Strong duality for generalized convex optimization problems*†

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Abstract. In this paper strong duality for nearly convex optimization problems is established. Three kind of conjugate dual problems are attached to the primal optimization problem: the Lagrange dual, the Fenchel dual and the Fenchel-Lagrange dual problems. Our main results show that under suitable conditions, the optimal objective values of these four problems coincide.

Key Words. Nearly convex set, nearly convex function, strong duality.

1. Introduction

In paper Ref. 1 Wanka and Boţ considered three types of conjugate dual problems for a constrained optimization problem (P): the well-known Lagrange and the Fenchel dual problems (denoted by (D_L) and (D_F) respectively) and a "combination" of the above two, called by the authors Fenchel-Lagrange dual problem (denoted by (D_{FL})). It is relatively easy to show that in each case the so-called "weak duality" holds, namely the optimal objective value $v(P)$ of the primal problem (P) is always greater than or equal to each of the optimal objective values of the considered dual problems. Moreover, among the optimal objective values of these three dual problems, $v(D_{FL})$ is the smallest. An interesting fact is that in general, an ordering between $v(D_L)$ and $v(D_F)$ cannot be established (for a counterexample see Ref. 1).

For both theoretical and practical reasons, one of the main issues in optimization theory is to find conditions which guarantee the so called "strong duality",

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*The first author was supported in part by Gottlieb Daimler- and Karl Benz-Stiftung (under 02-48/99).

†This paper has been performed while the second author visited the Chemnitz University of Technology with a grant provided by DAAD (Deutscher Akademischer Austauschdienst) Nr. A/02/12866.

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namely that the optimal objective values of the primal and the dual problems coincide. Usually this can be achieved under convexity assumptions on the sets and functions involved and some regularity conditions often called "constraint qualifications". In paper Ref. 1 of Wanka and Boţ it was shown that under the hypothesis of convexity and suitable constraint qualifications the strong duality holds for each dual problem.

The aim of the present paper is to weaken the convexity and the regularity assumptions considered in Ref. 1 in a way that the above mentioned strong duality results still hold. To do this, we assume that the sets and functions involved in problem (P) and its three duals are *nearly convex*, a kind of generalized convexity. This concept was first introduced for sets by Green and Gustin in Ref. 2 and some relevant properties have been studied for instance by Gherman and Soltan (Ref. 3) and Muntean (Ref. 4). Then Aleman (Ref. 5) defined this notion for functions (called by himself p-convexity). It has to be mentioned that the nearly convexity for functions is essentially weaker than usual convexity, as Example 3.1 (below) shows.

The paper is organized as follows. In Section 2 we define the primal optimization problem and its three conjugate dual problems, and show the basic relations between them. Section 3 contains our main results. After recalling the definitions of nearly convexity for sets and functions, we first establish the equality $v(D_F) = v(D_{FL})$ and then the equality $v(D_F) = v(P)$. Combining these relations with a simple property given before, we obtain sufficient conditions for the equality $v(P) = v(D_L) = v(D_F) = v(D_{FL})$.

2. The constrained optimization problem and its conjugate duals

2.1. Problem formulation. Let $X \subseteq \mathbb{R}^n$ be a nonempty set and $C \subseteq \mathbb{R}^k$ a nonempty closed convex cone with $C^* := \{c^* \in \mathbb{R}^k : c^{*T}c \geq 0, \forall c \in C\}$ its dual cone. Consider the (partial) ordering $\leq_C$ induced by C in $\mathbb{R}^k$, namely for $y, z \in \mathbb{R}^k$ we have that $y \leq_C z$, iff $z - y \in C$. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g = (g_1, \ldots, g_k)^T : \mathbb{R}^n \to \mathbb{R}^k$. The optimization problem which we investigate in this paper is the following

$$(P) \qquad \inf_{x \in G} f(x),$$

where

$$G = \{x \in X : g(x) \leq_C 0\}.$$

In the following we always suppose that the feasible set G is nonempty. Assume further that $dom(f) = X$, where $dom(f) := \{x \in \mathbb{R}^n : f(x) < +\infty\}$.

The problem (P) is said to be the primal problem and its optimal objective value is denoted by $v(P)$.

Definition 2.1. An element $\bar{x} \in G$ is said to be an optimal solution for (P) if $f(\bar{x}) = v(P)$.

The aim of this section is to construct different dual problems to (P). For this aim we will use an approach described in Ekeland and Temam (Ref. 6) which is based on the theory of conjugate functions. To do this, let us first consider the general optimization problem without constraints

$$(\mathrm{PG}) \qquad \inf_{x \in \mathbb{R}^n} F(x),$$

with F a mapping from $\mathbb{R}^n$ into $\overline{\mathbb{R}}$.

Definition 2.2. The function $F^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$F^*(p^*) = \sup_{x \in \mathbb{R}^n} \{ p^{*T} x - F(x) \}$$

is called the conjugate function of F.

Remark 2.1. By the assumptions we made for f we have

$$f^*(p^*) = \sup_{x \in \mathbb{R}^n} \{ p^{*T} x - f(x) \} = \sup_{x \in X} \{ p^{*T} x - f(x) \}.$$

The approach in Ref. 6 is based on the construction of a so-called perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, with the property that $\Phi(x, 0) = F(x)$ for each $x \in \mathbb{R}^n$. Here, $\mathbb{R}^m$ is the space of the perturbation variables. For each $p \in \mathbb{R}^m$ we obtain then a new optimization problem

$$(\mathrm{PG})_p \qquad \inf_{x \in \mathbb{R}^n} \Phi(x, p).$$

For $p \in \mathbb{R}^m$ the problem $(\mathrm{PG})_p$ is called the perturbed problem of (P).
By Definition 2.2, the conjugate of Φ is the function $\Phi^* : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$\begin{aligned} \Phi^*(x^*, p^*) &= \sup_{\substack{x \in \mathbb{R}^n \\ p \in \mathbb{R}^m}} \{ (x^*, p^*)^T (x, p) - \Phi(x, p) \} \\ &= \sup_{\substack{x \in \mathbb{R}^n \\ p \in \mathbb{R}^m}} \{ x^{*T} x + p^{*T} p - \Phi(x, p) \}. \end{aligned} \tag{1}$$

Now we can define the following optimization problem (cf. Ref. 6)

$$(\mathrm{DG}) \qquad \sup_{p^* \in \mathbb{R}^m} \{ -\Phi^*(0, p^*) \}.$$

The problem (DG) is called the dual problem of (PG) and its optimal objective value is denoted by $v(DG)$.

This approach has an important property: between the primal and the dual problem weak duality always holds. The following theorem proves this fact.

Theorem 2.1. (see Ekeland and Temam (Ref. 6)). The relation

$$-\infty \leq v(DG) \leq v(PG) \leq +\infty \tag{2}$$

always holds.

Our next aim is to show how we can apply this approach to the constrained optimization problem (P). Therefore let be $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ the function given by

$$F(x) = \begin{cases} f(x), & \text{if} \quad x \in G, \\ +\infty, & \text{otherwise.} \end{cases}$$

The primal problem (P) is then equivalent to

$$(\text{PG}) \qquad \inf_{x \in \mathbb{R}^n} F(x),$$

and since the perturbation function $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ satisfies $\Phi(x, 0) = F(x)$ for each $x \in \mathbb{R}^n$ we obtain that

$$\Phi(x, 0) = f(x), \quad \forall x \in G \tag{3}$$

and

$$\Phi(x, 0) = +\infty, \quad \forall x \in \mathbb{R}^n \setminus G. \tag{4}$$

In the following we will study for special choices of the perturbation function some dual problems of (P).

2.2. The Lagrange dual problem. For the beginning, let the function $\Phi_L : \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$ be defined by

$$\Phi_L(x, q) = \begin{cases} f(x), & \text{if} \quad x \in X, \quad g(x) \leqq_C q, \\ +\infty, & \text{otherwise,} \end{cases}$$

with $q \in \mathbb{R}^k$ the perturbation variable. It is obvious that the relations (3) and (4) are fulfilled. For the conjugate of Φ_L we have

$$\begin{aligned} \Phi_L^*(x^*, q^*) &= \sup_{\substack{x \in \mathbb{R}^n \\ q \in \mathbb{R}^k}} \{x^{*T} x + q^{*T} q - \Phi_L(x, q)\} \\ &= \sup_{\substack{x \in X, q \in \mathbb{R}^k \\ g(x) \leqq_C q}} \{x^{*T} x + q^{*T} q - f(x)\}. \end{aligned}$$

In order to calculate this expression we introduce the variable s instead of q by $s = q - g(x) \in C$. This implies

$$\begin{aligned}
\Phi_L^*(x^*, q^*) &= \sup_{x\in X, s\in C} \{x^{*T}x + q^{*T}[s + g(x)] - f(x)\} \\
&= \sup_{x\in X}\{x^{*T}x + q^{*T}g(x) - f(x)\} + \sup_{s\in C} q^{*T}s \\
&= \begin{cases} \sup\limits_{x\in X}\{x^{*T}x + q^{*T}g(x) - f(x)\}, & \text{if} \quad q^* \in -C^*, \\ +\infty, & \text{otherwise.} \end{cases}
\end{aligned}$$

As we have seen, the dual of (P) obtained by the perturbation function Φ_L is

$$(D_L) \qquad \sup_{q^*\in\mathbb{R}^k}\{-\Phi_L(0, q^*)\},$$

and since

$$\sup_{q^*\in -C^*}\{-\sup_{x\in X}[q^{*T}g(x) - f(x)]\} = \sup_{q^*\in -C^*}\{\inf_{x\in X}[-q^{*T}g(x) + f(x)]\},$$

the dual has the following form

$$(D_L) \quad \sup_{q^*\in C^*} \inf_{x\in X}[f(x) + q^{*T}g(x)]. \tag{5}$$

The problem (D_L) is actually the well-known Lagrange dual problem. Its optimal objective value is denoted by $v(D_L)$ and Theorem 2.1 implies

$$v(D_L) \leq v(P). \tag{6}$$

We are now interested to obtain dual problems for (P) different from the classical Lagrange problem.

2.3. The Fenchel dual problem. Let us consider the perturbation function $\Phi_F : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ given by

$$\Phi_F(x, p) = \begin{cases} f(x + p), & \text{if} \quad x \in G, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variable $p \in \mathbb{R}^n$. The relations (3) and (4) are also fulfilled and it holds

$$\begin{aligned}
\Phi_F^*(x^*, p^*) &= \sup_{\substack{x\in\mathbb{R}^n \\ p\in\mathbb{R}^n}}\{x^{*T}x + p^{*T}p - \Phi_F(x, p)\} \\
&= \sup_{\substack{x\in X, p\in\mathbb{R}^n \\ g(x)\leq_C 0}}\{x^{*T}x + p^{*T}p - f(x + p)\}.
\end{aligned}$$

For the new variable $r = x + p \in \mathbb{R}^n$, we have

$$\begin{aligned}\Phi_F^*(x^*, p^*) &= \sup_{\substack{x \in X, r \in \mathbb{R}^n \\ g(x) \leqq_C 0}} \{x^{*T}x + p^{*T}(r - x) - f(r)\} \\ &= \sup_{r \in \mathbb{R}^n} \{p^{*T}r - f(r)\} + \sup_{\substack{x \in X \\ g(x) \leqq_C 0}} \{(x^* - p^*)^T x\} \\ &= f^*(p^*) - \inf_{\substack{x \in X \\ g(x) \leqq_C 0}} \{(p^* - x^*)^T x\} = f^*(p^*) - \inf_{x \in G} \{(p^* - x^*)^T x\}.\end{aligned}$$

Now the dual of (P)

$$(D_F) \qquad \sup_{p^* \in \mathbb{R}^n} \{-\Phi_F^*(0, p^*)\}$$

can be written in the form

$$(D_F) \quad \sup_{p^* \in \mathbb{R}^n} \{-f^*(p^*) + \inf_{\substack{x \in X \\ g(x) \leqq_C 0}} p^{*T}x\}. \tag{7}$$

Let us call (D_F) the Fenchel dual problem and denote its optimal objective value by $v(D_F)$. The weak duality

$$v(D_F) \leq v(P) \tag{8}$$

is also fulfilled by Theorem 2.1.

2.4. The Fenchel-Lagrange dual problem. Another dual problem, different from (D_L) and (D_F), can be obtained considering the perturbation function as a combination of the functions Φ_L and Φ_F. Let this be defined by $\Phi_{FL} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$,

$$\Phi_{FL}(x, p, q) = \begin{cases} f(x + p), & \text{if} \quad x \in X, \quad g(x) \leqq_C q, \\ +\infty, & \text{otherwise,} \end{cases}$$

with the perturbation variables $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^k$. Φ_{FL} satisfies the relations (3) and (4) and its conjugate is

$$\begin{aligned}\Phi_{FL}^*(x^*, p^*, q^*) &= \sup_{\substack{x \in \mathbb{R}^n \\ p \in \mathbb{R}^n, q \in \mathbb{R}^k}} \{x^{*T}x + p^{*T}p + q^{*T}q - \Phi_{FL}(x, p, q)\} \\ &= \sup_{\substack{x \in X, g(x) \leqq_C q \\ p \in \mathbb{R}^n, q \in \mathbb{R}^k}} \{x^{*T}x + p^{*T}p + q^{*T}q - f(x + p)\}.\end{aligned}$$

Like in the previous subsections we introduce the new variables $r = x + p \in \mathbb{R}^n$ and $s = q - g(x) \in C$. Then we have

$$\begin{aligned}\Phi_{FL}^*(x^*, p^*, q^*) &= \sup_{\substack{r \in \mathbb{R}^n, s \in C \\ x \in X}} \{x^{*T}x + p^{*T}(r - x) + q^{*T}[s + g(x)] - f(r)\} \\ &= \sup_{r \in \mathbb{R}^n} \{p^{*T}r - f(r)\} + \sup_{s \in C} q^{*T}s + \sup_{x \in X} \{(x^* - p^*)^T x + q^{*T}g(x)\}.\end{aligned}$$

Computing the first two suprema we get again

$$\sup_{r\in\mathbb{R}^n}\{p^{*T}r - f(r)\} = f^*(p^*)$$

and

$$\sup_{s\in C} q^{*T}s = \begin{cases} 0, & \text{if} \quad q^* \in -C^*, \\ +\infty, & \text{otherwise.} \end{cases}$$

In this case the dual problem

$$(D_{FL}) \qquad \sup_{\substack{p^*\in\mathbb{R}^n \\ q^*\in\mathbb{R}^k}}\{-\Phi^*_{FL}(0,p^*,q^*)\}$$

is

$$(D_{FL}) \qquad \sup_{\substack{p^*\in\mathbb{R}^n \\ q^*\in -C^*}}\{-f^*(p^*) - \sup_{x\in X}[-p^{*T}x + q^{*T}g(x)]\}$$

or, equivalently,

$$(D_{FL}) \quad \sup_{\substack{p^*\in\mathbb{R}^n \\ q^*\in C^*}}\{-f^*(p^*) + \inf_{x\in X}[p^{*T}x + q^{*T}g(x)]\}. \tag{9}$$

We call (D_{FL}) the Fenchel-Lagrange dual problem and denote its optimal objective value by $v(D_{FL})$. By Theorem 2.1 the weak duality $v(D_{FL}) \leq v(P)$ is also true.

In the following we are going to give some relations existing between the optimal objective values of different dual problems we introduced above. For the sake of better understanding of the backgrounds of our duality approach we consider to recall the proofs of the following two propositions as being necessary. The first one refers to the problems (D_L) and (D_{FL}).

Proposition 2.1. (see Wanka and Boţ (Ref. 1)). The inequality $v(D_L) \geq v(D_{FL})$ holds.

Proof. Let $q^* \in C^*$ and $p^* \in \mathbb{R}^n$ be fixed. By the definition of the conjugate function we have for each $x \in X$

$$f^*(p^*) \geq p^{*T}x - f(x)$$

or, equivalently,

$$f(x) \geq p^{*T}x - f^*(p^*).$$

By adding $q^{*T}g(x)$ to both sides we obtain for each $x \in X$

$$f(x) + q^{*T}g(x) \geq -f^*(p^*) + p^{*T}x + q^{*T}g(x).$$

This means that for each $q^* \in C^*$ and $p^* \in \mathbb{R}^n$ it holds

$$\inf_{x\in X}[f(x) + q^{*T}g(x)] \geq -f^*(p^*) + \inf_{x\in X}[p^{*T}x + q^{*T}g(x)]. \tag{10}$$

We can calculate now the supremum over $p^* \in \mathbb{R}^n$ and $q^* \in C^*$ and this implies

$$\sup_{q^*\in C^*}\inf_{x\in X}[f(x) + q^{*T}g(x)] \geq \sup_{\substack{p^*\in\mathbb{R}^n\\ q^*\in C^*}}\{-f^*(p^*) + \inf_{x\in X}[p^{*T}x + q^{*T}g(x)]\}.$$

The last inequality is in fact $v(D_L) \geq v(D_{FL})$ and thus the proof is complete. □

The inequality in Proposition 2.1 can be strict in some situations (for an example see Ref. 1).

The next result states an inequality between the optimal objective values of the problems (D_F) and (D_{FL}).

Proposition 2.2. (see Wanka and Boţ (Ref. 1)). The inequality $v(D_F) \geq v(D_{FL})$ holds.

Proof. Let $p^* \in \mathbb{R}^n$ be fixed. Then for each $q^* \in C^*$ we have

$$\inf_{x\in X}[p^{*T}x + q^{*T}g(x)] \leq \inf_{\substack{x\in X\\ g(x)\leq_C 0}}[p^{*T}x + q^{*T}g(x)] \leq \inf_{\substack{x\in X\\ g(x)\leq_C 0}} p^{*T}x.$$

Then for every $p^* \in \mathbb{R}^n$

$$\sup_{q^*\in C^*}\inf_{x\in X}[p^{*T}x + q^{*T}g(x)] \leq \inf_{\substack{x\in X\\ g(x)\leq_C 0}} p^{*T}x. \tag{11}$$

By adding $-f^*(p^*)$ to both sides one obtains

$$-f^*(p^*) + \sup_{q^*\in C^*}\inf_{x\in X}[p^{*T}x + q^{*T}g(x)] \leq -f^*(p^*) + \inf_{\substack{x\in X\\ g(x)\leq_C 0}} p^{*T}x.$$

This last inequality implies

$$\sup_{\substack{p^*\in\mathbb{R}^n\\ q^*\in C^*}}\{-f^*(p^*) + \inf_{x\in X}[p^{*T}x + q^{*T}g(x)]\} \leq \sup_{p^*\in\mathbb{R}^n}\{-f^*(p^*) + \inf_{\substack{x\in X\\ g(x)\leq_C 0}} p^{*T}x\}$$

or, equivalently, $v(D_{FL}) \leq v(D_F)$. □

The inequality in Proposition 2.2 can also be strict in some situations (for an example see Ref. 1).

3. Strong duality for nearly convex programming problems

The aim of this section is to establish strong duality results for a class of generalized convex programming problems. In the first subsection we recall the concepts and some basic properties of nearly convex sets and nearly convex functions introduced by Green and Gustin (Ref. 2) and Aleman (Ref. 5), respectively. Then, in the second and third subsections we state and prove our results concerning duality for nearly convex optimization problems.

3.1. Nearly convex sets and functions. To start let us recall the following definition.

Definition 3.1. A subset $S \subseteq \mathbb{R}^m$ is called *nearly convex* if there exists a constant $0 < \alpha < 1$ such that for each $x, y \in S$ follows that $\alpha x + (1-\alpha)y \in S$.

Obviously, each convex set is nearly convex, but the contrary is not true since for instance the set $\mathbb{Q} \subset \mathbb{R}$ of all rational numbers is nearly convex (with $\alpha = 1/2$) but not a convex set. It is interesting to remark that the set $\mathbb{R} \setminus \mathbb{Q}$ of all irrational numbers is not a nearly convex set (see Frenk and Kassay (Ref. 7) or Breckner and Kassay (Ref. 8)).

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^n \to \mathbb{R}^k$ be given functions, and let D, E be nonempty subsets of $\mathbb{R}^n$ such that $D \subseteq dom(f)$. We denote the epigraph of f on D by $epi(f; D)$, i. e. the set $\{(x, r) \in D \times \mathbb{R} : f(x) \leq r\}$. Furthermore, if $C \subseteq \mathbb{R}^k$ is a nonempty convex cone, the epigraph of g on E with respect to the cone C will be the set

$$epi_C(g; E) := \{(x, v) \in E \times \mathbb{R}^k : g(x) \leq_C v\},$$

where $\leq_C$ denotes the partial ordering relation induced by C (see Section 2).

Now one can define the following concepts.

Definition 3.2. The function f is said to be nearly convex on D if $epi(f; D)$ is a nearly convex set. Moreover, the vector-valued function g is said to be nearly convex on E with respect to the cone C if $epi_C(g; E)$ is a nearly convex set.

Observe that by the above definition the function f is nearly convex on D if and only if there exists $0 < \alpha < 1$ such that for every $x, y \in D$ we have

$$\alpha x + (1-\alpha)y \in D$$

and

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

Similarly, g is nearly convex on E with respect to the cone C if and only if there exists $0 < \gamma < 1$ such that for every $x, y \in E$ we have

$$\gamma x + (1-\gamma)y \in E$$

and

$$g(\gamma x + (1-\gamma)y) \leq_C \gamma g(x) + (1-\gamma)g(y).$$

It is obvious that in case D or/and E are convex sets and f or/and g are convex functions in the usual sense on D and E, respectively, then they are also nearly convex. In case D or/and E are nearly convex sets but not convex, we can easily define nearly convex functions on them which obviously are not convex functions. But an interesting fact is that it is possible to give an example for nearly convex, but not convex function defined on a convex set. This is related to the functional equation of Cauchy.

Example 3.1. Let $F : \mathbb{R} \to \mathbb{R}$ be any discontinuous solution of the Cauchy functional equation, i. e. F satisfies

$$F(x+y) = F(x) + F(y), \quad \forall x, y \in \mathbb{R}.$$

(Such a solution exists, see Ref. 9.)

It is easy to deduce that

$$F\left(\frac{x+y}{2}\right) = \frac{F(x)+F(y)}{2}, \quad \forall x, y \in \mathbb{R}.$$

Therefore F is nearly convex on $\mathbb{R}$ with constant $1/2$. However, F is not convex (even more: there is no interval in $\mathbb{R}$ on which F is convex) due to the lack of continuity.

Next we recall some basic properties of nearly convex sets which we need for our results.

Lemma 3.1. (see Aleman (Ref. 5)). The following properties hold

(i) The closure $cl(S)$ and the relative interior $ri(S)$ of every nearly convex set S is convex ($ri(S)$ may be empty);

(ii) For every nearly convex set S the set

$$\Omega_S := \{t \in [0,1] : \forall x, y \in S \Rightarrow tx + (1-t)y \in S\}$$

is dense within the interval $[0,1]$;

(iii) For every nearly convex set S, every $x \in cl(S)$ and $y \in ri(S)$ we have that $tx + (1-t)y \in ri(S)$ for each $0 \leq t < 1$.

3.2. The equivalence of the dual problems (D_F) and (D_{FL}). In this subsection we will prove that in case of a nearly convex programming problem the optimal objective values of the Fenchel dual problem (D_F) and the Fenchel-Langrange dual problem (D_{FL}) are equal, provided some regularity conditions hold. To do this we need some auxiliary results which we list below.

Lemma 3.2. (Frenk and Kassay (Ref. 7), Theorem 3.1). If $M \subseteq \mathbb{R}^m$ is a nonempty convex set and if $0 \notin M$, then the sets M and $\{0\}$ can be properly separated. Moreover, the normal vector $y^* \neq 0$ of the separating hyperplane belongs to $aff(M)$ (the affine hull of M).

Lemma 3.3. (Frenk and Kassay (Ref. 7), Theorem 3.2). Let $M \subseteq \mathbb{R}^m$ be a nonempty set and $K \subseteq \mathbb{R}^m$ a nonempty convex cone such that $ri(M + K) \neq \emptyset$. Then it follows that $ri(M + K) = ri(M + ri(K))$.

Lemma 3.4. (Frenk and Kassay (Ref. 10)). Let $M \subseteq \mathbb{R}^m$ and $K \subseteq \mathbb{R}^m$ a nonempty convex cone. If for some $v_0 \in aff(M)$ the relation

$$M \subseteq v_0 + aff(K), \tag{12}$$

holds, then we have

$$ri(cl(M + K)) = M + ri(K).$$

Observe that for an arbitrary set $M \subseteq \mathbb{R}^m$ with $0 \in aff(M)$ it follows that $aff(M) = lin(M)$, where $lin(M)$ denotes the linear hull of the set M. For $K \subseteq \mathbb{R}^m$ a nonempty convex cone, since $0 \in cl(K) \subseteq aff(K)$, we always have that $aff(K) = lin(K)$. Therefore in relation (12) we can write either $aff(K)$ or $lin(K)$, not making any difference.

Observe that in case $int(K) \neq \emptyset$ follows $aff(K) = lin(K) = \mathbb{R}^m$, hence condition (12) in Lemma 3.4 is automatically satisfied.

As in the first section let $X \subseteq \mathbb{R}^n$ be a nonempty set, $C \subseteq \mathbb{R}^k$ a nonempty closed convex cone with C^* its dual cone, and let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $dom(f) = X$ and $g = (g_1, \ldots, g_k)^T : \mathbb{R}^n \to \mathbb{R}^k$ be given functions. In order to prove the equality $v(D_F) = v(D_{FL})$ we need the following lemma.

Lemma 3.5. Suppose that the vector-valued function g is nearly convex on X with respect to the closed convex cone C. Furthermore, suppose that there exists an element $y_0 \in aff(g(X))$ such that

$$g(X) \subseteq y_0 + aff(C) \tag{13}$$

and the (Slater type) regularity condition

$$0 \in g(X) + ri(C) \tag{14}$$

holds.

Let $p^* \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ be given elements. If the system

$$\begin{cases} p^{*T}x - \beta < 0 \\ g(x) \in -ri(C) \\ x \in X \end{cases}$$

has no solution, then there exists $q^* \in C^*$ such that

$$p^{*T}x - \beta + q^{*T}g(x) \geq 0, \quad \forall x \in X. \tag{15}$$

Proof. Define the vector-valued function $F : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^k$ given by $F(x) := (p^{*T}x - \beta, g(x))$ and let K be the closed convex cone $[0, +\infty) \times C$. It is easy to check that F is a nearly convex function on the set X with respect to the cone K. We show that condition (12) with $M := F(X)$ is implied by (13). Indeed, for the element $y_0 \in aff(g(X))$ there exist, by the definition of affine hull, $\lambda_1, ..., \lambda_m \in \mathbb{R}$ with $\sum_{j=1}^m \lambda_j = 1$ and $x_1, ..., x_m \in X$ such that $y_0 = \sum_{j=1}^m \lambda_j g(x_j)$. Then it follows that the pair (r_0, y_0) belongs to $aff(F(X))$, where $r_0 := p^{*T}(\sum_{j=1}^m \lambda_j x_j) - \beta$. Moreover, since $aff(K) = \mathbb{R} \times aff(C)$, (12) is satisfied with $v_0 := (r_0, y_0)$ and $M := F(X)$. Therefore, by Lemma 3.4 we have

$$ri(cl(F(X) + K)) = F(X) + ri(K).$$

Obviously, by our hypothesis, $(0, 0) \notin F(X) + ri(K)$. Since F is nearly convex on X with respect to K one can easily verify that the set $F(X) + K$ (the "epirange" of F) is a nearly convex set. Therefore, by Lemma 3.1 (i), $cl(F(X) + K)$ is a convex set, which implies that $ri(cl(F(X)+K))$ is a nonempty convex set (see for instance Rockafellar (Ref. 11)). Hence, by Lemma 3.2 the sets $\{(0, 0)\}$ and $ri(cl(F(X)+K))$ can be properly separated. The convexity of $cl(F(X)+K)$ implies $aff(ri(cl(F(X) + K))) = aff(cl(F(X) + K)) = aff(F(X) + K)$ (see for instance Ref. 11, Theorem 6.2), therefore, also by Lemma 3.2, the normal vector $(\lambda_0^*, q_0^*) \neq (0, 0)$ of the separating hyperplane belongs to $aff(F(X) + K)$. This means that $\lambda_0^* \in \mathbb{R}$, $q_0^* \in aff(g(X) + C)$ and again by the convexity of $cl(F(X) + K)$ follows that the same hyperplane separates the sets $\{(0, 0)\}$ and $cl(F(X) + K)$. This means that

$$\lambda_0^*(p^{*T}x - \beta + r) + q_0^{*T}(g(x) + c) \geq 0, \quad \forall x \in X, r \geq 0, c \in C. \tag{16}$$

A standard technique shows that $\lambda_0^* \geq 0$ and $q_0^* \in C^*$. In particular, for $r := 0$ and $c := 0$ (C is a closed convex cone, hence $0 \in C$) in (16) we obtain

$$\lambda_0^*(p^{*T}x - \beta) + q_0^{*T}g(x) \geq 0, \quad \forall x \in X. \tag{17}$$

Now we show that $\lambda_0^* \neq 0$. Supposing the contrary, (16) implies

$$q_0^{*T}(g(x)+c) \geq 0, \quad \forall x \in X, c \in C. \tag{18}$$

We show that in our hypothesis

$$g(X) + ri(C) = ri(g(X) + C). \tag{19}$$

Indeed, again by (13), Lemma 3.4 implies

$$ri(cl(g(X) + C)) = g(X) + ri(C). \tag{20}$$

Since $ri(ri(S)) = ri(S)$ for any set $S \subseteq \mathbb{R}^m$, taking the ri operator in both sides we obtain

$$ri(cl(g(X) + C)) = ri(g(X) + ri(C)). \tag{21}$$

Next we prove that $ri(g(X)+C) \neq \emptyset$. Since obviously $g(X)+ri(C) \neq \emptyset$, relations (20) and (21) imply that $ri(g(X) + ri(C)) \neq \emptyset$. By the obvious inclusion $g(X) + ri(C) \subseteq g(X) + C$, taking into account that $aff(g(X) + ri(C)) = aff(g(X)) + aff(ri(C)) = aff(g(X)) + aff(C) = aff(g(X) + C)$ (observe that the relation $aff(ri(C)) = aff(C)$ holds since C is a convex cone and therefore, $ri(C) \neq \emptyset$), we obtain $ri(g(X)+ri(C)) \subseteq ri(g(X)+C)$, hence $ri(g(X)+C)$ is also nonempty. Applying now Lemma 3.3 it follows that $ri(g(X) + ri(C)) = ri(g(X) + C)$ and this, together with (21) and (20), implies (19).

In virtue of (19), the regularity condition (14) means that

$$0 \in ri(g(X) + C). \tag{22}$$

Consequently, there exists an $\varepsilon > 0$ such that for each $y \in aff(g(X) + C)$ with $\|y\| \leq \varepsilon$ it follows that

$$y \in g(X) + C. \tag{23}$$

Let $v^* := -\frac{\varepsilon}{\|q_0^*\|} q_0^*$ (observe that $q_0^* \neq 0$, and therefore $v^* \neq 0$, otherwise we obtain a contradiction with $(\lambda_0^*, q_0^*) \neq (0, 0)$, due to the fact that we are supposing that $\lambda_0^* = 0$). By (18) one obtains

$$v^{*T}(g(x)+c) \leq 0, \quad \forall x \in X, c \in C. \tag{24}$$

Relation (22) implies in particular that $0 \in g(X)+C$, hence $aff(g(X)+C) = lin(g(X) + C)$. Then obviously $v^* \in aff(g(X) + C)$ and since $\|v^*\| = \varepsilon$, by (23) we obtain that $v^* \in g(X) + C$. Thus there exist $\bar{x} \in X$ and $\bar{c} \in C$ such that $v^* = g(\bar{x})+\bar{c}$ which together with (24) implies $v^* = 0$, a contradiction. This shows that $\lambda_0^* \neq 0$ and dividing relation (17) with λ_0^* we obtain (15) with $q^* := (1/\lambda_0^*)q_0^*$. This completes the proof. $\square$

Now we are ready to prove the equality between the optimal objective values of problems (D_F) and (D_{FL}).

Theorem 3.1. Suppose that $g : \mathbb{R}^n \to \mathbb{R}^k$ is a nearly convex function on the set $X \subseteq \mathbb{R}^n$ with respect to the closed convex cone $C \subseteq \mathbb{R}^k$. If the constraint qualifications (13) and (14) hold, then $v(D_F) = v(D_{FL})$.

Proof. For $p^* \in \mathbb{R}^n$ fixed we first prove that

$$\sup_{q^* \in C^*} \inf_{x \in X} [p^{*T} x + q^{*T} g(x)] = \inf_{x \in G} p^{*T} x. \tag{25}$$

Let $\beta := \inf\limits_{x \in G} p^{*T} x$. Since $G \neq \emptyset$, $\beta \in [-\infty, +\infty)$.

If $\beta = -\infty$, then by (11) it follows that

$$\sup_{q^* \in C^*} \inf_{x \in X} [p^{*T} x + q^{*T} g(x)] = -\infty = \inf_{x \in G} p^{*T} x.$$

Suppose now that $-\infty < \beta < +\infty$. It is easy to check that the system

$$\begin{cases} p^{*T} x - \beta < 0 \\ g(x) \in -C \\ x \in X \end{cases}$$

has no solutions. Therefore the system

$$\begin{cases} p^{*T} x - \beta < 0 \\ g(x) \in -ri(C) \\ x \in X \end{cases}$$

has no solutions too. By Lemma 3.5 there exists an element $q^* \in C^*$ satisfying

$$p^{*T} x - \beta + q^{*T} g(x) \geq 0, \quad \forall x \in X. \tag{26}$$

The latter relation implies

$$\sup_{q^* \in C^*} \inf_{x \in X} [p^{*T} x + q^{*T} g(x)] \geq \beta,$$

which together with (11) leads to (25).

To finish our proof we have for $p^* \in \mathbb{R}^n$ to add in both sides of (25) $-f^*(p^*)$. Then we obtain

$$-f^*(p^*) + \sup_{q^* \in C^*} \inf_{x \in X} [p^{*T} x + q^{*T} g(x)] = -f^*(p^*) + \inf_{\substack{x \in X \\ g(x) \leq_C 0}} p^{*T} x.$$

Now taking the supremum in both sides over $p^* \in \mathbb{R}^n$ we obtain the equality $v(D_F) = v(D_{FL})$. This completes the proof. □

3.3. Strong duality for the Fenchel dual problem (D_F)**.** In this subsection we will prove that the optimal objective values of the primal problem (P) and the Fenchel dual problem (D_F) are equal for nearly convex programming problems under some suitable conditions. For this purpose, as in the previous subsection, we need some auxiliary results.

Lemma 3.6. Let $S \subseteq \mathbb{R}^m$ be a nearly convex set. Then the relative interior $ri(S)$ of the set S is nonempty if and only if

$$ri(cl(S)) \subseteq S. \tag{27}$$

Proof. First suppose that (27) holds. Since the affine hull $aff(S)$ of S is a closed set we have that $aff(S) = aff(cl(S))$. Moreover, since by Lemma 3.1 (i) $cl(S)$ is a convex set, then $ri(cl(S))$ is a nonempty convex set (see for instance Ref. 11). Therefore,

$$aff(cl(S)) = aff(ri(cl(S)))$$

(see also relation (5) in (Ref. 7)), hence

$$aff(S) = aff(ri(cl(S))).$$

This allows us to take the ri operation in both sides of (27) and we obtain that $ri(ri(cl(S))) \subseteq ri(S)$. Since $ri(ri(M)) = ri(M)$ for each set $M \subseteq \mathbb{R}^m$ the latter leads to $\emptyset \neq ri(cl(S)) \subseteq ri(S)$. Thus $ri(S) \neq \emptyset$. For the reverse implication let $x \in ri(cl(S))$ be an arbitrary element. Then by definition there exists $\varepsilon > 0$ such that, denoting by B the closed unit ball of $\mathbb{R}^m$, we have

$$(x + \varepsilon B) \cap aff(cl(S)) \subseteq cl(S). \tag{28}$$

Choose an arbitrary element $x' \in ri(S)$ and let $0 < t < 1$ be such that

$$\frac{t}{1-t}\|x - x'\| < \varepsilon.$$

Then for the element

$$z := \frac{1}{1-t}x + \frac{-t}{1-t}x'$$

we clearly have that $z \in aff(cl(S))$ on one hand and

$$\|z - x\| = \|\frac{-t}{1-t}x' + \frac{t}{1-t}x\| = \frac{t}{1-t}\|x - x'\| < \varepsilon,$$

on the other hand. Thus, by (28) follows that $z \in cl(S)$. Since $x' \in ri(S)$ by Lemma 3.1 (iii) one has

$$x = tx' + (1-t)z \in ri(S) \subseteq S,$$

and the proof is complete. $\square$

Remark 3.1. As it can be seen, we have proved in fact that for $S \subseteq \mathbb{R}^m$ a nearly convex set $ri(S) \neq \emptyset$ if and only if $ri(cl(S)) \subseteq ri(S)$. Since the reverse inclusion holds obviously we obtain the following property: for a nearly convex set $S \subseteq \mathbb{R}^m$ $ri(S) \neq \emptyset$ if and only if $ri(S) = ri(cl(S))$.

Now consider again the set X, the closed convex cone C and the functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $dom(f) = X$ and $g = (g_1, \ldots, g_k)^T : \mathbb{R}^n \to \mathbb{R}^k$. As in the previous (sub)sections, let G be the feasible set for the primal problem (P). Then we have the following lemma.

Lemma 3.7. Suppose that the functions f and g are nearly convex on the set X and $a := \inf_{x \in G} f(x) > -\infty$. If

(i) $ri(epi(f)) \neq \emptyset$

and

(ii) $ri(G) \neq \emptyset$,

then

$$ri(cl(epi(f))) \cap [ri(cl(G)) \times (-\infty, a)] = \emptyset. \tag{29}$$

Proof. Assume by contradiction that there exists $(x, \mu) \in ri(cl(epi(f))) \cap [ri(cl(G)) \times (-\infty, a)]$. By our assumptions and Lemma 3.6 we obtain that $(x, \mu) \in epi(f)$, $x \in G$ and $\mu < a$. This means that $x \in G$ and $f(x) \leq \mu < a$, which contradicts the definition of a. $\square$

Now we are ready to prove the strong duality result concerning problem (D_F).

Theorem 3.2. Suppose that the functions f and g are nearly convex on the set X and the constraint qualifications (13) and (14) hold. Assume further that (i) and (ii) in Lemma 3.7 are satisfied. Then $v(P) = v(D_F)$.

Moreover, if $a := \inf_{x \in G} f(x) > -\infty$, then the dual problem (D_F) has an optimal solution.

Proof. If $a = -\infty$, then (8) (weak duality) implies that $v(D_F) = -\infty$. Hence, assume that $a > -\infty$. By Lemma 3.7 we have that

$$ri(cl(epi(f))) \cap [ri(cl(G)) \times (-\infty, a)] = \emptyset. \tag{30}$$

Denoting $A := cl(epi(f))$ and $B := cl(G) \times (-\infty, a]$, by our hypothesis and Lemma 3.1 (i) these sets are convex and, by (30),

$$ri(A) \cap ri(B) = \emptyset.$$

By the well-known separation theorem in finite dimensional spaces (see for instance Ref. 11, Theorem 11.3) the sets A and B can be properly separated, that is, there exists a vector $(p^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$ and $b \in \mathbb{R}$ such that

$$p^{*T}x + \mu^*\mu \leq b \leq p^{*T}y + \mu^*r, \quad \forall (x,\mu) \in A,\, (y,r) \in B \tag{31}$$

and

$$\inf\{p^{*T}x + \mu^*\mu : (x,\mu) \in A\} < \sup\{p^{*T}y + \mu^*r : (y,r) \in B\}. \tag{32}$$

It is easy to see that $\mu^* \leq 0$. Let us show that $\mu^* \neq 0$. For this, suppose by contradiction that $\mu^* = 0$. By (31) we have that $b \leq p^{*T}y$ for each $y \in cl(G)$. On the other hand, since $G \subseteq X$ we have that $p^{*T}x \leq b$ for each $x \in G$ (also by (31)) and consequently $p^{*T}x \leq b$ for each $x \in cl(G)$. Therefore,

$$p^{*T}x = b, \quad \forall x \in cl(G). \tag{33}$$

Now by the inequality (32) one can choose an element $(\bar{x}, \bar{\mu}) \in cl(epi(f))$ such that $p^{*T}\bar{x} < b$. Thus there exists $(x_1, \mu_1) \in epi(f)$ such that $p^{*T}x_1 < b$ or, in other words, there exists an element $x_1 \in X$ for which $p^{*T}x_1 < b$.

By our assumption (14) follows that there exists $x_2 \in X$ with $g(x_2) \in -ri(C)$. This means that $x_2 \in G$ and therefore $p^{*T}x_2 = b$ on one hand and there exists $\varepsilon > 0$ such that

$$(g(x_2) + \varepsilon B) \cap aff(-C) \subseteq -C, \tag{34}$$

on the other hand. Choose a number $0 < \bar{t} < 1$ such that

$$t' \| g(x_1) - g(x_2) \| < \varepsilon, \quad \forall\, 0 < t' < \bar{t}.$$

By our assumption (13) we have for every $t' \in \mathbb{R}$ that

$$t'(g(x_1) - g(x_2)) \in lin(C)$$

which together with $g(x_2) \in -ri(C) \subseteq lin(C)$ leads to

$$g(x_2) + t'(g(x_1) - g(x_2)) \in lin(C) = lin(-C) = aff(-C), \quad \forall t' \in \mathbb{R}.$$

The latter implies in virtue of (34) that

$$g(x_2) + t'(g(x_1) - g(x_2)) \in -C, \quad \forall\, 0 < t' < \bar{t}. \tag{35}$$

By Lemma 3.1 (ii) follows that the set $\Omega_{epi_C(g)}$ consisting of all $t \in [0,1]$ such that for each $x, y \in X$ we have

$$tx + (1-t)y \in X$$

and

$$g(tx + (1-t)y) \leq_C tg(x) + (1-t)g(y)$$

is dense in $[0, 1]$. Therefore one can choose a number $\tilde{t} \in \Omega_{epi_C(g)}$ such that $0 < \tilde{t} < \bar{t}$. Then we have by (35) that

$$g(\tilde{t}x_1 + (1-\tilde{t})x_2) \in -C + \tilde{t}g(x_1) + (1-\tilde{t})g(x_2) =$$

$$-C + g(x_2) + \tilde{t}(g(x_1) - g(x_2)) \subseteq -C + (-C) \subseteq -C.$$

The latter relation shows that the element $\tilde{t}x_1 + (1-\tilde{t})x_2 \in G$, hence by (33) we obtain

$$p^{*T}(\tilde{t}x_1 + (1-\tilde{t})x_2) = b. \tag{36}$$

On the other hand, $p^{*T}x_1 < b$ and $p^{*T}x_2 = b$ imply

$$p^{*T}(\tilde{t}x_1 + (1-\tilde{t})x_2) = \tilde{t}p^{*T}x_1 + (1-\tilde{t})p^{*T}x_2 < b,$$

which contradicts (36). This contradiction shows that $\mu^* \neq 0$ and therefore $\mu^* < 0$.

Now dividing relation (31) by $-\mu^*$ one obtains

$$p_0^{*T}x - \mu \leq b_0 \leq p_0^{*T}y - r, \quad \forall (x, \mu) \in A,\ (y, r) \in B, \tag{37}$$

where $p_0^* := \frac{1}{-\mu^*}p^*$ and $b_0 := \frac{1}{-\mu^*}b$. Since for every $x \in X$ the pair $(x, f(x)) \in epi(f)$ we obtain by (37) that

$$p_0^{*T}x - f(x) \leq b_0, \quad \forall x \in X,$$

and taking the supremum of the left hand side over all $x \in X$ we get

$$f^*(p_0^*) \leq b_0. \tag{38}$$

Similarly, since for every $x \in G$ the pair $(x, a) \in cl(G) \times (-\infty, a]$, also by (37) we obtain

$$b_0 \leq p_0^{*T}x - a, \quad \forall x \in G,$$

therefore,

$$a + b_0 \leq \inf_{x \in G} p_0^{*T}x. \tag{39}$$

Combining relations (38) and (39) it follows

$$a \leq -f^*(p_0^*) + \inf_{x \in G} p_0^{*T}x$$

or, in other words,

$$\inf_{x\in G} f(x) \le -f^*(p_0^*) + \inf_{x\in G} p_0^{*T}x \le \sup_{p^*\in\mathbb{R}^n} [-f^*(p^*) + \inf_{x\in G} p^{*T}x] = v(D_F). \qquad (40)$$

Hence we have shown that $v(P) \le v(D_F)$ and since the reverse inequality holds by (8) (weak duality), it means that in (40) we must have equality. This shows that the strong Fenchel duality holds on one hand, and p_0^* is a solution of problem (D_F), on the other hand. This completes the proof. □

Now combining Theorems 3.1 and 3.2 with Proposition 2.1 we obtain the following result which states that under suitable conditions the optimal objective values of the primal problem and its three dual problems defined in Section 2 are equal, or, in other words, strong duality between these problems holds.

Theorem 3.3. Suppose that the assumptions of Theorem 3.2 hold. Then

$$v(P) = v(D_L) = v(D_F) = v(D_{FL}).$$

Moreover, if $a := \inf_{x\in G} f(x) > -\infty$ then all dual problems (D_L), (D_F) and (D_{FL}) have optimal solutions.

Proof. By Theorems 3.1 and 3.2 we obtain that $v(D_{FL}) = v(D_F) = v(P)$. Moreover, by (6) and Proposition 2.1, $v(P) \ge v(D_L) \ge v(D_{FL})$. This means that $v(P) = v(D_L)$ and the first part of the proof is complete.

If $a := \inf_{x\in G} f(x) > -\infty$ then by Theorem 3.2 follows that (D_F) has an optimal solution $p_0^* \in \mathbb{R}^n$. By repeating the first part of the proof of Theorem 3.1 we obtain for this vector p_0^* that there exists $q_0^* \in C^*$ satisfying (26). It is easy to see now that the pair (p_0^*, q_0^*) is an optimal solution of (D_{FL}) and, furthermore, q_0^* is an optimal solution of (D_L). Therefore, the proof is complete.□

Remark 3.2. A careful analysis of the proof shows that our assumption in Theorem 3.2 (and consequently, in Theorem 3.3) concerning nearly convexity of functions f and g does not require that they have the same (nearly convexity) constant (see Definitions 3.1 and 3.2). In particular for $X \subseteq \mathbb{R}^n$ a convex set, the situation when the dense sets $\Omega_{epi(f)} \subset [0,1]$ and $\Omega_{epi_C(g)} \subset [0,1]$ are disjoint may occur.

We shall consider a nearly convex optimization problem for which the strong duality (Theorem 3.3) holds as follows.

Let $\mathbb{Q} \subset \mathbb{R}$ be the set of all rational numbers and $X := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\, x_2 > 0\} \cup \{(x_1, 0) \in \mathbb{R}^2 : x_1 \ge 0,\, x_1 \in \mathbb{Q}\} \cup \{(0, x_2) \in \mathbb{R}^2 : x_2 \ge 0,\, x_2 \in \mathbb{Q}\}$. This set is clearly nearly convex with constant $1/2$. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary convex function (with finite values) and define $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ as $f(x) := F(x)$ for $x \in X$ and $f(x) := \infty$ otherwise. Then clearly f is a nearly convex function on $\mathbb{R}^2$ (with constant 1/2). Also, $ri(epi(f)) = int(epi(f)) \neq \emptyset$ since for instance the vector $(1, 1, f(1, 1) + 1)$ belongs to it. Therefore, condition (i) of Lemma 3.7 is satisfied.

Let $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x_1, x_2) := x_1 + x_2 - 1$. Then g is an affine function (and therefore a convex function) and for $G = \{x \in X : g(x) \leq 0\}$ we have $ri(G) = int(G) \neq \emptyset$ (take for instance the element $(1/4, 1/4) \in int(G)$). Hence, condition (ii) of Lemma 3.7 is also satisfied. It is also immediate that the regularity conditions (13) and (14) are satisfied for $C := [0, \infty)$. Furthermore, $a := \inf_{x \in G} f(x) = \inf_{x \in G} F(x) < \infty$ since F being convex on $\mathbb{R}^2$ is continuous on $cl(G)$.

Now consider the following optimization problem

$$\inf\{f(x) : x \in \mathbb{R}^2,\, g(x) \leq 0\},$$

which is a nearly convex (but not convex) optimization problem on $\mathbb{R}^2$. Since this problem is equivalent to

$$\inf_{x \in G} f(x)$$

where

$$G = \{x \in X : g(x) \leq 0\}$$

and, as we have seen, the latter satisfies all conditions of Theorem 3.3, this theorem can be applied and we obtain the strong duality result.

4. Conclusions

Finally, let us recall the most important new and original results of this paper and suggest some possible future research on this subject.

For the optimization problem (P) (cf. Section 2.1) we have introduced three different dual problems based on a conjugacy and perturbation approach, the classical Lagrange dual (D_L), the Fenchel dual (D_F) and a new dual called Fenchel-Lagrange dual problem (D_{FL}).

The last-mentioned has been introduced for the first time by two of the authors in the paper Ref. 1. The former duality results in Ref. 1 have been established under classical convexity and regularity conditions. But within the present paper we could exceed those bounds supposing so-called nearly convexity assumptions and a new type of constraint qualification (cf. (13) and (14)).

Thus we are able to point out in Theorem 3.1 that the optimal objective function values $v(D_F)$ and $v(D_{FL})$ of the Fenchel and the Fenchel-Lagrange dual problems, respectively, coincide if the function g defining the set of constraints is nearly convex and the constraint qualifications (13) and (14) are fulfilled. The proof is essentially based on Lemma 3.5 which states a solvability condition for an inequality system including a nearly convex function. Independently of its consequence for the duality, this assertion is interesting for itself as a new result for the characterization of a solution of a general inequality system.

In Theorem 3.2 we proved under nearly convexity conditions for the objective function f and the constraint function g, the constraint qualifications (13), (14) and two further natural assumptions (cf. (i) and (ii) of Lemma 3.7) that there is strong duality between the original primal problem (P) and the Fenchel dual problem (D_F) indicating the coincidence of the optimal values of both problems $v(P) = v(D_F)$.

Afterwards we have verified that under the hypotheses of Theorem 3.2 the optimal objective values of the primal problem and of the three dual problems are equal $v(P) = v(D_L) = v(D_F) = v(D_{FL})$. Furthermore, when this value is finite the three dual problems have solutions.

With these considerations we have shown a way how strong duality results can be generalized to a kind of nonconvex programming problems with nearly convex functions. As we have seen this requires also a new kind of constraint qualification.

We think that in the future these basic investigations will make possible to treat also more general nonconvex mathematical programming problems, e.g. such ones with composed nearly convex functions as objective and constraint functions, respectively. Another direction of such kind of future research could be the duality for multiobjective programming problems with nearly convex functions. We could also imagine that some types of fractional programming problems where the occurring convex (concave) functions would be replaced by nearly convex (concave) functions represent an interesting area of thinking about duality.

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