Duality for multiobjective optimization problems with convex objective functions and D.C. constraints

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Abstract. In this paper we provide a duality theory for multiobjective optimization problems with convex objective functions and finitely many D.C. constraints. In order to do this, we study first the duality for a scalar convex optimization problem with inequality constraints defined by extended real-valued convex functions. For a family of multiobjective problems associated to the initial one we determine then, by means of the scalar duality results, their multiobjective dual problems. Finally, we consider as a special case the duality for the convex multiobjective optimization problem with convex constraints.

Key words: Multiobjective optimization, D.C. constraints, Conjugate duality, Optimality conditions

1. Introduction

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In the recent years, different duality theories have been provided for optimization problems with a difference of two convex functions in either the objective function or the constraints, or both. It has been observed that convex duality theory can be used for such nonconvex problems in order to construct dual problems with a zero duality gap (see for instance [3], [5], [6], [7], [8], [9]).

In the present work, our main purpose is to develop a duality theory for a multiobjective optimization problem with a convex objective function and finitely many D.C. constraints. By using the approach presented in [3], we express the feasible set in terms of Legendre-Fenchel conjugates of the data functions.

The basic and fruitful idea for the study of the duality for the multiobjective problem is to associate a scalar optimization problem and to establish, by means of the conjugacy approach (cf. [1], [12]), a suitable scalar dual problem. We derive the strong duality and the optimality conditions which later are used to obtain duality assertions for the primal multiobjective problem.

Following the same scheme, similar duality results are established for the multiobjective problem with a convex objective function and strict inequality D.C. constraints.

Finally, we consider as a special case of the initial problem, the multiobjective problem with a convex objective function and convex inequality constraints. For this problem the results concerning duality generalize those obtained in the past (see [10], [11], [13]]).

2. The formulation of the problem

The multiobjective optimization problem with D.C. constraints, which we consider here, is

$$(P) \quad \text{v-}\min_{x \in \mathcal{A}} f(x),$$

$$\mathcal{A} = \{x \in X : g_i(x) - h_i(x) \leq 0, i \in I_m\},$$

$$f(x) = (f_1(x), \ldots, f_k(x))^T,$$

where $f_i : X \to \overline{\mathbb{R}}$, $i = 1, ..., k$, are proper convex functions and $g_i, h_i : X \to \overline{\mathbb{R}}$, $i \in I_m = \{1, ..., m\}$, are extended real-valued convex functions on the real Hausdorff locally convex vector space X. Let $g : X \to \overline{\mathbb{R}}^m$ be the following vector function $g(x) = (g_1(x), ..., g_m(x))^T$. Moreover, we assume that the functions $h_i, i \in I_m$, are subdifferentiable on the feasible set of (P).

For the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, let us adopt the following conventions (see [4])

$$\begin{aligned} (+\infty) - (+\infty) = (-\infty) - (-\infty) = (+\infty) + (-\infty) \\ = (-\infty) + (+\infty) = +\infty \end{aligned} \tag{1}$$

and

$$0 \times (+\infty) = +\infty, 0 \times (-\infty) = 0. \tag{2}$$

Of course, for $r > 0$, we set $r(+\infty) = +\infty, r(-\infty) = -\infty$, and, for $r < 0$, $r(+\infty) = -\infty$ and $r(-\infty) = +\infty$.

By (2), for a function $f : X \to \overline{\mathbb{R}}$, we have $0f = \delta_{dom(f)}$, where $\delta_{dom(f)}$ is the indicator function of the set $dom(f) = \{x \in X : f(x) < +\infty\}$.

The notation "v-min" refers to a vector minimum problem. For this kind of problems different notions of solutions are known. We consider in this paper the so-called Pareto-efficient and properly efficient solutions.

DEFINITION 2.1 *An element $\bar{x} \in \mathcal{A}$ is said to be Pareto-efficient (or efficient)* *with respect to* (P) *if*

$$f(\bar{x}) \geq_{R^k_+} f(x), \quad \textit{for} \quad x \in \mathcal{A}, \quad \textit{implies} \quad f(\bar{x}) = f(x).$$

DEFINITION 2.2 ([2]) *An element* $\bar{x} \in \mathcal{A}$ *is said to be properly efficient with respect to* (P) *if it is efficient and there exists a scalar* $M > 0$ *such that, for each* $x \in \mathcal{A}$ *and each* $i \in \{1, ..., k\}$ *satisfying* $f_i(x) < f_i(\bar{x})$*, there exists* $j \in \{1, ..., k\}$ *such that* $f_j(x) > f_j(\bar{x})$ *and*

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M.$$

One may notice that the hypotheses concerning g_i and h_i, $i \in I_m$, are the same as those of the D.C. optimization problem considered by Martinez-Legaz and Volle in [3]. They have shown that the feasible set of the problem (P) can be written in the following way (cf. Lemma 2.1 in [3])

$$\mathcal{A} = \bigcup_{\substack{h_i^*(x_i^*) - g_i^*(x_i^*) \leq 0, \\ i=1,...,m}} \{x \in X : h_i^*(x_i^*) + g_i(x) - \langle x_i^*, x \rangle \leq 0, i \in I_m\}. \tag{3}$$

Here, h_i^* and g_i^* are the conjugate functions of h_i and g_i, respectively, for $i = 1, ..., m$. Let us recall briefly this notion. To each extended real-valued function $f : X \to \overline{\mathbb{R}}$ corresponds its conjugate function $f^* : X^* \to \overline{\mathbb{R}}$,

$$f^*(x^*) = \sup_{x \in X}\{\langle x^*, x \rangle - f(x)\},$$

for any $x^* \in X^*$, where X^* is the topological dual space of X. As usual, $\langle \cdot, \cdot \rangle$ is the bilinear pairing between X^* and X.

Using this result, we introduce for each $x^* = (x_1^*, ..., x_m^*)$, $x_i^* \in X^*, i \in I_m$, such that $h_i^*(x_i^*) - g_i^*(x_i^*) \leq 0, i \in I_m$, the multiobjective optimization problem

$$(P_{x^*}) \quad \underset{x \in \mathcal{A}_{x^*}}{\text{v-min}}\, f(x),$$

$$\mathcal{A}_{x^*} = \{x \in X : h_i^*(x_i^*) + g_i(x) - \langle x_i^*, x \rangle \leq 0, i \in I_m\},$$

$$f(x) = (f_1(x), \ldots, f_k(x))^T.$$

Let us notice that

$$\mathcal{A} = \bigcup_{\substack{x^* = (x_1^*, ..., x_m^*), \\ h_i^*(x_i^*) - g_i^*(x_i^*) \leq 0, \\ i = 1, ..., m}} \mathcal{A}_{x^*}.$$

When $\mathcal{A}$ is a non-empty set, from the assumptions we made at the beginning of this section, we have that the functions $h_i, i = 1, ..., m$ have proper conjugates and that $\bigcap_{i=1}^{m} dom(g_i) \neq \emptyset$.

The following two obvious assertions show the connection between the solutions of the problems (P) and (P_{x^*}).

PROPOSITION 2.1 *If $x \in \mathcal{A}$ is Pareto-efficient to (P), then there exists an $x^* = (x_1^*, ..., x_m^*)$, $x_i^* \in X^*$, $h_i^*(x_i^*) - g_i^*(x_i^*) \leq 0, i \in I_m$, such that x is Pareto-efficient to (P_{x^*}).*

PROPOSITION 2.2 *If $x \in \mathcal{A}$ is properly efficient to (P), then there exists an $x^* = (x_1^*, ..., x_m^*)$, $x_i^* \in X^*$, $h_i^*(x_i^*) - g_i^*(x_i^*) \leq 0, i \in I_m$, such that x is properly efficient to (P_{x^*}).*

3. Duality for the extended real-valued scalar optimization problem

In this section we deal with the duality for the following scalar optimization problem

$$(P_s) \quad \inf F(x),$$
$$G_i(x) \leq 0, i \in I_m,$$

with $F : X \to \overline{\mathbb{R}}$ and $G_i : X \to \overline{\mathbb{R}}$, $i \in I_m = \{1, ..., m\}$, being extended real-valued convex functions. Moreover we assume that F is proper.

It is noted that problem (P_s) can be reformulated as a problem with an extended real-valued convex objective function, but without constraint. However the study of (P_s) will help us to establish the optimality conditions for a D.C. optimization problem in this paper. Using the conjugacy approach we construct a dual problem to (P_s) and give a constraint qualification which guarantees the strong duality, namely, that the optimal objective values of the primal and dual problem are equal and the dual has an optimal solution.

In order to do this, let us consider the perturbation function $\Phi : X \times X \times \mathbb{R}^m \to \overline{\mathbb{R}}$,

$$\Phi(x, p, q) = \begin{cases} F(x + p), & \text{if } G_i(x) \leq q_i, i \in I_m, \\ +\infty, & \text{otherwise,} \end{cases}$$

with $p \in X$ and $q = (q_1, ..., q_m)^T \in \mathbb{R}^m$ being the perturbation variables.

A dual problem to (P_s) is given by the following formula (cf. [1])

$$(D_s) \quad \sup_{\substack{p^* \in X^*, \\ q^* \in \mathbb{R}^m}} \{-\Phi^*(0, p^*, q^*)\},$$

where $p^* \in X^*$, $q^* \in \mathbb{R}^m$ are the dual variables and Φ^* is the conjugate function of Φ.

Using the properties of conjugate functions, it can be shown (see [1]) that $\inf(P_s) \geq \sup(D_s)$, meaning that the optimal objective value of the primal problem is greater than or equal to the optimal objective value of the dual. This implies that the weak duality always holds. In order to obtain the strong duality ($\inf(P_s) = \max(D_s)$) we consider the following constraint qualification.

$$(CQ_s) \left| \begin{array}{l} \text{There exists } x' \in dom(F) \text{ such that } F \text{ is continuous at } x' \text{ and} \\ G_i(x') < 0, \text{ for } i \in I_m. \end{array} \right.$$

Below, under the constraint qualification (CQ_s), we obtain a sufficient condition for the strong duality between (P_s) and (D_s), which is a special case of Proposition 2.1 in [1].

THEOREM 3.1 *Let the constraint qualification* (CQ_s) *be fulfilled. Then the dual problem* (D_s) *has an optimal solution and the strong duality holds, i.e.* $\inf(P_s) = \max(D_s)$.

Proof. The constraint qualification (CQ_s) being fulfilled, it follows that $\inf(P_s) \leq F(x') < +\infty$.

We distinguish now between the cases $\inf(P_s) = -\infty$ and $\inf(P_s) \in \mathbb{R}$.

If $\inf(P_s) = -\infty$, by the weak duality, it follows that $\sup(D_s) = -\infty$. This implies that, for $p^* \in X^*$ and $q^* \in \mathbb{R}^m$,

$$-\Phi^*(0, p^*, q^*) = \sup(D_s) = -\infty.$$

In this case, each pair $(p^*, q^*) \in X^* \times \mathbb{R}^m$ is an optimal solution of the dual and $\inf(P_s) = \max(D_s) = -\infty$.

Let us assume now that $\inf(P_s) \in \mathbb{R}$. One can notice that the function Φ is convex.

Let $\varepsilon > 0$. By the continuity of F at x', there exists an open neighborhood $V \subseteq X$ of 0 such that $\forall p \in V$

$$|F(x'+p) - F(x')| < \varepsilon. \tag{4}$$

On the other hand, (CQ_s) being fulfilled, there exists an $\delta > 0$ such that $G_i(x') \leq -\delta, \forall i \in I_m$. Then the set $V \times (-\delta, +\delta)^m$ is an open neighborhood of $(0,0)$ in $X \times \mathbb{R}^m$ and, for each $p \in V$ and $q \in (-\delta, +\delta)^m$, it holds (by (4))

$$|\Phi(x', p, q) - \Phi(x', 0, 0)| = |F(x'+p) - F(x')| < \varepsilon.$$

This implies that the function $(p,q) \mapsto \Phi(x', p, q)$ is continuous at $(0,0) \in X \times \mathbb{R}^m$. The stability criterion introduced in Proposition III.2.3 in [1] is fulfilled and therefore the problem (P_s) is stable. Then from Proposition III.2.2 in [1] we have that (P_s) is normal and the dual problem (D_s) has optimal solutions. By the equivalence $(i) \Leftrightarrow (iii)$ in Proposition III.2.1 in [1] (cf. Remark 2.3 in [1] this equivalence is true if Φ is a proper convex function) we obtain further that the optimal objective values of the primal and dual are equal. □

The final form of the dual (D_s) can now be found by calculating the conjugate function of Φ. In [12] and [13] we proved that this leads to the following formulation for the dual problem of (P_s)

$$(D_s) \quad \sup_{\substack{p \in X^*, q \in \mathbb{R}^m, \\ q \geqq 0}} \left\{ -F^*(p) - (q^T G)^*(-p) \right\},$$

where $G(x) = (G_1(x), ..., G_m(x))^T$. Here, the conventions (1), (2) and the related calculus rules are crucial.

REMARK 3.1 Let us notice that if for an $i = 1, ..., m$, $G_i(x) = -\infty$ for some $x \in \bigcap_{i=1}^{m} dom(G_i)$, then q_i must be 0 in (D_s), because otherwise $(q^T G)^* \equiv +\infty$ and this q does not contribute to the supremum.

REMARK 3.2 One may notice that the duality scheme for scalar optimization problems presented above is different from that one used by Martinez-Legaz and Volle in [3].

Using the strong duality result between (P_s) and (D_s) we can derive now the following optimality conditions.

THEOREM 3.2

(a) *Let us assume that the constraint qualification* (CQ_s) *is fulfilled and let* $\bar{x}$ *be an optimal solution to* (P_s)*. Then there exists* $(\bar{p}, \bar{q}) \in X^* \times \mathbb{R}^m, \bar{q} \geqq 0$*, an optimal solution to* (D_s)*, such that the following optimality conditions are satisfied*

$$(i) \quad F^*(\bar{p}) + F(\bar{x}) = \langle \bar{p}, \bar{x} \rangle,$$

$$(ii) \quad \bar{q}^T G(\bar{x}) = 0,$$

$$(iii) \quad (\bar{q}^T G)^*(-\bar{p}) = \langle -\bar{p}, \bar{x} \rangle.$$

(b) *Let* $\bar{x}$ *be admissible to* (P_s) *and* $(\bar{p}, \bar{q})$ *be admissible to* (D_s)*, satisfying* (i)*,* (ii) *and* (iii)*. Then* $\bar{x}$ *is an optimal solution to* (P_s)*,* $(\bar{p}, \bar{q})$ *is an optimal* *solution to (D_s) and the strong duality holds*

$$F(\bar{x}) = -F^*(\bar{p}) - (\bar{q}^T G)^*(-\bar{p}).$$

Proof.

(a) The function F being proper, we have that $\inf(P_s)$ is finite. Then, by Theorem 3.1, follows that there exists an optimal solution to (D_s) $(\bar{p}, \bar{q}) \in X^* \times \mathbb{R}^m, \bar{q} \geqq 0$ such that $F(\bar{x}) = -F^*(\bar{p}) - (\bar{q}^T G)^*(-\bar{p})$ or, equivalently,

$$F(\bar{x}) + F^*(\bar{p}) - \langle \bar{p}, \bar{x} \rangle + \langle \bar{p}, \bar{x} \rangle + (\bar{q}^T G)^*(-\bar{p}) = 0. \tag{5}$$

This implies that $F^*(\bar{p}) \in \mathbb{R}$ and $(\bar{q}^T G)^*(-\bar{p}) \in \mathbb{R}$.

On the other hand, we have

$$-\infty < -(\bar{q}^T G)^*(-\bar{p}) \leq \langle \bar{p}, \bar{x} \rangle + \bar{q}^T G(\bar{x}).$$

Considering $I(x) = \{i \in I_m : G_i(x) = -\infty\}$, it follows that $\bar{q}_i = 0$ if $i \in I(\bar{x})$. Therefore holds

$$\begin{aligned} -\infty < \; & \inf_{x \in X} \left[\langle \bar{p}, x \rangle + \bar{q}^T G(x) \right] && \leq \langle \bar{p}, \bar{x} \rangle + \bar{q}^T G(\bar{x}) \\ = \; & \langle \bar{p}, \bar{x} \rangle + \sum_{i \in I_m \setminus I(\bar{x})} \bar{q}_i G_i(\bar{x}) && \leq \langle \bar{p}, \bar{x} \rangle. \end{aligned} \tag{6}$$

Finally, from (5), (6) and taking Young's inequality $F(\bar{x}) + F^*(\bar{p}) \geq \langle \bar{p}, \bar{x} \rangle$ into consideration we have

$$F^*(\bar{p}) + F(\bar{x}) = \langle \bar{p}, \bar{x} \rangle,$$

$$-(\bar{q}^T G)^*(-\bar{p}) = \inf_{x \in X} \left[\langle \bar{p}, x \rangle + \bar{q}^T G(x) \right] = \langle \bar{p}, \bar{x} \rangle$$

and

$$\bar{q}^T G(\bar{x}) = 0, \text{ with } \bar{q}_i = 0, \text{ for } i \in I(\bar{x}).$$

(b) The conclusion follows by doing all the calculations and transformations from (a) in the reverse direction.

□

Let us denote by $\prod_{i=1}^{m} \{h_i^* - g_i^* \leq 0\}$ the set of those $x^* = (x_1^*, ..., x_m^*)$, $x_i^* \in X^*, i \in I_m$, with the property that $h_i^*(x_i^*) - g_i^*(x_i^*) \leq 0, i \in I_m$.

Returning to the vectorial case, for every $x^* = (x_1^*, ..., x_m^*) \in \prod_{i=1}^{m} \{h_i^* - g_i^* \leq 0\}$, let us associate to the multiobjective problem (P_{x^*}) the following scalar problem

$$(P_{sx^*}) \quad \inf \sum_{j=1}^{k} \lambda_j f_j(x),$$
$$\tilde{g}_i(x) := g_i(x) + h_i^*(x_i^*) - \langle x_i^*, x \rangle \leq 0, i \in I_m,$$

with $\lambda = (\lambda_1, ..., \lambda_k)^T \in int(\mathbb{R}_+^k)$ fixed.

Moreover, we assume the following constraint qualification.

(CQ_{sx^*}) | There exists $x' \in \bigcap_{j=1}^{k} dom(f_j)$ such that f_j is continuous at x', $j = 1, ..., k$, and $\tilde{g}_i(x') < 0$ for $i \in I_m$.

Following the same scheme as in the first part of this section, a dual problem to (P_{sx^*}) is given by

$$(D_{sx^*}) \quad \sup_{\substack{p\in X^*, q\in\mathbb{R}^m,\\ q\geqq 0}} \left\{ -\left(\sum_{j=1}^k \lambda_j f_j\right)^*(p) - (q^T\tilde{g})^*(-p)\right\},$$

where $\tilde{g}(x) = (\tilde{g}_1(x), ..., \tilde{g}_m(x))^T$. One may notice that even the case of extended-valued constrained functions $\tilde{g}_i, i \in I_m$ in (P_{sx^*}) is covered by the duality theory developed in the first part of the section. This situation has been taken into consideration when we assumed that $G_i, i \in I_m$ in (P_s) were also extended real-valued functions.

By Remark 3.3 in [3] we have that

$$\left(\sum_{j=1}^k \lambda_j f_j\right)^*(p) = \min_{p=\sum\limits_{j=1}^k \tilde{p}_j} \sum_{j=1}^k (\lambda_j f_j)^*(\tilde{p}_j),$$

and, using the properties of the conjugate functions, we obtain for the dual of (P_{sx^*}) the following formulation

$$(D_{sx^*}) \quad \sup_{\substack{p_j\in X^*, j=1,...,k,\\ q\in\mathbb{R}^m, q\geqq 0}} \left\{ -\sum_{j=1}^k \lambda_j f_j^*(p_j) - (q^T\tilde{g})^*\left(-\sum_{j=1}^k \lambda_j p_j\right)\right\}.$$

Theorem 3.1 implies the following strong duality theorem for (P_{sx^*}).

THEOREM 3.3 *If the constraint qualification (CQ_{sx^*}) is fulfilled, then the dual problem (D_{sx^*}) has an optimal solution and the strong duality holds, i.e.* $\inf(P_{sx^*}) = \max(D_{sx^*})$.

Next, we give the optimality conditions for the problems (P_{sx^*}) and (D_{sx^*}).

THEOREM 3.4

(a) *Let the constraint qualification* (CQ_{sx^*}) *be fulfilled and let* $\bar{x}$ *be an optimal solution to* (P_{sx^*}). *Then there exists an optimal solution to* (D_{sx^*}) $(\bar{p}, \bar{q}), \bar{p} = (\bar{p}_1, ..., \bar{p}_k)$, $\bar{p}_i \in X^*, i = 1, ..., k$, $\bar{q} \geqq 0$ *such that the following optimality conditions are satisfied*

$$(i) \qquad f_j^*(\bar{p}_j) + f_j(\bar{x}) = \langle \bar{p}_j, \bar{x} \rangle, j = 1, ..., k,$$

$$(ii) \qquad \bar{q}^T g(\bar{x}) + \sum_{i=1}^{m} \bar{q}_i h_i^*(x_i^*) = \left\langle \sum_{i=1}^{m} \bar{q}_i x_i^*, \bar{x} \right\rangle,$$

$$(iii) \qquad \inf_{x \in X} \left[\left\langle \sum_{j=1}^{k} \lambda_j \bar{p}_j - \sum_{i=1}^{m} \bar{q}_i x_i^*, x \right\rangle + \bar{q}^T g(x) \right] = \left\langle \sum_{j=1}^{k} \lambda_j \bar{p}_j, \bar{x} \right\rangle - \sum_{i=1}^{m} \bar{q}_i h_i^*(x_i^*).$$

(b) *Let* $\bar{x}$ *be admissible to* (P_{sx^*}) *and* $(\bar{p}, \bar{q})$ *be admissible to* (D_{sx^*}), *satisfying* (i), (ii) *and* (iii). *Then* $\bar{x}$ *is an optimal solution to* (P_{sx^*}), $(\bar{p}, \bar{q})$ *is an optimal solution to* (D_{sx^*}) *and the strong duality holds*

$$\sum_{j=1}^{k} \lambda_j f_j(\bar{x}) = - \sum_{j=1}^{k} \lambda_j f_j^*(\bar{p}_j) - (\bar{q}^T \tilde{g})^* \left(- \sum_{j=1}^{k} \lambda_j \bar{p}_j \right).$$

REMARK 3.3 The relation (iii) in Theorem 3.4 is equivalent to

$$(\bar{q}^T g)^* \left(\sum_{i=1}^{m} \bar{q}_i x_i^* - \sum_{j=1}^{k} \lambda_j \bar{p}_j \right) = \sum_{i=1}^{m} \bar{q}_i h_i^*(x_i^*) - \left\langle \sum_{j=1}^{k} \lambda_j \bar{p}_j, \bar{x} \right\rangle. \qquad (7)$$

4. Duality for the multiobjective problem with D.C. constraints

For the functions $f_j, j = 1, ..., k$, let us impose the additional hypothesis of continuity over the set $\bigcap\limits_{j=1}^{k} dom(f_j)$, which is assumed to be non-empty. We denote by $g : X \to \overline{\mathbb{R}}^m$ the vector function $g(x) = (g_1(x), ..., g_m(x))^T$.

In section 2 we have introduced for each $x^* = (x_1^*, ..., x_m^*),\ x_i^* \in X^*, i \in I_m$, with the property that $h_i^*(x_i^*) - g_i^*(x_i^*) \leq 0, i \in I_m$, the following multiobjective optimization problem

$$(P_{x^*}) \quad \underset{x \in \mathcal{A}_{x^*}}{\text{v-min}}\, f(x),$$

$$\mathcal{A}_{x^*} = \left\{x \in X : h_i^*(x_i^*) + g_i(x) - \langle x_i^*, x\rangle \leq 0, i \in I_m\right\},$$

$$f(x) = (f_1(x), \ldots, f_k(x))^T.$$

We associate to each (P_{x^*}) a multiobjective optimization problem (D_{x^*}) and, by means of this family of multiobjective problems, we will formulate two theorems concerning the duality for the problem (P). The dual problem is

$$(D_{x^*}) \quad \underset{(p,q,\lambda,t) \in \mathcal{B}_{x^*}}{\text{v-max}}\ h_{x^*}(p, q, \lambda, t),$$

with

$$h_{x^*}(p, q, \lambda, t) = \begin{pmatrix} h_{x^*1}(p, q, \lambda, t) \\ \vdots \\ h_{x^*m}(p, q, \lambda, t) \end{pmatrix},$$

$$h_{x^*j}(p, q, \lambda, t) = -f_j^*(p_j) - \left((q^j)^T g\right)^* \left(-\frac{1}{k\lambda_j} \sum_{j=1}^{k} \lambda_j p_j + \sum_{i=1}^{m} q_i^j x_i^*\right) + t_j,$$

for $j = 1, \dots, k$, the dual variables

$$p = (p_1, \dots, p_k), q = (q^1, \dots, q^k), \lambda = (\lambda_1, \dots, \lambda_k)^T, t = (t_1, \dots, t_k)^T,$$

$$p_j \in X^*, \quad q^j \in \mathbb{R}^m, \quad \lambda_j \in \mathbb{R}, \quad t_j \in \mathbb{R}, \quad j = 1, \dots, k,$$

and the set of constraints

$$\mathcal{B}_{x^*} = \left\{ (p, q, \lambda, t) : \lambda \in int(\mathbb{R}^k_+), \sum_{j=1}^{k} \lambda_j q^j \geqq 0, \sum_{j=1}^{k} \lambda_j t_j = \sum_{j=1}^{k} \sum_{i=1}^{m} \lambda_j q^j_i h^*_i(x^*_i) \right\}.$$

The following constraint qualification has been introduced by Martinez-Legaz and Volle in [3].

(CQ) | For each $x^* = (x^*_1, ..., x^*_m) \in \prod_{i=1}^{m} \{h^*_i - g^*_i \leq 0\}$, there exists $x' \in \bigcap_{j=1}^{k} dom(f_j)$, such that $g_i(x') + h^*_i(x^*_i) - \langle x^*_i, x' \rangle < 0$, $\forall i \in I_m$.

The constraint qualification (CQ) will be used later for the characterization of the properly efficient solutions of the problem (P). First, we prove a weak duality-type theorem.

THEOREM 4.1 *There is no $x \in \mathcal{A}$, no $x^* = (x^*_1, ..., x^*_m) \in \prod_{i=1}^{m} \{h^*_i - g^*_i \leq 0\}$, with the property that $x \in \mathcal{A}_{x^*}$, and no $(p, q, \lambda, t) \in \mathcal{B}_{x^*}$ such that $f_j(x) \leq h_{x^*j}(p, q, \lambda, t)$, for $j = 1, ..., k$, and $f_i(x) < h_{x^*i}(p, q, \lambda, t)$, for at least one $i \in \{1, ..., k\}$.*

Proof. Let $x \in \mathcal{A}$ and $x^* = (x^*_1, ..., x^*_m) \in \prod_{i=1}^{m} \{h^*_i - g^*_i \leq 0\}$ be such that $x \in \mathcal{A}_{x^*}$. By (3) we have that an element x^* with these properties always exists.

For a $(p,q,\lambda,t) \in \mathcal{B}_{x^*}$ let us assume that $f_j(x) \leq h_{x^*j}(p,q,\lambda,t)$, for $j = 1,...,k$, and $f_i(x) < h_{x^*i}(p,q,\lambda,t)$, for at least one $i \in \{1,...,k\}$. This implies that

$$\sum_{j=1}^{k} \lambda_j f_j(x) < \sum_{j=1}^{k} \lambda_j h_{x^*j}(p,q,\lambda,t). \tag{8}$$

On the other hand, we have

$$\sum_{j=1}^{k} \lambda_j h_{x^*j}(p,q,\lambda,t) = -\sum_{j=1}^{k} \lambda_j f_j^*(p_j) + \sum_{j=1}^{k} \lambda_j t_j$$

$$-\sum_{j=1}^{k} \lambda_j \left((q^j)^T g\right)^* \left(-\frac{1}{k\lambda_j}\sum_{j=1}^{k} \lambda_j p_j + \sum_{i=1}^{m} q_i^j x_i^*\right) \leq \sum_{j=1}^{k} \lambda_j f_j(x)$$

$$-\left\langle \sum_{j=1}^{k} \lambda_j p_j, x \right\rangle + \sum_{j=1}^{k}\sum_{i=1}^{m} \lambda_j q_i^j h_i^*(x_i^*) + \sum_{j=1}^{k} \lambda_j \left((q^j)^T g\right)(x)$$

$$+\left\langle \sum_{j=1}^{k} \lambda_j p_j - \sum_{j=1}^{k}\sum_{i=1}^{m} \lambda_j q_i^j x_i^*, x \right\rangle = \sum_{j=1}^{k} \lambda_j f_j(x) + \sum_{i=1}^{m} \left(\sum_{j=1}^{k} \lambda_j q^j\right)_i h_i^*(x_i^*)$$

$$+\sum_{i=1}^{m} \left(\sum_{j=1}^{k} \lambda_j q^j\right)_i g_i(x) - \sum_{i=1}^{m} \left\langle \left(\sum_{j=1}^{k} \lambda_j q^j\right)_i x_i^*, x \right\rangle$$

$$= \sum_{j=1}^{k} \lambda_j f_j(x) + \left(\sum_{j=1}^{k} \lambda_j q^j\right)^T \tilde{g}(x),$$

where $\tilde{g}(x) = (\tilde{g}_1(x), ..., \tilde{g}_m(x))^T$, $\tilde{g}_i(x) = g_i(x) + h_i^*(x_i^*) - \langle x_i^*, x\rangle$, $i \in I_m$. Here, $\left(\sum\limits_{j=1}^{k} \lambda_j q^j\right)_i$ is the i-th component of the vector $\sum\limits_{j=1}^{k} \lambda_j q^j \in \mathbb{R}^m$.

Because of $x \in \mathcal{A}_{x^*}$ and $(p,q,\lambda,t) \in \mathcal{B}_{x^*}$ we have $\tilde{g}_j(x) \leq 0$, for $j \in I_m$, and $\sum\limits_{j=1}^{k} \lambda_j q^j \geqq 0$. This implies that $\left(\sum\limits_{j=1}^{k} \lambda_j q^j\right)^T \tilde{g}(x) \leq 0$, and therefore follows that

$$\sum_{j=1}^{k} \lambda_j h_{x^*j}(p,q,\lambda,t) \leq \sum_{j=1}^{k} \lambda_j f_j(x).$$

This last inequality contradicts relation (8) and then the assertion of the theorem must be true. □

The following theorem gives a characterization of the properly efficient solutions of (P) by means of the Pareto-efficient elements of (D_{x^*}), for an $x^* = (x_1^*, ..., x_m^*) \in \prod_{i=1}^{m} \{h_i^* - g_i^* \leq 0\}$.

THEOREM 4.2. *Let the constraint qualification (CQ) be fulfilled and let $\bar{x}$ be a properly efficient solution to (P). Then there exists an $x^* = (x_1^*, ..., x_m^*) \in \prod_{i=1}^{m} \{h_i^* - g_i^* \leq 0\}$ and a Pareto-efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_{x^*}$ to the dual (D_{x^*}) such that $f(\bar{x}) = h_{x^*}(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$.*

Proof. Assume $\bar{x}$ to be properly efficient to (P). This implies that $\bar{x} \in \mathcal{A}$. By Proposition 2.2 there exists an $x^* = (x_1^*, ..., x_m^*) \in \prod_{i=1}^{m} \{h_i^* - g_i^* \leq 0\}$ such that $\bar{x}$ is properly efficient to (P_{x^*}).

The feasible set of (P_{x^*}), $\mathcal{A}_{x^*}$, is a convex set and the objective function of (P_{x^*}) is a convex function. Then follows that $\bar{x}$ to be properly efficient to (P_{x^*}) can be expressed via scalarization (see [2]). Therefore exists a vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_k)^T \in int(\mathbb{R}_+^k)$ such that $\bar{x}$ solves the scalar problem

$$(P_{sx^*}) \quad \inf_{x \in \mathcal{A}_{x^*}} \sum_{j=1}^{k} \bar{\lambda}_j f_j(x).$$

The constraint qualification (CQ) is fulfilled and this implies that for $x^* = (x_1^*, ..., x_m^*)$ the constraint qualification (CQ_{sx^*}) is also fulfilled. Under this hypotheses, Theorem 3.3 assures the existence of an optimal solution $(\tilde{p}, \tilde{q})$ to the dual of (P_{sx^*}) and Theorem 3.4 states that the optimality conditions (i), (ii) and

(iii) are satisfied.

Let us construct now by means of $\bar{x}$ and $(\tilde{p}, \tilde{q})$ a Pareto-efficient solution $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ to (D_{x^*}). Let $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_k)^T$ be the vector introduced above. We consider for $j = 1, \dots, k, \bar{p}_j := \tilde{p}_j$ and $\bar{p} := (\bar{p}_1, \dots \bar{p}_k) = (\tilde{p}_1, \dots \tilde{p}_k) = \tilde{p}$. It remains to define $\bar{q} = (\bar{q}^1, \dots \bar{q}^k)$ and $\bar{t} = (\bar{t}_1, \dots \bar{t}_k)^T$.

These are defined in the following way, for $j = 1, \dots, k$,

$$\begin{aligned} \bar{q}^j &:= \frac{1}{k\bar{\lambda}_j}\tilde{q} \in \mathbb{R}^m, \\ \bar{t}_j &:= \langle \bar{p}_j, \bar{x} \rangle + \left((\bar{q}^j)^T g\right)^* \left(-\frac{1}{k\bar{\lambda}_j} \sum_{j=1}^{m} \bar{\lambda}_j \bar{p}_j + \sum_{i=1}^{m} \bar{q}_i^j x_i^* \right) \in \mathbb{R}. \end{aligned} \tag{9}$$

For the new element $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ holds

$$\bar{\lambda} \in int(\mathbb{R}^k_+), \sum_{j=1}^{k} \bar{\lambda}_j (\bar{q})^j = \tilde{q} \geqq 0$$

and

$$\sum_{j=1}^{k} \bar{\lambda}_j \bar{t}_j = \left\langle \sum_{j=1}^{m} \bar{\lambda}_j \bar{p}_j, \bar{x} \right\rangle + \sum_{j=1}^{k} \bar{\lambda}_j \left(\frac{1}{k\bar{\lambda}_j} \tilde{q}^T g \right)^* \left(-\frac{1}{k\bar{\lambda}_j} \left(\sum_{j=1}^{k} \bar{\lambda}_j \bar{p}_j - \sum_{i=1}^{m} \tilde{q}_i x_i^* \right) \right)$$

$$= \left\langle \sum_{j=1}^{k} \bar{\lambda}_j \bar{p}_j, \bar{x} \right\rangle + \sum_{j=1}^{k} \bar{\lambda}_j \frac{1}{k\bar{\lambda}_j} (\tilde{q}^T g)^* \left(\sum_{i=1}^{m} \tilde{q}_i x_i^* - \sum_{j=1}^{k} \bar{\lambda}_j \bar{p}_j \right)$$

$$= \left\langle \sum_{j=1}^{k} \bar{\lambda}_j \bar{p}_j, \bar{x} \right\rangle + (\tilde{q}^T g)^* \left(\sum_{i=1}^{m} \tilde{q}_i x_i^* - \sum_{j=1}^{k} \bar{\lambda}_j \bar{p}_j \right)$$

$$= \left\langle \sum_{j=1}^{k} \bar{\lambda}_j \bar{p}_j, \bar{x} \right\rangle + \sum_{i=1}^{m} \tilde{q}_i h_i^*(x_i^*) - \left\langle \sum_{j=1}^{k} \bar{\lambda}_j \bar{p}_j, \bar{x} \right\rangle \quad \text{(by (7))}$$

$$= \sum_{i=1}^{m} \tilde{q}_i h_i^*(x_i^*) = \sum_{j=1}^{k} \sum_{i=1}^{m} \bar{\lambda}_j \bar{q}_i^j h_i^*(x_i^*).$$

Therefore, $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ is feasible to (D_{x^*}). In order to finish the proof, it remains to show that the values of the objective functions on these elements are equal, i.e. $f(\bar{x}) = h_{x^*}(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$.

What we actually prove is that $f_j(\bar{x}) = h_{x^*j}(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$, for each $j = 1, \ldots, k$. By using relation (i) from Theorem 3.4 and the equalities in (9), for $j = 1, ..., k$, holds

$$\begin{aligned} h_{x^*j}(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) &= -f_j^*(\bar{p}_j) - \left((\bar{q}^j)^T g\right)^* \left(-\frac{1}{k\bar{\lambda}_j} \sum_{j=1}^{k} \bar{\lambda}_j \bar{p}_j + \sum_{i=1}^{m} \bar{q}_i^j x_i^*\right) + \bar{t}_j \\ &= -f_j^*(\bar{p}_j) + \langle \bar{p}_j, \bar{x} \rangle = f_j(\bar{x}). \end{aligned}$$

The fact that $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$ is Pareto-efficient to (D_{x^*}) is given by Theorem 4.1. □

5. Duality for the multiobjective problem with strict inequalities D.C. constraints

The next problem which we treat in this paper is the multiobjective optimization problem with a convex objective function and strict inequalities D.C. constraints

$$(P^{si}) \quad \operatorname*{v-min}_{x \in \mathcal{A}^{si}} f(x),$$

$$\mathcal{A}^{si} = \{x \in X : g_i(x) - h_i(x) < 0, i \in I_m\},$$

$$f(x) = (f_1(x), \ldots, f_k(x))^T,$$

where $f_j : X \to \overline{\mathbb{R}}$, $j = 1, ..., k$, are proper convex functions, continuous on the set $\bigcap\limits_{j=1}^{k} dom(f_j)$, which is assumed to be non-empty, and $g_i, h_i : X \to \overline{\mathbb{R}}$, $i = 1, ..., m$, are extended real-valued convex functions. We assume, as in the previous sections, that the functions $h_i, i \in I_m$, are subdifferentiable on the feasible set of (P^{si}).

Martinez-Legaz and Volle have also shown that the feasible set of the problem (P^{si}) can be written in the following way (cf. Lemma 5.1 in [3])

$$\mathcal{A}^{si} = \bigcup_{\substack{h_i^*(x_i^*) - g_i^*(x_i^*) < 0, \\ i=1,...,m}} \{x \in X : h_i^*(x_i^*) + g_i(x) - \langle x_i^*, x\rangle < 0, i \in I_m\}. \tag{10}$$

Starting from this result, we introduce for every $x^* = (x_1^*, ..., x_m^*)$, $x_i^* \in X^*, i \in I_m$, with the property that $h_i^*(x_i^*) - g_i^*(x_i^*) < 0, i \in I_m$, the following multiobjective optimization problem

$$(P_{x^*}^{si}) \quad \operatorname*{v\text{-}min}_{x \in \mathcal{A}_{x^*}^{si}} f(x),$$

$$\mathcal{A}_{x^*}^{si} = \{x \in X : h_i^*(x_i^*) + g_i(x) - \langle x_i^*, x\rangle < 0, i \in I_m\},$$

$$f(x) = (f_1(x), \ldots, f_k(x))^T.$$

The next two results, similar to Propositions 2.1 and 2.2, are also true.

PROPOSITION 5.1 *If $x \in \mathcal{A}^{si}$ is Pareto-efficient to (P^{si}), then there exists $x^* = (x_1^*, ..., x_m^*)$, $x_i^* \in X^*, i \in I_m$, $h_i^*(x_i^*) - g_i^*(x_i^*) < 0, i \in I_m$ such that x is Pareto-efficient to $(P_{x^*}^{si})$.*

PROPOSITION 5.2. *If $x \in \mathcal{A}^{si}$ is properly efficient to (P^{si}), then there exists an $x^* = (x_1^*, ..., x_m^*)$, $x_i^* \in X^*, i \in I_m$, $h_i^*(x_i^*) - g_i^*(x_i^*) < 0, i \in I_m$ such that x is properly efficient to $(P_{x^*}^{si})$.*

For an $x^* = (x_1^*, ..., x_m^*)$, $x_i^* \in X^*, i \in I_m$ such that $h_i^*(x_i^*) - g_i^*(x_i^*) < 0, i \in I_m$, we associate to the multiobjective problem $(P_{x^*}^{si})$ the following scalar problem

$$(P_{sx^*}^{si}) \quad \inf \sum_{j=1}^{k} \lambda_j f_j(x)$$
$$\tilde{g}_i(x) = g_i(x) + h_i^*(x_i^*) - \langle x_i^*, x \rangle < 0, i \in I_m,$$

with $\lambda = (\lambda_1, ..., \lambda_k) \in int(\mathbb{R}_+^k)$ fixed.

If the constraint qualification (CQ_{sx^*}) is fulfilled, then by Lemma 5.2 in [3] follows

$$\inf(P_{sx^*}^{si}) = \inf(P_{sx^*}).$$

Therefore we can consider as a dual problem to $(P_{sx^*}^{si})$ the same optimization problem as for (P_{sx^*})

$$(D_{sx^*}) \quad \sup_{\substack{p_j \in X^*, j=1,...,k, \\ q \in \mathbb{R}^m, q \geqq 0}} \left\{ -\sum_{j=1}^{k} \lambda_j f_j^*(p_j) - (q^T \tilde{g})^* \left(-\sum_{j=1}^{k} \lambda_j p_j \right) \right\}.$$

Let us present for $(P_{sx^*}^{si})$ the strong duality theorem and the optimality conditions.

THEOREM 5.1 *If the constraint qualification* (CQ_{sx^*}) *is fulfilled, then the dual problem* (D_{sx^*}) *has an optimal solution and the strong duality holds, i.e.* $\inf(P_{sx^*}^{si}) = \inf(P_{sx^*}) = \max(D_{sx^*})$.

THEOREM 5.2

(a) Let the constraint qualification (CQ_{sx^*}) *be fulfilled and let* $\bar{x}$ *be an optimal*

solution to $(P^{si}_{sx^*})$*. Then there exists an optimal solution to* (D_{sx^*}) $(\bar{p},\bar{q}), \bar{p} = (\bar{p}_1,...,\bar{p}_k)$*,* $\bar{p}_i \in X^*, i = 1,...,k$*,* $\bar{q} \geqq 0$ *such that the following optimality conditions are satisfied*

$$\begin{array}{ll} (i) & f_j^*(\bar{p}_j) + f_j(\bar{x}) = \langle \bar{p}_j, \bar{x}\rangle, j = 1,...,k, \\ (ii) & \bar{q} = 0, \\ (iii) & \sum\limits_{j=1}^{k} \lambda_j \bar{p}_j = 0. \end{array}$$

(b) Let $\bar{x}$ *be admissible to* $(P^{si}_{sx^*})$ *and* $(\bar{p},\bar{q})$ *be admissible to* (D_{sx^*})*, satisfying (i),(ii) and (iii). Then* $\bar{x}$ *is an optimal solution to* $(P^{si}_{sx^*})$*,* $(\bar{p},\bar{q})$ *is an optimal solution to* (D_{sx^*}) *and the strong duality holds*

$$\sum_{j=1}^{k} \lambda_j f_j(\bar{x}) = -\sum_{j=1}^{k} \lambda_j f_j^*(\bar{p}_j) - (\bar{q}^T \tilde{g})^* \left(-\sum_{j=1}^{k} \lambda_j \bar{p}_j \right).$$

Proof.

(a) If $\bar{x}$ is an optimal solution to $(P^{si}_{sx^*})$, then $\bar{x}$ is also an optimal solution to (P_{sx^*}). By Theorem 3.4 follows that there exists an optimal solution to (D_{sx^*}) $(\bar{p},\bar{q}), \bar{p} = (\bar{p}_1,...,\bar{p}_k)$, $\bar{p}_i \in X^*, i = 1,...,k$, $\bar{q} \geqq 0$ such that

$$f_j^*(\bar{p}_j) + f_j(\bar{x}) = \langle \bar{p}_j, \bar{x}\rangle, j = 1,...,k,$$

$$\bar{q}^T \tilde{g}(\bar{x}) = 0,$$

and

$$\inf_{x\in X} \left[\left\langle \sum_{j=1}^{k} \lambda_j \bar{p}_j, x \right\rangle + \bar{q}^T \tilde{g}(x) \right] = \left\langle \sum_{j=1}^{k} \lambda_j \bar{p}_j, \bar{x} \right\rangle.$$

But, for $\bar{x}$ being feasible to the problem $(P^{si}_{sx^*})$, follows that either $\tilde{g}_i(\bar{x}) =$

$-\infty$ or $\tilde{g}_i(\bar{x}) < 0$, for every $i \in I_m$. The relations (1) and (2) give us that $\bar{q}$ must be 0.

Therefore, by the third equality, we have

$$\inf_{x \in X} \left\langle \sum_{j=1}^{k} \lambda_j \bar{p}_j, x \right\rangle = \left\langle \sum_{j=1}^{k} \lambda_j \bar{p}_j, \bar{x} \right\rangle \in \mathbb{R},$$

which is possible just if $\sum_{j=1}^{k} \lambda_j \bar{p}_j = 0$. So, (i)-(iii) are proved.

(b) The conclusion follows by doing all the calculations and transformations from (a) in the reverse direction.

□

Inspired by the optimality conditions presented above, let us introduce the following multiobjective optimization problem

$$(D^{si}) \quad \underset{(p,\lambda,t)\in\mathcal{B}^{si}}{\text{v-max}}\, h^{si}(p,\lambda,t),$$

with

$$h^{si}(p,\lambda,t) = \begin{pmatrix} h_1^{si}(p,\lambda,t) \\ \vdots \\ h_m^{si}(p,\lambda,t) \end{pmatrix},$$

$$h_j^{si}(p,\lambda,t) = -f_j^*(p_j) + t_j,$$

for $j = 1, \ldots, k$, the dual variables

$$p = (p_1, \ldots, p_k), \lambda = (\lambda_1, \ldots, \lambda_k)^T, t = (t_1, \ldots, t_k)^T,$$

$$p_j \in X^*, \quad \lambda_j \in \mathbb{R}, \quad t_j \in \mathbb{R}, \quad j = 1, \ldots, k,$$

and the set of constraints

$$\mathcal{B}^{si} = \left\{ (p, \lambda, t) : \lambda \in int(\mathbb{R}^k_+), \sum_{j=1}^{k} \lambda_j p_j = 0, \sum_{j=1}^{k} \lambda_j t_j = 0 \right\}.$$

We show now that for the problems (P^{si}) and (D^{si}) the weak and strong duality theorems in their classical formulations hold.

THEOREM 5.3 *There is no* $x \in \mathcal{A}^{si}$ *and no* $(p, \lambda, t) \in \mathcal{B}^{si}$ *such that* $f_j(x) \leq h_j^{si}(p, \lambda, t)$*, for* $j = 1, ..., k$*, and* $f_i(x) < h_i^{si}(p, \lambda, t)$*, for at least one* $i \in \{1, ..., k\}$*.*

Proof. Let be $x \in \mathcal{A}$ and $(p, \lambda, t) \in \mathcal{B}^{si}$ such that $f_j(x) \leq h_j^{si}(p, \lambda, t)$, for $j = 1, ..., k$, and $f_i(x) < h_i^{si}(p, \lambda, t)$, for at least one $i \in \{1, ..., k\}$. This implies that

$$\sum_{j=1}^{k} \lambda_j f_j(x) < \sum_{j=1}^{k} \lambda_j h_j^{si}(p, \lambda, t). \tag{11}$$

On the other hand, we have

$$\begin{aligned} \sum_{j=1}^{k} \lambda_j h_j^{si}(p, \lambda, t) &= -\sum_{j=1}^{k} \lambda_j f_j^*(p_j) + \sum_{j=1}^{k} \lambda_j t_j \\ &= -\sum_{j=1}^{k} \lambda_j f_j^*(p_j) + \left\langle \sum_{j=1}^{k} \lambda_j p_j, x \right\rangle \\ &= \sum_{j=1}^{k} \lambda_j \left(-f_j^*(p_j) + \langle p_j, x \rangle \right) \leq \sum_{j=1}^{k} \lambda_j f_j(x). \end{aligned}$$

The last inequality contradicts relation (11) and this implies that the assertion of the theorem must be true. □

THEOREM 5.4 *Let* $\bar{x}$ *be a properly efficient solution to* (P^{si})*. Then there exists a Pareto-efficient solution* $(\bar{p}, \bar{\lambda}, \bar{t}) \in \mathcal{B}^{si}$ *to the dual* (D^{si}) *such that* $f(\bar{x}) =$

$h^{si}(\bar{p}, \bar{\lambda}, \bar{t})$.

Proof. Assume $\bar{x}$ to be properly efficient to (P^{si}). This implies that $\bar{x} \in \mathcal{A}^{si}$. By Proposition 5.2 follows that there exists an $x^* = (x_1^*, ..., x_m^*) \in \prod_{i=1}^{m} \{h_i^* - g_i^* < 0\}$ such that $\bar{x}$ is properly efficient to $(P_{x^*}^{si})$. Here $\prod_{i=1}^{m} \{h_i^* - g_i^* < 0\}$ represents the set of those $x^* = (x_1^*, ..., x_m^*)$, $x_i^* \in X^*, i \in I_m$, with the property that $h_i^*(x_i^*) - g_i^*(x_i^*) < 0, i \in I_m$.

The feasible set of $(P_{x^*}^{si})$, $\mathcal{A}_{x^*}^{si}$, is a convex set and the objective function of $(P_{x^*}^{si})$ is a convex function. The fact that $\bar{x}$ is properly efficient to $(P_{x^*}^{si})$ can be then expressed via scalarization (see [2]). Therefore, there exists a vector $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_k)^T \in int(\mathbb{R}_+^k)$ such that $\bar{x}$ solves the scalar problem

$$(P_{sx^*}^{si}) \quad \inf_{x \in \mathcal{A}_{x^*}^{si}} \sum_{j=1}^{k} \bar{\lambda}_j f_j(x).$$

On the other hand, one may notice by considering the definition of $\mathcal{A}_{x^*}^{si}$ that $\bar{x}$ automatically fulfills the constraint qualification (CQ). So, $\bar{x}$ also fulfills the constraint qualification (CQ_{sx^*}), for $x^* = (x_1^*, ..., x_m^*)$. Applying Theorem 5.1 we get an optimal solution $(\tilde{p}, \tilde{q})$ to the dual of $(P_{sx^*}^{si})$ and Theorem 5.2 states that for this solution the optimality conditions (i), (ii) and (iii) are satisfied.

Let us construct now, by means of $\bar{x}$ and $(\tilde{p}, \tilde{q})$, a Pareto-efficient solution $(\bar{p}, \bar{\lambda}, \bar{t})$ to (D^{si}). Let $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_k)^T$ be the vector obtained above. We consider for $j = 1, \ldots, k, \bar{p}_j := \tilde{p}_j$, and $\bar{p} := (\bar{p}_1, \ldots \bar{p}_k) = (\tilde{p}_1, \ldots \tilde{p}_k) = \tilde{p}$. It holds $\sum_{j=1}^{m} \bar{\lambda}_j \bar{p}_j = \sum_{j=1}^{m} \bar{\lambda}_j \tilde{p}_j = 0$.

For $j = 1, \ldots, k$, let us define $\bar{t}_j := \langle \bar{p}_j, \bar{x} \rangle$.

For the new element $(\bar{p}, \bar{\lambda}, \bar{t})$ holds $\bar{\lambda} \in int(\mathbb{R}^k_+)$, $\sum\limits_{j=1}^{k} \bar{\lambda}_j \bar{p}_j = 0$ and

$$\sum_{j=1}^{k} \bar{\lambda}_j \bar{t}_j = \left\langle \sum_{j=1}^{m} \bar{\lambda}_j \bar{p}_j, \bar{x} \right\rangle = 0,$$

which implies that $(\bar{p}, \bar{\lambda}, \bar{t})$ is feasible to the problem (D^{si}). It remains to show that the values of the objective functions are equal, i.e. $f(\bar{x}) = h^{si}(\bar{p}, \bar{\lambda}, \bar{t})$.

Using relation (i) in Theorem 5.2 we get

$$h^{si}_j(\bar{p}, \bar{\lambda}, \bar{t}) = -f^*_j(\bar{p}_j) + \bar{t}_j = -f^*_j(\bar{p}_j) + \langle \bar{p}_j, \bar{x} \rangle = f_j(\bar{x}), j = 1, ..., k,$$

and the equality is proved.

The fact that $(\bar{p}, \bar{\lambda}, \bar{t})$ is Pareto-efficient to (D^{si}) follows from Theorem 5.3.□

REMARK 5.1 One may notice that Theorem 5.4 holds without being necessary to assume the fulfillment of any constraint qualification.

6. The case $h_i = 0, i \in I_m$

In the last section of the paper we consider in the formulation of both multiobjective optimization problems (P) and (P^{si}) that $h_i = 0$, for $i \in I_m$. We assume, in fact, that both problems have convex inequality constraints. Obviously, the assumption of subdifferentiability for $h_i = 0$, $i \in I_m$, is fulfilled. Then, the primal multiobjective optimization problems become

$$(P_0) \quad \text{v-}\min_{x \in \mathcal{A}} f(x),$$

$$\mathcal{A} = \{x \in X : g_i(x) \leq 0, i \in I_m\},$$

$$f(x) = (f_1(x), \ldots, f_k(x))^T,$$

and, respectively,

$$(P_0^{si}) \quad \text{v-}\min_{x \in \mathcal{A}} f(x),$$

$$\mathcal{A} = \{x \in X : g_i(x) < 0, i \in I_m\},$$

$$f(x) = (f_1(x), \ldots, f_k(x))^T.$$

Looking at the formulation of the dual problem (D^{si}), one may notice that this does not depend on $x^* = (x_1^*, ..., x_m^*) \in \prod_{i=1}^{m} \{h_i^* - g_i^* < 0\}$. It implies that (D^{si}) can be considered as a dual multiobjective problem for the problem (P_0^{si}). Moreover, the weak duality theorem (Theorem 5.3) and the strong duality theorem (Theorem 5.4) are still true.

More interesting is to see what happens, in this case, with the family of problems (D_{x^*}). The feasible set of (P) being non-empty, it must hold, for $i \in I_m$, $g_i \neq +\infty$ and therefore $g_i^*(x_i^*) > -\infty$, $\forall x_i^* \in X^*$.

If $x^* = (x_1^*, ..., x_m^*) \in \prod_{i=1}^{m} \{h_i^* - g_i^* \leq 0\}$, by (1) and (2) follows that $h_i^*(x_i^*) < +\infty$, $\forall x_i^* \in X^*$, $i \in I_m$.

On the other hand, for $i \in I_m$, if $h_i(x) = 0$, then $h_i^*(x_i^*) = \sup_{x \in X} \langle x_i^*, x\rangle < +\infty$ if and only if $x_i^* = 0$. In this case, $h_i^*(x_i^*) = 0, i \in I_m$, and $x^* = (0, ..., 0) \in X^* \times ... \times X^*$. This implies that the dual multiobjective dual (D_0) does not depend anymore on x^*

$$(D_0) \quad \text{v-}\max_{(p,q,\lambda,t) \in \mathcal{B}_0} h_0(p, q, \lambda, t),$$

with

$$h_0(p,q,\lambda,t) = \begin{pmatrix} h_{01}(p,q,\lambda,t) \\ \vdots \\ h_{0m}(p,q,\lambda,t) \end{pmatrix},$$

$$h_{0j}(p,q,\lambda,t) = -f_j^*(p_j) - \left((q^j)^T g\right)^* \left(-\frac{1}{k\lambda_j}\sum_{j=1}^{k}\lambda_j p_j\right) + t_j,$$

for $j = 1,\ldots,k$, the dual variables

$$p = (p_1,\ldots,p_k), q = (q^1,\ldots,q^k), \lambda = (\lambda_1,\ldots,\lambda_k)^T, t = (t_1,\ldots,t_k)^T,$$

$$p_j \in X^*, \quad q^j \in \mathbb{R}^m, \quad \lambda_j \in \mathbb{R}, \quad t_j \in \mathbb{R}, \quad j = 1,\ldots,k,$$

and the set of constraints

$$\mathcal{B}_0 = \left\{(p,q,\lambda,t) : \lambda \in int(\mathbb{R}^k_+), \sum_{j=1}^{k}\lambda_j q^j \geqq 0, \sum_{j=1}^{k}\lambda_j t_j = 0\right\}.$$

The constraint qualification (CQ), which assures the existence of the strong duality, becomes

$$(CQ_0) \left| \text{ There exists } x' \in \bigcap_{j=1}^{k} dom(f_j), \text{ such that } g_i(x') < 0, \forall i \in I_m. \right.$$

Under these assumptions, instead of the Theorems 4.1 and 4.2, we get in this convex case the following weak duality and, respectively, strong duality theorems. We want to stress here that they have been also obtained in [13] in the context of the study of the duality for multiobjective convex optimization problems with cone inequality constraints.

THEOREM 6.1. *There is no* $x \in \mathcal{A}$ *and no* $(p, q, \lambda, t) \in \mathcal{B}_0$ *such that* $f_j(x) \leq h_{0j}(p, q, \lambda, t)$, *for* $j = 1, ..., k$, *and* $f_i(x) < h_{0i}(p, q, \lambda, t)$, *for at least one* $i \in \{1, ..., k\}$.

THEOREM 6.2 *Assume that the constraint qualification* (CQ_0) *is fulfilled. Let* $\bar{x}$ *be a properly efficient element to* (P). *Then there exists a Pareto-efficient solution* $(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_0$ *to the dual* (D_0) *such that* $f(\bar{x}) = h_0(\bar{p}, \bar{q}, \bar{\lambda}, \bar{t})$.

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