AN ANALYSIS OF SOME DUAL PROBLEMS IN MULTIOBJECTIVE OPTIMIZATION (II)

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In the first part of this study we have introduced six different multiobjective dual problems to a general multiobjective optimization problem, for which we presented weak as well as strong duality assertions. Afterwards, we derived some inclusion results for the image sets of three of these problems.

The aim of this second part is to complete our investigations by studying the relations between all six multiobjective dual problems. Moreover, conditions under which the dual problems are equivalent are given.

The results are illustrated by some examples.

A general scheme containing the relations between the six multiobjective duals and other duals mentioned in the literature is derived.

Keywords: Multiobjective dual problems; Sets of maximal elements; Weak and strong duality

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1 Preliminaries

The aim of this paper is to continue the investigations of the relationships between different dual problems in the theory of multiobjective optimization. In the first part of this study we have considered the following primal multiobjective problem

$$(P) \quad \underset{x \in \mathcal{A}}{\text{v-min}}\, f(x),$$

$$\mathcal{A} = \left\{ x \in \mathbb{R}^n : g(x) = (g_1(x), \ldots, g_k(x))^T \underset{K}{\leqq} 0 \right\},$$

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where $f(x) = (f_1(x), \dots, f_m(x))^T$ and $f_i : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $i = 1, ..., m$, are proper functions, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, ..., k$, and $K \subseteq \mathbb{R}^k$ is assumed to be a convex closed cone with $intK \neq \emptyset$, defining a partial ordering according to $x_2 \underset{K}{\leqq} x_1$ if and only if $x_1 - x_2 \in K$. We consider Pareto-efficient and properly efficient solutions with respect to the ordering cone $\mathbb{R}^m_+$.

To that problem we have associated six dual problems and proved the existence of weak duality and, under the fulfillment of (A_f), (A_g) and (A_{CQ}), the existence of strong duality. Let us recall this three assumptions, which play an important role also in this second part

$$(A_f) \left| \text{ the functions } f_i,\ i = 1, ..., m, \text{ are convex and } \bigcap_{i=1}^{m} ri(dom f_i) \neq \emptyset, \right.$$

$$(A_g) \left| \begin{array}{l} \text{the function } g \text{ is convex relative to the cone } K, \text{ i.e. } \forall x_1, x_2 \in \\ \mathbb{R}^n,\ \forall \lambda \in [0,1], \lambda g(x_1) + (1-\lambda) g(x_2) - g(\lambda x_1 + (1-\lambda) x_2) \in K, \end{array} \right.$$

$$(A_{CQ}) \left| \text{ there exists } x' \in \bigcap_{i=1}^{m} ri(dom f_i) \text{ such that } g(x') \in -intK. \right.$$

After proving the existence of weak and strong duality, we related the image sets of three of these duals, (D_1), (D_α), $\alpha \in \mathcal{F}$, and (D_{FL}), denoted by $D_1 = h^1(\mathcal{B}_1)$, $D_\alpha = h^\alpha(\mathcal{B}_\alpha)$, $\alpha \in \mathcal{F}$, and, respectively, $D_{FL} = h^{FL}(\mathcal{B}_{FL})$, to each other. Here, $\mathcal{B}_1$, $\mathcal{B}_\alpha$ and $\mathcal{B}_{FL}$ denote the admissible sets of the dual problems (D_1), (D_α) and (D_{FL}), and h^1, h^α and h^{FL} are the corresponding vector-valued dual objective functions.

We denote by $\mathcal{F}$ the following set

$$\mathcal{F} = \left\{ \alpha : int\mathbb{R}^m_+ \to \mathbb{R}^m_+ \ :\ \begin{array}{l} \alpha(\lambda) = (\alpha_1(\lambda), ..., \alpha_m(\lambda))^T, \quad \text{such that} \\ \sum\limits_{i=1}^{m} \lambda_i \alpha_i(\lambda) = 1,\ \forall \lambda = (\lambda_1, ..., \lambda_m)^T \in int\mathbb{R}^m_+ \end{array} \right\},$$

and, so, the family of problems (D_α), $\alpha \in \mathcal{F}$, generalizes the dual multiobjective problem introduced by us in [1].

We showed that, for every $\alpha \in \mathcal{F}$, it holds

$$D_1 \cap \mathbb{R}^m \subsetneqq D_\alpha \cap \mathbb{R}^m \subsetneqq D_{FL}, \tag{1. 1}$$

where the notation " $\subsetneqq$ " means that the inclusions in (1. 1) may be strict. This assertion remains valid even if (A_f), (A_g) and (A_{CQ}) are fulfilled.

On the other hand, we have proved that the sets of the maximal elements of these three sets are equal, i.e., for every $\alpha \in \mathcal{F}$,

$$vmaxD_1 = vmaxD_\alpha = vmaxD_{FL}. \tag{1. 2}$$

Here, by $vmaxA$ we denote the set of maximal elements of a set $A \subseteq \mathbb{R}^m$ with respect to the partial ordering given by $\mathbb{R}^m_+$.

In order to continue this analysis, let us denote the image sets of the problems (D_F), (D_L) and (D_P) by $D_F := h^F(\mathcal{B}_F)$, $D_L := h^L(\mathcal{B}_L)$ and $D_P := h^P(\mathcal{B}_P)$, respectively, where $\mathcal{B}_F$, $\mathcal{B}_L$ and $\mathcal{B}_P$ denote the admissible sets of the dual problems (D_F), (D_L) and (D_P), and h^F, h^L and h^P are the corresponding vector-valued dual objective functions. We mention that in the objective space we use the cone $\mathbb{R}^m_+$, and, in this situation, one can observe that the multiobjective dual problem (D_L) is actually the problem introduced by Jahn in [2] and [3].

We start our investigations by proving the existence of some relations of inclusion between the sets D_{FL}, D_F, D_L and D_P, in the general case. By giving some counter-examples we also show that, unfortunately, a relation like in (1. 2) does not hold. On the other hand, we show under which conditions the sets become identical. Obviously, in this case, they will also have the same maximal elements.

In the second part of the paper we include in our study the multiobjective duals introduced by Nakayama in [4], [5] and Weir and Mond in [6], [7] and [8].

2 The relations of inclusion between D_{FL}, D_F, D_L and D_P

For the beginning, let us notice that during this section we work in the general case. Obviously, from the definition of the multiobjective duals in the first part of the study, it follows that D_{FL}, D_F, D_L, D_P are subsets of $\mathbb{R}^m$.

PROPOSITION 2.1

(a) It holds $D_{FL} \subseteq D_F$.

(b) It holds $D_{FL} \subseteq D_L$.

Proof

(a) Let be $d = (d_1, ..., d_m)^T \in D_{FL}$. Then there exist $p_i \in \mathbb{R}^n, i = 1, ..., m$, $q \underset{K^*}{\geqq} 0$ and $\lambda \in int\mathbb{R}^m_+$ such that

$$\sum_{i=1}^m \lambda_i d_i \leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) - (q^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i \right).$$

By the definition of the conjugate function, we have

$$\begin{aligned} \sum_{i=1}^m \lambda_i d_i &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x \in \mathbb{R}^n} \left[\left(\sum_{i=1}^m \lambda_i p_i \right)^T x + q^T g(x) \right] \\ &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x \in \mathcal{A}} \left[\left(\sum_{i=1}^m \lambda_i p_i \right)^T x + q^T g(x) \right] \end{aligned}$$

$$\begin{aligned} &\leq -\sum_{i=1}^{m}\lambda_i f_i^*(p_i) + \inf_{x\in\mathcal{A}}\left(\sum_{i=1}^{m}\lambda_i p_i\right)^T x \\ &= -\sum_{i=1}^{m}\lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^*\left(-\sum_{i=1}^{m}\lambda_i p_i\right). \end{aligned}$$

This means that $(p,\lambda,d) \in \mathcal{B}_F$ and $d = h^F(p,\lambda,d) \in h^F(\mathcal{B}_F) = D_F$.

(b) Like in (a), let be $d = (d_1,...,d_m)^T \in D_{FL}$. Again, there exist $p_i \in \mathbb{R}^n, i = 1,...,m$, $q \underset{K^*}{\geqq} 0$ and $\lambda \in int\mathbb{R}_+^m$ such that

$$\sum_{i=1}^{m}\lambda_i d_i \leq -\sum_{i=1}^{m}\lambda_i f_i^*(p_i) - (q^T g)^*\left(-\sum_{i=1}^{m}\lambda_i p_i\right).$$

Applying the inequality of Young (cf. [9]) for $f_i, i = 1,...,m$,

$$-f_i^*(p_i) \leq f_i(x) - p_i^T x, \forall x \in \mathbb{R}^n,$$

and for $q^T g$

$$-(q^T g)^*\left(-\sum_{i=1}^{m}\lambda_i p_i\right) \leq q^T g(x) + \left(\sum_{i=1}^{m}\lambda_i p_i\right)^T x, \forall x \in \mathbb{R}^n,$$

it holds

$$\sum_{i=1}^{m}\lambda_i d_i \leq \sum_{i=1}^{m}\lambda_i f_i(x) + q^T g(x), \forall x \in \mathbb{R}^n.$$

From here,

$$\sum_{i=1}^{m}\lambda_i d_i \leq \inf_{x\in\mathbb{R}^n}\left[\sum_{i=1}^{m}\lambda_i f_i(x) + q^T g(x)\right],$$

and this means that $(p,\lambda,d) \in \mathcal{B}_L$ and $d = h^L(p,\lambda,d) \in h^L(\mathcal{B}_L) = D_L$.

□

Example 2.1 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let us consider the functions $f_1, f_2 : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$, defined by

$$f_1(x) = \begin{cases} x, & \text{if } \quad x \in [0,+\infty), \\ +\infty, & \text{otherwise,} \end{cases}$$

$$f_2(x) = 0,$$

and

$$g(x)=\begin{cases}1-x^2, & \text{if} \quad x\in[0,+\infty),\\ 1, & \text{otherwise.}\end{cases}$$

For $p=(p_1,p_2)=(1,0)$, $\lambda=(1,1)^T$ and $d=(1,0)^T$, it holds

$$\lambda_1 d_1+\lambda_2 d_2 = 1 = -\lambda_1 f_1^*(p_1)-\lambda_2 f_2^*(p_2)+\inf_{g(x)\leq 0}(\lambda_1 p_1+\lambda_2 p_2)\,x,$$

and, so, we have that $d=(1,0)^T\in D_F$.

Let us show now that $d\notin D_{FL}$. If this were not true, then there would exist $\bar{p}=(\bar{p}_1,\bar{p}_2)$, $\bar{q}\geq 0$ and $\bar{\lambda}=(\bar{\lambda}_1,\bar{\lambda}_2)^T\in int\mathbb{R}^2_+$ such that

$$\bar{\lambda}_1\leq -\bar{\lambda}_1 f_1^*(\bar{p}_1)-\bar{\lambda}_2 f_2^*(\bar{p}_2)+\inf_{x\in\mathbb{R}}\left[\left(\bar{\lambda}_1\bar{p}_1+\bar{\lambda}_2\bar{p}_2\right)x+\bar{q}g(x)\right]. \qquad (2.\ 1)$$

In order to happen this, we must have $\bar{p}_2=0$ and $f_2^*(\bar{p}_2)=0$. Then, from (2. 1),

$$1\leq -f_1^*(\bar{p}_1)+\inf_{x\in\mathbb{R}}\left[\bar{p}_1 x+\frac{\bar{q}}{\bar{\lambda}_1}g(x)\right]. \qquad (2.\ 2)$$

In the case $\bar{q}>0$, we have that $\inf\limits_{x\in\mathbb{R}}\left[\bar{p}_1 x+\frac{\bar{q}}{\bar{\lambda}_1}g(x)\right]=-\infty$, which means that $\bar{q}$ must be 0. Then the inequality (2. 2) becomes

$$1\leq -f_1^*(\bar{p}_1)+\inf_{x\in\mathbb{R}}\left[\bar{p}_1 x\right],$$

and, so, it is obvious that $\bar{p}_1$ must be also 0. It remains that

$$1\leq -f_1^*(0)=\inf_{x\in\mathbb{R}}\left[f_1(x)\right]=\inf_{x\geq 0}x=0,$$

and this is a contradiction. In conclusion, $d=(1,0)\notin D_{FL}$, which means that $D_{FL}\subsetneq D_F$, i.e. the inclusion may be strict.

Example 2.2 Let be now $m=2, n=1, k=1$, $K=\mathbb{R}_+$, and the functions $f_1,f_2:\mathbb{R}\to\overline{\mathbb{R}}$, $g:\mathbb{R}\to\mathbb{R}$, introduced by

$$f_1(x)=\begin{cases}-x^2, & \text{if} \quad x\in[0,+\infty),\\ +\infty, & \text{otherwise,}\end{cases}$$

$$f_2(x)=0 \text{ and } g(x)=x^2-1.$$

For $q=1$, $\lambda=(1,1)^T$ and $d=(-1,0)^T$, it holds

$$\lambda_1 d_1+\lambda_2 d_2=-1=\inf_{x\in\mathbb{R}}\left[\lambda_1 f_1(x)+\lambda_2 f_2(x)+qg(x)\right],$$

and this implies that $d=(-1,0)^T\in D_L$.

Like in the previous example, let us show now that $d \notin D_{FL}$. If this were not true, then there would exist $\bar{p} = (\bar{p}_1, \bar{p}_2)$, $\bar{q} \geq 0$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int\mathbb{R}^2_+$ such that

$$-\bar{\lambda}_1 \leq -\bar{\lambda}_1 f_1^*(\bar{p}_1) - \bar{\lambda}_2 f_2^*(\bar{p}_2) + \inf_{x \in \mathbb{R}} \left[\left(\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2 \right) x + \bar{q} g(x) \right]. \tag{2. 3}$$

It holds $\bar{p}_2 = 0$, $f_2^*(\bar{p}_2) = 0$ and, from (2. 3),

$$-1 \leq -f_1^*(\bar{p}_1) + \inf_{x \in \mathbb{R}} \left[\bar{p}_1 x + \frac{\bar{q}}{\bar{\lambda}_1} g(x) \right]. \tag{2. 4}$$

But

$$-f_1^*(\bar{p}_1) = \inf_{x \in \mathbb{R}} [f_1(x) - \bar{p}_1 x] = \inf_{x \geq 0} \left[-x^2 - \bar{p}_1 x \right] = -\infty,$$

and this contradicts relation (2. 4). So, $d = (-1, 0)^T \notin D_{FL}$ and, from here, $D_{FL} \subsetneq D_F$, i.e. the inclusion $D_{FL} \subseteq D_L$ may be strict.

PROPOSITION 2.2

(a) It holds $D_F \subseteq D_P$.

(b) It holds $D_L \subseteq D_P$.

Proof

(a) Let be $d = (d_1, ..., d_m)^T \in D_F$. Then there exist $p_i \in \mathbb{R}^n, i = 1, ..., m$, and $\lambda \in int\mathbb{R}^m_+$ such that

$$\begin{aligned} \sum_{i=1}^m \lambda_i d_i &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^* \left(-\sum_{i=1}^m \lambda_i p_i \right) \\ &= -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x \in \mathcal{A}} \left(\sum_{i=1}^m \lambda_i p_i \right)^T x \\ &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \left(\sum_{i=1}^m \lambda_i p_i \right)^T x, \forall x \in \mathcal{A}. \end{aligned}$$

By the inequality of Young, we obtain

$$\sum_{i=1}^m \lambda_i d_i \leq \sum_{i=1}^m \lambda_i f_i(x), \forall x \in \mathcal{A},$$

or, equivalently,

$$\sum_{i=1}^m \lambda_i d_i \leq \inf_{x \in \mathcal{A}} \sum_{i=1}^m \lambda_i f_i(x).$$

This means that $(\lambda, d) \in \mathcal{B}_P$ and $d = h^P(\lambda, d) \in h^P(\mathcal{B}_P) = D_P$.

(b) Let be again $d = (d_1, ..., d_m)^T \in D_L$, $q \underset{K^*}{\geqq} 0$ and $\lambda \in int\mathbb{R}^m_+$ such that

$$\begin{aligned}\sum_{i=1}^m \lambda_i d_i &\leq \inf_{x\in\mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x)\right] \\ &\leq \inf_{x\in\mathcal{A}} \left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x)\right] \\ &\leq \inf_{x\in\mathcal{A}} \sum_{i=1}^m \lambda_i f_i(x).\end{aligned}$$

Like before, $(\lambda, d) \in \mathcal{B}_P$ and $d = h^P(\lambda, d) \in h^P(\mathcal{B}_P) = D_P$.

□

Remark 2.1 Let us consider again the problem in Example 2.2. We show that $d = (-1, 0)^T \in D_P$, but $d = (-1, 0)^T \notin D_F$. For $\lambda = (1, 1)^T$, it holds

$$\lambda_1 d_1 + \lambda_2 d_2 = 1 = \inf_{x\in\mathcal{A}} \left[\lambda_1 f_1(x) + \lambda_2 f_2(x)\right],$$

and, from here, we have $d = (-1, 0)^T \in D_P$.

Assuming that $d \in D_F$, there would exist then $\bar{p} = (\bar{p}_1, \bar{p}_2)$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int\mathbb{R}^2_+$ such that

$$-\bar{\lambda}_1 \leq -\bar{\lambda}_1 f_1^*(\bar{p}_1) - \bar{\lambda}_2 f_2^*(\bar{p}_2) + \inf_{g(x)\leq 0} \left(\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2\right) x. \qquad (2.\ 5)$$

In order to have this fulfilled, we must have $\bar{p}_2 = 0$ and $f_2^*(\bar{p}_2) = 0$. So, (2. 5) becomes

$$-1 \leq -f_1^*(\bar{p}_1) + \inf_{x\in[-1,1]} (\bar{p}_1 x).$$

Again, $-f_1^*(\bar{p}_1) = -\infty$ leads us to a contradiction. So, $d = (-1, 0)^T \notin D_F$, and, from here, $D_F \subsetneq D_P$, i.e. the inclusion $D_F \subseteq D_P$ may be strict.

Remark 2.2 We show now that, for the problem presented in Example 2.1, $d = (1, 0)^T \in D_P$, but $d = (1, 0)^T \notin D_L$. For $\lambda = (1, 1)^T$, it holds

$$\lambda_1 d_1 + \lambda_2 d_2 = 1 = \inf_{x\in\mathcal{A}} \left[\lambda_1 f_1(x) + \lambda_2 f_2(x)\right],$$

and, then, we have $d = (1, 0)^T \in D_P$.

Assuming $d \in D_L$ there would exist $\bar{q} \geq 0$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int\mathbb{R}^2_+$ such that

$$\bar{\lambda}_1 \leq \inf_{x\in\mathbb{R}} \left[\bar{\lambda}_1 f_1(x) + \bar{\lambda}_2 f_2(x) + \bar{q} g(x)\right] = \inf_{x\geq 0} \left[\bar{\lambda}_1 x + \bar{q}(1 - x^2)\right]. \qquad (2.\ 6)$$

Obviously, (2. 6) is true just if $\bar{q} = 0$ and, in this case, it becomes

$$\bar{\lambda}_1 \leq \inf_{x \geq 0} \left[\bar{\lambda}_1 x \right] = 0,$$

which is a contradiction. From here, $d = (1, 0)^T \notin D_L$, and, so, the inclusion $D_L \subseteq D_P$ may be also strict.

So far we have proved that

$$D_{FL} \subsetneq \begin{array}{c} D_F \\ D_L \end{array} \subsetneq D_P. \qquad (2.\ 7)$$

Remark 2.3 In the examples 2.1 and 2.2, one may observe that $(1, 0)^T \in D_F$, $(1, 0)^T \notin D_L$ and $(-1, 0)^T \in D_L$, $(-1, 0)^T \notin D_F$, respectively. This certifies the fact that in the general case it cannot be established any relation of inclusion between the sets D_F and D_L, similar to the ones asserted in the propositions 2.1 and 2.2.

From (1. 1) and (2. 7), we can conclude that, in the general case, it holds, for every $\alpha \in \mathcal{F}$,

$$D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} \subsetneq \begin{array}{c} D_F \\ D_L \end{array} \subsetneq D_P. \qquad (2.\ 8)$$

In chapter 5 of the first part of the study we proved that even if the inclusions in (1. 1) between D_1, $D_\alpha, \alpha \in \mathcal{F}$, and D_{FL} are strict, their sets of maximal elements are equal (cf. (1. 2)). We show now by some counter-examples that this result does not hold for the maximal elements sets of D_{FL}, D_F, D_L and D_P. Actually we show that there is no relation of inclusion between $vmaxD_{FL}$, $vmaxD_F$, $vmaxD_L$ and $vmaxD_P$.

Remark 2.4 Let us consider again the problem in Example 2.1. We showed that $d = (1, 0)^T \notin D_{FL}$ and this means that $d = (1, 0)^T \notin vmaxD_{FL}$. On the other hand, we have $d = (1, 0) \in D_F$ and, moreover, it can be proved that $d = (1, 0) \in vmaxD_F$. In conclusion, $vmaxD_F \nsubseteq vmaxD_{FL}$.

For the same example, let be now $\tilde{d} = (0, 0)^T$. It can be verified that $\tilde{d} \in vmaxD_{FL}$, which means that $\tilde{d} = (0, 0)^T \in D_{FL} \subseteq D_F$. But, because $d = (1, 0)^T \in D_F$, it follows that $\tilde{d} \notin vmaxD_F$. So, $vmaxD_{FL} \nsubseteq vmaxD_F$.

Example 2.3 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let be $f_1, f_2 : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$ such that

$$f_1(x) = \begin{cases} x^2, & \text{if} \quad x \in [0, +\infty), \\ +\infty, & \text{otherwise,} \end{cases}$$

$$f_2(x) = 0,$$

and

$$g(x) = \begin{cases} 1 - x^2, & \text{if} \quad x \in [0, +\infty), \\ 1, & \text{otherwise.} \end{cases}$$

For $q = 1$, $\lambda = (1,1)^T$ and $d = (1,0)^T$, we have $(q, \lambda, d) \in \mathcal{B}_L$ and $d \in D_L$. Moreover, $d \in vmaxD_L$. It can be also verified that $d \notin D_{FL}$ and, from here, $d = (1,0)^T \notin vmaxD_{FL}$. This means that $vmaxD_L \nsubseteq vmaxD_{FL}$.

On the other hand, it can be shown that $\tilde{d} = (0,0)^T \in vmaxD_{FL}$. But, Proposition 5.1 (b) implies that $\tilde{d} = (0,0)^T \in D_{FL} \subseteq D_L$. Obviously, $\tilde{d} \notin vmaxD_L$, otherwise it would contradict the maximality of $d = (1,0)^T$ in D_L. So, $vmaxD_{FL} \nsubseteq vmaxD_L$.

Remark 2.5 For the problem presented in Example 2.2, we have that $d = (-1,0)^T \in D_P$ and, moreover, $d \in vmaxD_P$. Because $d \notin D_F$, we also have that $d \notin vmaxD_F$. In conclusion, $vmaxD_P \nsubseteq vmaxD_F$.

In order to show that $vmaxD_F \nsubseteq vmaxD_P$, let us consider for $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, the functions $f_1, f_2 : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$ such that

$$f_1(x) = \begin{cases} x, & \text{if} \quad x \in (0, +\infty), \\ +\infty, & \text{otherwise,} \end{cases}$$

$$f_2(x) = 0, g(x) = x.$$

It is easy to verify that for $p = (0,0)$, $\lambda = (1,1)^T$ and $d = (0,0)^T$ the element (p, λ, d) belongs to $\mathcal{B}_F$, which gives us that $d = (0,0)^T \in D_F$. Moreover, $d = (0,0)^T \in vmaxD_F$.

By Proposition 2.2 (a) we have $d = (0,0)^T \in D_P$. But, for $\lambda = (1,1)^T$ and $\tilde{d} = (1,0)^T$, $(\lambda, \tilde{d}) \in \mathcal{B}_P$ and, from here, $\tilde{d} = (1,0)^T \in D_P$. So, $d = (0,0)^T \notin vmaxD_P$ and $vmaxD_F \nsubseteq vmaxD_P$.

Remark 2.6 Considering again the problem in Example 2.1, we have $d = (1,0)^T \in D_P$, and $d \notin D_L$. From here, $d \notin vmaxD_L$. Moreover, $d = (1,0)^T \in vmaxD_P$, which shows that $vmaxD_P \nsubseteq vmaxD_L$.

On the other hand, $\tilde{d} = (0,0)^T \in vmaxD_L$ and, by Proposition 2.2 (b), $\tilde{d} \in D_L \subseteq D_P$. Because, $d = (1,0)^T \in D_P$, it follows $\tilde{d} = (0,0)^T \notin vmaxD_P$. So, $vmaxD_L \nsubseteq vmaxD_P$.

In the general case we can conclude now that between the sets of maximal elements of D_{FL}, D_F, D_L and D_P a relation of equality or any relation of inclusion does not hold. In this situation, the only valid relation is the relation of inclusion (2. 7).

3 Conditions for the equality of the sets D_{FL}, D_F, D_L and D_P

Assuming the conditions (A_f), (A_g) and (A_{CQ}) are satisfied, we prove in this section that relation (2. 7) becomes an equality.

THEOREM 3.1 *Let the assumptions* (A_g) *and* (A_{CQ}) *be fulfilled. Then it holds* $D_{FL} = D_F$.

Proof By Proposition 2.1 (a) we have that $D_{FL} \subseteq D_F$.

Now let be $d \in D_F$. Then there exist $p = (p_1, ..., p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$ and $\lambda \in int\mathbb{R}^m_+$ such that $(p, \lambda, d) \in \mathcal{B}_F$, i.e.

$$\begin{aligned} \sum_{i=1}^{m} \lambda_i d_i &\leq -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^*\left(-\sum_{i=1}^{m} \lambda_i p_i\right) \\ &= -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) + \inf_{x\in\mathcal{A}} \left(\sum_{i=1}^{m} \lambda_i p_i\right)^T x. \end{aligned} \tag{3. 1}$$

But, (A_g) and (A_{CQ}) being fulfilled, it follows that for the scalar problem

$$(P_{\lambda p}) \ \inf_{x\in\mathcal{A}} \left(\sum_{i=1}^{m} \lambda_i p_i\right)^T x,$$

$$\mathcal{A} = \left\{ x \in \mathbb{R}^n : g(x) \underset{K}{\leqq} 0 \right\},$$

the strong duality holds (cf. Theorem 2.1 in part one). One of its dual problems (cf. [10]) is

$$(D_L^{\lambda p}) \ \sup_{q \underset{K^*}{\geqq} 0} \inf_{x\in\mathbb{R}^n} \left[\left(\sum_{i=1}^{m} \lambda_i p_i\right)^T x + q^T g(x) \right].$$

The strong duality theorem assures the existence of an element $\bar{q} \underset{K^*}{\geqq} 0$, that is a solution to $(D_L^{\lambda p})$ such that

$$\inf_{x\in\mathcal{A}} \left(\sum_{i=1}^{m} \lambda_i p_i\right)^T x = \inf(P_{\lambda p}) = \max(D_L^{\lambda p}) = \inf_{x\in\mathbb{R}^n} \left[\left(\sum_{i=1}^{m} \lambda_i p_i\right)^T x + \bar{q}^T g(x) \right]. \tag{3. 2}$$

From (3. 1) and (3. 2), we have

$$\begin{aligned}\sum_{i=1}^m \lambda_i d_i &\leq -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x\in\mathcal{A}} \left(\sum_{i=1}^m \lambda_i p_i\right)^T x \\ &= -\sum_{i=1}^m \lambda_i f_i^*(p_i) + \inf_{x\in\mathbb{R}^n} \left[\left(\sum_{i=1}^m \lambda_i p_i\right)^T x + \bar{q}^T g(x)\right] \\ &= -\sum_{i=1}^m \lambda_i f_i^*(p_i) - (\bar{q}^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i\right).\end{aligned}$$

This means that $(p, \bar{q}, \lambda, d) \in \mathcal{B}_{FL}$ and $d = h^{FL}(p, \bar{q}, \lambda, d) \in h^{FL}(\mathcal{B}_{FL}) = D_{FL}$.
□

Remark 3.1 For the problem presented in Example 2.2, one may observe that (A_g) and (A_{CQ}) are fulfilled, $d = (-1, 0)^T \in D_P$, but $d = (-1, 0)^T \notin D_{FL} = D_F$. We conclude that just these two assumptions are not sufficient to have equality between all the sets in (2. 7).

THEOREM 3.2 *Let the assumptions (A_f) and (A_g) be fulfilled. Then it holds $D_{FL} = D_L$.*

Proof By Proposition 2.1 (b) we have that $D_{FL} \subseteq D_L$.

Let be $d \in D_L$. Then there exist $q \underset{K^*}{\geqq} 0$ and $\lambda \in int\mathbb{R}^m_+$ such that $(p, \lambda, d) \in \mathcal{B}_L$, i.e.

$$\sum_{i=1}^m \lambda_i d_i \leq \inf_{x\in\mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x)\right]. \tag{3. 3}$$

Let us consider the function $k : \mathbb{R}^n \to \overline{\mathbb{R}}$, $k(x) = \sum_{i=1}^m \lambda_i f_i(x)$. We have $domk = \bigcap_{i=1}^m domf_i$ and, from (A_f), it follows $ri(domk) = \bigcap_{i=1}^m ri(domf_i) \neq \emptyset$ (cf. [11]). Let us also notice that $dom(q^T g) = \mathbb{R}^n$ and then, by Theorem 31.1 in [12], there exists $\tilde{p} \in \mathbb{R}^n$ such that

$$\inf_{x\in\mathbb{R}^n} \left[\sum_{i=1}^m \lambda_i f_i(x) + q^T g(x)\right] = -\left(\sum_{i=1}^m \lambda_i f_i\right)^* (\tilde{p}) + \inf_{x\in\mathbb{R}^n} \left[\tilde{p}^T x + q^T g(x)\right]. \tag{3. 4}$$

On the other hand, from Theorem 16.4 in [12], there exist $\bar{p}_i \in \mathbb{R}^n, i = 1, ..., m$, such that $\tilde{p} = \sum_{i=1}^m \lambda_i \bar{p}_i$ and

$$\left(\sum_{i=1}^m \lambda_i f_i\right)^* (\tilde{p}) = \sum_{i=1}^m \lambda_i f_i^*(\bar{p}_i).$$

From (3. 3) and (3. 4) we obtain

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i d_i &\leq \inf_{x \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x) \right] \\
&= -\left(\sum_{i=1}^{m} \lambda_i f_i \right)^* (\tilde{p}) + \inf_{x \in \mathbb{R}^n} \left[\tilde{p}^T x + q^T g(x) \right] \\
&= -\sum_{i=1}^{m} \lambda_i f_i^*(\bar{p}_i) + \inf_{x \in \mathbb{R}^n} \left[\left(\sum_{i=1}^{m} \lambda_i \bar{p}_i \right)^T x + q^T g(x) \right] \\
&= -\sum_{i=1}^{m} \lambda_i f_i^*(\bar{p}_i) - (q^T g)^* \left(-\sum_{i=1}^{m} \lambda_i \bar{p}_i \right).
\end{aligned}$$

This means that, for $\bar{p} = (\bar{p}_1, ..., \bar{p}_m)$, $(\bar{p}, q, \lambda, d) \in \mathcal{B}_{FL}$ and $d = h^{FL}(\bar{p}, q, \lambda, d) \in h^{FL}(\mathcal{B}_{FL}) = D_{FL}$. □

Example 3.1 For $m = 2, n = 2, k = 1$, $K = \mathbb{R}_+$, we consider the functions $f_1, f_2 : \mathbb{R}^2 \to \overline{\mathbb{R}}$, $g : \mathbb{R}^2 \to \mathbb{R}$, introduced by

$$f_1(x_1, x_2) = \begin{cases} x_2 & \text{if} \quad x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$X = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, \begin{array}{ll} 3 \leq x_2 \leq 4 & \text{for} \quad x_1 = 0 \\ 1 < x_2 \leq 4 & \text{for} \quad x_1 > 0 \end{array} \right\},$$

$$f_2(x_1, x_2) = 0 \text{ and } g(x_1, x_2) = x_1.$$

It can be observed that (A_f) and (A_g) are fulfilled, $d = (3, 0)^T \in D_P$, but $d = (3, 0)^T \notin D_{FL} = D_L$. Like in Remark 3.1, we can conclude that just the assumptions (A_f) and (A_g) are also not sufficient to have equality between all the sets in (2. 7). The next theorem shows when this really happens.

THEOREM 3.3 *Let the assumptions (A_f), (A_g) and (A_{CQ}) be fulfilled. Then it holds $D_{FL} = D_L = D_F = D_P$.*

Proof By the Theorems 3.1 and 3.2, we have $D_{FL} = D_L = D_F$. Let us prove now that $D_F = D_P$.

Proposition 2.2 (a) gives us that $D_F \subseteq D_P$. It remains to prove just that the reversed inclusion also holds.

Let be $d \in D_P$. Then there exists $\lambda \in int\mathbb{R}^m_+$ such that $(\lambda, d) \in \mathcal{B}_P$, i.e.

$$\sum_{i=1}^{m} \lambda_i d_i \leq \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x). \tag{3. 5}$$

Moreover, by (3. 5), and, since (A_f), (A_g) and (A_{CQ}) are true, it follows that the assumptions of the strong duality Theorem 2.1, presented in the first part of this study, are fulfilled. Considering for the primal problem

$$(P_\lambda) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x)$$

its dual

$$(D_F^\lambda) \quad \sup_{p_i \in \mathbb{R}^n, i=1,...,m} \left\{ -\sum_{i=1}^{m} \lambda_i f_i^*(p_i) - \chi_{\mathcal{A}}^* \left(-\sum_{i=1}^{m} \lambda_i p_i \right) \right\},$$

the last one has a solution. Then there exist $\bar{p}_i \in \mathbb{R}^n, i = 1, ..., m$, such that

$$\inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x) = -\sum_{i=1}^{m} \lambda_i f_i^*(\bar{p}_i) - \chi_{\mathcal{A}}^* \left(-\sum_{i=1}^{m} \lambda_i \bar{p}_i \right). \tag{3. 6}$$

From (3. 5) and (3. 6) we have

$$\sum_{i=1}^{m} \lambda_i d_i \leq \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x) = -\sum_{i=1}^{m} \lambda_i f_i^*(\bar{p}_i) - \chi_{\mathcal{A}}^* \left(-\sum_{i=1}^{m} \lambda_i \bar{p}_i \right),$$

which actually means that, for $\bar{p} = (\bar{p}_1, ..., \bar{p}_m)$, $(\bar{p}, \lambda, d) \in \mathcal{B}_F$ and $d = h^F(\bar{p}, \lambda, d)$ $\in h^F(\mathcal{B}_F) = D_F$. □

As a consequence of this last theorem we can affirm that, if (A_f), (A_g) and (A_{CQ}) are fulfilled, from (1. 1) and (2. 7) we have, for every $\alpha \in \mathcal{F}$,

$$D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} = D_L = D_F = D_P. \tag{3. 7}$$

This last relation, together with (1. 2), gives us for every $\alpha \in \mathcal{F}$,

$$vmax D_1 = vmax D_\alpha = vmax D_{FL} = vmax D_F = vmax D_L = vmax D_P, \tag{3. 8}$$

provided that (A_f), (A_g) and (A_{CQ}) hold.

4 Nakayama multiobjective duality

One of the first theories concerning duality for convex multiobjective problems was developed by Nakayama and has been described in [4], [5] and [13]. If we consider this theory for the primal problem (P), the dual introduced there becomes

$$(D_N) \quad \underset{(U,y) \in \mathcal{B}_N}{\text{v-max}} \; h^N(U, y),$$

$$h^N(U,y)=\begin{pmatrix} h_1^N(U,y) \\ \vdots \\ h_m^N(U,y)\end{pmatrix}=\begin{pmatrix} y_1 \\ \vdots \\ y_m\end{pmatrix},$$

with

$$h_j^N(U,y)=y_j, j=1,...,m,$$

the dual variables

$$U\in\mathcal{U}, y=(y_1,\ldots,y_m)^T\in\mathbb{R}^m,$$

$$\mathcal{U}=\{U : U \text{ is a } m\times k \text{ matrix such that } U\cdot K\subseteq\mathbb{R}_+^m\},$$

and the set of constraints

$$\mathcal{B}_N=\{(U,y): U\in\mathcal{U} \text{ and there is no } x\in\mathbb{R}^n, \text{ such that } y\gneqq f(x)+Ug(x)\}.$$

If $U=\begin{pmatrix} q_1^T \\ \vdots \\ q_m^T\end{pmatrix}\in\mathcal{U}, q_i\in\mathbb{R}^k, i=1,...,m,$ then for every $k\in K$, it must hold $(q_1^Tk,...,q_m^Tk)^T\in\mathbb{R}_+^m$. From here, for $i=1,...,m$, $q_i^Tk\geq 0, \forall k\in K$, which actually means that $q_i\in K^*$, for $i=1,...,m$. By this observation the dual (D_N) can be written, equivalently, in the following way

$$(D_N)\quad \underset{(q_1,\ldots,q_m,y)\in\mathcal{B}_N}{\text{v-max}}\, h^N(q_1,...,q_m,y),$$

$$h^N(q_1,...,q_m,y)=\begin{pmatrix} h_1^N(q_1,...,q_m,y) \\ \vdots \\ h_m^N(q_1,...,q_m,y)\end{pmatrix}=\begin{pmatrix} y_1 \\ \vdots \\ y_m\end{pmatrix},$$

with

$$h_j^N(q_1,...,q_m,y)=y_j, j=1,...,m,$$

the dual variables

$$q_i\in\mathbb{R}^k, i=1,...,m, y=(y_1,\ldots,y_m)^T\in\mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_N=\{(q_1,...,q_m,y): \quad q_i\underset{K^*}{\geqq}0, i=1,...,m, \text{ and there is no } x\in\mathbb{R}^n$$
$$\text{such that } y\gneqq f(x)+(q_1^Tg(x),...,q_m^Tg(x))^T\}.$$

The proofs of the next two theorems had been given in [4].

THEOREM 4.1 (weak duality for (D_N)) *There is no $x\in\mathcal{A}$ and no element $(q_1,...,q_m,y)\in\mathcal{B}_N$ fulfilling $h^N(q_1,...,q_m,y)\underset{\mathbb{R}_+^m}{\geqq}f(x)$ and $h^N(q_1,...,q_m,y)\neq f(x)$.*

THEOREM 4.2 (strong duality for (D_N)) *Assume that* (A_f), (A_g) *and* (A_{CQ}) *are fulfilled. If* $\bar{x}$ *is a properly efficient solution to* (P), *then there exists an efficient solution* $(\bar{q}_1, ..., \bar{q}_m, \bar{y}) \in \mathcal{B}_N$ *to the dual* (D_N) *and strong duality* $f(\bar{x}) = h^N(\bar{q}_1, ..., \bar{q}_m, \bar{y}) = \bar{y}$ *holds.*

In order to relate the dual (D_N) to the duals considered in the previous chapters, let us denote by $D_N := h^N(\mathcal{B}_N) \subseteq \mathbb{R}^m$ the image set of the Nakayama multiobjective dual.

PROPOSITION 4.3 *It holds* $D_L \subseteq D_N$.

Proof Let be $d = (d_1, ..., d_m)^T \in D_L$. Then there exist $q \underset{K^*}{\geqq} 0$ and $\lambda \in int\mathbb{R}^m_+$ such that $(q, \lambda, d) \in \mathcal{B}_L$, i.e.

$$\sum_{i=1}^{m} \lambda_i d_i \leq \inf_{x \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x) \right]. \tag{4. 1}$$

Let be, for $i = 1, ..., m$, $\bar{q}_i := \frac{1}{\sum\limits_{i=1}^{m} \lambda_i} q \underset{K^*}{\geqq} 0$.

We show now that $(\bar{q}_1, ..., \bar{q}_m, d) \in \mathcal{B}_N$. If this does not happen, then there exists $x' \in \mathbb{R}^n$ such that $d \geqq\hspace{-0.9em}\neq f(x') + (\bar{q}_1^T g(x'), ..., \bar{q}_m^T g(x'))^T$. It follows that $\sum\limits_{i=1}^{m} \lambda_i d_i > \sum\limits_{i=1}^{m} \lambda_i f_i(x') + q^T g(x')$, but this contradicts the inequality in (4. 1). From here we obtain that $(\bar{q}_1, ..., \bar{q}_m, d) \in \mathcal{B}_N$ and $d = h^N(\bar{q}_1, ..., \bar{q}_m, d) \in h^N(\mathcal{B}_N) = D_N$.
□

Example 4.1 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that $f_1(x) = x$, $f_2(x) = 1$ and $g(x) = -1$.

Considering $q_1 = q_2 = 0$ and $d = (1, 0)^T$, it is obvious that there is no $x \in \mathbb{R}^n$ such that

$$d = (1, 0)^T \geqq\hspace{-0.9em}\neq f(x) + (q_1 g(x), q_2 g(x))^T = (x, 1)^T.$$

This means that $d = (1, 0)^T \in D_N$.

On the other hand, we have $d \notin D_L$ and, so, $D_L \subsetneq D_N$, i.e. the inclusion $D_L \subseteq D_N$ may be strict.

Example 4.2 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let now be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that

$$f_1(x) = f_2(x) = x,$$

and

$$g(x) = \begin{cases} 1 - x^2, & \text{if} \quad x \in [0, +\infty), \\ 1, & \text{otherwise.} \end{cases}$$

The element $d = (1,1)^T$ belongs to D_F and D_P. We show now that $d \notin D_N$. If this were not true, then there would exist $\bar{q}_1, \bar{q}_2 \geq 0$ such that $(\bar{q}_1, \bar{q}_2, d) \in D_N$, or, equivalently,

$$d = (1,1)^T \gneqq (x + q_1 g(x), x + q_2 g(x))^T, \tag{4. 2}$$

would not hold for any $x \in \mathbb{R}$. But, for $i = 1, 2$, $\lim\limits_{x \to -\infty} (x + q_i g(x)) = -\infty$, which means that there exists $x' \in \mathbb{R}$ such that $x + q_1 g(x) < 1$ and $x + q_2 g(x) < 1$. This contradicts (4. 2). The conclusion is that, in general, $D_F \nsubseteq D_N$ and $D_P \nsubseteq D_N$.

Remark 4.1 For the problem introduced in Example 4.1, let us notice that (A_f), (A_g) and (A_{CQ}) are fulfilled. By Theorem 3.3, we have $D_L = D_F = D_P$, and, so, $d = (1,0)^T$ neither belongs to D_F, nor to D_P. But, we have shown that $d = (1,0)^T \in D_N$. We conclude that $D_N \nsubseteq D_F$ and $D_N \nsubseteq D_P$.

The last results allow us to extend the relation (2. 8) by introducing the set D_N. We get, for every $\alpha \in \mathcal{F}$,

$$D_1 \cap \mathbb{R}^m \subsetneqq D_\alpha \cap \mathbb{R}^m \subsetneqq D_{FL} \subsetneqq \begin{array}{c} D_F \subsetneqq D_P \\ D_L \subsetneqq \begin{array}{c} D_P \\ D_N \end{array} \end{array}. \tag{4. 3}$$

If (A_f), (A_g) and (A_{CQ}) are fulfilled, then from (3. 7) and Proposition 4.3 this relation becomes, for every $\alpha \in \mathcal{F}$,

$$D_1 \cap \mathbb{R}^m \subsetneqq D_\alpha \cap \mathbb{R}^m \subsetneqq D_{FL} = D_L = D_F = D_P \subsetneqq D_N. \tag{4. 4}$$

We remind that, if (A_f), (A_g) and (A_{CQ}) are fulfilled, then the maximal elements sets of the first six duals are equal (cf. (3. 8)). The following example shows that, even if the three assumptions are fulfilled, between $vmaxD_N$ and $vmaxD_P$ does not exist any relation of inclusion.

Example 4.3 For $m = 2, n = 2, k = 1$, $K = \mathbb{R}$, let be $f_1, f_2 : \mathbb{R}^2 \to \overline{\mathbb{R}}$, $g : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f_1(x_1, x_2) = \begin{cases} x_1 & \text{if} \quad x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$f_2(x_1, x_2) = \begin{cases} x_2 & \text{if} \quad x \in X, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$X = \left\{ x = (x_1, x_2)^T \in \mathbb{R}^2 : x_1, x_2 \geq 0 \text{ such that } x_2 > 0, \text{ if } x_1 \in [0,1) \right\},$$

and

$$g(x_1, x_2) = 0.$$

We notice that (A_f), (A_g) and (A_{CQ}) are fulfilled.

For $q_1 = q_2 = 0 \in K^* = \{0\}$ and $d = (1,0)^T$ it does not exist $x = (x_1, x_2)^T \in X$ such that $(1,0)^T \gneqq (x_1, x_2)^T$. This means that $(0,0,d) \in \mathcal{B}_N$ and $d \in D_N$.

Let us assume now that there exist $\bar{q}_1, \bar{q}_2 \in K^*$ and $\bar{d} \in \mathbb{R}^2$ such that $(\bar{q}_1, \bar{q}_2, \bar{d}) \in \mathcal{B}_N$ and $\bar{d} \gneqq d = (1,0)$. We have then $\bar{q}_1 = \bar{q}_2 = 0$ and for $\bar{x} = (1,0)^T \in X$ holds

$$(f_1(\bar{x}) + \bar{q}_1 g(\bar{x}), f_2(\bar{x}) + \bar{q}_2 g(\bar{x}))^T = (\bar{x}_1, \bar{x}_2)^T = (1,0)^T = d \lneqq \bar{d}.$$

It follows that $(\bar{q}_1, \bar{q}_2, \bar{d}) \notin \mathcal{B}_N$, which means that $d = (1,0)^T \in vmaxD_N$.

Let us assume now that $d \in D_P = D_L$. Then there exists $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in intR^2_+$ such that

$$\bar{\lambda}_1 = \bar{\lambda}_1 d_1 + \bar{\lambda}_2 d_2 \leq \inf_{x \in \mathcal{A}} \left[\bar{\lambda}_1 f_1(x) + \bar{\lambda}_2 f_2(x)\right] = \inf_{x \in X} \left(\bar{\lambda}_1 x_1 + \bar{\lambda}_2 x_2\right).$$

On the other hand, for $n \in \mathbb{N}^*$, $(\frac{1}{n}, \frac{1}{n})^T \in X$, it holds

$$\bar{\lambda}_1 \leq \bar{\lambda}_1 \frac{1}{n} + \bar{\lambda}_2 \frac{1}{n}, \forall n \in \mathbb{N}^*.$$

If $n \to +\infty$, then we must have $\bar{\lambda}_1 \leq 0$ and this is a contradiction. From here, $d = (1,0)^T \notin D_P$ and, obviously, $d = (1,0)^T \notin vmaxD_P$. In conclusion, $vmaxD_N \nsubseteq vmaxD_P$.

On the other hand, for $\lambda_1 = \lambda_2 = 1$ and $\tilde{d} = (0,0)^T$, we have $\tilde{d} = (0,0)^T \in D_P$ and, moreover, $\tilde{d} = (0,0)^T \in vmaxD_P$.

By Proposition 4.3, $\tilde{d} = (0,0)^T \in D_P \subseteq D_N$ and, because $d = (1,0)^T \in D_N$, it follows $\tilde{d} = (0,0)^T \notin vmaxD_N$. So, $vmaxD_P \nsubseteq vmaxD_N$.

Remark 4.2 In Proposition 5 in [5], Nakayama gives some necessary conditions to have

$$vminP = vmaxD_L = vmaxD_N, \tag{4. 5}$$

where $vminP$ represents the set of the Pareto-efficient solutions of the problem (P).

In order to have (4. 5), this proposition claims that (A_f), (A_g) and (A_{CQ}) must be fulfilled, the problem (P) must have at least one Pareto-efficient solution, all these Pareto-efficient solutions must be properly efficient and the set

$$G = \{(z,y) \in \mathbb{R}^m \times \mathbb{R}^k : \exists x \in X, \text{ s.t. } y \underset{\mathbb{R}^m_+}{\geqq} f(x), z \underset{K}{\geqq} g(x)\}$$

must be closed.

5 Wolfe multiobjective duality

The next multiobjective dual problem, that we treat in this paper is the Wolfe multiobjective dual also well-known in the literature. First it was introduced in the differentiable case by Weir in [6]. Its formulation for the nondifferentiable case can be found in [7] and it has been inspired by the Wolfe scalar dual problem for nondifferentiable optimization problems (cf. [14]).

The Wolfe multiobjective dual problem has the following formulation

$$(D_W) \quad \underset{(x,q,\lambda)\in\mathcal{B}_W}{\text{v-max}} h^W(x,q,\lambda),$$

$$h^W(x,q,\lambda) = \begin{pmatrix} h_1^W(x,q,\lambda) \\ \vdots \\ h_m^W(x,q,\lambda) \end{pmatrix},$$

with

$$h_j^W(x,q,\lambda) = f_j(x) + q^T g(x), j = 1, ..., m,$$

the dual variables

$$x \in \mathbb{R}^n, q \in \mathbb{R}^k, \lambda = (\lambda_1, \dots, \lambda_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_W = \{(x,q,\lambda) : \quad x \in \mathbb{R}^n, \lambda = (\lambda_1, \dots, \lambda_m)^T \in int\mathbb{R}^m_+, \sum_{i=1}^m \lambda_i = 1,$$
$$q \underset{K^*}{\geqq} 0, 0 \in \partial\left(\sum_{i=1}^m \lambda_i f_i\right)(x) + \partial(q^T g)(x)\}.$$

Here, for a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\partial f(\bar{x}) = \{x^* \in \mathbb{R}^n : f(x) - f(\bar{x}) \geq < x^*, x - \bar{x} > \forall x \in \mathbb{R}^n\}$ represents the subdifferential of f at the point $\bar{x} \in \mathbb{R}^n$.

The following two theorems represent the weak and strong duality theorems. Their proofs can be derived from [6] and [7].

THEOREM 5.1 (weak duality for (D_W)) *There is no $x \in \mathcal{A}$ and no element $(y,q,\lambda) \in \mathcal{B}_W$ fulfilling $h^W(y,q,\lambda) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^W(y,q,\lambda) \neq f(x)$.*

THEOREM 5.2 (strong duality for (D_W)) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists $\bar{q} \underset{K^*}{\geqq} 0$ and $\bar{\lambda} \in int\mathbb{R}^m_+$ such that $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_W$ is a properly efficient solution to the dual (D_W) and strong duality $f(\bar{x}) = h^W(\bar{x}, \bar{q}, \bar{\lambda})$ holds.*

Let us consider now $D_W := h^W(\mathcal{B}_W) \subseteq \mathbb{R}^m$. We study, in the general case, the relations between D_W and the image sets of the duals introduced so far.

PROPOSITION 5.3 *It holds $D_W \subseteq D_L$.*

Proof Let be $d = (d_1, ..., d_m)^T \in D_W$. Then there exists $(x, q, \lambda) \in \mathcal{B}_W$ such that $d = h^W(x, q, \lambda) = f(x) + (q^T g(x), ..., q^T g(x))^T$.

From here, it follows

$$\sum_{i=1}^{m} \lambda_i d_i = \sum_{i=1}^{m} \lambda_i f_i(x) + \left(\sum_{i=1}^{m} \lambda_i\right) q^T g(x) = \sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x). \qquad (5.\ 1)$$

On the other hand, because of $(x, q, \lambda) \in \mathcal{B}_W$, we have

$$0 \in \partial \left(\sum_{i=1}^{m} \lambda_i f_i\right)(x) + \partial(q^T g)(x),$$

which implies that

$$\sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x) \le \inf_{u \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(u) + q^T g(u)\right]. \qquad (5.\ 2)$$

From (5. 1) and (5. 2) we obtain

$$\sum_{i=1}^{m} \lambda_i d_i = \sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x) \le \inf_{u \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(u) + q^T g(u)\right],$$

which gives us $(q, \lambda, d) \in \mathcal{B}_L$ and $d = h^L(q, \lambda, d) \in h^L(\mathcal{B}_L) = D_L$. □

Example 5.1 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that $f_1(x) = f_2(x) = x^2$ and $g(x) = 0$.

For $q = 0 \in K^* = \{0\}$, $\lambda = (1, 1)^T$ and $d = (-1, -1)^T$ we have

$$\lambda_1 d_1 + \lambda_2 d_2 = -2 < 0 = \inf_{x \in \mathbb{R}} \left[x^2 + x^2\right] = \inf_{x \in \mathbb{R}} \left[\lambda_1 f_1(x) + \lambda_2 f_2(x) + q^T g(x)\right],$$

which implies that $d = (-1, -1)^T \in D_L$.

We will show now that $d = (1, -1)^T \notin D_W$. If this were not true, then there would exists $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_W$, with $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int\mathbb{R}^2_+$, $\bar{\lambda}_1 + \bar{\lambda}_2 = 1$, $\bar{q} \in K^* = \{0\}$ such that

$$d = (-1, -1)^T = (f_1(\bar{x}) + \bar{q} g(\bar{x}), f_2(\bar{x}) + \bar{q} g(\bar{x}))^T = (\bar{x}^2, \bar{x}^2)^T.$$

But, this is a contradiction and, so, $D_W \subsetneq D_L$, i.e. the inclusion may be strict. Moreover, by (4. 3), we have $D_P \nsubseteq D_W$ and $D_N \nsubseteq D_W$.

Example 5.2 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that $f_1(x) = f_2(x) = 0$ and $g(x) = 0$.

For $p = (0, 0)$, $q = 0 \in K^* = \{0\}$, $\lambda = \left(\frac{1}{2}, \frac{1}{2}\right)^T$, $t = (1, -1)^T$, it holds $d = (1, -1)^T \in D_1$. Otherwise, $d = (1, -1)^T \notin D_W$. So, $D_1 \cap \mathbb{R}^m \nsubseteq D_W$, whence,

$D_\alpha \cap \mathbb{R}^m \nsubseteq D_W$, $\alpha \in \mathcal{F}$, $D_{FL} \nsubseteq D_W$ and $D_F \nsubseteq D_W$.

Example 5.3 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that $f_1(x) = x^2 - 1$, $f_2(x) = 1 - x^2$ and $g(x) = 0$.

For $x = 0$, $q = 0$ and $\lambda = (\frac{1}{2}, \frac{1}{2})^T$ it holds $(x, q, \lambda) \in \mathcal{B}_W$ and $d = (-1, 1)^T = (f_1(0), f_2(0))^T \in D_W$.

Let us show now that $d \notin D_F$. If this were not true, then there would exist $\bar{p} = (\bar{p}_1, \bar{p}_2)$, $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int\mathbb{R}^2_+$ such that $(\bar{p}, \bar{\lambda}, d) \in \mathcal{B}_F$, i.e.

$$-\bar{\lambda}_1 + \bar{\lambda}_2 \leq -\bar{\lambda}_1 f_1^*(\bar{p}_1) - \bar{\lambda}_2 f_2^*(\bar{p}_2) + \inf_{x \in \mathbb{R}} \left(\bar{\lambda}_1 \bar{p}_1 + \bar{\lambda}_2 \bar{p}_2\right) x. \qquad (5.\ 3)$$

But, $f_2^*(\bar{p}_2) = \sup_{x \in \mathbb{R}} \{\bar{p}_2 x + x^2 - 1\} = +\infty$, and this contradicts the inequality in (5. 3). In conclusion, $D_W \nsubseteq D_F$, and, so, $D_W \nsubseteq D_{FL}$, $D_W \nsubseteq D_\alpha \cap \mathbb{R}^m$, $\alpha \in \mathcal{F}$, and $D_W \nsubseteq D_1 \cap \mathbb{R}^m$ (cf. (4. 3)).

By (4. 3), Proposition 5.3 and examples 5.1-5.3, we obtain in the general case the following scheme for every $\alpha \in \mathcal{F}$

$$\begin{array}{ccc} & & D_F \subsetneqq D_P \\ D_1 \cap \mathbb{R}^m \subsetneqq D_\alpha \cap \mathbb{R}^m \subsetneqq D_{FL} \subsetneqq & & D_L \subsetneqq \begin{array}{c} D_P \\ D_N \end{array} \\ & & D_W \subsetneqq D_L \subsetneqq \begin{array}{c} D_P \\ D_N \end{array} \end{array} . \qquad (5.\ 4)$$

For the last part of this section, let us assume that (A_f), (A_g) and (A_{CQ}) are fulfilled.

PROPOSITION 5.4 *It holds $D_W \subseteq D_1 \cap \mathbb{R}^m$.*

Proof Let be $d = (d_1, ..., d_m)^T \in D_W$. Then there exists $(x, q, \lambda) \in \mathcal{B}_W$ such that $d = h^W(x, q, \lambda)$. Because of

$$0 \in \partial \left(\sum_{i=1}^m \lambda_i f_i \right)(x) + \partial(q^T g)(x) = \sum_{i=1}^m \lambda_i \partial f_i(x) + \partial(q^T g)(x),$$

it follows that there exist $p_i \in \mathbb{R}^n, i = 1, ..., m$ such that $p_i \in \partial f_i(x), i = 1, ..., m$, and $-\sum_{i=1}^m \lambda_i p_i \in \partial(q^T g)(x)$. As a consequence follows (cf. [9])

$$f_i^*(p_i) + f_i(x) = p_i^T x, i = 1, ..., m, \qquad (5.\ 5)$$

and

$$(q^T g)^* \left(-\sum_{i=1}^m \lambda_i p_i \right) + q^T g(x) = \left(-\sum_{i=1}^m \lambda_i p_i \right)^T x. \qquad (5.\ 6)$$

Defining, for $j = 1, ..., m$,

$$t_j := p_j^T x + \left(- \sum_{i=1}^{m} \lambda_i p_i \right)^T x \in \mathbb{R},$$

then $\sum\limits_{i=1}^{m} \lambda_i t_i = 0$ and this means that $(p, q, \lambda, t) \in \mathcal{B}_1$, for $p = (p_1, ..., p_m)$. On the other hand, from (5. 5) and (5. 6) we have, for $j = 1, ..., m$,

$$\begin{aligned} h_j^1(p, q, \lambda, t) &= -f_j^*(p_j) - (q^T g)^* \left(-\frac{1}{\sum\limits_{i=1}^{m} \lambda_i} \sum_{i=1}^{m} \lambda_i p_i \right) + t_j \\ &= -f_j^*(p_j) - (q^T g)^* \left(- \sum_{i=1}^{m} \lambda_i p_i \right) + t_j \\ &= f_j(x) - p_j^T x + q^T g(x) - \left(- \sum_{i=1}^{m} \lambda_i p_i \right)^T x + t_j \\ &= f_j(x) + q^T g(x) = d_j. \end{aligned}$$

In conclusion, $d = h^1(p, q, \lambda, t) \in h^1(\mathcal{B}_1) = D_1$. □

Remark 5.1 For the problem described in Example 5.2 the assumptions (A_f), (A_g) and (A_{CQ}) are fulfilled and $d = (1, -1)^T \in D_1 \cap \mathbb{R}^2$, but $d \notin D_W$. This means that even in this case the inclusion $D_W \subseteq D_1 \cap \mathbb{R}^m$ may be strict.

So, if (A_f), (A_g) and (A_{CQ}) are fulfilled, then (5. 4) becomes, for every $\alpha \in \mathcal{F}$,

$$D_W \subsetneq D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} = D_F = D_L = D_P \subsetneq D_N. \qquad (5.\ 7)$$

Let us recall that in this situation we have, by (3. 8), the following equality for every $\alpha \in \mathcal{F}$

$$vmaxD_1 = vmaxD_\alpha = vmaxD_{FL} = vmaxD_F = vmaxD_L = vmaxD_P.$$

The next example shows that, even in this case, the sets $vmaxD_W$ and $vmaxD_P$ are in general not equal.

Example 5.4 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}$, let be $f_1, f_2 : \mathbb{R} \to \overline{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$ such that

$$f_1(x) = f_2(x) = \begin{cases} x^2, & \text{if } x \in (0, +\infty), \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$g(x) = 0.$$

It is obvious that (A_f), (A_g) and (A_{CQ}) are fulfilled. For $\lambda = (1,1)^T$ and $d = (0,0)^T$, we have $(\lambda, d) \in \mathcal{B}_P$ and $d \in D_P$. Moreover, $d \in vmaxD_P$.

We will show now that $d = (0,0)^T \notin D_W$. If this were not true, then there would exist $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_W$, with $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2)^T \in int\mathbb{R}^2_+$, $\bar{\lambda}_1 + \bar{\lambda}_2 = 1$, $\bar{q} \in K^* = \{0\}$ such that

$$d = (0,0)^T = (f_1(\bar{x}) + \bar{q}g(\bar{x}), f_2(\bar{x}) + \bar{q}g(\bar{x}))^T = (f_1(\bar{x}), f_2(\bar{x}))^T.$$

But, $f_1(x) = f_2(x) > 0, \forall x \in \mathbb{R}$, and this leads to a contradiction. From here we obtain that $d = (0,0)^T \notin D_W$ and, obviously, $d = (0,0)^T \notin vmaxD_W$.

6 Weir-Mond multiobjective duality

The last section of this work is dedicated to the study of the so-called Weir-Mond dual optimization problem. It has the following formulation (cf. [6] and [8])

$$(D_{WM}) \quad \underset{(x,q,\lambda)\in\mathcal{B}_{WM}}{\text{v-max}} h^{WM}(x,q,\lambda),$$

$$h^{WM}(x,q,\lambda) = \begin{pmatrix} h_1^{WM}(x,q,\lambda) \\ \vdots \\ h_m^{WM}(x,q,\lambda) \end{pmatrix},$$

with

$$h_j^{WM}(x,q,\lambda) = f_j(x), j = 1,...,m,$$

the dual variables

$$x \in \mathbb{R}^n, q \in \mathbb{R}^k, \lambda = (\lambda_1, \dots, \lambda_m)^T \in \mathbb{R}^m,$$

and the set of constraints

$$\mathcal{B}_{WM} = \{(x,q,\lambda) : \quad x \in \mathbb{R}^n, \lambda = (\lambda_1, \dots, \lambda_m)^T \in int\mathbb{R}^m_+, \sum_{i=1}^m \lambda_i = 1, q \underset{K^*}{\geqq} 0,$$
$$q^T g(x) \geq 0, 0 \in \partial\left(\sum_{i=1}^m \lambda_i f_i\right)(x) + \partial(q^T g)(x)\}.$$

The following theorems state the existence of weak and strong duality (cf. [6] and [8]).

THEOREM 6.1 (weak duality for (D_{WM})) *There is no $x \in \mathcal{A}$ and no element $(y,q,\lambda) \in \mathcal{B}_{WM}$ fulfilling $h^{WM}(y,q,\lambda) \underset{\mathbb{R}^m_+}{\geqq} f(x)$ and $h^{WM}(y,q,\lambda) \neq f(x)$.*

THEOREM 6.2 (strong duality for (D_{WM})) *Assume that (A_f), (A_g) and (A_{CQ}) are fulfilled. If $\bar{x}$ is a properly efficient solution to (P), then there exists $\bar{q} \underset{K^*}{\geqq} 0$ and $\bar{\lambda} \in int\mathbb{R}^m_+$ such that $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_{WM}$ is a properly efficient solution to the dual (D_{WM}) and strong duality $f(\bar{x}) = h^{WM}(\bar{x}, \bar{q}, \bar{\lambda})$ holds.*

Let be $D_{WM} := h^{WM}(\mathcal{B}_{WM}) \subseteq \mathbb{R}^m$. We are now interested in relating the image set D_{WM} to the image sets which appear in the relation (5. 4).

PROPOSITION 6.3 *It holds $D_{WM} \subseteq D_L$.*

Proof. Let be $d = (d_1, ..., d_m)^T \in D_{WM}$. Then there exists $(x, q, \lambda) \in \mathcal{B}_{WM}$ such that $d = h^{WM}(x, q, \lambda) = f(x)$. Because

$$0 \in \partial \left(\sum_{i=1}^{m} \lambda_i f_i \right)(x) + \partial(q^T g)(x),$$

we have

$$\sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x) \leq \inf_{u \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(u) + q^T g(u) \right].$$

On the other hand,

$$\sum_{i=1}^{m} \lambda_i d_i = \sum_{i=1}^{m} \lambda_i f_i(x) \leq \sum_{i=1}^{m} \lambda_i f_i(x) + q^T g(x),$$

which implies

$$\sum_{i=1}^{m} \lambda_i d_i \leq \inf_{u \in \mathbb{R}^n} \left[\sum_{i=1}^{m} \lambda_i f_i(u) + q^T g(u) \right].$$

So, $(q, \lambda, d) \in \mathcal{B}_L$ and $d = h^L(q, \lambda, d) \in h^L(\mathcal{B}_L) = D_L$. □

Remark 6.1 For the problem considered in Example 5.1 we have that $d = (-1, -1)^T \in D_L$ and $d \notin D_W$. In a similar way it can be shown that $d = (1, -1)^T \notin D_{WM}$. This means that the inclusion $D_{WM} \subseteq D_L$ may be strict. From here it also follows that $D_P \nsubseteq D_{WM}$ and $D_N \nsubseteq D_{WM}$ (cf. (4. 3)).

Remark 6.2 Let us consider now the problem in Example 5.2. Here, it holds $d = (1, -1) \in D_1$. But, one can verify that $d = (1, -1) \notin D_{WM}$, which implies that $D_1 \cap \mathbb{R}^m \nsubseteq D_{WM}$ and, from here, we have that $D_\alpha \cap \mathbb{R}^m \nsubseteq D_{WM}$, $\alpha \in \mathcal{F}$, $D_{FL} \nsubseteq D_{WM}$, $D_F \nsubseteq D_{WM}$ and $D_P \nsubseteq D_{WM}$.

Remark 6.3 For the problem in Example 5.3, we have $d = (-1, 1) \notin D_F$ and, obviously, $d = (-1, 1) \in D_{WM}$. So, it holds $D_{WM} \nsubseteq D_F$ and, as a consequence,

$D_{WM} \nsubseteq D_{FL}$, $D_{WM} \nsubseteq D_\alpha \cap \mathbb{R}^m$, $\alpha \in \mathcal{F}$, and $D_{WM} \nsubseteq D_1 \cap \mathbb{R}^m$.

Next we construct two other examples which show that between D_W and D_{WM} also does not exist any relation of inclusion.

Example 6.1 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that $f_1(x) = f_2(x) = 0$ and $g(x) = x^2 - 1$.

For $x = 0$, $q = 1$ and $\lambda = \left(\frac{1}{2}, \frac{1}{2}\right)^T$, it holds $(x, q, \lambda) \in \mathcal{B}_W$ and

$$d = (-1, -1)^T = (f_1(0) + qg_1(0), f_2(0) + qg_2(0))^T \in D_W.$$

Otherwise, $d \notin D_{WM}$, which means that $D_W \nsubseteq D_{WM}$.

Example 6.2 For $m = 2, n = 1, k = 1$, $K = \mathbb{R}_+$, let be $f_1, f_2 : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ such that $f_1(x) = x$, $f_2(x) = x$ and $g(x) = -x + 1$.

For $x = \frac{1}{2}$, $q = 1$ and $\lambda = (\frac{1}{2}, \frac{1}{2})^T$, it holds $qg\left(\frac{1}{2}\right) = \frac{1}{2} \geq 0$ and

$$\inf_{x \in \mathbb{R}} [\lambda_1 f_1(x) + \lambda_2 f_2(x) + qg(x)] = 1,$$

which means that $(x, q, \lambda) \in \mathcal{B}_{WM}$ and $d = \left(\frac{1}{2}, \frac{1}{2}\right)^T = (f_1(\frac{1}{2}), f_2(\frac{1}{2}))^T \in D_{WM}$.

Let us prove that $d \notin D_W$. If this were not true, then there would exist $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_W$ such that

$$d = \left(\frac{1}{2}, \frac{1}{2}\right)^T = (f_1(\bar{x}) + \bar{q}g(\bar{x}), f_2(\bar{x}) + \bar{q}g(\bar{x}))^T = (\bar{x} + \bar{q}(-\bar{x} + 1), \bar{x} + \bar{q}(-\bar{x} + 1))^T. \tag{6. 1}$$

Because $(\bar{x}, \bar{q}, \bar{\lambda}) \in \mathcal{B}_W$, we have

$$\inf_{x \in \mathbb{R}} [\bar{\lambda}_1 f_1(x) + \bar{\lambda}_2 f_2(x) + \bar{q}g(x)] = \bar{\lambda}_1 f_1(\bar{x}) + \bar{\lambda}_2 f_2(\bar{x}) + \bar{q}g(\bar{x}),$$

or, equivalently,

$$\inf_{x \in \mathbb{R}} [x + \bar{q}(-x + 1)] = \bar{x} + \bar{q}(-\bar{x} + 1).$$

This is true just if $\bar{q} = 1$. But, in this case, (6. 1) leads us to a contradiction.

In conclusion, $D_{WM} \nsubseteq D_W$.

In the general case, we get the following scheme for every $\alpha \in \mathcal{F}$

$$\begin{array}{r} D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} \subsetneq \begin{array}{l} D_F \subsetneq D_P \\ D_L \subsetneq \begin{array}{l} D_P \\ D_N \end{array} \end{array} \\ D_W \subsetneq D_L \subsetneq \begin{array}{l} D_P \\ D_N \end{array} \\ D_{WM} \subsetneq D_L \subsetneq \begin{array}{l} D_P \\ D_N \end{array} \end{array} . \tag{6. 2}$$

Let us now try to find out how is this scheme changing under the fulfillment of the assumptions (A_f), (A_g) and (A_{CQ}). From (5. 7) we have for every $\alpha \in \mathcal{F}$

$$D_W \subsetneq D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} = D_F = D_L = D_P \subsetneq D_N.$$

Remark 6.4 Let us notice that for the problem formulated in Example 6.1 (A_f), (A_g) and (A_{CQ}) are fulfilled. But, $D_W \nsubseteq D_{WM}$, which implies $D_1 \cap \mathbb{R}^m \nsubseteq D_{WM}$, $D_\alpha \cap \mathbb{R}^m \nsubseteq D_{WM}$, $\alpha \in \mathcal{F}$, and $D_{FL} = D_F = D_L = D_P \nsubseteq D_{WM}$.

Remark 6.5 For the problem presented in Example 6.2 we proved that $d = \left(\frac{1}{2}, \frac{1}{2}\right)^T \in D_{WM}$. By using some calculation techniques concerning conjugate functions, it can be also proved that $d = \left(\frac{1}{2}, \frac{1}{2}\right)^T \notin D_\alpha$, for every $\alpha \in \mathcal{F}$. In conclusion, $D_{WM} \nsubseteq D_\alpha \cap \mathbb{R}^m$, $\alpha \in \mathcal{F}$, and, from here, $D_{WM} \nsubseteq D_1 \cap \mathbb{R}^m$, even if (A_f), (A_g) and (A_{CQ}) are fulfilled.

By the last two remarks, using (5. 7), if (A_f), (A_g) and (A_{CQ}) are fulfilled, we get the following scheme for every $\alpha \in \mathcal{F}$

$$D_W \subsetneq D_1 \cap \mathbb{R}^m \subsetneq D_\alpha \cap \mathbb{R}^m \subsetneq D_{FL} = D_F = D_L = D_P \subsetneq D_N,$$

and

$$D_{WM} \subsetneq D_{FL} = D_F = D_L = D_P \subsetneq D_N,$$

and no other relation of inclusion holds between these sets.

Remark 6.6 For the problem in Example 5.4 we have $d = (0,0)^T \in vmaxD_P$, but $d \notin vmaxD_W$ and $d \notin vmaxD_{WM}$. This means that $vmaxD_P \nsubseteq vmaxD_W$ and $vmaxD_P \nsubseteq vmaxD_{WM}$ and we notice that, even if (A_f), (A_g) and (A_{CQ}) are fulfilled, these sets may be different.

Remark 6.7 The question concerning finding some sufficient or necessary conditions for which the sets $vmaxD_P$, $vmaxD_W$ and $vmaxD_{WM}$ coincide is still open.

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