A New Duality Approach for Multiobjective Convex Optimization Problems

Gert Wanka * Radu Ioan Boţ †

Abstract

This paper contains a new duality approach for general convex multiobjective programming problems. The vector objective function of the dual problem is represented in closed form by the conjugate functions of the primal objective functions and the functions describing the constraints. The basic idea is to establish a dual problem for the scalarized primal problem different from the dual problems usually considered in optimization, e.g. the Lagrange dual. But this dual problem based on a special perturbation and conjugacy has an adapted form allowing to construct a multiobjective dual problem in a natural way. Weak, strong and converse duality assertions are presented. Finally, some special cases show the applicability of the general approach.

Key words: multiobjective duality – conjugate duality – Pareto-efficiency – converse duality – multiobjective convex optimization

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1 Introduction

In this paper a new duality approach for general convex multiobjective programming problems (P) is submitted. It may be characterized as a rigorous application of conjugacy to such problems. In this treatment the objective function of the dual problem (D) can be represented in a closed form wherein the conjugate goal functions of the original (primal) problem as well as the conjugates of the functions describing the set of constraints appear in a clear and natural way. Also, the dual constraints adopt a very clear form of only two conditions, a simple bilinear inequality and a scalar product to be zero (cf. (4)). In this representation our dual problem differs from other known formulations of multiobjective dual problems as can be found in the optimization literature.

*Faculty of Mathematics, Technical University of Chemnitz, D-09107 Chemnitz, Germany, email: gert.wanka@mathematik.tu-chemnitz.de

†Faculty of Mathematics, Technical University of Chemnitz, D-09107 Chemnitz, Germany, email: radu.bot@mathematik.tu-chemnitz.de

Among the large number of papers and books dealing with different approaches to multiobjective duality we mention as a representative selection the books [8], [10], [11], [15], [16] and the papers [1], [3], [4], [6], [12], [13], [14], [18], [22], [23], [26] and [27].

Beside presentations in the sense of approaches for general formulated problems there are a lot of contributions devoted to duality of multiobjective programming problems of special type, as for example linear problems [7], [9], location and approximation problems [17], [19], [20], [22], portfolio optimization problems [21], fractional programming problems [2], [25] etc.

In section 2 we introduce the original convex multiobjective problem and give the basic and well known definition of Pareto-minimal efficient solutions followed by the definition of properly efficient solutions via linear scalarization.

The main and fruitful idea for construction the multiobjective dual problem is to use a dual problem (D_λ) to the scalarized multiobjective problem (P_λ) different from dual problems usually considered, in particular, different from the standard Lagrange dual problem. This dual problem (D_λ) results from a special perturbation of the primal problem and applying the Fenchel-Rockafellar duality concept based on conjugacy and perturbation. We derive strong duality and optimality conditions which later are used to obtain duality assertions for the original and dual multiobjective problem.

The scalar dual problem obtained in this way in section 4 really turns out to be a genuine form adapted for generating in a natural way a conjugate multiobjective dual problem (D) to the original problem (P) that allows to prove weak an strong duality (cf. Theorem 3 and Theorem 4).

Moreover, in section 5 a converse duality assertion will be verified (Theorem 5).

Finally, in section 6 some special cases with linear constraints which can be obtained from the presented general result are summarized.

Thus, from a more general point of view we rediscover and extend former results (cf. [23], [24]).

It remains an interesting topic for future research to discover the connections and relations to other duality concepts in multiobjective programming as developed in the papers mentioned above.

2 Problem formulation

We consider the following optimization problem with convex objective functions and convex constraints

$$(P) \quad \text{v-}\min_{x \in \mathcal{A}} f(x),$$

$$\mathcal{A} = \left\{ x \in \mathbb{R}^n : g(x) \underset{K}{\leqq} 0 \right\},$$

$$f(x) = (f_1(x), \ldots, f_m(x))^T,$$
$$g(x) = (g_1(x), \ldots, g_k(x))^T.$$

The functions $f_i, i = 1, \ldots, m$, mapping from $\mathbb{R}^n$ into $\mathbb{R} \cup \{\pm\infty\}$ and the functions $g_j, j = 1, \ldots, k$, mapping from $\mathbb{R}^n$ into $\mathbb{R}$ are convex and proper, with the property that $\bigcap\limits_{i=1}^{m} ri(dom f_i) \neq \emptyset$, where $ri(dom f_i)$ represents the relative interior of the set $dom f_i = \{x \in \mathbb{R}^n : f_i(x) < +\infty\}$, $i = 1, \ldots, m$. $K \subseteq \mathbb{R}^k$ is assumed to be a convex closed cone with $intK \neq \emptyset$, defining a partial ordering according to $x_1 \underset{K}{\geqq} x_2$ if and only if $x_1 - x_2 \in K$.

The problem (P) is a multiobjective optimization problem in the form of a vector minimum problem and for such kind of problems different notions of solutions are known. We will use in our paper the so-called minimal and properly minimal solutions. Now, let us recall the two solution concepts.

Definition 1. *An element $\bar{x} \in \mathcal{A}$ is said to be efficient (or minimal or Pareto-minimal) with respect to (P) if from*

$$f(\bar{x}) \underset{R^m_+}{\geqq} f(x) \quad \text{for} \quad x \in \mathcal{A} \quad \text{follows} \quad f(\bar{x}) = f(x).$$

Definition 2. *An element $\bar{x} \in \mathcal{A}$ is said to be properly efficient (or properly minimal) with respect to (P) if there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in intR^m_+$ (i.e. $\lambda_i > 0, i = 1, \ldots, m$) such that $\sum\limits_{i=1}^{m} \lambda_i f_i(\bar{x}) \leq \sum\limits_{i=1}^{m} \lambda_i f_i(x) \quad \forall x \in \mathcal{A}$.*

In Definition 1, $\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m : x_i \geq 0, i = 1, \ldots, m\}$ denotes the ordering cone of the non-negative elements of $\mathbb{R}^m$.

By these definitions, a properly efficient element turns out to be also an efficient one.

3 Duality for the scalarized problem

In order to study the duality for the multiobjective problem (P) we will study, at first, the duality for the scalarized problem

$$(P_\lambda) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x),$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)^T$ is a fixed vector in $int\mathbb{R}^m_+$.

To do this let us consider, for the beginning, a general optimization problem

$$(PG) \quad \inf_{x \in \mathcal{A}} \tilde{f}(x),$$

where $\tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is a convex and proper function.

We give, now, the dual of (PG) using a Fenchel-Rockafellar approach (cf. [5]) using the perturbation function

$$\Phi(x, \varphi, \gamma) = \begin{cases} \tilde{f}(x + \varphi) & , \text{ if } \quad g(x) \underset{K}{\leqq} \gamma \\ +\infty & , \text{ otherwise} \end{cases}$$

with the perturbation variables $\varphi \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}^k$.

We obtain the following dual problem

$$(DG) \quad \sup_{\substack{\tilde{p} \in \mathbb{R}^n \\ q \underset{K^*}{\geqq} 0}} \left\{ -\tilde{f}^*(\tilde{p}) + \inf_{x \in \mathbb{R}^n} \left[\tilde{p}^T x + q^T g(x) \right] \right\}.$$

Here, $\tilde{f}^*(\tilde{p}) = \sup_{x \in \mathbb{R}^n} \{\tilde{p}^T x - \tilde{f}(x)\}$ represents the value of the conjugate function of $\tilde{f}$ at $\tilde{p}$ and $K^* = \{q \in \mathbb{R}^k : q^T x \geq 0, \forall x \in K\}$ is the dual cone of K.

Remark 1.

(a) Using this approach it is possible to formulate different dual problems for (PG), but the problem formulated before is the most convenient one for the purpose to construct a dual of the multiobjective problem (P).

(b) One of the classical assumptions which assures the existence of strong duality (i.e $\inf(PG) = \max(DG)$) is that a constraint qualification is fulfilled. There exist different types of constraint qualifications. Let us consider the following so called Slater condition.

(CQ) | There exists an element $x' \in \bigcap_{i=1}^{m} ri(dom f_i)$ such that $g(x') \underset{K}{<} 0$ (i.e. $g(x') = (g_1(x'), \ldots, g_m(x'))^T \in -intK$)(cf. [5]).

(CQ) is sufficient to obtain strong duality for the problems (PG) and (DG).

Returning to (P_λ), for $\tilde{f}(x) = \sum_{i=1}^{m} \lambda_i f_i(x), x \in \mathbb{R}^n$, (DG) gives us the dual of the scalarized problem

$$(D_\lambda) \quad \sup_{\substack{\tilde{p} \in \mathbb{R}^n \\ q \underset{K^*}{\geqq} 0}} \left\{ -(\sum_{i=1}^{m} \lambda_i f_i)^*(\tilde{p}) + \inf_{x \in \mathbb{R}^n} \left[\tilde{p}^T x + q^T g(x) \right] \right\}.$$

Because of, $\bigcap_{i=1}^{m} ri(dom f_i) \neq \emptyset$, we have

$$(\sum_{i=1}^{m} \lambda_i f_i)^*(\tilde{p}) = \inf \left\{ \sum_{i=1}^{m} (\lambda_i f_i)^*(\tilde{p}_i) : \sum_{i=1}^{m} \tilde{p}_i = \tilde{p} \right\}$$

and the dual (D_λ) will be then

$$(D_\lambda) \quad \sup_{\substack{\tilde{p}\in\mathbb{R}^n, q \underset{K^*}{\geqq} 0 \\ \tilde{p}_i\in\mathbb{R}^n, \sum\limits_{i=1}^{m}\tilde{p}_i=\tilde{p}}} \left\{ -\sum_{i=1}^{m}(\lambda_i f_i)^*(\tilde{p}_i) + \inf_{x\in\mathbb{R}^n}\left[\tilde{p}^T x + q^T g(x)\right]\right\}.$$

Because $(\lambda_i f_i)^*(\tilde{p}_i) = \lambda_i f_i^*(\frac{\tilde{p}_i}{\lambda_i}), i = 1,\ldots,m$, we can make the substitutions $\frac{\tilde{p}_i}{\lambda_i} = p_i, i = 1,\ldots,m$ and $\tilde{p} = \sum\limits_{i=1}^{m}\lambda_i p_i$. Then, omitting $\tilde{p}$, we obtain for the dual of (P_λ)

$$(D_\lambda) \quad \sup_{\substack{p_i\in\mathbb{R}^n, i=1,\ldots,m \\ q \underset{K^*}{\geqq} 0}} \left\{ -\sum_{i=1}^{m}\lambda_i f_i^*(p_i) + \inf_{x\in\mathbb{R}^n}\left[(\sum_{i=1}^{m}\lambda_i p_i)^T x + q^T g(x)\right]\right\}.$$

It is well-known that perturbating the primal problem in different ways it is possible to obtain some other dual problems (including the Lagrange dual) for (P_λ). The reason why we considered the dual problem in this form is that, as we will see in Section 4, (D_λ) suggests us the form of the dual for the multiobjective problem (P).

According to Remark 1 (a), we can formulate the following strong duality theorem.

Theorem 1. *Let there exists an element* $x' \in \bigcap\limits_{i=1}^{m} ri(dom f_i)$ *such that* $g(x') \in -intK$ *(i.e. the constraint qualification* (CQ) *is fulfilled). Then the dual problem* (D_λ) *has a solution and strong duality* $\inf(P_\lambda) = \max(D_\lambda)$ *holds.*

For later investigations we need the optimality conditions regarding to the scalar problem (P_λ) and its dual (D_λ). The following theorem gives us these conditions.

Theorem 2.

(a) Let the constraint qualification (CQ) *be fulfilled and let* $\bar{x}$ *be a solution to* (P_λ). *Then there exists* $(\bar{p},\bar{q}), \bar{p} = (\bar{p}_1,\ldots,\bar{p}_m) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n, \bar{q} \underset{K^*}{\geqq} 0$, *solution for* (D_λ), *such that the following optimality conditions are satisfied*

$$(i) \quad f_i^*(\bar{p}_i) + f_i(\bar{x}) = \bar{p}_i^T\bar{x}, \quad i = 1,\ldots,m,$$

$$(ii) \quad \bar{q}^T g(\bar{x}) = 0,$$

$$(iii) \quad \inf_{x\in\mathbb{R}^n}\left[(\sum_{i=1}^{m}\lambda_i\bar{p}_i)^T x + \bar{q}^T g(x)\right] = (\sum_{i=1}^{m}\lambda_i\bar{p}_i)^T\bar{x}.$$

(b) Let $\bar{x}$ be admissible to (P_λ) and $(\bar{p},\bar{q})$ be admissible to (D_λ) satisfying (i),(ii) and (iii).

Then $\bar{x}$ is a solution to (P_λ), $(\bar{p},\bar{q})$ is a solution to (D_λ) and strong duality holds

$$\sum_{i=1}^{m}\lambda_i f_i(\bar{x}) = -\sum_{i=1}^{m}\lambda_i f_i^*(\bar{p}_i) + \inf_{x\in\mathbb{R}^n}\left[(\sum_{i=1}^{m}\lambda_i\bar{p}_i)^T x + \bar{q}^T g(x)\right].$$

Proof.

(a) Let $\bar{x}$ be a solution to (P_λ). By Theorem 1, there exists a solution $(\bar{p},\bar{q})$ to (D_λ), $\bar{p}=(\bar{p}_1,\ldots,\bar{p}_m)$, $\bar{q} \underset{K^*}{\geqq} 0$ such that the values of the objective functions are equal. This means that

$$\sum_{i=1}^{m}\lambda_i f_i(\bar{x}) = -\sum_{i=1}^{m}\lambda_i f_i^*(\bar{p}_i) + \inf_{x\in\mathbb{R}^n}\left[(\sum_{i=1}^{m}\lambda_i\bar{p}_i)^T x + \bar{q}^T g(x)\right]. \tag{1}$$

After some transformations (1) yields

$$\begin{aligned} 0 &= \sum_{i=1}^{m}\lambda_i\left[f_i(\bar{x}) + f_i^*(\bar{p}_i) - \bar{p}_i^T\bar{x}\right] + \left[-\bar{q}^T g(\bar{x})\right] \\ &+ \left\{(\sum_{i=1}^{m}\lambda_i\bar{p}_i)^T\bar{x} + \bar{q}^T g(\bar{x}) - \inf_{x\in\mathbb{R}^n}\left[(\sum_{i=1}^{m}\lambda_i\bar{p}_i)^T x + \bar{q}^T g(x)\right]\right\}. \end{aligned} \tag{2}$$

Because of $f_i(\bar{x}) + f_i^*(\bar{p}_i) - \bar{p}_i^T\bar{x} \geq 0, i=1,\ldots,m$ (the so-called Young inequality) and $\bar{q}^T g(\bar{x}) \leq 0$ it follows that all the terms of the sum in (2) must be equal to zero. This gives us the optimality conditions (i), (ii) and (iii).

(b) All the calculations and transformations done within part (a) may be carried out in the inverse direction starting from the conditions (i)-(iii). □

Remark 2. From (iii) in Theorem 2 (a) we have

$$\begin{aligned} -(\sum_{i=1}^{m}\lambda_i\bar{p}_i)^T\bar{x} &= -\inf_{x\in\mathbb{R}^n}\left[(\sum_{i=1}^{m}\lambda_i\bar{p}_i)^T x + \bar{q}^T g(x)\right] \\ &= \sup_{x\in\mathbb{R}^n}\left[(-\sum_{i=1}^{m}\lambda_i\bar{p}_i)^T x - (\bar{q}^T g)(x)\right]. \end{aligned}$$

This means that

$$(\bar{q}^T g)^*(-\sum_{i=1}^{m}\lambda_i\bar{p}_i) = -(\sum_{i=1}^{m}\lambda_i\bar{p}_i)^T\bar{x}. \tag{3}$$

4 The multiobjective dual problem

Now, we are able to formulate a multiobjective dual to (P). The dual (D) will be a vector maximum problem and therefore the efficient solutions in the sense of maximum are considered. The aim of this section is to present the weak and strong duality theorems.

A dual multiobjective optimization problem (D) to (P) is introduced by

$$(D) \quad \underset{(p,q,\lambda,t)\in\mathcal{B}}{\text{v-max}} \; h(p,q,\lambda,t)$$

$$h(p,q,\lambda,t) = \begin{pmatrix} h_1(p,q,\lambda,t) \\ \vdots \\ h_m(p,q,\lambda,t) \end{pmatrix}$$

with

$$h_j(p,q,\lambda,t) = -f_j^*(p_j) - (q_j^T g)^*(-\frac{1}{m\lambda_j}\sum_{i=1}^{m}\lambda_i p_i) + t_j, j = 1,\ldots,m,$$

the dual variables

$$p = (p_1,\ldots,p_m), q = (q_1,\ldots,q_m), \lambda = (\lambda_1,\ldots,\lambda_m)^T, t = (t_1,\ldots,t_m)^T,$$

$$p_i \in \mathbb{R}^n, \quad q_i \in \mathbb{R}^k, \quad \lambda_i \in \mathbb{R}, \quad t_i \in \mathbb{R}, \quad i = 1,\ldots,m$$

and the set of constraints

$$\mathcal{B} = \left\{ (p,q,\lambda,t) : \lambda \in int\mathbb{R}^m_+, \quad \sum_{i=1}^{m}\lambda_i q_i \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^{m}\lambda_i t_i = 0 \right\}. \tag{4}$$

Definition 3. *An element $(\bar{p},\bar{q},\bar{\lambda},\bar{t}) \in \mathcal{B}$ is said to be efficient (or maximal or Pareto-maximal) for (D) if from*

$$h(p,q,\lambda,t) \underset{R^m_+}{\geqq} h(\bar{p},\bar{q},\bar{\lambda},\bar{t}) \quad for \quad (p,q,\lambda,t) \in \mathcal{B}$$

follows $h(p,q,\lambda,t) = h(\bar{p},\bar{q},\bar{\lambda},\bar{t})$.

The following theorem states the weak duality assertion for the multiobjective problem (P).

Theorem 3. *There is no $x \in \mathcal{A}$ and no $(p,q,\lambda,t) \in \mathcal{B}$ fulfilling $h(p,q,\lambda,t) \underset{R^m_+}{\geqq} f(x)$ and $h(p,q,\lambda,t) \neq f(x)$.*

Proof. We assume that there exist $x \in \mathcal{A}$ and $(p,q,\lambda,t) \in \mathcal{B}$ such that $f_i(x) \leq h_i(p,q,\lambda,t), \forall i \in \{1,\dots,m\}$ and $f_j(x) < h_j(p,q,\lambda,t)$ for at least one $j \in \{1,\dots,m\}$. This means that

$$\sum_{i=1}^{m} \lambda_i f_i(x) < \sum_{i=1}^{m} \lambda_i h_i(p,q,\lambda,t). \tag{5}$$

On the other hand we have that

$$\sum_{i=1}^{m} \lambda_i h_i(p,q,\lambda,t) = -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - \sum_{i=1}^{m} \lambda_i (q_i^T g)^*(-\frac{1}{m\lambda_i}\sum_{i=1}^{m} \lambda_i p_i) + \sum_{i=1}^{m} \lambda_i t_i.$$

For f_i and $q_i^T g, i = 1,\dots,m$ we can apply the inequality of Young

$$\begin{aligned} -f_i^*(p_i) &\leq f_i(x) - p_i^T x \\ -(q_i^T g)^*(-\frac{1}{m\lambda_i}\sum_{i=1}^{m} \lambda_i p_i) &\leq q_i^T g(x) + (\frac{1}{m\lambda_i}\sum_{i=1}^{m} \lambda_i p_i)^T x \end{aligned}$$

and so, we obtain

$$\begin{aligned} \sum_{i=1}^{m} \lambda_i h_i(p,q,\lambda,t) &\leq \sum_{i=1}^{m} \lambda_i f_i(x) - (\sum_{i=1}^{m} \lambda_i p_i)^T x \\ &+ \sum_{i=1}^{m} \lambda_i \left[q_i^T g(x) + (\frac{1}{m\lambda_i}\sum_{i=1}^{m} \lambda_i p_i)^T x \right] \\ &= \sum_{i=1}^{m} \lambda_i f_i(x) + (\sum_{i=1}^{m} \lambda_i q_i)^T g(x) \\ &\leq \sum_{i=1}^{m} \lambda_i f_i(x). \end{aligned}$$

The inequality $\sum_{i=1}^{m} \lambda_i h_i(p,q,\lambda,t) \leq \sum_{i=1}^{m} \lambda_i f_i(x)$ contradicts relation (5). □

The following theorem expresses the so-called strong duality between the two multiobjective problems (P) and (D).

Theorem 4. *Assume the existence of an element* $x' \in \bigcap_{i=1}^{m} ri(dom f_i)$ *fulfilling* $g(x') \in -intK$. *Let* $\bar{x}$ *be a properly efficient element to* (P). *Then an efficient solution* $(\bar{p},\bar{q},\bar{\lambda},\bar{t}) \in \mathcal{B}$ *to the dual* (D) *exists and the strong duality* $f(\bar{x}) = h(\bar{p},\bar{q},\bar{\lambda},\bar{t})$ *is true.*

Proof. Assume $\bar{x}$ to be properly efficient to (P). From Definition 2 the existence of a corresponding vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T \in int\mathbb{R}^m_+$ follows such that $\bar{x}$ solves the scalar problem

$$(P_{\bar{\lambda}}) \ \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i f_i(x).$$

The constraint qualification (CQ) is fulfilled and, so, Theorem 1 assures the existence of a solution $(\tilde{p}, \tilde{q})$ to the dual of $(P_{\bar{\lambda}})$. Theorem 2 affirms that the optimailty conditions (i), (ii) and (iii) are satisfied.

Now, we will construct by means of $\bar{x}$ and $(\tilde{p}, \tilde{q})$ the efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ of (D). Let $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_m)^T$ be the vector given by the proper efficiency of $\bar{x}$. We consider for $i = 1, \ldots, m, \bar{p}_i = \tilde{p}_i$ and then we have $\bar{p} = (\bar{p}_1, \ldots \bar{p}_m) = (\tilde{p}_1, \ldots \tilde{p}_m) = \tilde{p}$. It remains to define $\bar{q} = (\bar{q}_1, \ldots \bar{q}_m)$ and $\bar{t} = (\bar{t}_1, \ldots \bar{t}_m)^T$.

Let for $i = 1, \ldots, m$,

$$\begin{aligned} \bar{q}_i &= \frac{1}{m\bar{\lambda}_i}\tilde{q} \in \mathbb{R}^k \\ \bar{t}_i &= \bar{p}_i^T \bar{x} + (\bar{q}_i^T g)^*\Big(-\frac{1}{m\bar{\lambda}_i}\sum_{i=1}^{m} \bar{\lambda}_i \bar{p}_i\Big) \in \mathbb{R}. \end{aligned} \tag{6}$$

For the new element $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ it holds $\bar{\lambda} \in int\mathbb{R}^m_+, \sum_{i=1}^{m} \bar{\lambda}_i \bar{q}_i = \tilde{q} \underset{K^*}{\geqq} 0$ and

$$\begin{aligned} \sum_{i=1}^{m} \bar{\lambda}_i \bar{t}_i &= \Big(\sum_{i=1}^{m} \bar{\lambda}_i \bar{p}_i\Big)^T \bar{x} + \sum_{i=1}^{m} \bar{\lambda}_i \left(\frac{1}{m\bar{\lambda}_i}\tilde{q}^T g\right)^* \Big(-\frac{1}{m\bar{\lambda}_i}\sum_{i=1}^{m} \bar{\lambda}_i \bar{p}_i\Big) \\ &= \Big(\sum_{i=1}^{m} \bar{\lambda}_i \bar{p}_i\Big)^T \bar{x} + \sum_{i=1}^{m} \bar{\lambda}_i \frac{1}{m\bar{\lambda}_i}(\tilde{q}^T g)^*\Big(-\sum_{i=1}^{m} \bar{\lambda}_i \bar{p}_i\Big) \\ &= \Big(\sum_{i=1}^{m} \bar{\lambda}_i \bar{p}_i\Big)^T \bar{x} + (\tilde{q}^T g)^*\Big(-\sum_{i=1}^{m} \bar{\lambda}_i \bar{p}_i\Big) \\ &= 0 \quad \text{(by (3))}. \end{aligned}$$

We proved that the element $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ is feasible for (D). It remains to show that the values of the objective functions are equal $f(\bar{x}) = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$.

Therefore we will prove that $f_i(\bar{x}) = h_i(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ for each $i = 1, \ldots, m$. For this we will use the relation (i) from Theorem 2 and the equations (6).

We obtain the following equalities

$$\begin{aligned} h_i(\bar{p},\bar{q},\bar{\lambda},\bar{t}) &= -f_i^*(\bar{p}_i) - (\bar{q}_i^T g)^*(-\frac{1}{m\bar{\lambda}_i}\sum_{i=1}^m \bar{\lambda}_i\bar{p}_i) + \bar{t}_i \\ &= -f_i^*(\bar{p}_i) - (\bar{q}_i^T g)^*(-\frac{1}{m\bar{\lambda}_i}\sum_{i=1}^m \bar{\lambda}_i\bar{p}_i) + \bar{p}_i^T\bar{x} \\ &+ (\bar{q}_i^T g)^*(-\frac{1}{m\bar{\lambda}_i}\sum_{i=1}^m \bar{\lambda}_i\bar{p}_i) = -f_i^*(\bar{p}_i) + \bar{p}_i^T\bar{x} = f_i(\bar{x}). \end{aligned}$$

The maximality of $(\bar{p},\bar{q},\bar{\lambda},\bar{t})$ is given by Theorem 3. □

5 The converse duality

In this section we will complete our investigations by the formulation of the converse duality theorem for (P) and (D).

Therefore we will introduce some new notations. For each $\lambda \in int\mathbb{R}^m_+$ let us denote

$$\mathcal{B}_\lambda = \left\{ (p,q,t) : \sum_{i=1}^m \lambda_i q_i \underset{K^*}{\geqq} 0, \sum_{i=1}^m \lambda_i t_i = 0 \right\},$$

$$p = (p_1,\ldots,p_m), q = (q_1,\ldots,q_m), t = (t_1,\ldots,t_m)^T,$$

$$p_i \in \mathbb{R}^m, \quad q_i \in \mathbb{R}^k, \quad t_i \in \mathbb{R}, \quad i = 1,\ldots,m.$$

Further, let be

$$\begin{aligned} M &= \{a \in \mathbb{R}^m : \exists \lambda \in int\mathbb{R}^m_+, \quad \exists (p,q,t) \in \mathcal{B}_\lambda \\ &\quad \text{such that} \quad \sum_{i=1}^m \lambda_i a_i = \sum_{i=1}^m \lambda_i h_i(p,q,\lambda,t)\}. \end{aligned}$$

For the proof of the converse duality theorem we need the following propositions.

Proposition 1. *It holds $h(\mathcal{B}) \cap \mathbb{R}^m = M$.*

Proof. Obviously, $h(\mathcal{B}) \cap \mathbb{R}^m \subseteq M$. We have to prove the inverse inclusion.

Therefore, let be $a \in M$. Then there exist $\lambda \in int\mathbb{R}^m_+$ and $(p,q,t) \in \mathcal{B}_\lambda$ such that $\sum\limits_{i=1}^m \lambda_i a_i = \sum\limits_{i=1}^m \lambda_i h_i(p,q,\lambda,t)$ or, equivalently,

$$\sum_{i=1}^m \lambda_i a_i = -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \sum_{i=1}^m \lambda_i (q_i^T g)^*(-\frac{1}{m\lambda_i}\sum_{i=1}^m \lambda_i p_i) + \sum_{i=1}^m \lambda_i t_i.$$

Let us define for $i = 1, \ldots, m$,

$$\bar{t}_i = a_i + f_i^*(p_i) + (q_i^T g)^*(-\frac{1}{m\lambda_i}\sum_{i=1}^m \lambda_i p_i) \in \mathbb{R}.$$

It is easy to observe that $\sum_{i=1}^m \lambda_i \bar{t}_i = \sum_{i=1}^m \lambda_i t_i = 0$ and this means that $(p, q, \lambda, \bar{t}) \in \mathcal{B}$. We also have for $i = 1, \ldots, m$,

$$a_i = -f_i^*(p_i) - (q_i^T g)^*(-\frac{1}{m\lambda_i}\sum_{i=1}^m \lambda_i p_i) + \bar{t}_i$$

and so, it follows that $a = h(p, q, \lambda, \bar{t}) \in h(\mathcal{B})$. In conclusion, $M \subseteq h(\mathcal{B}) \cap \mathbb{R}^m$ and the proof is complete. □

Proposition 2. *An element $\bar{a} \in \mathbb{R}^m$ is maximal in M if and only if for every $a \in M$ with corresponding $\lambda^a \in int\mathbb{R}^m_+$ and $(p^a, q^a, t^a) \in \mathcal{B}_{\lambda^a}$ it holds*

$$\sum_{i=1}^m \lambda_i^a \bar{a}_i \geq \sum_{i=1}^m \lambda_i^a a_i. \tag{7}$$

Proof. At first we show the sufficiency. Assume the existence of some $a \in M$ such that $a \in \bar{a} + \mathbb{R}^m_+ \setminus \{0\}$. For the corresponding $\lambda^a \in int\mathbb{R}^m_+$ it holds $\sum_{i=1}^m \lambda_i^a \bar{a}_i < \sum_{i=1}^m \lambda_i^a a_i$, which contradicts relation (7).

To prove the necessity, let us assume that there exists $b \in \mathbb{R}^m$, $b \in \bar{a} + \mathbb{R}^m_+ \setminus \{0\}$ and $a \in M$ with corresponding $\lambda^a \in int\mathbb{R}^m_+$ and $(p^a, q^a, t^a) \in \mathcal{B}_{\lambda^a}$ such that

$$\sum_{i=1}^m \lambda_i^a a_i \geq \sum_{i=1}^m \lambda_i^a b_i. \tag{8}$$

We will show that the assumption we made is false.

If equality holds in (8), $\sum_{i=1}^m \lambda_i^a a_i = \sum_{i=1}^m \lambda_i^a b_i$, then $b \in M$ and this contradicts the maximality of $\bar{a}$ in M.

If $\sum_{i=1}^m \lambda_i^a a_i > \sum_{i=1}^m \lambda_i^a b_i$, then we can choose $c \in \mathbb{R}^m$ such that $c_i > a_i$ and $c_i > b_i, i = 1, \ldots, m$.

Because

$$\sum_{i=1}^m \lambda_i^a c_i > \sum_{i=1}^m \lambda_i^a a_i > \sum_{i=1}^m \lambda_i^a b_i$$

there exists $r \in (0, 1)$ such that $\sum_{i=1}^m \lambda_i^a a_i = \sum_{i=1}^m \lambda_i^a [(1 - r)b_i + rc_i]$. This means that $(1 - r)b + rc \in M$.

On the other hand,

$$(1-r)b + rc = r(c-b) + b \in \mathbb{R}^m_+ \setminus \{0\} + \bar{a} + \mathbb{R}^m_+ \setminus \{0\} \subseteq \bar{a} + \mathbb{R}^m_+ \setminus \{0\}.$$

Our assumption was false because the last inclusion also contradicts the maximality of $\bar{a}$ in M.

Then, for each $b \in \bar{a} + \mathbb{R}^m_+ \setminus \{0\}$ and $a \in M$ with corresponding $\lambda^a \in int\mathbb{R}^m_+$ and $(p^a, q^a, t^a) \in \mathcal{B}_{\lambda^a}$ we must have

$$\sum_{i=1}^m \lambda_i^a b_i > \sum_{i=1}^m \lambda_i^a a_i. \tag{9}$$

From this last relation implies that for each $a \in M$ with corresponding $\lambda^a \in int\mathbb{R}^m_+$ and $(p^a, q^a, t^a) \in \mathcal{B}_{\lambda^a}$ it holds

$$\sum_{i=1}^m \lambda_i^a \bar{a}_i = \inf\left\{\sum_{i=1}^m \lambda_i^a b_i : b \in \bar{a} + \mathbb{R}^m_+ \setminus \{0\}\right\} \geq \sum_{i=1}^m \lambda_i^a a_i,$$

which finishes our proof. □

We are now so far to formulate the converse duality theorem.

Theorem 5. *Assume the constraint qualification (CQ) is fulfilled. Suppose that for each $\lambda \in int\mathbb{R}^m_+$ the following property holds.*

(C) | *If* $\inf_{x \in \mathcal{A}} \sum_{i=1}^m \lambda_i f_i(x) > -\infty$, *then there exists an element* $x_\lambda \in \mathcal{A}$ *such that* $\inf_{x \in \mathcal{A}} \sum_{i=1}^m \lambda_i f_i(x) = \sum_{i=1}^m \lambda_i f_i(x_\lambda)$.

(a) Then, for any efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ of (D), $h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in cl(f(\mathcal{A}) + \mathbb{R}^m_+)$ and there exists a properly efficient solution $\bar{x}_{\bar{\lambda}}$ of (P) such that

$$\sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}_{\bar{\lambda}}) - h_i(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})] = 0.$$

(b) If, additionally, $f(\mathcal{A})$ is $\mathbb{R}^m_+$-closed (i.e. $f(\mathcal{A}) + \mathbb{R}^m_+$ is closed), then there exists a properly efficient to (P) $\bar{x} \in \mathcal{A}$, such that

$$\sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}}) = \sum_{i=1}^m \bar{\lambda}_i f_i(\bar{x})$$

and

$$f(\bar{x}) = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}).$$

Proof.

(a) Let us denote $\bar{a} = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$. From the maximality of $\bar{a}$ in $h(\mathcal{B})$, we have that $\bar{a} \in h(\mathcal{B}) \cap \mathbb{R}^m$. By Proposition 1, it follows that $\bar{a}$ is maximal in M.

For the beginning, we will prove that $\bar{a} \in cl(f(\mathcal{A}) + \mathbb{R}^m_+)$.

Assume the contrary. Because $cl(f(\mathcal{A}) + \mathbb{R}^m_+)$ is convex and closed , by a well known separation theorem, there exist $\lambda^1 \in \mathbb{R}^m \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that

$$\sum_{i=1}^{m} \lambda_i^1 \bar{a}_i < \alpha \leq \sum_{i=1}^{m} \lambda_i^1 d_i, \quad \forall d \in cl(f(\mathcal{A}) + \mathbb{R}^m_+). \tag{10}$$

From this last inequality it is easy to observe that $\lambda^1 \in \mathbb{R}^m_+ \setminus \{0\}$.

That $\bar{a} \in M$ assures the existence of the corresponding $\lambda^{\bar{a}} \in int\mathbb{R}^m_+$ and $(p^{\bar{a}}, q^{\bar{a}}, t^{\bar{a}}) \in \mathcal{B}_{\lambda^{\bar{a}}}$ such that $\sum\limits_{i=1}^{m} \lambda_i^{\bar{a}} \bar{a}_i = \sum\limits_{i=1}^{m} \lambda_i^{\bar{a}} h_i(p^{\bar{a}}, q^{\bar{a}}, t^{\bar{a}})$. Like in the proof of Theorem 3 it holds

$$\sum_{i=1}^{m} \lambda_i^{\bar{a}} \bar{a}_i = \sum_{i=1}^{m} \lambda_i^{\bar{a}} h_i(p^{\bar{a}}, q^{\bar{a}}, t^{\bar{a}}) \leq \sum_{i=1}^{m} \lambda_i^{\bar{a}} d_i, \quad \forall d \in cl(f(\mathcal{A}) + \mathbb{R}^m_+). \tag{11}$$

Let now $s \in (0, 1)$ be fixed. Considering $\lambda^* = s\lambda^1 + (1 - s)\lambda^{\bar{a}} \in int\mathbb{R}^m_+$, from (10) and (11) follows

$$\sum_{i=1}^{m} \lambda_i^* \bar{a}_i < \sum_{i=1}^{m} \lambda_i^* d_i, \quad \forall d \in cl(f(\mathcal{A}) + \mathbb{R}^m_+)$$

which implies that for each $x \in \mathcal{A}$

$$\sum_{i=1}^{m} \lambda_i^* \bar{a}_i < \sum_{i=1}^{m} \lambda_i^* f_i(x). \tag{12}$$

Relation (12) guarantees that the assumption of condition (C) is fulfilled and this assures the existence of a solution $x_{\lambda^*} \in \mathcal{A}$ for the problem (P_{λ^*}). The constraint qualification (CQ) is also fulfilled and then we can construct, like in the proof of Theorem 4, a maximal efficient element $(p_{\lambda^*}, q_{\lambda^*}, \lambda^*, t_{\lambda^*})$ to (D) such that

$$f(x_{\lambda^*}) = h(p_{\lambda^*}, q_{\lambda^*}, \lambda^*, t_{\lambda^*}) \in h(\mathcal{B}) = M.$$

Using the maximality of $\bar{a} \in M$, by Proposition 2, we have that

$$\sum_{i=1}^{m} \lambda_i^* \bar{a}_i \geq \sum_{i=1}^{m} \lambda_i^* f_i(x_{\lambda^*}),$$

which contradicts the strict inequality (12).

This means that $\bar{a} = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in cl(f(\mathcal{A}) + \mathbb{R}^m_+)$. Then there exist sequences $(x^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ and $(k^n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^m_+$ with the property that $f(x^n) + k^n$ converges to $\bar{a}$.

On the other hand, by the proof of Theorem 3, for each $x \in \mathcal{A}$,

$$\sum_{i=1}^{m} \bar{\lambda}_i f_i(x) \geq \sum_{i=1}^{m} \bar{\lambda}_i \bar{a}_i.$$

Considering, again, condition (C), there exists a properly efficient solution $\bar{x}_{\bar{\lambda}} \in \mathcal{A}$ for (P) such that $\inf\limits_{x\in\mathcal{A}} \sum\limits_{i=1}^{m} \bar{\lambda}_i f_i(x) = \sum\limits_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}})$.

For this solution $\bar{x}_{\bar{\lambda}}$ the following sequence of inequalities holds

$$\begin{aligned} \sum_{i=1}^{m} \bar{\lambda}_i \bar{a}_i &\leq \sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}}) = \inf_{x\in\mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i f_i(x) \\ &\leq \sum_{i=1}^{m} \bar{\lambda}_i (f_i(x^n) + k_i^n), \quad \forall n \in \mathbb{N}. \end{aligned}$$

Letting $n \to +\infty$, it follows that

$$\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}}) = \sum_{i=1}^{m} \bar{\lambda}_i \bar{a}_i = \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$$

and the first part is proved.

(b) If $f(\mathcal{A}) + \mathbb{R}^m_+$ is closed, then, by the first part of the proof, $\bar{a} \in cl(f(\mathcal{A}) + \mathbb{R}^m_+) = f(\mathcal{A}) + \mathbb{R}^m_+$. According to the weak duality theorem, Theorem 3, we have $\bar{a} \in f(\mathcal{A})$ and this implies the existence of an element $\bar{x} \in \mathcal{A}$ such that $f(\bar{x}) = \bar{a} = h(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$. It is obvoius that $\bar{x}$ is properly efficient and that holds $\sum\limits_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}}) = \sum\limits_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x})$. □

6 Some special cases

Recently, in [23] and [24], we studied the duality for a multiobjective optimization problem with convex objective functions and linear inequality constraints. The aim of this last section is to show that the dual obtained in the both papers is actually a special case of the general dual problem (D).

6.1 Special case I

Let us consider $g : \mathbb{R}^n \to \mathbb{R}^k$, defined by $g(x) = Ax + b$, with A being a $k \times n$ matrix and $b \in \mathbb{R}^k, b \neq 0$.

We have now the following primal problem

$$(P_1) \;\; \underset{x \in \mathcal{A}_1}{\text{v-min}} \, f(x)$$

with

$$\mathcal{A}_1 = \left\{ x \in \mathbb{R}^n : Ax + b \underset{K}{\leqq} 0 \right\}$$

and

$$f(x) = (f_1(x), \ldots, f_m(x))^T.$$

Using the linearity of g, we can calculate the conjugate of $q_i^T g, i = 1, \ldots, m$

$$\begin{aligned}
(q_i^T g)^*(-\frac{1}{m\lambda_i} \sum_{i=1}^m \lambda_i p_i) &= \sup_{x \in \mathbb{R}^n} \left[(-\frac{1}{m\lambda_i} \sum_{i=1}^m \lambda_i p_i)^T x - q_i^T (Ax + b) \right] \\
&= -q_i^T b + \sup_{x \in \mathbb{R}^n} \left[\left(-A^T q_i - \frac{1}{m\lambda_i} \sum_{i=1}^m \lambda_i p_i \right)^T x \right] \\
&= \begin{cases} -q_i^T b & , \text{ if } \;\; A^T q_i + \frac{1}{m\lambda_i} \sum\limits_{i=1}^m \lambda_i p_i = 0 \\ +\infty & , \text{ otherwise.} \end{cases}
\end{aligned}$$

Thus, the dual of (P_1) is

$$(D_1) \quad \underset{(p,q,\lambda,t) \in \mathcal{B}_1}{\text{v-max}} \; h(p,q,\lambda,t) = \begin{pmatrix} -f_1^*(p_1) + q_1^T b + t_1 \\ \vdots \\ -f_m^*(p_m) + q_m^T b + t_m \end{pmatrix}$$

with the set of constraints

$$\begin{aligned}
\mathcal{B}_1 \;\; = \;\; \{ (p,q,\lambda,t) : \lambda \in int\mathbb{R}^m_+, \;\; & \sum_{i=1}^m \lambda_i q_i \underset{K^*}{\geqq} 0, \;\; \sum_{i=1}^m \lambda_i t_i = 0, \\
& A^T q_i + \frac{1}{m\lambda_i} \sum_{i=1}^m \lambda_i p_i = 0, \quad i = 1, \ldots, m \}.
\end{aligned}$$

Next we will prove that the images of h of the sets $\mathcal{B}_1$ and

$$\begin{aligned}
\mathcal{B}_1{}' \;\; = \;\; \{ (p,q',\lambda,t') : \lambda \in int\mathbb{R}^m_+, \;\; & \sum_{i=1}^m \lambda_i q_i' \underset{K^*}{\geqq} 0, \;\; \sum_{i=1}^m \lambda_i t_i' = 0, \\
& \sum_{i=1}^m \lambda_i (A^T q_i' + p_i) = 0 \}
\end{aligned}$$

coincide (i.e. $h(\mathcal{B}_1) = h(\mathcal{B}_1{}'))$.

It is only to show that $h(\mathcal{B}_1{}') \subseteq h(\mathcal{B}_1)$ because the inverse inclusion is obvious. Therefore, let be $(p, q', \lambda, t') \in \mathcal{B}_1{}'$. Considering for $i = 1, \ldots, m$, $q_i = \frac{1}{m\lambda_i} \sum\limits_{i=1}^{m} \lambda_i q_i'$ and $t_i = t_i' + q_i'^T b - \left(\frac{1}{m\lambda_i} \sum\limits_{i=1}^{m} \lambda_i q_i' \right)^T b$, we obtain an element $(p, q, \lambda, t) \in \mathcal{B}_1$ such that $h(p, q', \lambda, t') = h(p, q, \lambda, t) \in h(\mathcal{B}_1)$.

The dual of (P_1) then is equivalent to the following problem

$$(D_1) \quad \underset{(p,q',\lambda,t') \in \mathcal{B}_1{}'}{\text{v-max}} \; h(p, q', \lambda, t') = \begin{pmatrix} -f_1^*(p_1) + q_1'^T b + t_1' \\ \vdots \\ -f_m^*(p_m) + q_m'^T b + t_m' \end{pmatrix}.$$

The last step is to show that the dual (D_1) actually is the multiobjective optimization problem

$$(D_1) \quad \underset{(p,\delta,\lambda) \in \tilde{\mathcal{B}}_1}{\text{v-max}} \; \tilde{h}_1(p, \delta, \lambda) = \begin{pmatrix} -f_1^*(p_1) + \delta_1^T b \\ \vdots \\ -f_m^*(p_m) + \delta_m^T b \end{pmatrix}$$

with

$$\tilde{\mathcal{B}}_1 = \{(p, \delta, \lambda) : \lambda \in int\mathbb{R}_+^m, \quad \sum_{i=1}^{m} \lambda_i \delta_i \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^{m} \lambda_i (A^T \delta_i + p_i) = 0\}.$$

For $q_i' = \delta_i$ and $t_i' = 0, i = 1, \ldots, m$ it is easy to observe that $\tilde{h}_1(\tilde{\mathcal{B}}_1) \subseteq h(\mathcal{B}_1{}')$. To show the inverse inclusion, let be $(p, q', \lambda, t') \in \mathcal{B}_1{}'$. Because $b \neq 0$ we can consider a vector $\gamma \in \mathbb{R}^k$ such that $\gamma^T b = 1$. Now, defining for $i = 1, \ldots, m$, $\delta_i = q_i' + t_i' \gamma$, it holds $\delta_i^T b = q_i'^T b + t_i'$ and $\sum\limits_{i=1}^{m} \lambda_i \delta_i = \sum\limits_{i=1}^{m} \lambda_i q_i'$. This means that (p, δ, λ) belongs to $\tilde{\mathcal{B}}_1$ and that $h(p, q', \lambda, t') = \tilde{h}_1(p, \delta, \lambda) \in \tilde{h}_1(\tilde{\mathcal{B}}_1)$.

6.2 Special case II

In this part we will consider the same problem like before but for the case $b = 0$. The first steps are the same like for the previous case and this means that the dual for

$$(P_2) \quad \underset{x \in \mathcal{A}_2}{\text{v-min}} \, f(x)$$

with

$$\mathcal{A}_2 = \left\{ x \in \mathbb{R}^n : Ax \underset{K}{\leqq} 0 \right\}$$

and

$$f(x) = (f_1(x), \ldots, f_m(x))^T$$

will be

$$(D_2) \quad \underset{(p,q',\lambda,t')\in \mathcal{B}_2{}'}{\text{v-max}} \; h(p,q',\lambda,t') = \begin{pmatrix} -f_1^*(p_1) + t_1' \\ \vdots \\ -f_m^*(p_m) + t_m' \end{pmatrix}$$

with

$$\begin{aligned} \mathcal{B}_2{}' \;=\; & \{(p,q',\lambda,t') : \lambda \in int\mathbb{R}^m_+, \quad \sum_{i=1}^m \lambda_i q_i' \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i t_i' = 0, \\ & \sum_{i=1}^m \lambda_i (A^T q_i' + p_i) = 0\}. \end{aligned}$$

Substituting $\gamma = \sum\limits_{i=1}^m \lambda_i q_i'$ we obtain the equivalent dual problem

$$(D_2) \quad \underset{(p,\gamma,\lambda,t)\in \tilde{\mathcal{B}}_2}{\text{v-max}} \; \tilde{h}_2(p,\gamma,\lambda,t) = \begin{pmatrix} -f_1^*(p_1) + t_1 \\ \vdots \\ -f_m^*(p_m) + t_m \end{pmatrix}$$

with

$$\tilde{\mathcal{B}}_2 = \{(p,\gamma,\lambda,t) : \lambda \in int\mathbb{R}^m_+, \quad \gamma \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i t_i = 0, \quad -A^T\gamma = \sum_{i=1}^m \lambda_i p_i\}.$$

6.3 Special case III

In the last part of the paper we will study the following multiobjective optimization problem (cf. [23], [24])

$$(P_3) \quad \underset{x\in\mathcal{A}_3}{\text{v-min}} \, f(x),$$

$$\mathcal{A}_3 = \left\{ x \in \mathbb{R}^n : x \underset{K_0}{\geqq} 0, Ax + b \underset{K_1}{\leqq} 0 \right\},$$

$$f(x) = (f_1(x), \ldots, f_m(x))^T.$$

Let A be a $k \times n$ matrix, $b \in \mathbb{R}^k, b \neq 0$, $K_0 \subseteq \mathbb{R}^n$ and $K_1 \subseteq \mathbb{R}^k$ two convex closed cones. Considering the $(k+n) \times n$ matrix $\bar{A} = \begin{pmatrix} A \\ -I_n \end{pmatrix}$, the vector $\bar{b} = \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathbb{R}^{k+n}, \bar{b} \neq 0$ and the convex closed cone $K = K_1 \times K_0 \in \mathbb{R}^{k+n}$ we can represent the feasible set of (P_3) as being $\mathcal{A}_3 = \left\{ x \in \mathbb{R}^n : \bar{A}x + \bar{b} \underset{K}{\leqq} 0 \right\}$ and then we can reduce the problem (P_3) to the problem studied as the special case I.

The dual of (P_3) will be

$$(D_3) \quad \underset{(p,\bar{\delta},\lambda)\in\bar{\mathcal{B}}_3}{\text{v-max}} \bar{h}_3(p,\bar{\delta},\lambda) = \begin{pmatrix} -f_1^*(p_1) + \bar{\delta}_1^T\bar{b} \\ \vdots \\ -f_m^*(p_m) + \bar{\delta}_m^T\bar{b} \end{pmatrix}$$

with

$$\bar{\mathcal{B}}_3 = \{(p,\bar{\delta},\lambda) : \lambda \in int\mathbb{R}^m_+, \quad \sum_{i=1}^m \lambda_i\bar{\delta}_i \underset{K^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i(\bar{A}^T\bar{\delta}_i + p_i) = 0\}$$

and the dual variables $p = (p_1,\ldots,p_m) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ and $\bar{\delta} = (\delta^1,\delta^2) \in \mathbb{R}^k \times \mathbb{R}^n$.

Remarking that $\bar{\delta}_i^T\bar{b} = \delta_i^{1T}b$ and $\bar{A}^T\bar{\delta}_i = A^T\delta_i^1 - \delta_i^2, i = 1,\ldots,m,$ we obtain for the dual of (P_3) the following problem

$$(D_3) \quad \underset{(p,\delta_1,\delta_2,\lambda)\in\bar{\mathcal{B}}_3}{\text{v-max}} \bar{h}_3(p,\delta_1,\delta_2,\lambda) = \begin{pmatrix} -f_1^*(p_1) + \delta_1^{1T}b \\ \vdots \\ -f_m^*(p_m) + \delta_m^{1T}b \end{pmatrix}$$

with

$$\begin{aligned} \bar{\mathcal{B}}_3 \quad = \quad & \{(p,\delta_1,\delta_2,\lambda) : \lambda \in int\mathbb{R}^m_+, \quad \sum_{i=1}^m \lambda_i\delta_i^1 \underset{K_1^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i\delta_i^2 \underset{K_0^*}{\geqq} 0, \\ & \sum_{i=1}^m \lambda_i(A^T\delta_i^1 + p_i) = \sum_{i=1}^m \lambda_i\delta_i^2\} \end{aligned}$$

or, equivalently,

$$(D_3) \quad \underset{(p,\delta,\lambda)\in\tilde{\mathcal{B}}_3}{\text{v-max}} \tilde{h}_3(p,\delta,\lambda) = \begin{pmatrix} -f_1^*(p_1) + \delta_1^Tb \\ \vdots \\ -f_m^*(p_m) + \delta_m^Tb \end{pmatrix}$$

with

$$\tilde{\mathcal{B}}_3 = \{(p,\delta,\lambda) : \lambda \in int\mathbb{R}^m_+, \quad \sum_{i=1}^m \lambda_i\delta_i \underset{K_1^*}{\geqq} 0, \quad \sum_{i=1}^m \lambda_i(A^T\delta_i + p_i) \underset{K_0^*}{\geqq} 0\}.$$

The dual (D_3) exactly is the problem obtained in [23] and [24].

Remark 3.

(a) For the problem (P_3), the constraint qualification (CQ) assumes the existence of an element $x' \in \mathbb{R}^n$ such that $x' \in intK_0$ and $Ax' + b \in -intK_1$. But, as we have proved in [23] and [24], for strong duality it is enough to assume the existence of an element $x' \in \mathbb{R}^n$ such that $x' \in K_0$ and $Ax' + b \in -intK_1$.

(b) A special case of the problem (P_3) has been considered in [17], the functions $f_i, i = 1, \ldots, m$, being norms. The converse duality theorem presented there is false. The converse duality theorem proved in Section 5 corrects that theorem and is valid in that very general case.

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