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THE RIEMANNIAN MANIFOLD OF ALL RIEMANNIAN METRICS

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Introduction

If M is a (not necessarily compact) smooth finite dimensional manifold, the space $\mathcal{M} = \mathcal{M}(M)$ of all Riemannian metrics on it can be endowed with a structure of an infinite dimensional smooth manifold modeled on the space $\mathcal{D}(S^2T^*M)$ of symmetric $\binom{0}{2}$-tensor fields with compact support, in the sense of [Michor, 1980]. The tangent bundle of $\mathcal{M}$ is $T\mathcal{M} = C^\infty(S^2_+T^*M) \times \mathcal{D}(S^2T^*M)$ and a smooth Riemannian metric can be defined by

$$G_g(h,k) = \int_M \operatorname{tr}(g^{-1}hg^{-1}k)\operatorname{vol}(g).$$

In this paper we study the geometry of $(\mathcal{M}, G)$ by using the ideas developed in [Michor, 1980].

With that differentiable structure on $\mathcal{M}$ it is possible to use variational principles and so we start in section 2 by computing geodesics as the curves in $\mathcal{M}$ minimizing the energy functional. From the geodesic equation, the covariant derivative of the Levi-Civita connection can be obtained, and that provides a direct method for computing the curvature of the manifold.

Christoffel symbol and curvature turn out to be pointwise in M and so, although the mappings involved in the definition of the Ricci tensor and the scalar curvature have no trace, in our case we can define the concepts of "Ricci like curvature" and "scalar like curvature".

The pointwise character mentioned above allows us in section 3, to solve explicitly the geodesic equation and to obtain the domain of definition of the

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exponential mapping. That domain turns out to be open for the topology considered on $\mathcal{M}$ and the exponential mapping is a diffeomorphism onto its image which is also explicitly given. In the L^2-topology given by G itself this domain is, however, nowhere open. Moreover, we prove that it is, in fact, a real analytic diffeomorphism, using [Kriegl-Michor, 1990]. We think that this exponential mapping will be a very powerful tool for further investigations of the stratification of orbit space of $\mathcal{M}$ under the diffeomorphism group, and also the stratification of principal connections modulo the gauge group.

In section 4 Jacobi fields of an infinite dimensional Riemannian manifold are defined as the infinitesimal geodesic variations and we show that they must satisfy the Jacobi Equation. For the manifold $(\mathcal{M}, G)$ the existence of Jacobi fields, with any initial conditions, is obtained from the results about the exponential mapping in section 3. Uniqueness and the fact that they are exactly the solutions of the Jacobi Equation follows from its pointwise character. We finally give the expresion of the Jacobi fields.

For fixed $x \in M$, there exists a family of homothetic Riemannian metrics in the finite dimensional manifold $S^2_+ T^*_x M$ whose geodesics are given by the evaluation of the geodesics of $(\mathcal{M}, G)$. The relationship between the geometry of $(\mathcal{M}, G)$ and that of these manifolds is explained in each case and it is used to visualize the exponential mapping. Nevertheless, in this paper, we have not made use of these manifolds to obtain the results, every computation having been made directly on the infinite dimensional manifold.

Metrics on $S^2_+ T^*_x M$ for three dimensional manifolds M which are similar to ours but have different signatures were considered by [DeWitt, 1967]. He computed the curvature and the geodesics and gave some ideas on how to use them to determine the distance between two 3-geometries, but without considering explicitly the infinite dimensional manifold of all Riemannian metrics on a given manifold.

The topology of $\mathcal{M}(M)$, under the assumption that M is compact, orientable, without boundary, was studied by [Ebin, 1970] who treated G in the context of Sobolev completions of mapping spaces and computed the Levi-Civita connection. In the same context and under the same assumptions, the curvature and the geodesics have been computed in [Freed-Groisser, 1989].

The explicit formulas of the three papers just mentioned are the same as in this paper.

We want to thank A. Montesinos Amilibia for producing the computer image of figure 1.

1. The general setup

1.1. The space of Riemannian metrics. Let M be a smooth second countable finite dimensional manifold. Let S^2T^*M denote the vector bundle of all symmetric $\binom{0}{2}$-tensors on M and let $S^2_+T^*M$ be the open subset of all the positive definite ones. Then the space $\mathcal{M}(M) = \mathcal{M}$ of all Riemannian metrics is the space of sections $C^\infty(S^2_+T^*M)$ of this fiber bundle. It is open in the space of sections $C^\infty(S^2T^*M)$ in the Whitney C^∞-topology, in which the latter space is, however, not a topological vector space, since $\frac{1}{n}h$ converges to 0 if and only if h has compact support. So the space $\mathcal{D} = \mathcal{D}(S^2T^*M)$ of sections with compact support is the largest topological vector space contained in the topological group $(C^\infty(S^2T^*M), +)$, and the trace of the Whitney C^∞-topology on it coincides with the inductive limit topology

$$\mathcal{D}(S^2T^*M) = \varinjlim_K C^\infty_K(S^2T^*M),$$

where $C^\infty_K(S^2T^*M)$ is the space of all sections with support contained in K and where K runs through all compact subsets of M.

So we declare the path components of $C^\infty(M, S^2_+T^*M)$ for the Whitney C^∞-topology also to be open. We get a topology which is finer than the Whitney topology, where each connected component is homeomorphic to an open subset in $\mathcal{D} = \mathcal{D}(S^2T^*M)$. So $\mathcal{M} = C^\infty(S^2_+T^*M)$ is a smooth manifold modeled on nuclear (LF)-spaces, and the tangent bundle is given by $T\mathcal{M} = \mathcal{M} \times \mathcal{D}$.

1.2. Remarks. The main reference for the infinite dimensional manifold structures is [Michor, 1980]. But the differential calculus used there is not completely up to date, the reader should consult [Frölicher-Kriegl, 1988], whose calculus is more natural and much easier to apply. There a mapping between locally convex spaces is smooth if and only if it maps smooth curves to smooth curves. See also [Kriegl-Michor, 1990] for a setting for real analytic mappings along the same lines and applications to manifolds of mappings.

As a final remark let us add that the differential structure on the space $\mathcal{M}$ of Riemannian metrics is not completely satisfying, if M is not compact. In fact $C^\infty(S^2T^*M)$ is a topological vector space with the compact C^∞-topology, but the space $\mathcal{M} = C^\infty(S^2_+T^*M)$ of Riemannian metrics is not open in it. Nevertheless, we will see later that the exponential mapping for the natural Riemannian metric on $\mathcal{M}$ is defined also for some tangent vectors which are not in $\mathcal{D}$. This is an indication that the most natural setting for manifolds of mappings is based on the compact C^∞-topology, but that one loses existence of charts. In [Michor, 1985] a setting for infinite dimensional manifolds is presented which is based on an axiomatic structure of smooth curves instead of charts.

1.3. The metric. The tangent bundle of the space $\mathcal{M} = C^\infty(S^2_+T^*M)$ of Riemannian metrics is $T\mathcal{M} = \mathcal{M} \times \mathcal{D} = C^\infty(S^2_+T^*M) \times \mathcal{D}(S^2T^*M)$. We identify the vector bundle S^2T^*M with the subbundle

$$\{\ell \in L(TM, T^*M) : \ell^t = \ell\}$$

of $L(TM, T^*M)$, where the transposed is given by the composition

$$\ell^t : TM \xrightarrow{i} T^{**}M \xrightarrow{\ell^*} T^*M.$$

Then the fiberwise inner product on S^2T^*M induced by $g \in \mathcal{M}$ is given by the expression $\langle h, k\rangle_g := \operatorname{tr}(g^{-1}hg^{-1}k)$, so a smooth Riemannian metric on $\mathcal{M}$ is given by

$$G_g(h, k) = \int_M \operatorname{tr}(g^{-1}hg^{-1}k)\operatorname{vol}(g),$$

where $\operatorname{vol}(g)$ is the positive density defined by the local formula $\operatorname{vol}(g) = \sqrt{\det g}\,dx$. We call this the *canonical Riemannian metric* on $\mathcal{M}$, since it is invariant under the action of the diffeomorphism group $\operatorname{Diff}(M)$ on the space $\mathcal{M}$ of metrics. The integral is defined since h, k have compact support. The metric is positive definite, $G_g(h, h) \geq 0$ and $G_g(h, h) = 0$ only if $h = 0$. So G_g defines a linear injective mapping from the tangent space $T_g\mathcal{M} = \mathcal{D}(S^2T^*M)$ into its dual $\mathcal{D}(S^2T^*M)'$, the space of distributional densities with values in the dual bundle S^2TM. This linear mapping is, however, never surjective, so G is only a *weak* Riemannian metric. The tangent space $T_g\mathcal{M} = \mathcal{D}(S^2T^*M)$ is a pre-Hilbert space, whose completion is a Sobolev space of order 0, depending on g if M is not compact.

1.4. Remark. Since G is only a weak Riemannian metric, all objects which are only implicitly given, a priori lie in the Sobolev completions of the relevant spaces. In particular this applies to the formula

$$\begin{aligned}2G(\xi, \nabla_\eta\zeta) =& \xi G(\eta, \zeta) + \eta G(\zeta, \xi) - \zeta G(\xi, \eta)\\ &+ G([\xi, \eta], \zeta) + G([\eta, \zeta], \xi) - G([\zeta, \xi], \eta),\end{aligned}$$

which a priori gives only uniqueness but not existence of the Levi Civita covariant derivative.

2. Geodesics, Levi Civita connection, and curvature

2.1 The covariant derivative. Since we will need later the covariant derivative of vector fields along a geodesic for the derivation of the Jacobi equation, we present here a careful description of the notion of the covariant derivative, which is valid in infinite dimensions. Here $\mathcal{M}$ might be any infinite dimensional manifold, modeled on locally convex spaces. If we are given a horizontal bundle, complementary to the vertical one, in $T^2\mathcal{M}$, with the usual properties of a linear connection, then the projection from $T^2\mathcal{M}$ to the vertical bundle $V(T\mathcal{M})$ along the horizontal bundle, followed by the vertical projection $V(T\mathcal{M}) \to T\mathcal{M}$, defines the *connector* $K : T^2\mathcal{M} \to T\mathcal{M}$, which has the following properties:

(1) It is a left inverse to the vertical lift mapping with any foot point.
(2) It is linear for both vector bundle structures on T^2M.
(3) The connection is symmetric (torsionfree) if and only if $K \circ \kappa = K$, where κ is the canonical flip mapping on the second tangent bundle.

If a connector K is given, the covariant derivative is defined as follows: Let $f : \mathcal{N} \to \mathcal{M}$ be a smooth mapping, let $s : \mathcal{N} \to T\mathcal{M}$ be a vector field along f and let $X_x \in T_x\mathcal{N}$. Then

$$\nabla_{X_x} s := (K \circ Ts)(X_x).$$

In a chart the Christoffel symbol is related to the connector by

$$K(g, h; k, \ell) = (g, \ell - \Gamma_g(h, k))$$

.

We want to state one property, which is usually stated rather clumsily in the literature: If $f_1 : \mathcal{P} \to \mathcal{N}$ is another smooth mapping and $Y_y \in T_y\mathcal{P}$, then we have $\nabla_{Y_y}(s \circ f_1) = \nabla_{T_y(f_1)Y_y} s$. Equivalently, if vector fields $Y \in \mathfrak{X}(\mathcal{P})$ and $X \in \mathfrak{X}(\mathcal{N})$ are f_1-related, then $\nabla_Y(s \circ f_1) = (\nabla_X s) \circ f_1$.

If $V(t)$ is a vector field along a smooth curve $g(t)$, we have $\nabla_{\partial_t} V = \frac{\partial}{\partial t} V - \Gamma_g(g_t, V)$, in local coordinates.

If $c : \mathbb{R}^2 \to \mathcal{M}$ is a smooth mapping for a symmetric connector K we have

$$\begin{aligned}
\nabla_{\partial_t} \frac{\partial}{\partial s} c(t, s) &= K \circ T(Tc \circ \partial_s) \circ \partial_t = K \circ T^2 c \circ T(\partial_s) \circ \partial_t \\
&= K \circ \kappa \circ T^2 c \circ T(\partial_s) \circ \partial_t = K \circ T^2 c \circ \kappa \circ T(\partial_s) \circ \partial_t \\
&= K \circ T^2 c \circ T(\partial_t) \circ \partial_s = \nabla_{\partial_s} \frac{\partial}{\partial t} c(t, s),
\end{aligned}$$

which will be used for Jacobi fields.

2.2. Let $t \mapsto g(t)$ be a smooth curve in $\mathcal{M}$: so $g : \mathbb{R} \times M \to S^2_+T^*M$ is smooth and by the choice of the topology on $\mathcal{M}$ made in 1.1 the curve $g(t)$ varies only in a compact subset of M, locally in t, by [Michor, 1980, 4.4.4, 4.11, and 11.9]. Then its energy is given by

$$\begin{aligned} E_a^b(g) :&= \tfrac{1}{2}\int_a^b G_g(g_t, g_t)dt \\ &= \tfrac{1}{2}\int_a^b \int_M \operatorname{tr}(g^{-1}g_tg^{-1}g_t)\operatorname{vol}(g)\,dt, \end{aligned}$$

where $g_t = \frac{\partial}{\partial t} g(t)$.

Now we consider a variation of this curve, so we assume now that $(t,s) \mapsto g(t,s)$ is smooth in all variables and locally in (t,s) it only varies within a compact subset in M — this is again the effect of the topology chosen in 1.1. Note that $g(t,0)$ is the old $g(t)$ above.

2.3. Lemma. *In the setting of 2.2 we have the first variation formula*

$$\begin{aligned} &\frac{\partial}{\partial s}|_0 E_a^b(g(\quad, s)) = G_g(g_t, g_s)|_{t=a}^{t=b} + \\ &\qquad + \int_a^b G_g\big(-g_{tt} + g_tg^{-1}g_t + \tfrac{1}{4}\operatorname{tr}(g^{-1}g_tg^{-1}g_t)g - \tfrac{1}{2}\operatorname{tr}(g^{-1}g_t)g_t, g_s\big)\,dt. \end{aligned}$$

Proof. We have

$$\frac{\partial}{\partial s}|_0 E_a^b(g(\quad, s)) = \frac{\partial}{\partial s}|_0 \frac{1}{2}\int_a^b \int_M \operatorname{tr}(g^{-1}g_tg^{-1}g_t)\operatorname{vol}(g)dt.$$

We may interchange $\frac{\partial}{\partial s}|_0$ with the first integral since this is finite dimensional analysis, and we may interchange it with the second one, since $\int_M$ is a continuous linear functional on the space of all smooth densities with compact support on M, by the chain rule. Then we use that tr_* is linear and continuous, $d(\operatorname{vol})(g)h = \frac{1}{2}\operatorname{tr}(g^{-1}h)\operatorname{vol}(g)$, and that $d((\quad)^{-1})_*(g)h = -g^{-1}hg^{-1}$ and partial integration. □

2.4 The geodesic equation. By lemma 2.3 the curve $t \mapsto g(t)$ is a geodesic if and only if we have

$$\begin{aligned} g_{tt} &= g_tg^{-1}g_t + \tfrac{1}{4}\operatorname{tr}(g^{-1}g_tg^{-1}g_t)\,g - \tfrac{1}{2}\operatorname{tr}(g^{-1}g_t)\,g_t \\ &= \Gamma_g(g_t, g_t), \end{aligned}$$

where the *Christoffel symbol* $\Gamma : \mathcal{M}\times\mathcal{D}\times\mathcal{D}\to\mathcal{D}$ is given by symmetrisation

$$\begin{aligned}\Gamma_g(h,k) = \tfrac12 hg^{-1}k + \tfrac12 kg^{-1}h+ &\\ + \tfrac14\operatorname{tr}(g^{-1}hg^{-1}k)\,g - \tfrac14\operatorname{tr}(g^{-1}h)\,k - \tfrac14\operatorname{tr}(g^{-1}k)\,h.\end{aligned}$$

The sign of Γ is chosen in such a way that the horizontal subspace of $T^2\mathcal{M}$ is parameterized by $(x,y;z,\Gamma_x(y,z))$. If instead of the obvious framing we use $T\mathcal{M}=\mathcal{M}\times\mathcal{D}\ni(g,h)\mapsto(g,g^{-1}h)=:(g,H)\in\{g\}\times\mathcal{D}(L_{sym,g}(TM,TM))\subset \mathcal{M}\times\mathcal{D}(L(TM,TM))$, the Christoffel symbol looks like

$$\bar\Gamma_g(H,K)=\tfrac12(HK+KH)+\tfrac14\operatorname{tr}(HK)Id-\tfrac14\operatorname{tr}(H)K-\tfrac14\operatorname{tr}(K)H,$$

and the geodesic equation for $H(t):=g^{-1}g_t$ becomes

$$H_t=\frac{\partial}{\partial t}|_0(g^{-1}g_t)=\tfrac14\operatorname{tr}(HH)Id-\tfrac12\operatorname{tr}(H)H.$$

2.5 The curvature. In the setting of 2.1, for vector fields X, $Y\in\mathfrak{X}(\mathcal{N})$ and a vector field $s:\mathcal{N}\to T\mathcal{M}$ along $f:\mathcal{N}\to\mathcal{M}$ we have

$$\begin{aligned}R(X,Y)s &= (\nabla_{[X,Y]}-[\nabla_X,\nabla_Y])s\\ &= (K\circ TK - K\circ TK\circ\kappa_{T\mathcal{M}})\circ T^2s\circ TX\circ Y,\end{aligned}$$

which in local coordinates reduces to the usual formula

$$R(h,k)\ell = d\Gamma(h)(k,\ell)-d\Gamma(k)(h,\ell)-\Gamma(h,\Gamma(k,\ell))+\Gamma(k,\Gamma(h,\ell)).$$

A global derivation of this formula can be found in [Kainz-Michor, 1987]

2.6 Proposition. *The Riemannian curvature for the canonical Riemannian metric on the manifold $\mathcal{M}$ of all Riemannian metrics is given by*

$$\begin{aligned}g^{-1}R_g(h,k)\ell = \tfrac14[[H,K],L] &\\ &+\frac{\dim M}{16}(\operatorname{tr}(KL)H-\operatorname{tr}(HL)K)\\ &+\frac{1}{16}(\operatorname{tr}(H)\operatorname{tr}(L)K-\operatorname{tr}(K)\operatorname{tr}(L)H)\\ &+\frac{1}{16}(\operatorname{tr}(K)\operatorname{tr}(HL)-\operatorname{tr}(H)\operatorname{tr}(KL))Id.\end{aligned}$$

Proof. This is a long but elementary computation using the formula from 2.5 and

$$\begin{aligned}d\Gamma(h)(k,\ell) = &-\tfrac12 kg^{-1}hg^{-1}\ell-\tfrac12\ell g^{-1}hg^{-1}k-\tfrac14\operatorname{tr}(g^{-1}hg^{-1}kg^{-1}\ell)g\\ &-\tfrac14\operatorname{tr}(g^{-1}kg^{-1}hg^{-1}\ell)g+\tfrac14\operatorname{tr}(g^{-1}kg^{-1}\ell)h\\ &+\tfrac14\operatorname{tr}(g^{-1}hg^{-1}k)\ell+\tfrac14\operatorname{tr}(g^{-1}hg^{-1}\ell)k. \quad\square\end{aligned}$$

2.7. Ricci curvature for the Riemannian space $(\mathcal{M}, G)$ **does not exist**, since the mapping $k \mapsto R_g(h,k)\ell$ is just the push forward of the section by a certain tensor field, a differential operator of order 0. If this is not zero, it induces a topological linear isomorphism between certain infinite dimensional subspaces of $T_g\mathcal{M}$, and is therefore never of trace class.

2.8. Ricci like curvature. But we may consider the pointwise trace of the tensorial operator $k \mapsto R_g(h,k)\ell$ which we call the *Ricci like curvature* and denote by

$$\operatorname{Ric}_g(h,\ell)(x) := \operatorname{tr}(k_x \mapsto R_g(h_x,k_x)\ell_x).$$

Proposition. *The Ricci like curvature of* $(\mathcal{M}, G)$ *is given by*

$$\begin{aligned}\operatorname{Ric}_g(h,\ell) &= \frac{4+n(n+1)}{32}(\operatorname{tr}(H)\operatorname{tr}(L) - n\operatorname{tr}(HL))\\ &= -\frac{n}{32}(4+n(n+1))\langle h_0,\ell\rangle_g,\end{aligned}$$

where $h_0 := h - \frac{1}{n}\operatorname{tr}(H)g$.

Proof. We compute the pointwise trace $\operatorname{tr}(k_x \mapsto R_g(h_x,k_x)\ell_x)$ and use the following

Lemma. *For* H, K, *and* $L \in L_{sym}(\mathbb{R}^n,\mathbb{R}^n)$ *we have*

$$\operatorname{tr}(K \mapsto [[H,K],L]) = \operatorname{tr}(H)\operatorname{tr}(L) - n\operatorname{tr}(HL). \quad \square$$

We can define a $\binom{0}{2}$-tensor field Ric on $\mathcal{M}$ by

$$\operatorname{Ric}(\xi,\eta)(g) := \int_M \operatorname{Ric}_g(\xi(g),\eta(g))\operatorname{vol}(g)$$

for ξ, $\eta \in \mathfrak{X}(\mathcal{M})$, and $g \in \mathcal{M}$. Then by the proposition we have

$$\operatorname{Ric}(\xi,\eta) = -\frac{n}{32}(4+n(n+1))G(\xi_0,\eta),$$

where ξ_0 is the vector field given by $\xi_0(g) := \xi(g) - \frac{1}{n}\operatorname{tr}(g^{-1}\xi(g))g$.

2.9. Scalar like curvature. By 2.8 there is a unique $\binom{1}{1}$-tensor field $\overline{\mathrm{Ric}}$ on $\mathcal{M}$ such that $G(\overline{\mathrm{Ric}}(\xi),\eta) = \mathrm{Ric}(\xi,\eta)$ for all vector fields ξ, $\eta \in \mathfrak{X}(\mathcal{M})$, which is given by $\overline{\mathrm{Ric}}(\xi) = -\frac{n}{32}(4+n(n+1))\xi_0$. Again, for every $g \in \mathcal{M}$ the corresponding endomorphism of $T_g\mathcal{M}$ is a differential operator of order 0 and is never of trace class. But we may again form its pointwise trace as a linear vector bundle endomorphism on $S^2T^*M \to M$, which is a function on M. We call it the *scalar like curvature* of $(\mathcal{M}, G)$ and denote it by Scal_g. It turns out to be the constant $c(n)$ depending only on the dimension n of M, because the endomorphism involved is just the projection onto a hyperplane. We have

$$c(n) = -\frac{n}{32}(4+n(n+1))(\frac{n(n+1)}{2} - 1).$$

Remark. For fixed $\tilde{g} \in \mathcal{M}$ and $x \in M$ the expression

$$G^{x,\tilde{g}}(h_x, k_x) := \langle h, k\rangle_g(x)\sqrt{\det(\tilde{g}^{-1}g)(x)}$$

gives a Riemannian metric on $S^2_+T^*_xM$. It is not difficult to see that the Ricci like curvature of $(\mathcal{M}, G)$ at x is just the Ricci curvature of the family of homothetic metrics on $S^2_+T^*_xM$ obtained by varying $\tilde{g}$, and that the scalar curvature of $G^{x,\tilde{g}}$ equals the function $c(n)/\sqrt{\det(\tilde{g}^{-1}g)(x)}$ on $S^2_+T^*_xM$.

3. Analysis of the exponential mapping

3.1. The geodesic equation 2.4 is an ordinary differential equation and the evolution of $g(t)(x)$ depends only on $g(0)(x)$ and $g_t(0)(x)$ and stays in $S^2_+T^*_xM$ for each $x \in M$.

The geodesic equation can be solved explicitly and we have

3.2. Theorem. *Let $g^0 \in \mathcal{M}$ and $h \in T_{g^0}\mathcal{M} = \mathcal{D}$. Then the geodesic in $\mathcal{M}$ starting at g^0 in the direction of h is the curve*

$$g(t) = g^0 e^{(a(t)Id + b(t)H_0)},$$

where H_0 is the traceless part of $H := (g^0)^{-1}h$ (i.e. $H_0 = H - \frac{\mathrm{tr}(H)}{n}Id$) and where $a(t)$ and $b(t) \in C^\infty(M)$ are defined as follows:

$$a(t) = \tfrac{2}{n}\log\left((1+\tfrac{t}{4}\,\mathrm{tr}(H))^2 + \tfrac{n}{16}\,\mathrm{tr}(H_0^2)t^2\right)$$

$$b(t) = \begin{cases} \dfrac{4}{\sqrt{n\,\mathrm{tr}(H_0^2)}}\,\mathrm{arctg}\left(\dfrac{\sqrt{n\,\mathrm{tr}(H_0^2)}\,t}{4+t\,\mathrm{tr}(H)}\right) & \text{where } \mathrm{tr}(H_0^2) \neq 0 \\[2ex] \dfrac{t}{1+\frac{t}{4}\,\mathrm{tr}(H)} & \text{where } \mathrm{tr}(H_0^2) = 0. \end{cases}$$

Here arctg *is taken to have values in* $(-\frac{\pi}{2}, \frac{\pi}{2})$ *for the points of the manifold where* $\operatorname{tr}(H) \geq 0$*, and on a point where* $\operatorname{tr}(H) < 0$ *we define*

$$\operatorname{arctg}\left(\frac{\sqrt{n\operatorname{tr}(H_0^2)}\,t}{4 + t\operatorname{tr}(H)}\right) = \begin{cases} \operatorname{arctg} \ \textit{in } [0, \frac{\pi}{2}) & \textit{for } t \in [0, -\frac{4}{\operatorname{tr}(H)}) \\ \frac{\pi}{2} & \textit{for } t = -\frac{4}{\operatorname{tr}(H)} \\ \operatorname{arctg} \ \textit{in } (\frac{\pi}{2}, \pi) & \textit{for } t \in (-\frac{4}{\operatorname{tr}(H)}, \infty). \end{cases}$$

Let $N^h := \{x \in M : H_0(x) = 0\}$*, and if* $N^h \neq \emptyset$ *let* $t^h := \inf\{\operatorname{tr}(H)(x) : x \in N^h\}$*. Then the geodesic* $g(t)$ *is defined for* $t \in [0, \infty)$ *if* $N^h = \emptyset$ *or if* $t^h \geq 0$*, and it is only defined for* $t \in [0, -\frac{4}{t^h})$ *if* $t^h < 0$*.*

Proof. Check that $g(t)$ is a solution of the pointwise geodesic equation. Computations leading to this solution can be found in [Freed-Groisser, 1989]. □

3.3. The exponential mapping. For $g^0 \in S^2_+ T^*_x M$ we consider the sets

$$\begin{aligned} U_{g^0} &:= S^2 T^*_x M \setminus (-\infty, -\frac{4}{n}]\, g^0, \\ L_{sym,g^0}(T_xM, T_xM) &:= \{\ell \in L(T_xM, T_xM) : g^0(\ell X, Y) = g^0(X, \ell Y)\}, \\ L^+_{sym,g^0}(T_xM, T_xM) &:= \{\ell \in L_{sym,g^0}(T_xM, T_xM) : \ell \text{ is positive } \}, \\ U'_{g^0} &:= L_{sym,g^0}(T_xM, T_xM) \setminus (-\infty, -\frac{4}{n}]\, Id_{T_xM}, \end{aligned}$$

and the fiber bundles over $S^2_+ T^* M$

$$\begin{aligned} U &:= \bigcup \left\{\{g^0\} \times U_{g^0} : g^0 \in S^2_+ T^* M\right\}, \\ L_{sym} &:= \bigcup \left\{\{g^0\} \times L_{sym,g^0}(T_xM, T_xM) : g^0 \in S^2_+ T^* M\right\}, \\ L^+_{sym} &:= \bigcup \left\{\{g^0\} \times L^+_{sym,g^0}(T_xM, T_xM) : g^0 \in S^2_+ T^* M\right\}, \\ U' &:= \bigcup \left\{\{g^0\} \times U'_{g^0} : g^0 \in S^2_+ T^* M\right\}. \end{aligned}$$

Then we consider the mapping $\Phi : U \to S^2_+ T^* M$ which is given by the following composition

$$U \xrightarrow{\sharp} U' \xrightarrow{\varphi} L_{sym} \xrightarrow{\exp} L^+_{sym} \xrightarrow{\flat} S^2_+ T^* M,$$

where $\sharp(g^0,h) := (g^0,(g^0)^{-1}h)$ is a fiber respecting diffeomorphism, where $\varphi(g^0,H) := (g^0, a(1)Id + b(1)H_0)$ comes from theorem 3.2, where the usual exponential mapping $\exp : L_{sym,g^0} \to L^+_{sym,g^0}$ is a diffeomorphism (see for instance [Greub-Halperin-Vanstone, 1972, page 26]) with inverse log, and where $\flat(g^0,H) := g^0 H$, a diffeomorphism for fixed g^0.

We now consider the mapping $(pr_1,\Phi) : U \to S^2_+T^*M \times_M S^2_+T^*M$. From the expression of $b(1)$ it is easily seen that the image of (pr_1,Φ) is contained in the following set:

$$\begin{aligned}V :&= \left\{(g^0, g^0 \exp H) : \operatorname{tr}(H_0^2) < \frac{1}{n}(4\pi)^2\right\}\\&= \left\{(g^0,g) : \operatorname{tr}\left(\left(\log((g^0)^{-1}g) - \frac{\operatorname{tr}(\log((g^0)^{-1}g))}{n} Id\right)^2\right) < \frac{(4\pi)^2}{n}\right\}\end{aligned}$$

Then $(pr_1,\Phi) : U \to V$ is a diffeomorphism, since the mapping $(g^0,A) \to (g^0,\psi(A))$ is an inverse to $\varphi : U' \to \big\{(g^0,A) : \operatorname{tr}(A_0^2) < \frac{1}{n}(4\pi)^2\big\}$, where ψ is given by:

$$\psi(A) = \begin{cases} \dfrac{4}{n}\left(e^{\frac{\operatorname{tr}(A)}{4}}\cos\left(\dfrac{\sqrt{n\operatorname{tr}(A_0^2)}}{4}\right) - 1\right)Id & \\ \qquad + \dfrac{4}{\sqrt{n\operatorname{tr}(A_0^2)}} e^{\frac{\operatorname{tr}(A)}{4}} \sin\left(\dfrac{\sqrt{n\operatorname{tr}(A_0^2)}}{4}\right) A_0 & \text{if } A_0 \neq 0 \\ \dfrac{4}{n}\left(e^{\frac{\operatorname{tr}(A)}{4}} - 1\right) Id & \text{otherwise.}\end{cases}$$

3.4. Theorem. *In the setting of 3.3 the exponential mapping* Exp_{g^0} *is a real analytic diffeomorphism between the open subsets*

$$\begin{aligned}\mathcal{U}_{g^0} &:= \{h \in \mathcal{D}(S^2T^*M) : (g^0,h)(M) \subset U\}\\ \mathcal{V}_{g^0} &:= \{g \in C^\infty(S^2_+T^*M) : (g^0,g)(M) \subset V, g - g^0 \in \mathcal{D}(S^2T^*M)\}\end{aligned}$$

and it is given by

$$\operatorname{Exp}_{g^0}(h) = \Phi \circ (g^0,h).$$

The mapping $(\pi_{\mathcal{M}}, \operatorname{Exp}) : T\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ *is a real analytic diffeomorphism from the open neighborhood of the zero section*

$$\mathcal{U} := \{(g^0,h) \in C^\infty(S^2_+T^*M) \times \mathcal{D}(S^2T^*M) : (g^0,h)(M) \subset U\}$$

onto the open neighborhood of the diagonal

$$\begin{aligned}\mathcal{V} := \{(g^0, g) \in C^\infty(S^2_+ T^*M) \times C^\infty(S^2_+ T^*M) : (g^0, g)(M) \subset V,\\ g - g^0 \textit{ has compact support } \}.\end{aligned}$$

All these sets are maximal domains of definition for the exponential mapping and its inverse.

Proof. Since $\mathcal{M}$ is a disjoint union of chart neighborhoods, it is trivially a real analytic manifold, even if M is not supposed to carry a real analytic structure.

From the consideration in 3.3 it follows that $\mathrm{Exp} = \Phi_*$ and $(\pi_{\mathcal{M}}, \mathrm{Exp})$ are just push forwards by smooth fiber respecting mappings of sections of bundles. So by [Michor, 1980, 8.7] they are smooth and this applies also to their inverses.

To show that these mappings are real analytic, by [Kriegl-Michor, 1990] we have to check that they map real analytic curves into real analytic curves. So we may just invoke the description [Kriegl-Michor, 1990, 7.7.2] of real analytic curves in spaces of smooth sections:

For a smooth vector bundle (E, p, M) a curve $c : \mathbb{R} \to C^\infty(E)$ is real analytic if and only if $\hat{c} : \mathbb{R} \times M \to E$ satisfies the following condition:

(1) *For each n there is an open neighborhood U_n of $\mathbb{R} \times M$ in $\mathbb{C} \times M$ and a (unique) C^n-extension $\tilde{c} : U_n \to E_{\mathbb{C}}$ such that $\tilde{c}(\quad, x)$ is holomorphic for all $x \in M$.*

In statement (1) the space of sections $C^\infty(E)$ is equipped with the compact C^∞-topology. So we have to show that (1) remains true for the space $C^\infty_c(E)$ of smooth sections with compact support with its inductive limit topology.

This is easily seen since we may first exchange $C^\infty(E)$ by the closed linear subspace $C^\infty_K(E)$ of sections with support in a fixed compact subset; then we just note that (1) is invariant under passing to the strict inductive limit in question.

Now it is clear that Φ has a fiberwise extension to a holomorphic germ since Φ is fiber respecting from an open subset in a vector bundle and is fiberwise a real analytic mapping. So the push forward Φ_* maps real analytic curves to real analytic curves. $\square$

3.5. Remarks. The domain $\mathcal{U}_{g^0}$ of definition of the exponential mapping does not contain any ball centered at 0 for the norm derived from G_{g^0}.

Note that Exp_{g^0} is in fact defined on the set

$$\mathcal{U}'_{g^0} := \{h \in C^\infty(S^2 T^*M) : (g^0, h)(M) \subset U\}$$

which is not contained in the tangent space for the differentiable structure we use. Recall now the remarks from 1.2. If we equip $\mathcal{M}$ with the compact C^∞-topology, then $\mathcal{M}$ it is not open in $C^\infty(S^2T^*M)$. The tangent space is then the set of all tangent vectors to curves in $\mathcal{M}$ which are smooth in $C^\infty(S^2T^*M)$, which is probably not a vector space. The integral in the definition of the canonical Riemannian metric might not converge on all these tangent vectors.

The approach presented here is clean and conceptually clear, but some of the concepts have larger domains of definition.

3.6. Visualizing the exponential mapping. Let us fix a point $x \in M$ and let us consider the space $S^2_+T^*_xM = L^+(T_xM, T^*_xM)$ of all positive definite symmetric inner products on T_xM. If we fix an element $\tilde{g} \in S^2_+T^*_xM$, we may define a Riemannian metric G on $S^2_+T^*_xM$ by

$$G_g(h,k) := \operatorname{tr}(g^{-1}hg^{-1}k)\sqrt{\det(\tilde{g}^{-1}g)}$$

for $g \in S^2_+T^*_xM$ and $h,\, k \in T_g(S^2_+T^*_xM) = S^2T^*_xM$. The variational method used in section 2 leading to the geodesic equation shows that the geodesic starting at g^0 in the direction $h \in S^2T^*_xM$ is given by

$$g(t) = g^0 e^{(a(t)Id + b(t)H_0)}$$

in the setting of theorem 3.2. We have the following diffeomorphisms

$$L_{sym,g^0}(T_xM,T_xM) \xrightarrow{\exp} L^+_{sym,g^0}(T_xM,T_xM) \xrightarrow{\flat=(g^0)_*} S^2_+T^*_xM,$$
$$H \mapsto \exp(H) = e^H \mapsto g^0e^H,$$

the manifolds $(S^2_+T^*_xM, G)$ and $(L_{sym,g^0}(T_xM,T_xM), ((g^0)_* \circ \exp)^*G)$ are isometric. For $L \in L_{sym,g^0}(T_xM,T_xM)$ we have by the affine structure $T_L L_{sym,g^0}(T_xM,T_xM) = L_{sym,g^0}(T_xM,T_xM)$ and we get

$$((g^0)_* \circ \exp)^*G)_L(H,K) = \sqrt{\det(\tilde{g}^{-1}g^0)}\langle H, \mathcal{A}_L(K)\rangle,$$

where $\langle H,K\rangle = \operatorname{tr}(HK)$ and

$$\mathcal{A}_L = e^{\frac{1}{2}\operatorname{tr}(L)} \sum_{k=0}^{\infty} \frac{2(\operatorname{ad}(L))^{2k}}{(2k+2)!}.$$

The geodesic on $L_{sym,g^0}(T_xM, T_xM)$ for that metric starting at 0 in the direction H is then given by

$$L(t) := a(t)Id + b(t)H_0.$$

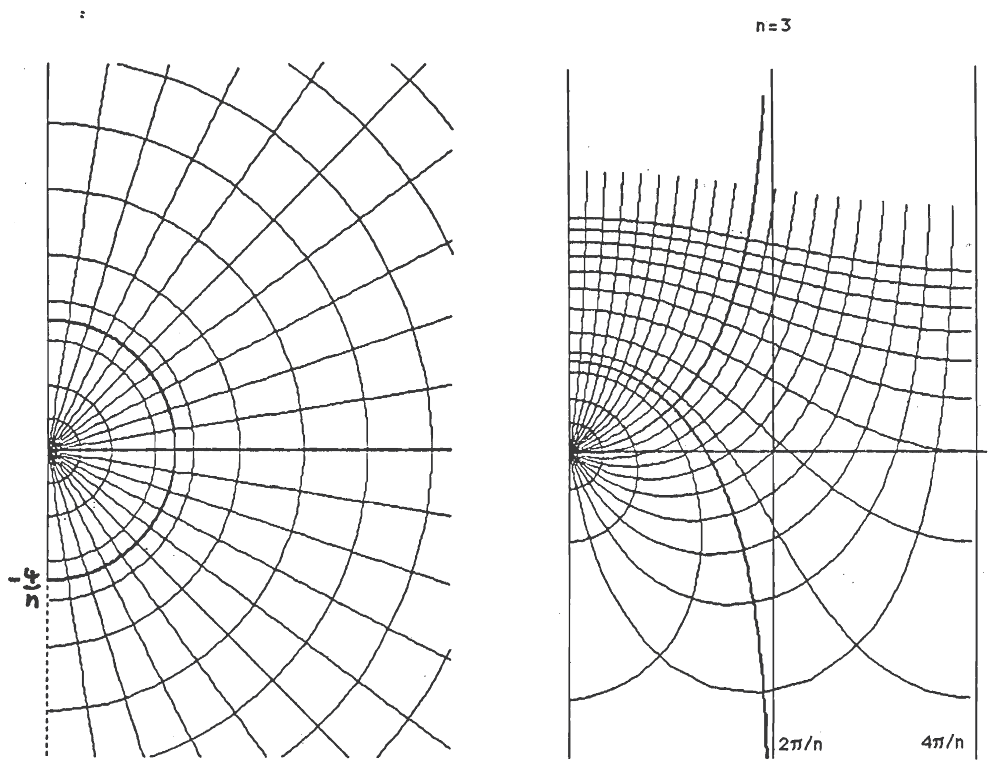

Figure 1

Let us choose now a g^0-orthonormal basis of T_xM and let $\tilde{g} = n^{\frac{2}{n}} g^0$. Then the exponential mapping at g^0 of $(S^2_+ T^*_x M, G)$ can be viewed as the exponential mapping at 0 of $(Mat_{sym}(n, \mathbb{R}), \langle\ \ ,\ \ \rangle')$, where for symmetric matrices A, B, and C we have

$$\langle A, B\rangle'_C = \frac{1}{n}\langle A, \mathcal{A}_C(B)\rangle,$$

where the scale factor is chosen in such a way that we have $\langle Id, Id\rangle'_0 = 1$. This exponential mapping is defined on the set

$$\mathcal{U}'' := \{A \in Mat_{sym}(n, \mathbb{R}) : A_0 \neq 0 \text{ or } A = \lambda Id \text{ with } \lambda > -4\}$$

by the formula

$$\begin{aligned}\exp_0(A) = \tfrac{2}{n}\log\left((1+\tfrac{1}{4}\operatorname{tr}(A))^2 + \tfrac{n}{16}\operatorname{tr}(A_0^2)\right) Id \\ + \frac{4}{\sqrt{n\operatorname{tr}(A_0^2)}}\operatorname{arctg}\left(\frac{\sqrt{n\operatorname{tr}(A_0^2)}}{4+\operatorname{tr}(A)}\right) A_0.\end{aligned}$$

If A is traceless (i.e. $A_0 = A$) and if P is the plane in $Mat_{sym}(n,\mathbb{R})$ through 0, Id, and A, then $\operatorname{Exp}_0(P\cap\mathcal{U}'')\subset P$ and we can view at a 2-dimensional picture of this exponential mapping. If we normalize A in such a way that $\operatorname{tr}(A_0^2)=n$ the exponential mapping is just the diffeomorphism

$$\mathbb{R}^2\setminus(\{0\}\times(-\infty,-\tfrac{4}{n}]) \to (-\frac{4\pi}{n},\frac{4\pi}{n})\times\mathbb{R}$$

$$\begin{cases} u(x,y) = \tfrac{4}{n}\operatorname{arctg}\left(\dfrac{nx}{4+ny}\right) \\ v(x,y) = \tfrac{2}{n}\log\left(\tfrac{1}{16}\left((4+ny)^2+n^2x^2\right)\right). \end{cases}$$

Here arctg is taken to have values in $(0,\pi)$ for $x\geq 0$ and to have values in $(-\pi,0)$ for $x\leq 0$. The images of the straight lines and the circles can be seen in figure 1. They correspond respectively to the images of the geodesics and to the level sets of the distance function $\operatorname{dist}(0,\quad)^{-1}(r)$. For $r<\frac{4}{n}$ they are exactly the geodesic spheres.

It is known that for a finite dimensional Riemannian manifold, if for some point the exponential map is defined in the whole tangent space, then every other point can be joined with that by a minimizing geodesic (Hopf-Rinow-theorem). Here we have a nice example where, although only a half line lacks from the domain of definition of the exponential mapping, it is far from being surjective.

4. Jacobi fields

4.1. The concept of Jacobi fields. Let $(\mathcal{M},G)$ be an infinite dimensional Riemannian manifold which admits a smooth Levi Civita connection, and let $c:[0,a]\to\mathcal{M}$ be a geodesic segment. By a *geodesic variation* of c we mean a smooth mapping $\alpha:[0,a]\times(-\varepsilon,\varepsilon)\to\mathcal{M}$ such that for each fixed $s\in(-\varepsilon,\varepsilon)$ the curve $t\mapsto\alpha(t,s)$ is a geodesic and $\alpha(t,0)=c(t)$.

Definition. A vector field ξ along a geodesic segment $c:[0,a]\to\mathcal{M}$ is called a *Jacobi field* if and only if it is an infinitesimal geodesic variation of c, i.e.

if there exists a geodesic variation $\alpha : [0,a] \times (-\varepsilon,\varepsilon) \to \mathcal{M}$ of c such that $\xi(t) = \frac{\partial}{\partial s}|_0 \alpha(t,s)$.

In 2.1 and 2.5 we have set up all the machinery necessary for the usual proof that any Jacobi field ξ along a geodesic c satisfies the *Jacobi equation*

$$\nabla_{\partial_t}\nabla_{\partial_t}\xi = R(\xi, c')c'.$$

In a finite dimensional manifold solutions of the Jacobi equation are Jacobi fields, but in infinite dimensions one has in general neither existence nor uniqueness of ordinary differential equations, nor an exponential mapping.

Nevertheless for the manifold $(\mathcal{M}(M), G)$ we will show existence and also uniqueness of solutions of the Jacobi equation for given initial conditions, and that each solution is a Jacobi field.

4.2. Lemma. *Let $g(t)$ be a geodesic in $\mathcal{M}(M)$. Then for a vector field ξ along g the Jacobi equation has the following form:*

$$\begin{aligned}\xi_{tt} = &-g_t g^{-1}\xi g^{-1} g_t + g_t g^{-1}\xi_t + \xi_t g^{-1} g_t + \tfrac12 \operatorname{tr}(g^{-1}g_t g^{-1}\xi_t)\, g \\ &- \tfrac12 \operatorname{tr}(g^{-1}g_t g^{-1}g_t g^{-1}\xi)\, g + \tfrac12 \operatorname{tr}(g^{-1}g_t g^{-1}\xi)\, g_t - \tfrac12 \operatorname{tr}(g^{-1}\xi_t)\, g_t \\ &+ \tfrac14 \operatorname{tr}(g^{-1}g_t g^{-1}g_t)\, \xi - \tfrac12 \operatorname{tr}(g^{-1}g_t)\, \xi_t.\end{aligned}$$

Proof. From 2.1 we have $\nabla_{\partial_t}\xi = \xi_t - \Gamma_g(g_t,\xi)$, thus

$$\begin{aligned}\nabla_{\partial_t}\nabla_{\partial_t}\xi &= \xi_{tt} - \Gamma(g_t,\xi)_t - \Gamma(g_t,\xi_t) + \Gamma(g_t,\Gamma(g_t,\xi)) \\ &= \xi_{tt} - d\Gamma(g_t)(g_t,\xi) - \Gamma(g_{tt},\xi) - 2\Gamma(g_t,\xi_t) + \Gamma(g_t,\Gamma(g_t,\xi)).\end{aligned}$$

On the other hand we have by 2.5

$$R(\xi,g_t)g_t = d\Gamma(\xi)(g_t,g_t) - d\Gamma(g_t)(\xi,g_t) - \Gamma(\xi,\Gamma(g_t,g_t)) + \Gamma(g_t,\Gamma(\xi,g_t)).$$

So ξ satisfies the Jacobi equation if and only if

$$\xi_{tt} = d\Gamma(\xi)(g_t,g_t) + 2\Gamma(g_t,\xi_t).$$

By plugging in formula 2.4 for Γ and the formula for $d\Gamma$ in the proof of 2.6, the result follows. $\square$

4.3. Lemma. *For any geodesic $g(t)$ and for any k and $\ell \in T_{g(0)}\mathcal{M}(M)$ there exists a unique vector field $\xi(t)$ along $g(t)$ which is a solution of the Jacobi equation with $\xi(0) = k$ and $(\nabla_{\partial_t}\xi)(0) = \ell$.*

Proof. From lemma 4.2 above we see that the Jacobi equation is pointwise with respect to M, and that $\xi(t)$ satisfies the Jacobi equation if and only if at each $x \in M$ is a Jacobi field of the associated finite dimensional manifold treated in section 3. The result then follows from the properties of the finite dimensional ordinary differential equation involved. $\square$

4.4. Theorem. *For any geodesic $g(t)$ and for any k and $\ell \in T_{g(0)}\mathcal{M}(M)$ there exists a unique Jacobi field $\xi(t)$ along $g(t)$ with initial values $\xi(0) = k$ and $(\nabla_{\partial_t}\xi)(0) = \ell$.*

In particular the solutions of the Jacobi equation are exactly the Jacobi fields.

Proof. We have uniqueness since Jacobi fields satisfy the Jacobi equation and by lemma 4.3. Now we prove existence. Let $h = g'(0)$ and $g^0 = g(0)$. Since k has compact support, there is an $\varepsilon > 0$ such that for $s \in (-\varepsilon, \varepsilon)$ the tensor field $\lambda(s) = g^0 + sk$ is still a Riemannian metric on M. Let $\tilde{\ell} := \ell + \Gamma_{g^0}(k, h)$, and then $W(s) := h + s\tilde{\ell}$ is a vector field along λ which satisfies $W(0) = h = g'(0)$ and $(\nabla_{\partial_s} W)(0) = \ell$. Now we consider $\alpha(t, s) := \operatorname{Exp}_{\lambda(s)}(tW(s))$, which is defined for all (t, s) such that $(\lambda(t), tW(s))$ belongs to the open set $\mathcal{U}$ of $T\mathcal{M}$ defined in 3.4.

Since h, k, and ℓ have all compact support we have: if $g(t)$ is defined on $[0, \infty)$, then ε can be chosen so small that $\alpha(t, s)$ is defined on $[0, \infty) \times (-\varepsilon, \varepsilon)$; and if $g(t)$ is defined only on $[0, -\frac{4}{t^h})$ then for each $\delta > 0$ there is an $\varepsilon > 0$ such that α is defined on $[0, -\frac{4}{t^h} - \delta) \times (-\varepsilon, \varepsilon)$.

Then $J(t) := \frac{\partial}{\partial s}|_0 \alpha(t, s)$ is a Jacobi field for every geodesic segment and satisfies

$$\begin{aligned}
J(0) &= \frac{\partial}{\partial s}|_0 \alpha(0, s) = \frac{\partial}{\partial s}|_0 \operatorname{Exp}(0_{\lambda(s)}) = \frac{\partial}{\partial s}|_0 \lambda(s) = k \\
(\nabla_{\partial_t} J)(0) &= \nabla_{\partial_t}|_0 (t \mapsto \frac{\partial}{\partial s}|_0 \alpha(t, s)) \\
&= \nabla_{\partial_s}|_0 (t \mapsto \frac{\partial}{\partial t}|_0 \alpha(t, s)) = (\nabla_{\partial_s} W)(0) = \ell,
\end{aligned}$$

where we used 2.1. $\square$

4.5. Since we will need it later we continue here with the explicit expression of α. As $\lambda(s)^{-1}W(s) = (Id + sK)^{-1}(H + s\tilde{L})$, where as usual $K = (g^0)^{-1}k$,

$H = (g^0)^{-1}h$, and $\tilde{L} = (g^0)^{-1}\tilde{\ell}$, from the expression in 3.3 of the exponential mapping we get

$$\begin{aligned}
&\alpha(t,s) = \lambda(s)e^{Q(t,s)}, \qquad \text{where}\\
&Q(t,s) = (a(t,s)Id + b(t,s)((Id+sK)^{-1}(H+s\tilde{L}) - \frac{c(s)}{n}\,Id),
\end{aligned}$$

where $c(s) = \operatorname{tr}((Id+sK)^{-1}(H+s\tilde{L}))$ and where $a(t,s)$ and $b(t,s)$ are given by

$$\begin{aligned}
a(t,s) &= \tfrac{2}{n}\log\Big(\tfrac{1}{16}\big((4+tc(s))^2 + t^2nd(s)\big)\Big),\\
b(t,s) &= \tfrac{4}{\sqrt{nd(s)}}\operatorname{arctg}\left(\frac{\sqrt{nd(s)}t}{4+tc(s)}\right), \quad \text{where}\\
d(s) &= \operatorname{tr}((\lambda(s)^{-1}W(s))_0^2) = f(s) - \frac{c(s)^2}{n}, \quad \text{and}\\
f(s) &= \operatorname{tr}\left(((Id+sK)^{-1}(H+s\tilde{L}))^2\right).
\end{aligned}$$

In fact $b(t,s)$ should be defined with the same care as in the explicit formula for the geodesics in 3.2. This is omitted here.

Before computing the Jacobi fields let us introduce some notation. For every point in M the mapping $(H,K)\mapsto \operatorname{tr}(HK)$ is an inner product, thus the quadrilinear mapping

$$T(H,K,L,N) := \operatorname{tr}(HL)\operatorname{tr}(KN) - \operatorname{tr}(HN)\operatorname{tr}(KL)$$

is an algebraic curvature tensor, [Kobayashi-Nomizu, I, page 198]. We will also use

$$S(H,K) := T(H,K,H,K) = \operatorname{tr}(H^2)\operatorname{tr}(K^2) - \operatorname{tr}(HK)^2.$$

Let $P(t) := g(t)^{-1}g'(t)$ then it is easy to see from 3.2 that

$$P(t) = e^{-\frac{1}{2}na(t)}\left(\frac{4\operatorname{tr}(H) + nt\operatorname{tr}(H^2)}{4n}Id + H_0\right).$$

We denote by $P(t)^\perp$ the $\binom{1}{1}$-tensor field of trace $e^{-\frac{1}{2}na(t)}$ in the plane through 0, Id, and H_0 which at each point is orthogonal to $P(t)$ with respect to the inner product $\operatorname{tr}(HK)$ from above.

4.6. Lemma. *In the setting of 4.5 we have*

$$\frac{\partial}{\partial s}|_0 Q(t,s) = \frac{\operatorname{tr}(H\hat{L})}{\operatorname{tr}(H^2)} tP(t) + \frac{T(\hat{L},H,Id,H)}{\operatorname{tr}(H^2)} tP(t)^{\perp} \\ + b(t)\left(-\frac{T(H,Id,\hat{L},Id)}{S(H,Id)}H_0 + \hat{L}_0\right),$$

where $\hat{L} := -KH + \tilde{L}$.

Proof. From the expression of $Q(t,s)$ we have

$$\frac{\partial}{\partial s}|_0 Q(t,s) = \left(\frac{\partial}{\partial s}|_0 a(t,s) Id + \frac{\partial}{\partial s}|_0 b(t,s) H_0 + b(t,0)\hat{L}_0\right).$$

Now,

$$\begin{aligned} \frac{\partial}{\partial s}|_0 a(t,s) &= \frac{2}{n}\frac{8tc'(0) + nt^2 f'(0)}{16 + 8tc(0) + nt^2 f(0)} \\ \frac{\partial}{\partial s}|_0 b(t,s) &= -\frac{d'(0)}{2d(0)}\, b(t) + \frac{2}{d(0)}\frac{(4+tc(0))td'(0) - 2t^2c'(0)d(0)}{16 + 8tc(0) + nt^2 f(0)}. \end{aligned}$$

From the definitions of c, d, and f we have

$$\begin{aligned} c(0) &= \operatorname{tr}(H), \qquad c'(0) = \operatorname{tr}(\hat{L}) \\ d(0) &= \operatorname{tr}(H_0^2) = \tfrac{1}{n} S(H,Id), \\ f(0) &= \operatorname{tr}(H^2), \qquad f'(0) = 2\operatorname{tr}(H\hat{L}), \\ d'(0) &= \tfrac{2}{n}\, T(H,Id,\hat{L},Id) \end{aligned}$$

and $16 + 8tc(0) + nt^2 f(0) = 16\, e^{\frac{na(t)}{2}}$, and so we get

$$\begin{aligned} \frac{\partial}{\partial s}|_0 a(t,s) &= \frac{1}{4n} e^{-\frac{1}{2}na(t)}(4t\operatorname{tr}(\hat{L}) + nt^2\operatorname{tr}(H\hat{L})), \\ \frac{\partial}{\partial s}|_0 b(t,s) &= -\frac{T(H,Id,\hat{L},Id)}{S(H,Id)} b(t) \\ &\quad + \frac{e^{-\frac{1}{2}na(t)}t}{4S(H,Id)}\big((4 + t\operatorname{tr}(H))T(H,Id,\hat{L},Id) - t\operatorname{tr}(\hat{L})S(H,Id)\big). \end{aligned}$$

If we collect all terms and compute a while we get the result. $\square$

4.7. Theorem. *Let $g(t)$ be the geodesic in $(\mathcal{M}(M),G)$ starting from g^0 in the direction $h\in T_{g^0}\mathcal{M}(M)$. For each $k,\ell\in T_{g^0}\mathcal{M}(M)$ the Jacobi field $J(t)$ along $g(t)$ with initial conditions $J(0)=k$ and $(\nabla_{\partial_t}J)(0)=\ell$ is given by*

$$\begin{aligned}J(t)&=\frac{\operatorname{tr}(H\hat{L})}{\operatorname{tr}(H^2)}\,t\,g'(t)+\frac{\operatorname{tr}(\hat{L})\operatorname{tr}(H^2)-\operatorname{tr}(\hat{L}H)\operatorname{tr}(H)}{\operatorname{tr}(H^2)}\,t\,(g'(t))^{\perp}\\&\quad+b(t)g(t)\left(-\frac{\operatorname{tr}(H_0\hat{L}_0)}{\operatorname{tr}(H_0^2)}H_0+\hat{L}_0\right)\\&\quad+g(t)\sum_{m=1}^{\infty}\frac{(-\operatorname{ad}(b(t)H))^m}{(m+1)!}(b(t)\hat{L})+k(g^0)^{-1}g(t),\end{aligned}$$

where $H=(g^0)^{-1}h$, $K=(g^0)^{-1}k$, $L=(g^0)^{-1}\ell$, $\hat{L}=-KH+L+(g^0)^{-1}\Gamma_{g^0}(k,h)$, and $(g'(t))^{\perp}=g(t)P(t)^{\perp}$.

Proof. By theorem 4.4 we have

$$\begin{aligned}J(t)&=\frac{\partial}{\partial s}|_0\alpha(t,s)\qquad\text{and then}\\&=g^0\frac{\partial}{\partial s}|_0 e^{Q(t,s)}+k\,e^{Q(t,0)}\\&=g^0e^{Q(t,0)}\left(\sum_{m=0}^{\infty}\frac{(-\operatorname{ad}(b(t)H))^m}{(m+1)!}(\frac{\partial}{\partial s}|_0Q(t,s))\right)+k\,e^{Q(t,0)}.\end{aligned}$$

The result now follows from lemma 4.6 $\square$

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