POISSON BIVECTORS ON INFINITE DIMENSIONAL MANIFOLDS

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Abstract. We show that, on a smoothly paracompact convenient manifold M modeled on a convenient space with the bornological approximation property, the dual map of a Poisson bracket factors as a smooth section of the vector bundle $L^2_{\text{skew}}(T^*M, \mathbb{R})$.

1. Introduction

For a finite-dimensional smooth manifold M, it is well known that a Poisson bracket on $C^\infty(M)$ induces a Poisson tensor $\pi \in \Gamma(\wedge^2 TM)$ (see [2], [17]). But for a manifold modeled on a locally convex space or on a convenient space it is not automatic (see [16], [4]). In [1], a Poisson structure is constructed on a Hilbert manifold such that the Poisson bracket of two functions depends on its second-order jets, hence the dual map of the Poisson bracket cannot be factored as a smooth section of $L^2_{\text{skew}}(T^*M, \mathbb{R})$. In this article, we assume that the modeling space E of a smoothly paracompact manifold M has the bornological approximation property (see [7, Definition 6.6]). Then, if $\{-,-\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a Poisson bracket on $C^\infty(M)$, we show that the dual map $\widehat{\{\ ,\ \}}'$ factors as a smooth section of the vector bundle $L^2_{\text{skew}}(T^*M, \mathbb{R})$.

1.1. **Notations and Terminology.** Throughout this article, we utilize the smooth calculus in the convenient setting of global analysis, as developed in [7].

Let M be a smooth manifold modeled on a convenient vector space E. This implies we have a smooth atlas $M \supset U_\alpha \xrightarrow{u_\alpha} u_\alpha(U_\alpha) \subset E$ where all transition mappings $u_{\alpha\beta} : u_\beta(U_\alpha \cap U_\beta) \to u_\alpha(U_\alpha \cap U_\beta)$ are smooth in the convenient sense. We explicitly assume that M is a *smoothly paracompact* manifold. Consequently, M admits smooth partitions of unity, which allows for the global gluing of locally defined smooth functions and sections.

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We denote by $\pi_M : TM \to M$ the kinematic tangent bundle of M, which is locally trivialized by the induced charts $TU_\alpha \xrightarrow{Tu_\alpha} u_\alpha(U_\alpha) \times E \subset E \times E$. The space of smooth vector fields on M, denoted $\mathfrak{X}(M)$, is the Lie algebra of smooth sections of the tangent bundle. The subspace of compactly supported vector fields is denoted by $\mathfrak{X}_c(M)$. The cotangent bundle is denoted by $T^*M \to M$, where each fiber T^*_xM is the bounded dual space of the kinematic tangent space T_xM.

For function spaces, $C^\infty(M, \mathbb{R})$ denotes the commutative algebra of smooth, real-valued functions on M equipped with the convenient structure (see [7, Definition 3.6, Definition 3.11]) .

We denote the space of smooth test functions with compact support as $C^\infty_c(\mathbb{R}^n)$. Its continuous dual is the space of scalar distributions, denoted by $\mathcal{D}'(\mathbb{R}^n)$. Extending this to vector bundles, the space of distributional sections (or currents) of a bundle $E \to M$, such as the space of distributional 1-forms, is denoted by $\Gamma_{\mathcal{D}'}(T^*M)$.

Bounded bilinear mappings on convenient spaces are linearized using the Mackey-completed bornological tensor product, denoted by $\widehat{\otimes}_\beta$. However, when working with nuclear spaces, such as the (LF)-space $\mathfrak{X}_c(\mathbb{R}^n)$, we can identify $\widehat{\otimes}_\beta$ with the completed projective tensor product, denoted by $\widehat{\otimes}_\pi$ (see [15, Proposition 50.5]).

In the infinite-dimensional Banach or convenient setting, the natural target space for a Poisson bivector is not automatically $\bigwedge^2 TM$). Instead, we work with the smooth vector bundle $L^2_{\text{skew}}(T^*M, \mathbb{R}) \to M$, whose fiber over $x \in M$ consists of bounded, skew-symmetric bilinear forms on the cotangent space T^*_xM (see [10, section 3.1]). A global bivector field in this context is a smooth section $P \in \Gamma(L^2_{\text{skew}}(T^*M, \mathbb{R}))$.

2. The general setting

2.1. Poisson brackets. Let M be a smooth convenient manifold. A *Poisson bracket* on M is a bounded bilinear mapping

$$\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

that satisfies the following properties:

- **Skew-symmetry:** $\{f, g\} = -\{g, f\}$
- **Leibniz rule:** $\{f, gh\} = \{f, g\}h + g\{f, h\}$
- **Jacobi identity:** $\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$

for all $f, g, h \in C^\infty(M)$. The Poisson bracket is said to be of *first order* if the mapping $f \mapsto \{f, g\}$ is a linear differential operator of the first order. Explicitly imposing this condition is crucial in the infinite-dimensional setting, as the algebra $C^\infty(M)$ admits derivations that correspond to differential operators of order two or higher (see [7, 28.3]).

Using the Mackey-completed bornological tensor product we can linearize bounded bilinear mappings; see [7, 5.7]. Consequently, the Poisson bracket factors through the skew-symmetric subspace to yield a bounded linear operator

$$C^\infty(M)\bar{\otimes}_\beta C^\infty(M) \supset \bigwedge^2 C^\infty(M) \xrightarrow{\widehat{\{\,,\,\}}} C^\infty(M).$$

Taking the dual of this operator, we obtain the the map

$$C^\infty(M)' \xrightarrow{\widehat{\{\,,\,\}}'} \big(\bigwedge^2 C^\infty(M)\big)' = L^2_{\text{skew}}(C^\infty(M), \mathbb{R}).$$

2.2. **Theorem.** *Let M be a smoothly paracompact convenient manifold modeled on a convenient vector space E, such that one of the following holds:*

(a) *E has the bornological approximation property (see* [7, 6.6]*), or*
(b) *E is a Montel space, and it has the approximation property, i.e. $E' \otimes E$ is dense in $L_p(E,E)$, where $L_p(E,E)$ denotes the space of all continuous maps from E to E with the topology of uniform convergence on pre-compact subsets of E.*

Then the dual mapping of a Poisson bracket factors as follows:

$$\begin{tikzcd} C^\infty(M)' \arrow[r, "\widehat{\{\,,\,\}}'"] & L^2_{skew}(C^\infty(M), \mathbb{R}) \\ M \arrow[u, "\mathrm{ev}"] \arrow[r, "P"] & L^2_{skew}(T^*M, \mathbb{R}) \arrow[u, "L^2_{skew}(\mathrm{ev}\circ d, \mathbb{R})"'] \end{tikzcd}$$

where $\mathrm{ev}_x(f) = f(x)$ and $(L^2_{skew}(\mathrm{ev}\circ d, \mathbb{R})\bar{P}_x)(f,g) := \bar{P}_x(df_x, dg_x)$. Moreover, the Schouten-Nijenhuis bracket described in [10] *is applicable to P, and $[P,P] = 0$.*

2.3. **Remark.** [7, 6.14] *The following convenient spaces have the bornological approximation property:*

- *The space $\Gamma(M \leftarrow F)$ of smooth sections of any smooth finite dimensional vector bundle $F \to M$ with separable base M.*
- *The space $\Gamma(M \leftarrow F)$ of smooth sections with compact support in any finite dimensional vector bundle $F \to M$ with separable base M.*
- *The Fréchet space of holomorphic sections $\Gamma_{\mathcal{H}}(M \leftarrow E)$ of a holomorphic vector bundle over a complex Stein manifold M.*

Proof. We have to show that for any $x \in M$ and $f, g \in C^\infty(M)$ the expression $\{f,g\}_x$ depends only on df_x and dg_x in $T^*_x M$. By skew symmetry it suffices to do this for f, say.
Step 1: *If $f = 0$ on an open set $U \subset M$, then also $\{f,g\} = 0$ on U for any g.* Since M is smoothly paracompact, for any $y \in U$ there exists $h \in C^\infty(M)$ such that $h(y) = 0$ and $h|_{M\setminus U} = 1$. Then $hf = f$ and $\{f,g\} = \{hf,g\} = h\{f,g\} + \{h,g\}f$ vanish at y.

Step 2: Assume that $df_x = 0$. Without loss of generality, we can also assume that $f(x) = 0$ since for any constant C we have $\{C, g\} = 0$. By step 1 we may replace M by a smooth chart V centered at x which is diffeomorphic to an absolutely convex neighborhood of $x = 0$ in the modeling convenient vector space E. Let $h \in C^\infty(M)$ be a function with support in V with $h = 1$ in an open neighborhood U of 0 in V. Then $(h^2 f - f)|_U = 0$ thus by step 1 also $\{hf - f, g\}|_U = 0$. In V we have

$$f(y) = f(0) + df_0(y) + \int_0^1 d^2 f(ty)(y, y)(1 - t)dt = \int_0^1 d^2 f(ty)(y, y)(1 - t)dt$$

Let us denote the remainder term by

$$R(y) = \int_0^1 d^2 f(ty)(y, y)(1 - t)dt. \tag{1}$$

For a fixed $g \in C^\infty(U)$ and $x \in U$, the mapping $f \mapsto \{f, g\}(x)$ defines a bounded point derivation $D_{g,x} \in C^\infty(U)'$ at x. We aim to show that $D_{g,0}(R) = 0$, meaning the bracket depends only on the first derivative $df(0)$.

Because $D_{g,0}$ is a bounded linear functional on the convenient vector space $C^\infty(U)$, and the curve $t \mapsto (y \mapsto d^2 f(ty)(y, y))$ is smooth into $C^\infty(U)$, $D_{g,0}$ commutes with the Riemann integral over the compact interval $[0, 1]$. Thus, we have

$$D_{g,0}(R) = \int_0^1 D_{g,0}(R_t)(1 - t)dt, \tag{2}$$

where $R_t(y) := d^2 f(ty)(y, y)$. It suffices to show that $D_{g,0}(R_t) = 0$ for all $t \in [0, 1]$.

Without loss of generality, we assume $x = 0$.

In case (a), because E has the bornological approximation property, the identity operator $id_E \in L(E, E)$ can be approximated by elements in $E' \otimes E$, that means that there exists a net of finite-dimensional linear operators $L_i \in E' \otimes E$ converging to $id_E \in L(E, E)$ uniformly on bounded subsets of E.

Since the second derivative $d^2 f(ty)$ is a bounded bilinear form, we can approximate $R_t(y)$ by the net of functions

$$R_{t,i}(y) := d^2 f(ty)(L_i y, y). \tag{3}$$

We claim that $R_{t,i} \to R_t$ in the convenient topology of $C^\infty(U)$. Let $c \in C^\infty(\mathbb{R}, U)$ be a smooth curve, then $R_t \circ c, R_{t,i} \circ c \in C^\infty(\mathbb{R})$; we need to show that $R_{t,i} \circ c$ and all its derivatives converge uniformly on compact intervals in R to $R_{t,i} \circ c$. Let $[a, b] \subset \mathbb{R}$ be a compact interval. The map $\mathbb{R} \to E'$ defined by $s \mapsto d^2 f(tc(s))(-, c(s))$ is a smooth curve and is thus continuous (by [7, 1.8], e.g.), so the image C of $[a, b]$ under this map is compact, hence the image is weakly bounded, but since a convenient space E is quasibarrelled, the image is equicontinuous in E'; see [13, III 4.2]. This means that

$$|d^2(f \circ c)(ty)(v, y)| < \epsilon$$

for $y \in C, v \in V$, where V is an open 0-neighborhood of E. Hence,

$$|R_{t,i} \circ c(y) - R_t \circ c(y)| < \epsilon$$

for $y \in [a, b]$. Therefore, $R_{t,i} \circ c \to R_t \circ c$ uniformly on compact intervals in $\mathbb{R}$.

We now show that for every $k \in \mathbb{N}$, the sequence of derivatives $\frac{d^k}{ds^k}(R_{t,i} \circ c)$ converges uniformly to $\frac{d^k}{ds^k}(R_t \circ c)$ on compact intervals. By repeated application of the chain rule (see [7, Theorem 3.18]), the k-th derivative $\frac{d^k}{ds^k}(R_{t,i} \circ c)(s)$ expands as a finite sum of terms of the form

$$d^m f(tc(s))(v_1(s), \ldots, v_m(s)),$$

where each argument $v_r(s)$ is constructed via multilinear combinations of the vectors

$$c(s), c'(s), \ldots, c^{(k)}(s) \quad \text{and} \quad L_i c(s), L_i c'(s), \ldots, L_i c^{(k)}(s).$$

Fix a compact interval $[a, b] \subset \mathbb{R}$. Because $c : \mathbb{R} \to U$ is a smooth curve into the convenient vector space E, the collection of all its relevant derivatives over this interval,

$$B = \{c^{(j)}(s) \mid s \in [a, b],\ 0 \leq j \leq k\},$$

constitutes a bounded subset of E. Since E satisfies the bornological approximation property, the net of finite-rank operators L_i converges to id_E uniformly on bounded subsets of E. Consequently, we have

$$L_i c^{(j)}(s) \to c^{(j)}(s) \quad \text{uniformly for } s \in [a, b] \text{ and } 0 \leq j \leq k.$$

Furthermore, the smoothness of f implies that its derivative $d^m f : U \to L^m(E; \mathbb{R})$ maps bounded subsets of U to bounded subsets of the space of bounded multilinear forms $L^m(E; \mathbb{R})$. Because the convenient vector space E is quasibarrelled, these bounded sets of multilinear forms are equicontinuous. The uniform convergence of $L_i c^{(j)}(s)$ within these equicontinuous families of multilinear forms ensures that every term in the expansion of $\frac{d^k}{ds^k}(R_{t,i} \circ c)(s)$ converges uniformly on $[a, b]$ to the corresponding term of $\frac{d^k}{ds^k}(R_t \circ c)(s)$. We therefore conclude that

$$\frac{d^k}{ds^k}(R_{t,i} \circ c) \to \frac{d^k}{ds^k}(R_t \circ c) \quad \text{uniformly on } [a, b].$$

Thus, $R_{t,i} \to R_t$ in the convenient topology of $C^\infty(U)$. Now, since L_i has finite rank, it can be written as $L_i(y) = \sum_{j=1}^{N_i} e'_{ij}(y) e_{ij}$ for some $e_{ij} \in E$ and $e'_{ij} \in E'$. Consequently,

$$R_{t,i}(y) = \sum_{j=1}^{N_i} e'_{ij}(y) \cdot d^2 f(ty)(e_{ij}, y). \tag{4}$$

Applying the derivation $D_{g,0}$ to this finite sum of products yields:

$$D_{g,0}(R_{t,i}) = \sum_{j=1}^{i} D_{g,0}(h_{ij}) g_{ij}(0) + D_{g,0}(g_{ij}) h_{ij}(0).$$

where $h_{ij}(y) := e'_{ij}(y), g_{ij}(y) := d^2f(y)(e_{ij}, y)$ are smooth functions on a suitable 0-neighborhood $V \subseteq E$.

Note that as $d^2f(0)$ is bilinear, $g_{ij}(0) = d^2f(0)(e_{ij}, 0) = 0$, and because e'_{ij} is a linear functional, $h_{ij}(0) = e'_{ij}(0) = 0$. Therefore, every term in the sum vanishes, yielding $D_{g,0}(R_{t,i}) = 0$ for all $i \in I$. Passing to the limit over the net, we conclude $D_{g,0}(R_t) = 0$, and thus $D_{g,0}(R) = 0$.

In case (b), when E has the usual approximation property, we can find a net $L_i \in E' \otimes E$ of finite rank operators converges to the identity operator $id_E \in L(E, E)$ uniformly on compact subsets of E. But since E is a Montel space, this convergence is the same as the uniform convergence on bounded subsets of E, the remaining proof follow thereby. □

2.4. **Remark.** *Examples of Montel space with the approximation property are spaces of smooth sections of finite-dimensional vector bundles with compact base, or even noncompact base when one considers inductive limits of all subspaces of sections with compact support; spaces of holomorphic sections of complex vector bundles; but in general not their dual spaces.*

2.5. **Corollary.** *If M is a smoothly paracompact manifold modeled on a nuclear Fréchet (or a Silva space) then the dual mapping of a Poisson bracket factors as a smooth section of $L^2_{skew}(T^*M, \mathbb{R}) \cong \wedge^2 TM$.*

In the corollary, $\wedge^2 TM$ denotes the smooth vector bundle with the fiber $\wedge^2 T_xM$ over $x \in M$ and $\wedge^2 T_xM$ denotes the space of the skew-symmetric tensor product of T_xM with itself completed in the projective tensor product topology.

2.6. **Remark.** *Note that if M is modeled on a nuclear Fréchet space (see* [14]*) or on a nuclear (LF) space (see* [9]*), similar arguments are used to show the topological isomorphism between the space of continuous derivations on $C^\infty(M)$ and the space of smooth vector fields $\mathfrak{X}(M)$.*

The following proposition establishes that infinite-dimensional Banach spaces cannot satisfy the bornological approximation property.

2.7. **Proposition.** *Let E be a Banach space. The identity operator id_E lies in the bornological closure of the space of finite-rank operators $E' \otimes E$ in $L(E, E)$ if and only if $\dim(E) < \infty$. Consequently, E satisfies the bornological approximation property if and only if it is finite-dimensional.*

Proof. Let E be a Banach space with norm $\|\cdot\|_E$ and closed unit ball $B_E = \{x \in E \mid \|x\|_E \leq 1\}$.

The space of bounded linear operators $L(E, E)$ is endowed with the locally convex topology of uniform convergence on bounded subsets of E. Because E is a Banach space, every bounded subset $S \subset E$ is contained in λB_E for some scalar $\lambda > 0$. Therefore, the topology of uniform convergence on bounded sets on $L(E, E)$ coincides with the norm topology induced by the uniform operator norm

$$\|T\|_{op} = \sup_{x \in B_E} \|Tx\|_E. \tag{5}$$

Assume E possesses the bornological approximation property. By definition, id_E belongs to the bornological closure of $E' \otimes E$ in $L(E, E)$. Because the topology on $L(E, E)$ is normable, the bornological closure coincides with the topological closure. Thus,

$$id_E \in \overline{E' \otimes E}^{\|\cdot\|_{op}}. \tag{6}$$

Since every finite-rank operator is compact and the space $\mathcal{K}(E)$ of compact operators is closed in $(L(E, E), \|\cdot\|_{op})$, we obtain

$$\overline{E' \otimes E}^{\|\cdot\|_{op}} \subset \mathcal{K}(E).$$

This implies $id_E \in \mathcal{K}(E)$.

The identity operator id_E is compact if and only if the image of the closed unit ball, $id_E(B_E) = B_E$, is a relatively compact set in E. Since B_E is closed, B_E must be compact. By Riesz's Theorem, the closed unit ball of a Banach space is compact if and only if the space is finite-dimensional. Hence, $\dim(E) < \infty$.

Conversely, assume $\dim(E) < \infty$. The operator id_E has finite rank, implying $id_E \in E' \otimes E$. It trivially lies in its own bornological closure, so E satisfies the bornological approximation property. $\square$

3. Examples

3.1. **The Poisson structure on the dual of a Lie algebra.** Let $\mathfrak{g}$ be a convenient Lie algebra; i.e., $\mathfrak{g}$ is a convenient vector space together with a bounded Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. By the universal property of the bornological tensor product, the Lie bracket extends to a linear bounded mapping

$$\begin{tikzcd} \mathfrak{g} \times \mathfrak{g} \arrow[r, "{[\ ,\ ]}"] \arrow[d] & \mathfrak{g} \\ \bigwedge^2 \mathfrak{g} \arrow[ur, "\widetilde{[\ ,\ ]}"'] & \end{tikzcd}$$

The dual mapping

$$P := \widetilde{[\ ,\ ]}^* : \mathfrak{g}^* \to (\bigwedge^2 \mathfrak{g})^* = L^2_{\text{skew}}(\mathfrak{g}; \mathbb{R})$$

is the candidate for the Lie-Poisson structure on the dual of the Lie algebra. It induces a Lie bracket on the space

$$C^\infty_P(\mathfrak{g}^*) = \{f \in C^\infty(\mathfrak{g}^*) : df \text{ factors as } i_{\mathfrak{g}} \circ \widetilde{df} \text{ for } \widetilde{df} \in C^\infty(\mathfrak{g}^*, \mathfrak{g})\}$$

with the convenient vector space structure induced by the linear mappings

$$C^\infty_P(\mathfrak{g}^*) \xrightarrow{\text{incl}} C^\infty(\mathfrak{g}^*,\mathbb{R}), \quad C^\infty_P(\mathfrak{g}^*) \xrightarrow{d} C^\infty(\mathfrak{g}^*,\mathfrak{g})\,.$$

Similarly to [7, 48.6] we have the following result:

3.2. **Lemma.** *For $f\in C^\infty(\mathfrak{g}^*,\mathbb{R})$ we have (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3), where:*

1. *$df:\mathfrak{g}^*\to\mathfrak{g}^{**}$ actually factors as a smooth mapping $\mathfrak{g}^*\xrightarrow{df}\mathfrak{g}\xrightarrow{i_\mathfrak{g}}\mathfrak{g}^{**}$.*
2. *Each iterated derivative $d^kf_x\in L^k_{sym}(\mathfrak{g}^*,\mathbb{R})$ has the property that*
$$d^kf_x(\quad,y_2,\dots,y_k)\in\mathfrak{g} \text{ for all } x,y_2,\dots,y_k\in\mathfrak{g}^*,$$
and that the mapping
$$\prod^k\mathfrak{g}^*\to\mathfrak{g},\qquad \text{given by}\qquad (x,y_2,\dots,y_k)\mapsto df_x(\quad,y_2,\dots,y_k)\,,$$
is smooth. By the symmetry of higher derivatives, this is then true for all entries of d^kf_x, for all x.
3. *f has a smooth P-gradient $\operatorname{grad}^P f=P\circ df\in\mathfrak{X}(\mathfrak{g}^*)=C^\infty(\mathfrak{g}^*,\mathfrak{g}^*)$ which is given by $P_x(df_x,y)=\langle y,\operatorname{grad}^P f(x)\rangle$.*

This looks like the usual finite dimensional version if $\mathfrak{g}$ is reflexive.

For non-reflexive $\mathfrak{g}$ let $i=i_\mathfrak{g}:\mathfrak{g}\to\mathfrak{g}^{**}$ be canonical embedding into the bidual. Then we may ask for a linear lift $\tilde P$ fitting into the following diagram

$$\begin{tikzcd}
\mathfrak{g}^* \arrow[r, "P:=\widetilde{[\,,\,]}^*"] \arrow[dr, dashed, "\tilde P"'] & (\bigwedge^2\mathfrak{g})^* \arrow[r, equal] & L^2_{\text{skew}}(\mathfrak{g};\mathbb{R}) \\
& (\bigwedge^2\mathfrak{g}^{**})^* \arrow[u, "\bigwedge^2(i_\mathfrak{g})^*"'] \arrow[r, equal] & L^2_{\text{skew}}(\mathfrak{g}^{**};\mathbb{R}) \arrow[u, "L^2_{\text{skew}}(i_\mathfrak{g};\mathbb{R})"']
\end{tikzcd}$$

If $\mathfrak{g}$ has predual $\mathfrak{g}^p$ with $(\mathfrak{g}^p)*=\mathfrak{g}$ as a convenient vector space, we get such a natural lift: Let $i_{\mathfrak{g}^p}:\mathfrak{g}^p\to(\mathfrak{g}^p)^{**}=\mathfrak{g}^*$ be the canonical injection, $(i_{\mathfrak{g}^p})^*:\mathfrak{g}^{**}\to\mathfrak{g}$, so that

$$\begin{tikzcd}
\mathfrak{g}^* \arrow[r, "P:=\widetilde{[\,,\,]}^*"] \arrow[dr, dashed, "\tilde P"'] & (\bigwedge^2\mathfrak{g})^* \arrow[r, equal] \arrow[d, "(\bigwedge^2(i_{\mathfrak{g}^p})^*)^*"] & L^2_{\text{skew}}(\mathfrak{g};\mathbb{R}) \arrow[d, "L^2_{\text{skew}}((i_{\mathfrak{g}^p})^*,\mathbb{R})"] \\
& (\bigwedge^2\mathfrak{g}^{**})^* \arrow[r, equal] & L^2_{\text{skew}}(\mathfrak{g}^{**};\mathbb{R})
\end{tikzcd}$$

3.3. **Example: Lie algebras of vector fields on $\mathbb{R}^n$.** We first consider the Lie algebra $\mathfrak{X}_c(\mathbb{R}^n)$ of vector fields with compact support, a nuclear (LF)-space. By [15, Theorem 51.6] we have $L(\mathfrak{X}_c(\mathbb{R}^n),\mathfrak{X}_c(\mathbb{R}^n)')=\mathcal{D}'(\mathbb{R}^n)^n\widehat{\otimes}_\pi\mathcal{D}'(\mathbb{R}^n)^n$. Thus, we get the Lie Poisson structure:

$$P:=-\widetilde{[\,,\,]}^*:\mathcal{D}'(\mathbb{R}^n)^n=\Gamma_{\mathcal{D}'}(T^*\mathbb{R}^n)\to(\bigwedge^2\mathfrak{X}_c(\mathbb{R}^n))^*=\bigwedge^2\mathcal{D}'(\mathbb{R}^n)^n.$$

It fits the setup of section 1: thanks to nuclearity, $\mathfrak{X}_c(\mathbb{R}^n)$ has the approximation property (see [6, 22.2, Corollary 2]), and since $\mathfrak{X}_c(\mathbb{R}^n)$ is reflexive $D'(\mathbb{R}^n) = \mathfrak{X}_c(\mathbb{R}^n)'$ also has the approximation property by [7, Lemma 6.1]. We compute $P(A)$ for a distributional 1-form (a 1-co-current) $A = \sum_j A_j.dx^j \in \Gamma_{\mathcal{D}'}(T^*\mathbb{R}^n)$. It suffices to compute its action on $X \otimes Y$ for $X = \sum_i X^i\partial_i, Y = \sum_i Y^i\partial_i \in \mathfrak{X}_c(\mathbb{R}^n)$. By definition, the pairing is given by $\langle P(A), X \otimes Y\rangle = \langle A, -[X,Y]\rangle$. Expanding the Lie bracket in coordinates, we have

$$\langle P(A), X\otimes Y\rangle = \sum_{i,j}\langle A_j, Y^i\partial_i X^j - X^i\partial_i Y^j\rangle = \sum_{i,j}\left(\langle A_i, (\partial_j X^i)Y^j\rangle - \langle A_j, X^i(\partial_i Y^j)\rangle\right).$$

At this stage, the test functions are evaluated at a single point $x \in \mathbb{R}^n$. To represent this as an evaluation on the product space $\mathbb{R}^n \times \mathbb{R}^n$, we utilize the diagonal embedding $\mathrm{diag} : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$, mapping $x \mapsto (x, x)$. This induces a continuous linear pullback $\mathrm{diag}^* : C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$. By writing the pointwise products as restrictions of two-variable functions, we obtain

$$P\left(\sum_k A_k dx^k\right) = \sum_{i,j}\left((\mathrm{diag}_* A_i)\frac{\partial}{\partial x^j} - (\mathrm{diag}_* A_j)\frac{\partial}{\partial y^i}\right) dx^i \otimes dy^j$$

$$= \sum_{i,j}\left(\langle A_i, \mathrm{diag}^*\left(\frac{\partial}{\partial x^j}X^i(x)Y^j(y)\right)\rangle - \langle A_j, \mathrm{diag}^*\left(\frac{\partial}{\partial y^i}X^i(x)Y^j(y)\right)\rangle\right).$$

The transpose of the pullback is the pushforward map $\mathrm{diag}_* : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, defined by the duality pairing $\langle \mathrm{diag}_* T, \Phi\rangle := \langle T, \mathrm{diag}^*\Phi\rangle$. Applying this shift, we move the evaluation from $\mathbb{R}^n$ to the product space $\mathbb{R}^n \times \mathbb{R}^n$:

$$P\left(\sum_k A_k dx^k\right) = \sum_{i,j}\left(\langle \mathrm{diag}_* A_i, \frac{\partial}{\partial x^j}(X^i(x)Y^j(y))\rangle - \langle \mathrm{diag}_* A_j, \frac{\partial}{\partial y^i}(X^i(x)Y^j(y))\rangle\right).$$

Observe that the bilinear mapping defining the Lie bracket $[\cdot,\cdot] : \mathfrak{X}_c(\mathbb{R}^n) \times \mathfrak{X}_c(\mathbb{R}^n) \to \mathfrak{X}_c(\mathbb{R}^n)$ is a local differential operator. Consequently, the continuous bilinear functional $(X, Y) \mapsto \langle P(A), X\otimes Y\rangle$ vanishes whenever $\mathrm{supp}(X)\cap\mathrm{supp}(Y) = \emptyset$. This implies that the associated bi-distribution $P(A) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)^{n\times n}$ has its support strictly confined to the diagonal $\Delta \subset \mathbb{R}^n \times \mathbb{R}^n$.

By the structure theorem for distributions supported on submanifolds (cf. Hörmander [5, Theorem 5.2.3], based on [5, Theorem 2.3.5]), any distribution $U \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying $\mathrm{supp}(U) \subseteq \Delta$ admits a representation as a locally finite sum of transverse derivatives of pushed-forward distributions. Thus, for any test function $\Phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, the action of U takes the form

$$\langle U, \Phi\rangle = \sum_{|\alpha|\le m}\langle \mathrm{diag}_* T_\alpha, D^\alpha\Phi\rangle \tag{7}$$

where $T_\alpha \in \mathcal{D}'(\mathbb{R}^n)$, α is a multi-index, and D^α are differential operators transverse to Δ.

For our specific terms, this representation holds for $m = 1$, with the operators D^α given by the partial derivatives $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^i}$. Thus, we have

$$P\left(\sum_k A_k dx^k\right) = \sum_{i,j} \left\langle \left((\operatorname{diag}_* A_i)\frac{\partial}{\partial x^j} - (\operatorname{diag}_* A_j)\frac{\partial}{\partial y^i} \right), X^i(x)Y^j(y) \right\rangle. \tag{8}$$

Consequently, we obtain the final expression for the Poisson bivector:

$$P\left(\sum_k A_k dx^k\right) = \sum_{i,j} \left((\operatorname{diag}_* A_i)\frac{\partial}{\partial x^j} - (\operatorname{diag}_* A_j)\frac{\partial}{\partial y^i} \right) dx^i \otimes dy^j. \tag{9}$$

Analogous results also hold for the Lie algebras $\mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n)$ of rapidly decreasing smooth vector fields, and for the Lie algebra $\mathfrak{X}(\mathbb{R}^n)$ of all smooth vector fields, since they are also nuclear. For other Lie algebras like $\mathfrak{X}_{\mathcal{B}}(\mathbb{R}^n)$ consisting of smooth vector fields that are bounded together with all of their derivatives or $\mathfrak{X}_{H^\infty}(\mathbb{R}^n)$, the space of Sobolev vector fields whose components and all derivatives are square-integrable (in the notation from [11]) formula (9) for P still holds, but the image is not in the space of summable skew multi vector fields, since the spaces are not nuclear. The same holds for the Lie algebras based on Denjoy-Carleman ultradifferentiable functions on $\mathbb{R}^n$ of Roumieu and Beurling type discussed in [8], depending on whether they are nuclear or not. For the non-nuclear ones, the Poisson structure is, however, a bounded skew multi vector field on the dual of the Lie algebra.

All the above also holds if we replace $\mathbb{R}^n$ by an open subset therein, and formula (9) remains valid; this paves the way to carry this over to manifolds. We summarize this as:

3.4. **Proposition.** *Let N be a finite-dimensional smooth manifold. Then the Poisson structure on the dual $\mathfrak{X}(N)'$ of the Lie algebra $\mathfrak{X}(N)$ of all smooth vector fields on N, given as*

$$P = -\widetilde{[\ ,\ ]}^* : \mathfrak{X}(N)' = \Gamma_{\mathcal{D}'}(T^*N) \to (\textstyle\bigwedge^2 \mathfrak{X}_c(N))' = \textstyle\bigwedge^2 \mathfrak{X}_c(N)'$$

is given by formula (9) on each open chart of N.

The same holds for all other suitable Lie algebras of vector fields: For example, for the Lie algebra $\mathfrak{X}_\mu(M)$ of divergence free vector fields on a (say) compact oriented manifold (M, μ) with positive smooth volume form μ, the dual space is

$$\mathfrak{X}_\mu(M)' = \Gamma_{\mathcal{D}'}(T^*M)/d(\mathcal{D}'(M)),$$

and formula (9) factors to the quotient.

In different topological settings, Poisson brackets on the continuous dual $\mathfrak{g}'$ of an infinite-dimensional Lie algebra $\mathfrak{g}$ have also been discussed by Glöckner [3] and by Neeb, Sahlmann, and Thiemann [12].

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