+1)/(m^r)$ equals

$$(-1)^{\binom{(N-n)/2}{2}+\frac{1}{2}r(N-n)}2^{\binom{n}{2}+(N-n-m)r}\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!$$

$$\times\frac{1}{\left(r+\frac{N-n-2}{2}\right)!^{(N-n)/2}\left(\frac{N+n-2}{2}\right)!^{(N-n)/2}}\frac{\prod_{i=1}^{(N+n)/2}(i-1)!}{\prod_{i=1}^{n}(N-n+2i-1)!}$$

$$\times\prod_{i=1}^{r}\frac{\left(\frac{N-n}{2}+i-1\right)!\,(n+m-r+2i-1)!\,\left(\frac{n+m-r}{2}+i\right)_{(N-n)/2}}{(m+i-1)!\,(N-m-r+2i-1)!}$$

$$\times\sum_{0\le\ell_1<\ell_2<\dots<\ell_{(N-n)/2}\le r+\frac{N-n-2}{2}}(-1)^{\sum_{i=1}^{(N-n)/2}\ell_i}\left(\prod_{1\le i<j\le\frac{N-n}{2}}(\ell_i-\ell_j)^2\right)$$

$$\cdot\prod_{i=1}^{\frac{N-n}{2}}\binom{\frac{N-n-2}{2}+r}{\ell_i}\frac{\left(\frac{N+n}{2}-\ell_i\right)_{\ell_i}\left(\frac{n+m-r+1}{2}-i\right)_{r+i-\ell_i-1}\left(\frac{N-m-r+2}{2}-i\right)_{r+i-\ell_i-1}}{\left(\frac{N+m-r+2}{2}-i\right)_{r+i-\ell_i-1}},$$

and there is a similar statement if $N-n$ is odd.

In the case of a full staircase (i.e., $n = N$), the formula reduces to DeWitt's original result.

Corollary

The number of standard Young tableaux of shape $(n, n-1, \dots, 1)/(m^r)$ equals

$$2^{\binom{n}{2}-rm}\left(\binom{n+1}{2}-mr\right)!\prod_{i=1}^{n}\frac{(i-1)!}{(2i-1)!}\\ \times\prod_{i=1}^{r}\frac{(n+m-r+2i-1)!\,(i-1)!}{(m+i-1)!\,(n-m-r+2i-1)!},$$

The "next" case:

Corollary

The number of standard Young tableaux of shape $(n+1, n, \dots, 2)/(m^r)$ *equals*

$$2^{\binom{n}{2}-(m-1)r}\left(\binom{n+2}{2}-mr-1\right)!\prod_{i=1}^{n}\frac{(i-1)!}{(2i)!}$$
$$\times\prod_{i=1}^{r}\frac{(n+m-r+2i-1)!\,(i-1)!}{(m+i-1)!\,(n-m-r+2i)!}$$
$$\times\sum_{\ell=0}^{r}(-1)^{r-\ell}\binom{r}{\ell}\frac{(n-\ell+1)_\ell\left(\frac{n+m-r}{2}\right)_{r-\ell}\left(\frac{n-m-r+1}{2}\right)_{r-\ell}}{\left(\frac{n+m-r+1}{2}\right)_{r-\ell}}.$$

In general:
The number of standard Young tableaux of shape $(N, N-1, \dots, N-n)/(m^r)$ equals an $\lceil (N-n)/2 \rceil$-fold hypergeometric sum.

John Stembridge:

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`I think her approach is much simpler;`

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`I think her approach is much simpler;`
`but I don't think it would extend to the ``next`
`case'' you mention.`

The second application: Differential operators

(joint work with Andreas Juhl)

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A GJMS[1]-operator P_{2N}, $N \geq 1$, is a specific rule which associates to any pseudo-Riemannian manifold (M, g) a differential operator of the form

$$P_{2N}(g) = \Delta_g^N + \text{lower-order terms},$$

where $\Delta_g = -\delta_g d$ is the Laplace–Beltrami operator of g.

[1] C. R. Graham, R. Jenne, L. J. Mason and G. A. J. Sparling (1992)

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Our pseudo-Riemannian manifold is the Möbius spheres

$$\mathbb{S}^{q,p} = \mathbb{S}^q \times \mathbb{S}^p$$

with the signature (q, p)-metric $g_{\mathbb{S}^q} - g_{\mathbb{S}^p}$ given by the round metrics on the factors.

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For an r-tuple $I=(I_1,\dots,I_r)$ of positive integers, we write $|I|:=I_1+\dots+I_r$ and

$$P_{2I}=P_{2I_1}\circ\dots\circ P_{2I_r}.$$

For $N\ge 1$, Juhl looked at sums of the form

$$\sum_{|I|=N} m_I P_{2I}$$

with the multiplicities m_I, for $I=(I_1,\dots,I_r)$, defined by

$$m_I=-(-1)^r|I|!\,(|I|-1)!\prod_{j=1}^{r}\frac{1}{I_j!\,(I_j-1)!}\prod_{j=1}^{r-1}\frac{1}{I_j+I_{j+1}}.$$

The second application: Differential operators

Here is the first set of relations found by Juhl:

Conjecture

On $\mathbb{S}^{q,p}$, we have

$$\sum_{|I|=2N} m_I P_{2I} = (2N)!\,(2N-1)!\,\left(\frac{1}{2} - B^2 - C^2\right),\ N \ge 1$$

and

$$\sum_{|I|=2N+1} m_I P_{2I} = (2N+1)!\,(2N)!\,(-B^2 + C^2),\ N \ge 0.$$

Here,

$$B^2 = -\Delta_{\mathbb{S}^q} + \left(\frac{q-1}{2}\right)^2 \quad \textit{and} \quad C^2 = -\Delta_{\mathbb{S}^p} + \left(\frac{p-1}{2}\right)^2.$$

Recall:

$$m_I = -(-1)^r |I|!\,(|I|-1)! \prod_{j=1}^{r} \frac{1}{I_j!\,(I_j-1)!} \prod_{j=1}^{r-1} \frac{1}{I_j + I_{j+1}}.$$

So far, so mysterious . . .

So far, so mysterious ...

It is well-known that

$$P_{4N} = \prod_{j=1}^{N} \left((B^2 - C^2)^2 - 2(2j-1)^2(B^2 + C^2) + (2j-1)^4\right)$$

and

$$P_{4N+2} = (-B^2 + C^2) \prod_{j=1}^{N} \left((B^2 - C^2)^2 - 2(2j)^2(B^2 + C^2) + (2j)^4\right).$$

So far, so mysterious ...

It is well-known that

$$P_{4N} = \prod_{j=1}^{N} (B+C+(2j-1))(B-C-(2j-1))(B+C-(2j-1))(B-C+(2j-1))$$

and

$$P_{4N+2} = (-B^2+C^2) \prod_{j=1}^{N} (B+C+2j)(B-C-2j)(B+C-2j)(B-C+2j).$$

So far, so mysterious . . .

It is well-known that

$$P_{2N} = 2^{2N} \left((C+B+1-N)/2\right)_N \left((C-B+1-N)/2\right)_N ,$$

So far, so mysterious ...

It is well-known that

$$P_{2N} = 2^{2N}\left((C+B+1-N)/2\right)_N \left((C-B+1-N)/2\right)_N ,$$

Writing $X = C + B$ and $Y = C - B$, this becomes

$$P_{2N} = 2^{2N}\left((X+1-N)/2\right)_N \left((Y+1-N)/2\right)_N .$$

The second application: Differential operators

The conjecture again:

Conjecture

On $\mathbb{S}^{q,p}$, we have

$$\sum_{|I|=2N} m_I P_{2I} = \frac{1}{2}(2N)!\,(2N-1)!\,\left(1 - X^2 - Y^2\right),\ N \ge 1,$$

and

$$\sum_{|I|=2N+1} m_I P_{2I} = (2N+1)!\,(2N)!\,XY,\ N \ge 0,$$

where

$$P_{2N} = 2^{2N}\,((X+1-N)/2)_N\,((Y+1-N)/2)_N\,.$$

Recall:

$$m_I = -(-1)^r |I|!\,(|I|-1)! \prod_{j=1}^{r} \frac{1}{I_j!\,(I_j-1)!} \prod_{j=1}^{r-1} \frac{1}{I_j + I_{j+1}}.$$

The second application: Differential operators

Lemma

For all non-negative integers A and B, we have

$$\begin{aligned}((X+1-A)/2)_A\,((X+1-B)/2)_B \\ &= \sum_{j=0}^{\lfloor (A+B)/2\rfloor} (-1)^j \frac{(-A/2)_j\,(-B/2)_j\,(-(A+B)/2)_j}{j!} \\ &\qquad \times ((X+1-A-B+2j)/2)_{A+B-2j}\,.\end{aligned}$$

The second application: Differential operators

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Lemma

For all non-negative integers A and B, we have

$$\big((X+1-A)/2\big)_A\,\big((X+1-B)/2\big)_B = \sum_{j=0}^{\lfloor (A+B)/2\rfloor} (-1)^j \frac{(-A/2)_j\,(-B/2)_j\,(-(A+B)/2)_j}{j!} \times \big((X+1-A-B+2j)/2\big)_{A+B-2j}\,.$$

Proof.

In hypergeometric notation, the sum on the right-hand side reads

$$\big((X+1-A-B)/2\big)_{A+B}\;{}_3F_2\!\left[\begin{matrix}-(A+B)/2,-A/2,-B/2\\ (1+X-A-B)/2,(1-X-A-B)/2\end{matrix};1\right].$$

The ${}_3F_2$-series is balanced and can hence be summed by means of the Pfaff–Saalschütz summation formula. □

The conjecture again:

Conjecture

On $\mathbb{S}^{q,p}$, we have

$$\sum_{|I|=2N} m_I P_{2I} = \frac{1}{2}(2N)!\,(2N-1)!\,\left(1 - X^2 - Y^2\right),\ N \ge 1,$$

and

$$\sum_{|I|=2N+1} m_I P_{2I} = (2N+1)!\,(2N)!\,XY,\ N \ge 0,$$

where

$$P_{2N} = 2^{2N}\left((X+1-N)/2\right)_N\left((Y+1-N)/2\right)_N.$$

Recall:

$$m_I = -(-1)^r |I|!\,(|I|-1)! \prod_{j=1}^{r} \frac{1}{I_j!\,(I_j-1)!} \prod_{j=1}^{r-1} \frac{1}{I_j + I_{j+1}}.$$

The proof strategy: induction! First evaluate the partial sum, where in

$$P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_r}$$

the last component is fixed.

The second application: Differential operators

Lemma

For all positive integers $a < N$, the partial sum

$$S(N,a) = \sum_{J:|J|+a=N} m_{(J,a)} P_{2J}$$

satisfies

$$S(N,a) = \binom{N-1}{a-1} \sum_{k=0}^{\lfloor (N-a)/2 \rfloor} \sum_{l=0}^{\lfloor (N-a)/2 \rfloor} (-1)^{N+k+l+a} 2^{2N-2k-2l-2a}$$

$$\cdot \left((X+1-N+a+2k)/2\right)_{N-a-2k} \left((Y+1-N+a+2l)/2\right)_{N-a-2l}$$

$$\cdot \frac{(-N+a)_{2k}\,(-N+a)_{2l}\,(-N/2)_k\,(-N/2)_l}{k!\,l!}$$

$$\times {}_4F_3\left[\begin{matrix} -\frac{1}{2}, -k, -l, \frac{1}{2}-\frac{N}{2} \\ -\frac{N}{2}, \frac{a}{2}-\frac{N}{2}, \frac{1}{2}+\frac{a}{2}-\frac{N}{2} \end{matrix}; 1\right].$$

The proof of the lemma is somewhat tedious . . .

By Lemma A.2 with $X = -a - 1$ and $M = N - a - 2s + 1$, this is equal to

$$\frac{(-1)^{N-a-2s-1}}{(-N+2s)_{N-a-2s}}((-N+2s)/2)_{s_1-s}((-N+2s)/2)_{s_2-s} + \chi(s_1 = s_2 = (N-a)/2) \cdot 2^{-N+a+2s}$$

$$= -\frac{a!}{(N-2s)!}\frac{(-N/2)_{s_1}\,(-N/2)_{s_2}}{(-N/2)_s^2} + \chi(s_1 = s_2 = (N-a)/2) \cdot 2^{-N+a+2s},$$

where $\chi(\mathcal{S}) = 1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S}) = 0$ otherwise. If we substitute this in (2.15), then we obtain

$$\sum_{s_1=0}^{\lfloor (N-a)/2\rfloor}\sum_{s_2=0}^{\lfloor (N-a)/2\rfloor}\big((X+1-N+a+2s_1)/2\big)_{N-a-2s_1}\big((Y+1-N+a+2s_2)/2\big)_{N-a-2s_2}$$
$$\cdot (-1)^{N+s_1+s_2+a}2^{2N-2s_1-2s_2-2a}\frac{(N-1)!}{(a-1)!\,(N-a)!}$$
$$\cdot \frac{(-N+a)_{2s_1}\,(-N+a)_{2s_2}\,(-N/2)_{s_1}\,(-N/2)_{s_2}}{s_1!\,s_2!}$$
$$\cdot \sum_{s=0}^{s_1}\frac{(-1/2)_s\,(-s_1)_s\,(-s_2)_s\,N!\,(N-a-2s)!}{s!\,(N-2s)!\,(-N/2)_s^2\,(N-a)!}$$
$$+ \chi(N-a \text{ is even}) \cdot \frac{N!\,(N-1)!}{a!\,(a-1)!\,((N-a)/2)!^2}$$
$$\times \sum_{s=0}^{(N-a)/2}\frac{2^{-N+a+2s}(-1/2)_s\,(-(N-a)/2)_s^2\,(N-a-2s)!}{s!}.$$

Here, the first sum is, upon rewriting, exactly equal to the right-hand side of (2.10) (except that s_1 and s_2 took over the role of k and l). On the other hand, if we write the second sum in hypergeometric notation, we obtain

$$\chi(N-a \text{ is even}) \cdot \frac{N!\,(N-1)!}{a!\,(a-1)!\,((N-a)/2)!^2}$$
$$\times \sum_{s=0}^{(N-a)/2}\frac{2^{-N+a+2s}(-1/2)_s\,(-(N-a)/2)_s^2\,(N-a-2s)!}{s!}$$

The proof of the lemma is somewhat tedious ...
but it is not difficult. (The only summation formulas needed for simplification are the Chu–Vandermonde summation formula, the binomial theorem, and simple difference calculus.)

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Now the conjecture can be proved:

$$\sum_{|I|=N} m_I P_{2I} = P_{2N} + \sum_{a=1}^{N-1} S(N,a) P_{2a}.$$

One uses the lemma for the expansion of $S(N,a)$, applies the multiplication lemma, has to go through some more pages of the kind ...

The second application: Differential operators

$$\big((Y+1-a)/2\big)_a \big((Y+1-N+a+2l)/2\big)_{N-a-2l}$$
$$= \sum_{j_2=0}^{\lfloor (N-2l)/2\rfloor} (-1)^{j_2} \frac{(-a/2)_{j_2}\,(-(N-a-2l)/2)_{j_2}\,(-(N-2l)/2)_{j_2}}{j_2!}$$
$$\cdot \big((Y+1-N+2l+2j_2)/2\big)_{N-2l-2j_2}.$$

We use these in (2.18) and, in addition, perform the index transformation $s_1 = k + j_1$ and $s_2 = l + j_2$. Thus, the left-hand side in (2.16) can be written in the form

$$\sum_{s_1=0}^{\lfloor N/2\rfloor}\sum_{s_2=0}^{\lfloor N/2\rfloor}\sum_{a=1}^{N}(-1)^{N+s_1+s_2+a}2^{2N-2k-2l}\binom{N-1}{a-1}$$
$$\cdot \big((X+1-N+2s_1)/2\big)_{N-2s_1}\big((Y+1-N+2s_2)/2\big)_{N-2s_2}$$
$$\cdot \sum_{k=0}^{\lfloor (N-a)/2\rfloor}\sum_{l=0}^{\lfloor (N-a)/2\rfloor}\frac{(-a/2)_{s_1-k}\,(-a/2)_{s_2-l}\,(-(N-a)/2)_{s_1}\,(-(N-a)/2)_{s_2}}{(s_1-k)!\,(s_2-l)!\,(-(N-a)/2)_k\,(-(N-a)/2)_l}$$
$$\cdot \frac{(-N+a)_{2k}\,(-N+a)_{2l}\,(-N/2)_{s_1}\,(-N/2)_{s_2}}{k!\,l!}\,{}_4F_3\left[\begin{matrix}-\frac12,-k,-l,\frac12-\frac N2\\ -\frac N2,\frac a2-\frac N2,\frac12+\frac a2-\frac N2\end{matrix};1\right]. \quad (2.19)$$

In this expression, we now concentrate on the terms involving the summation index k only:

$$\sum_{k=0}^{\lfloor (N-a)/2\rfloor} 2^{-2k}\frac{(-a/2)_{s_1-k}\,(-N+a)_{2k}\,(-k)_s}{k!\,(s_1-k)!\,(-(N-a)/2)_k}$$
$$= \sum_{k=0}^{s_1}\frac{(-a/2)_{s_1-k}\,(-(N-a-1)/2)_k\,(-k)_s}{k!\,(s_1-k)!}, \quad (2.20)$$

where s stands for the summation index of the ${}_4F_3$-series in (2.19). Because of the term $(-k)_s$ in the numerator of the summand, we may start the summation at $k = s$ (instead of at $k = 0$). Hence, if we write this sum in hypergeometric notation, we obtain

$$\frac{(-a/2)_{s_1-s}\,(-(N-a-1)/2)_s\,(-s)_s}{\dots}\,{}_2F_1\left[\begin{matrix}-s_1+s,\frac12-\frac N2+\frac a2+s\\ \dots\end{matrix};1\right].$$

The second application: Differential operators

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but it is not difficult. (The only summation formulas needed for simplification are the Chu–Vandermonde summation formula, the binomial theorem, and simple difference calculus.)
Now the conjecture can be proved:

$$\sum_{|I|=N} m_I P_{2I} = P_{2N} + \sum_{a=1}^{N-1} S(N,a) P_{2a}.$$

One uses the lemma for the expansion of $S(N,a)$, applies the multiplication lemma, has to go through some more pages of the kind ...
until one arrives at the desired conclusion. □

The second application: Differential operators

Theorem

On $\mathbb{S}^{q,p}$, we have

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and

$$\sum_{|I|=2N+1} m_I P_{2I} = (2N+1)!\,(2N)!\,XY, \ N \geq 0,$$

where

$$P_{2N} = 2^{2N}\left((X+1-N)/2\right)_N \left((Y+1-N)/2\right)_N.$$

Recall:

$$m_I = -(-1)^r |I|!\,(|I|-1)! \prod_{j=1}^{r} \frac{1}{I_j!\,(I_j-1)!} \prod_{j=1}^{r-1} \frac{1}{I_j + I_{j+1}}.$$

A second theorem providing relations for the GJMS-operators:

Theorem

On $\mathbb{S}^{q,p}$*, we have*

$$\sum_{|I|=N} m_I \frac{P_{2I}(1)}{\frac{n}{2} - I_{last}} = N!\,(N-1)!\,\sum_{M=0}^{N}(-1)^M \binom{\frac{q}{2}}{M}\binom{\frac{p}{2}}{N-M}$$

for all $N \geq 1$.

A belated

Happy Birthday!