A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces

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Abstract. In this paper we present a new regularity condition for the subdifferential sum formula of a convex function with the precomposition of another convex function with a continuous linear mapping. This condition is formulated by using the epigraphs of the conjugates of the functions involved and turns out to be weaker than the generalized interior-point regularity conditions given so far in the literature. Moreover, it provides a weak sufficient condition for Fenchel duality regarding convex optimization problems in infinite dimensional spaces. As an application, we discuss the strong conical hull intersection property (CHIP) for a finite family of closed convex sets.

Key Words. regularity condition, subdifferential sum formula, Fenchel duality, strong conical hull intersection property

AMS subject classification. 49N15, 90C25, 90C46

1 Introduction

A nowadays challenge in the convex analysis is to give weaker regularity conditions for the subdifferential sum formula of a convex function with the precomposition of another convex function with a continuous linear mapping in infinite dimensional spaces. Among the large number of works which deal with this subject we want to mention [14], [17], [18], [20], [21], [23], [24], where different so-called generalized interior-point regularity conditions have been introduced. Concerning the subdifferential sum formula of two convex functions, which is a particular case of the problem presented above, let us mention the paper of Attouch and Brézis [1] and the very recent articles [7] and [8]. The popularity

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of these regularity conditions is brought by the central role played by them in the theory of duality for convex optimization problems. Moreover, they provide sufficient conditions which guarantee the strong conical hull intersection property (CHIP) for a finite family of closed convex sets. The strong CHIP is proved to be useful in the study of best approximation problems (cf. [2], [3], [9], [10], [11], [17]).

In this paper we introduce a new regularity condition for the subdifferential sum formula of a convex function with the precomposition of another convex function with a continuous linear mapping in locally convex spaces. This condition is formulated by using the epigraphs of the conjugates of the functions involved and turns out to be weaker than the generalized interior-point regularity conditions given so far in the literature. Moreover, it generalizes the so-called dual regularity condition introduced by Burachik and Jeyakumar in [7] and [8]. Then we succeed to further weaken this new regularity condition and to obtain a sufficient condition which still guarantees strong duality between a convex optimization problem and its Fenchel dual problem, namely that the optimal objective values of the primal and dual are equal and the dual has an optimal solution. The converse duality, namely the situation when the optimal objective values of the primal and dual are equal but the primal problem has an optimal solution, is also studied. As an application, we discuss the strong conical hull intersection property for a finite family of closed convex sets.

The paper is organized as follows. In the next section we present some definitions and preliminary results. In Section 3 we give the announced general regularity condition for the subdifferential sum formula and we deal with some particular cases of it. Section 4 is devoted to the study of the Fenchel duality and Section 5 deals with the so-called converse duality. A short concluding section and the list of references close the paper.

2 Notations and preliminary results

In this section we describe the notations used throughout this paper and present some preliminary results. Let X be a nontrivial locally convex space and X^* the continuous dual space of X. X^* will be endowed with the weak* topology $w(X^*, X)$ and $\langle x^*, x\rangle$ will denote the value at $x \in X$ of the continuous linear functional $x^* \in X^*$. For a set $D \subseteq X$ we shall denote the *closure*, the *interior*, the *affine hull* and the *linear hull* of D by $cl(D)$, $int(D)$, $aff(D)$ and $lin(D)$, respectively.

Furthermore, for the nonempty set $D \subseteq X$, the *indicator function* $\delta_D : X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_D(x) = \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{otherwise,} \end{cases}$$

while the *support function* $\sigma_D : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined by $\sigma_D(x^*) = \sup_{x \in D} \langle x^*, x \rangle$. Considering now a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we denote by

$$dom(f) = \{x \in X : f(x) < +\infty\}$$

its *effective domain* and by

$$epi(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$$

its *epigraph*. Moreover, by $cl(f)$ we denote the *lower semicontinuous envelope* of f, namely the function whose epigraph is the closure of $epi(f)$ in $X \times \mathbb{R}$. We say that $f : X \to \mathbb{R} \cup \{+\infty\}$ is *proper* if $dom(f) \neq \emptyset$. The (Fenchel-Moreau) *conjugate function* of f is $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^*(p) = \sup_{x \in X} \{\langle p, x \rangle - f(x)\}$$

and the *subdifferential* f at $x \in dom(f)$ is the following set

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x \rangle, \forall y \in X\}.$$

Next we introduce some further notations used within this paper.

Definition 2.1 *Let M_1, M_2, N_1, N_2 be nonempty sets and $f : M_1 \to M_2$, $g : N_1 \to N_2$ be some given functions. We denote by $f \times g : M_1 \times N_1 \to M_2 \times N_2$ the function defined in the following way*

$$f \times g(m, n) = (f(m), g(n)), \forall (m, n) \in M_1 \times N_1.$$

Definition 2.2 *Let the functions $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i = 1, ..., m$, be given. The function $f_1 \square \cdots \square f_m : X \to \mathbb{R} \cup \{\pm\infty\}$ defined by*

$$f_1 \square \cdots \square f_m(x) = \inf \left\{ \sum_{i=1}^{m} f_i(x_i) : \sum_{i=1}^{m} x_i = x \right\}$$

is called the infimal convolution function of $f_1, ..., f_m$. We say that $f_1 \square \cdots \square f_m$ is exact at $x \in X$ if there exist some $x_i \in X, i = 1, ..., m$, such that $f_1 \square \cdots \square f_m(x) = f_1(x_1) + ... + f_m(x_m)$. Furthermore, we call $f_1 \square \cdots \square f_m$ exact if it is exact at every $x \in X$.

Definition 2.3 *Let X and Y be nontrivial locally convex spaces, $A : X \to Y$ be a linear continuous mapping and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given function. The function $Af : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by*

$$Af(y) = \inf\{f(x) : Ax = y\}$$

is called the marginal function of f through A. By convention, if $\{x \in X : Ax = y\}$ is empty, then $Af(y) = +\infty$.

The following two results characterize the epigraph of the conjugate of the sum of two functions, whereby the first one is a consequence of the Rockafellar-Moreau theorem (cf. [19], [22]).

Theorem 2.1 *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous functions such that $dom(f) \cap dom(g) \neq \emptyset$. Then the following relation holds*

$$epi((f+g)^*) = cl(epi(f^* \Box g^*)) = cl(epi(f^*) + epi(g^*)). \tag{1}$$

Remark 1. One may notice that the second equality in (1) remains true even considering the closure in the product topology of $(X^*, \tau) \times \mathbb{R}$, where τ is an arbitrary topology on X^*.

Proposition 2.2 *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper functions such that $dom(f) \cap dom(g) \neq \emptyset$. Then the following statements are equivalent:*

(i) $epi((f+g)^*) = epi(f^*) + epi(g^*)$;

(ii) $(f+g)^* = f^* \Box g^*$ and $f^* \Box g^*$ is exact at every $p \in X^*$.

Proof. "$(i) \Rightarrow (ii)$" Let be $p \in X^*$. By the definition of the conjugate function, we have

$$(f+g)^*(p) \leq \sup_{x \in X}\{\langle u, x\rangle - f(x)\} + \sup_{x \in X}\{\langle p-u, x\rangle - g(x)\} =$$

$$f^*(u) + g^*(p-u), \forall u \in X^*.$$

If $(f+g)^*(p) = +\infty$, then (ii) is fulfilled. In case $(f+g)^*(p) < +\infty$, we have that $(p, (f+g)^*(p)) \in epi((f+g)^*) = epi(f^*) + epi(g^*)$. By (i), there exist $(q, s) \in epi(f^*)$ and $(r, t) \in epi(g^*)$ such that $p = q + r$ and $(f+g)^*(p) = s + t$. Therefore $f^*(q) \leq s$, $g^*(p-q) \leq t$ and $f^*(q) + g^*(p-q) \leq (f+g)^*(p)$. This proves (ii).

"$(ii) \Rightarrow (i)$" Let be $(q, s) \in epi(f^*)$ and $(r, t) \in epi(g^*)$. Then

$$(f+g)^*(q+r) \leq \sup_{x \in X}\{\langle q, x\rangle - f(x)\} + \sup_{x \in X}\{\langle r, x\rangle - g(x)\} =$$

$$f^*(q) + g^*(r) \leq s + t,$$

which implies that $(q + r, s + t) \in epi((f+g)^*)$. Therefore $epi(f^*) + epi(g^*) \subseteq epi((f+g)^*)$.

Taking now $(p, w) \in epi((f+g)^*)$, we have $(f+g)^*(p) \leq w$. By (ii), there exists $u \in X^*$ such that $f^*(u) + g^*(p-u) \leq w$. Then the element (p, w) can be written as

$$(p, w) = (u, f^*(u)) + (p-u, w - f^*(u)),$$

which belongs to $epi(f^*) + epi(g^*)$. Thus $epi((f+g)^*) = epi(f^*) + epi(g^*)$. $\square$

The next result has been proved by Fitzpatrick and Simons in [13].

Theorem 2.3 (see Theorem 2.7 in [13]) *Let X and Y be nontrivial locally convex spaces, $A : X \to Y$ a linear continuous mapping and $g : Y \to \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous function such that $g \circ A$ is proper on X. Then*

$$epi((g \circ A)^*) = cl(epi(A^* g^*)), \tag{2}$$

where the closure is taken in the product topology of $(X^*, \tau) \times \mathbb{R}$, for every locally convex topology τ on X^* giving X as dual.

Remark 2. Significant choices for τ are the weak* topology $w(X^*, X)$ on X^* or the norm topology of X^* in case X is a reflexive Banach space.

In Theorem 2.4 we give a further characterization for the epigraph of $A^* g^*$. In relation (3) $id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, $id_{\mathbb{R}}(r) = r, \forall r \in \mathbb{R}$ denotes the identity mapping on $\mathbb{R}$.

Theorem 2.4 *Let X and Y be nontrivial locally convex spaces, τ an arbitrary topology on X^*, $A : X \to Y$ a linear continuous mapping and $g : Y \to \mathbb{R} \cup \{+\infty\}$ a proper function. Then*

$$cl(epi(A^* g^*)) = cl(A^* \times id_{\mathbb{R}}(epi(g^*))), \tag{3}$$

where the closure is taken in the product topology of $(X^*, \tau) \times \mathbb{R}$. Here $A^* \times id_{\mathbb{R}}(epi(g^*))$ stands for the image of the function $A^* \times id_{\mathbb{R}} : Y^* \times \mathbb{R} \to X^* \times \mathbb{R}$ over the set $epi(g^*)$.

Proof. First, let be $(x^*, r) \in A^* \times id_{\mathbb{R}}(epi(g^*))$. Then there exists $y^* \in Y^*$ such that $A^* y^* = x^*$ and $(y^*, r) \in epi(g^*)$. From here it follows

$$A^* g^*(x^*) = \inf\{g^*(y^*) : A^* y^* = x^*\} \leq r$$

and so $(x^*, r) \in epi(A^* g^*)$. Thus the inclusion $A^* \times id_{\mathbb{R}}(epi(g^*)) \subseteq epi(A^* g^*)$ is certain.

In the second part of the proof we show that $epi(A^* g^*) \subseteq cl(A^* \times id_{\mathbb{R}}(epi(g^*)))$ and this will lead us to the desired result. To this end, let be $(x^*, r) \in epi(A^* g^*)$,

$\mathcal{V}(x^*)$ an open neighborhood of x^* in τ and $\varepsilon > 0$. Because

$$A^*g^*(x^*) = \inf\{g^*(y^*) : A^*y^* = x^*\} \leq r < r + \frac{\varepsilon}{2},$$

there exists a $y^* \in Y^*$ such that $A^*y^* = x^*$ and $g^*(y^*) \leq r + \frac{\varepsilon}{2}$. Thus $(x^*, r + \frac{\varepsilon}{2}) \in A^* \times id_{\mathbb{R}}(epi(g^*))$ and, on the other hand, $(x^*, r + \frac{\varepsilon}{2}) \in \mathcal{V}(x*) \times (r - \varepsilon, r + \varepsilon)$. The open neighborhood $\mathcal{V}(x*)$ and $\varepsilon > 0$ being arbitrary chosen, it follows that $(x^*, r) \in cl(A^* \times id_{\mathbb{R}}(epi(g^*)))$. □

Taking in (2) and (3) the closure in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$ we obtain the following equality

$$epi((g \circ A)^*) = cl(epi(A^*g^*)) = cl(A^* \times id_{\mathbb{R}}(epi(g^*))). \tag{4}$$

Considering nontrivial, locally convex spaces X and Y, X^* endowed with the weak* topology $w(X^*, X)$, $A : X \to Y$ a linear continuous mapping, $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ proper, convex and lower semicontinuous functions such that $A(dom(f)) \cap dom(g) \neq \emptyset$, we have (cf. Theorem 2.1 and relation (4))

$$epi((f + g \circ A)^*) = cl(epi(f^*) + epi((g \circ A)^*)) = cl(epi(f^*) + cl(A^* \times id_{\mathbb{R}}(epi(g^*)))),$$

which is nothing else than (because $cl(E + cl(F)) = cl(E + F)$, for any arbitrary sets E and F)

$$epi((f + g \circ A)^*) = cl(epi(f^*) + A^* \times id_{\mathbb{R}}(epi(g^*))). \tag{5}$$

Inspired by the last relation we introduce the following regularity condition

(RC_A) : $epi(f^*) + A^* \times id_{\mathbb{R}}(epi(g^*))$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$.

One can notice that the regularity condition (RC_A) is equivalent to

$$epi((f + g \circ A)^*) = epi(f^*) + A^* \times id_{\mathbb{R}}(epi(g^*)). \tag{6}$$

3 The subdifferential sum formula and some applications

In this section we establish the subdifferential sum formula of a convex function with the precomposition of another convex function with a continuous linear mapping, assuming (RC_A) is fulfilled. Further we show that (RC_A) provides a generalization for the dual condition recently introduced by Burachik and Jeyakumar (cf. [7], [8]) and, furthermore, that is weaker than some generalized interior-point regularity conditions given so far in the literature. We conclude the section by deriving a sufficient condition for the strong conical hull intersection property (CHIP) in locally convex spaces. The main result of this section follows.

Theorem 3.1 *Let X and Y be nontrivial locally convex spaces, $A : X \to Y$ a linear continuous mapping, $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ proper, convex and lower semicontinuous functions such that $A(dom(f)) \cap dom(g) \neq \emptyset$. Then*

(i) (RC_A) is fulfilled if and only if $\forall x^* \in X^*$,

$$(f + g \circ A)^*(x^*) = \inf\{f^*(x^* - A^*y^*) + g^*(y^*) : y^* \in Y^*\}$$

and the infimum is attained.

(ii) If (RC_A) is fulfilled, then $\forall x \in dom(f) \cap A^{-1}(dom(g))$,

$$\partial(f + g \circ A)(x) = \partial f(x) + A^*\partial g(Ax).$$

Proof.

(i) "$\Rightarrow$" Assume that (RC_A) is fulfilled and let $x^* \in X^*$. For all $x \in X$ and $y^* \in Y^*$ we have (by the so-called Young-Fenchel inequality)

$$\begin{aligned} f^*(x^* - A^*y^*) + g^*(y^*) &\geq \langle x^* - A^*y^*, x\rangle - f(x) + \langle y^*, Ax\rangle - g(Ax) \\ &= \langle x^*, x\rangle - f(x) - g(Ax) \end{aligned}$$

and therefore

$$\inf\{f^*(x^* - A^*y^*) + g^*(y^*) : y^* \in Y^*\} \geq (f + g \circ A)^*(x^*). \tag{7}$$

If $(f + g \circ A)^*(x^*) = +\infty$, then the conclusion follows. In case $(f + g \circ A)^*(x^*) < +\infty$, we have that $(x^*, (f + g \circ A)^*(x^*)) \in epi((f + g \circ A)^*)$. The regularity condition (RC_A) being fulfilled, there exist $(u^*, r) \in epi(f^*)$ and $(v^*, s) \in A^* \times id_{\mathbb{R}}(epi(g^*))$ such that $x^* = u^* + v^*$ and $(f + g \circ A)^*(x^*) = r + s$. Thus there exists a $y^* \in Y^*$ such that $A^*y^* = v^*$ and $g^*(y^*) \leq s$, which implies

$$f^*(x^* - A^*y^*) + g^*(y^*) = f^*(u^*) + g^*(y^*) \leq r + s = (f + g \circ A)^*(x^*).$$

This delivers the desired result.

"$\Leftarrow$" In order to prove that (RC_A) is fulfilled, it is enough to show that the equality in (6) holds. For the beginning, let be $(u^*, r) \in epi(f^*)$ and $(v^*, s) \in A^* \times id_{\mathbb{R}}(epi(g^*))$. Thus there exists a $y^* \in Y^*$ such that $A^*y^* = v^*$ and $g^*(y^*) \leq s$. By (7) we have

$$(f + g \circ A)^*(u^* + v^*) \leq f^*(u^*) + g^*(y^*) \leq r + s,$$

which is nothing else than $(u^*+v^*, r+s) \in epi((f+g\circ A)^*)$. Therefore the inclusion $epi(f^*)+A^*\times id_{\mathbb{R}}(epi(g^*)) \subseteq epi((f+g\circ A)^*)$ is always satisfied.

In order to show the opposite inclusion, let $(x^*, r) \in epi((f+g\circ A)^*)$ or, equivalently, $(f+g\circ A)^*(x^*) \leq r$. The relation $(f+g\circ A)^*(x^*) = \inf\{f^*(x^*-A^*y^*)+g^*(y^*) : y^* \in Y^*\}$ being fulfilled and the infimum being attained for every $x^* \in X^*$, there exists a $y^* \in Y^*$ such that $f^*(x^*-A^*y^*)+g^*(y^*) \leq r$. The element (x^*, r) can be written in the following way

$$\begin{aligned}(x^*, r) &= (x^*-A^*y^*, f^*(x^*-A^*y^*)) + (A^*y^*, r-f^*(x^*-A^*y^*)) \\ &\in epi(f^*)+A^*\times id_{\mathbb{R}}(epi(g^*))\end{aligned}$$

and so $epi((f+g\circ A)^*) = epi(f^*)+A^*\times id_{\mathbb{R}}(epi(g^*))$.

(ii) Assume that (RC_A) is fulfilled and let $x \in dom(f)\cap A^{-1}(dom(g))$ be fixed. If $u^* \in \partial f(x)$ and $v^* \in \partial g(Ax)$, then $\forall y \in X$

$$\begin{aligned}\langle u^*+A^*v^*, y-x\rangle &= \langle u^*, y-x\rangle + \langle v^*, Ay-Ax\rangle \\ &\leq f(y)-f(x)+g(Ay)-g(Ax) \\ &= (f+g\circ A)(y)-(f+g\circ A)(x).\end{aligned}$$

By the definition of the subdifferential we get that $u^*+A^*v^* \in \partial(f+g\circ A)(x)$.

On the other hand, let $x^* \in \partial(f+g\circ A)(x)$. This is nothing else than $(f+g\circ A)^*(x^*)+(f+g\circ A)(x) = \langle x^*, x\rangle$. By (i), there exists a $y^* \in Y^*$ such that

$$\langle x^*, x\rangle - (f+g\circ A)(x) = (f+g\circ A)^*(x^*) = f^*(x^*-A^*y^*)+g^*(y^*)$$

or, equivalently,

$$f^*(x^*-A^*y^*)+f(x)-\langle x^*-A^*y^*, x\rangle + g^*(y^*)+g(Ax)-\langle y^*, Ax\rangle = 0.$$

Noticing that the following Young-Fenchel inequalities

$$f^*(x^*-A^*y^*)+f(x) \geq \langle x^*-A^*y^*, x\rangle$$

and

$$g^*(y^*)+g(Ax) \geq \langle y^*, Ax\rangle$$

are always fulfilled for all $x^* \in X^*$ and for all $y^* \in Y^*$, we get that

$$f^*(x^*-A^*y^*)+f(x) \geq \langle x^*-A^*y^*, x\rangle \Leftrightarrow x^*-A^*y^* \in \partial f(x)$$

and

$$g^*(y^*)+g(Ax) = \langle y^*, Ax\rangle \Leftrightarrow y^* \in \partial g(Ax).$$

Thus $x^* \in \partial f(x)+A^*\partial g(Ax)$, which concludes the proof. $\square$

In case $X = Y$ and $A = id_X$ is the identity mapping on X, $A^* \times id_{\mathbb{R}}$ becomes the identity mapping on $X^* \times \mathbb{R}$ and the regularity condition (RC_A) can be rewritten in the following way

$$(RC): \quad epi(f^*) + epi(g^*) \text{ is closed in the product topology of } (X^*, w(X^*, X)) \times \mathbb{R}.$$

Theorem 3.1 leads to the following result.

Theorem 3.2 *Let X be a nontrivial locally convex space and $f, g : X \to \mathbb{R} \cup \{+\infty\}$ proper, convex and lower semicontinuous functions such that $dom(f) \cap dom(g) \neq \emptyset$. Then*

(i) (RC) is fulfilled if and only if $\forall x^* \in X^*$,

$$(f + g)^*(x^*) = \inf\{f^*(x^* - y^*) + g^*(y^*) : y^* \in Y^*\}$$

and the infimum is attained.

(ii) If (RC) is fulfilled, then $\forall x \in dom(f) \cap dom(g)$,

$$\partial(f + g)(x) = \partial f(x) + \partial g(x).$$

Remark 3. Let us notice that the statement (ii) in Theorem 3.2 has been obtained by Burachik and Jeyakumar (cf. Theorem 3.1 in [7]; see also [8]) in case X is a Banach space.

Next we show that (RC_A) is implied by some generalized interior-point regularity conditions given in the literature. To arrive there, we need to introduce the following notions first.

For a subset $D \subseteq X$, the *core* of D is defined by $core(D) = \{d \in D : \forall x \in X \ \exists \varepsilon > 0 : \forall \lambda \in [-\varepsilon, \varepsilon] \ d + \lambda x \in D\}$. The core of D relative to $aff(D)$ is called the *intrinsic core* of D and is written $icr(D)$ (cf. [15]). For a convex subset $D \subseteq X$, the *strong quasi-relative interior* of D is the set of those $x \in D$ for which $cone(D - x)$ is a closed subspace and is written $sqri(D)$ (cf. [16]). Consider now the following generalized interior-point regularity conditions:

(i) : $\exists x' \in dom(f)$ such that $Ax' \in int(dom(g))$;

(ii) : $0 \in core(dom(g) - A(dom(f)))$ (cf. [18]);

(iii) : $0 \in sqri(dom(g) - A(dom(f)))$ (cf. [20]);

(iv) : $0 \in icr(dom(g) - A(dom(f)))$ and $aff(dom(g) - A(dom(f)))$ is a closed subspace (cf. [14]).

The following relation holds between them (cf. [14])

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv).$$

Gowda and Teboulle in [14] and Rodrigues in [20] (see also the paper of Rodrigues and Simons [21]) have proved that in case X and Y are Banach spaces, respectively, Fréchet spaces the regularity conditions enumerated above ensure the subdifferential sum formula. On the other hand, by Theorem 2.6 in [20] (see also [21]), one has that if $0 \in sqri(dom(g) - A(dom(f)))$ (which becomes in case A is the identity mapping on X the regularity condition of Attouch-Brézis in [1]), then $\forall x^* \in X^*$,

$$(f + g \circ A)^*(x^*) = \inf\{f^*(x^* - A^*y^*) + g^*(y^*) : y^* \in Y^*\}$$

and the infimum is attained. By Theorem 3.1 (i) this is nothing else than that (RC_A) must be fulfilled. Therefore the regularity conditions $(i) - (iv)$ imply (RC_A).

Remark 4. A very comprehensive result which gives different regularity conditions for the subdifferential sum formula, including also the generalized interior-point regularity conditions enumerated above, is Theorem 2.8.3 in [24] (see also [23]). One can deduce from the mentioned theorem that (RC_A) is implied by each of the ten regularity conditions given there. The most general regularity condition in Theorem 2.8.3 in [24] follows:

(v) : there exist $\lambda_0 \in \mathbb{R}, B$ a bounded subset in X and V_0 a balanced and closed neighborhood of 0 in $lin(dom(g) - A(dom(f)))$ such that $V_0 \subseteq A(\{x \in X : f(x) \leq \lambda_0\} \cap B) - \{y \in Y : g(y) \leq \lambda_0\}$.

We show that (RC_A) is actually weaker even than (v). This means that (RC_A) is weaker than all the regularity conditions given in Theorem 2.8.3 in [24] and, as a consequence, that (RC_A) is weaker than $(i) - (iv)$. Let be $X = Y = \mathbb{R}$, $A = id_{\mathbb{R}}$ the identity mapping on $\mathbb{R}$, $f = \delta_{[0,+\infty)}$ and $g = \delta_{(-\infty,0]}$. One can easily see that $epi(f^*) + epi(g^*) = \mathbb{R} \times [0, +\infty)$, which is a closed set in $\mathbb{R}^2$. Assume that (v) is also fulfilled. Then there would exist a $\lambda_0 \geq 0$, B a bounded subset in $\mathbb{R}$ and V_0 a balanced and closed neighborhood of 0 in $lin(dom(g) - A(dom(f))) = \mathbb{R}$ such that

$$V_0 \subseteq [0, +\infty) \cap B - (-\infty, 0] = [0, +\infty) \cap B + [0, +\infty) \subseteq [0, +\infty).$$

It is obvious that the relation above leads to a contradiction.

In the last part of the section we formulate a sufficient condition which guarantees the strong conical hull intersection property for a finite family of closed convex sets. Therefore we give first a preliminary result which can be derived from Theorem 3.1 by taking $f \equiv 0$. For this choice of the function f we have that $epi(f^*) = \{0\} \times [0, +\infty)$, $\partial f(x) = 0, \forall x \in X$ and the regularity condition (RC_A) becomes:

> $\{0\} \times [0, +\infty) + A^* \times id_{\mathbb{R}}(epi(g^*))$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$.

We prove that

$$\{0\} \times [0, +\infty) + A^* \times id_{\mathbb{R}}(epi(g^*)) = A^* \times id_{\mathbb{R}}(epi(g^*)). \tag{8}$$

First it is obvious that $A^* \times id_{\mathbb{R}}(epi(g^*)) \subseteq \{0\} \times [0, +\infty) + A^* \times id_{\mathbb{R}}(epi(g^*))$. In order to show the opposite inclusion, let be $r \geq 0$ and $(v^*, s) \in A^* \times id_{\mathbb{R}}(epi(g^*))$. Then there exists a $y^* \in Y^*$ such that $A^* y^* = v^*$ and $g^*(y^*) \leq s$. From here we have that $g^*(y^*) \leq r + s$ and so $(v^*, r + s) \in A^* \times id_{\mathbb{R}}(epi(g^*))$. This means that (8) is true and thus Theorem 3.1 leads to the following result.

Theorem 3.3 *Let X and Y be nontrivial locally convex spaces, $A : X \to Y$ a linear continuous mapping and $g : Y \to \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous function such that $g \circ A$ is proper. Then*

(i) $A^* \times id_{\mathbb{R}}(epi(g^*))$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$ if and only if $\forall x^* \in X^*$,

$$(g \circ A)^*(x^*) = \inf\{g^*(y^*) : A^* y^* = x^*\}$$

and the infimum is attained.

(ii) If $A^* \times id_{\mathbb{R}}(epi(g^*))$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$, then $\forall x \in A^{-1}(dom(g))$,

$$\partial(g \circ A)(x) = A^* \partial g(Ax).$$

Remark 5. From Theorem 2.3 and the proof of Theorem 2.4 follows that $A^* \times id_{\mathbb{R}}(epi(g^*))$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$ if and only if

$$A^* \times id_{\mathbb{R}}(epi(g^*)) = epi(A^* g^*) = epi((g \circ A)^*).$$

Let us recall now the definition of the strong conical hull intersection property for a finite family of closed convex sets.

Definition 3.1 *Let $C_1, ..., C_m$ be closed convex subsets of X with $C = \bigcap\limits_{i=1}^{m} C_i \neq \emptyset$. We say that $\{C_1, ..., C_m\}$ has the strong conical hull intersection property (CHIP), if $\forall x \in C$*

$$N_C(x) = \sum_{i=1}^{m} N_{C_i}(x),$$

where $N_C(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \forall y \in C\}$ *represents the normal cone of* C *at* x.

The notion of strong CHIP has been introduced by Deutsch, Li and Ward (cf. [11]) in Hilbert spaces and has proved to be useful for dealing with best approximation problems and in the conjugate duality theory (cf. [2], [9], [10], [11]). Obviously, $\{C_1, ..., C_m\}$ has the strong CHIP if and only if

$$\partial(\delta_C)(x) = \sum_{i=1}^{m} \partial(\delta_{C_i})(x), \forall x \in C.$$

Particularizing Theorem 3.3 we give a sufficient condition for the strong CHIP for $\{C_1, ..., C_m\}$, which is assumed to be a family of closed convex subsets in the nontrivial locally convex space X. Therefore let $Y = X^m$, $A : X \to X^m$, $Ax = (x, ..., x)$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ defined by

$$g(x_1, ..., x_m) = \delta_{\prod\limits_{i=1}^{m} C_i}(x_1, ..., x_m) = \begin{cases} 0, & \text{if } x_i \in C_i, i = 1, ..., m, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then $A^* : (X^*)^m \to X^*$ and $g^* : (X^*)^m \to \mathbb{R} \cup \{+\infty\}$ turn out to be $A^*(x_1^*, ..., x_m^*) = \sum\limits_{i=1}^{m} x_i^*$ and $g^*(x_1^*, ..., x_m^*) = \sum\limits_{i=1}^{m} \delta_{C_i}^*(x_i^*)$, respectively, for $(x_1^*, ..., x_m^*) \in (X^*)^m$.

In order to determine the set $A^* \times id_{\mathbb{R}}(epi(g^*))$, we consider an arbitrary element (v^*, s) belonging to it. This happens if and only if there exists $(x_1^*, ..., x_m^*) \in (X^*)^m$ such that $A^*(x_1^*, ..., x_m^*) = v^*$ and $g^*(x_1^*, ..., x_m^*) \leq s$, which is further equivalent to $\sum\limits_{i=1}^{m} x_i^* = v^*$ and $\sum\limits_{i=1}^{m} \delta_{C_i}^*(x_i^*) \leq s$. It can be easily shown that the last two relations take place if and only if $(v^*, s) \in epi(\delta_{C_1}^*) + ... + epi(\delta_{C_m}^*)$. In conclusion

$$A^* \times id_{\mathbb{R}}(epi(g^*)) = \sum_{i=1}^{m} epi(\delta_{C_i}^*). \tag{9}$$

On the other hand, we have $\forall x \in C$,

$$(g \circ A)(x) = \begin{cases} 0, & \text{if } x \in \bigcap\limits_{i=1}^{m} C_i, \\ +\infty, & \text{otherwise,} \end{cases} = \delta_{\bigcap\limits_{i=1}^{m} C_i}(x) = \delta_C(x)$$

and (see, for instance, [17])

$$\partial g(Ax) = \{(x_1^*, ..., x_m^*) : x_i^* \in \partial(\delta_{C_i})(x), i = 1, ..., m\}.$$

So $A^* \partial g(Ax) = \sum\limits_{i=1}^{m} \partial(\delta_{C_i})(x)$ and Theorem 3.3 (ii) provides the following result.

Corollary 3.4 *Let X be a nontrivial locally convex space and $C_1, ..., C_m$ be closed convex subsets of X with $C = \bigcap_{i=1}^{m} C_i \neq \emptyset$. If $\sum_{i=1}^{m} epi(\delta^*_{C_i})$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$, then $\{C_1, ..., C_m\}$ has the strong CHIP.*

Remark 6. For $m = 2$, a similar result has been given by Burachik and Jeyakumar for X a Banach space (cf. Theorem 3.1 in [8]). On the other hand, Ng and Song have given a sufficient condition for strong CHIP in case X is a Fréchet space and a generalized interior-point regularity condition is fulfilled (cf. Theorem 4.3 in [17]). By Remark 4, it turns out that Corollary 3.4 improves the result of Ng and Song, to more general spaces and more general regularity condition.

4 A regularity condition for Fenchel duality

In this section we introduce a further regularity condition which guarantees the existence of strong duality between a convex optimization problem and its Fenchel dual, namely that the optimal objective values of the primal and of the dual are equal and the dual has an optimal solution. This new regularity condition called (FRC_A) turns out to be weaker than (RC_A). Then we specialize (FRC_A) for convex optimization problems over a infinite intersection of closed convex sets.

For X and Y nontrivial locally convex spaces, $A : X \to Y$ a linear continuous mapping, $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ proper, convex and lower semicontinuous functions such that $A(dom(f)) \cap dom(g) \neq \emptyset$, we consider the following convex optimization problem

$$(P_A) \quad \inf_{x \in X} \{f(x) + g(Ax)\}.$$

The Fenchel dual problem to (P_A) is

$$(D_A) \quad \sup_{y^* \in Y^*} \{-f^*(-A^*y^*) - g^*(y^*)\}$$

and assuming that (RC_A) is fulfilled, Theorem 3.1 (i) (taking $x^* = 0$) guarantees strong duality between (P_A) and (D_A). Let us denote by $v(P_A)$ and $v(D_A)$ the optimal objective values of (P_A) and (D_A), respectively.

Let the regularity condition (FRC_A) be

$$(FRC_A): \quad \begin{array}{l} f^*\square A^*g^* \text{ is lower semicontinuous and } epi(f^*\square A^*g^*) \cap (\{0\} \times \mathbb{R}) = \\ (epi(f^*) + A^* \times id_{\mathbb{R}}(epi(g^*))) \cap (\{0\} \times \mathbb{R}). \end{array}$$

Theorem 4.1 *If (FRC_A) is fulfilled, then $v(P_A) = v(D_A)$ and (D_A) has an optimal solution.*

Proof. Taking in (7) $x^* = 0$, it holds

$$v(P_A) = -(f + g \circ A)^*(0) \geq v(D_A) \geq -f^*(-A^*y^*) - g^*(y^*), \forall y^* \in Y^*.$$

If $v(P_A) = -\infty$, then the conclusion follows. Assume now that $v(P_A) > -\infty$. By the theorems 2.1-2.3 we obtain

$$epi((f + g \circ A)^*) = cl(epi(f^*) + epi(g \circ A)^*)) = cl(epi(f^*) + cl(epi(A^*g^*)))$$

$$= cl(epi(f^*) + epi(A^*g^*)) = cl(epi(f^* \square A^*g^*)),$$

which is nothing else than $(f + g \circ A)^* = cl(f^* \square A^*g^*)$. The regularity condition (FRC_A) being fulfilled, we have actually that $(f+g\circ A)^* = f^* \square A^*g^*$ and, because of $-v(P_A) = (f + g \circ A)^*(0)$, we get

$$(0, -v(P_A)) \in (epi(f^*) + A^* \times id_{\mathbb{R}}(epi(g^*))) \cap (\{0\} \times \mathbb{R}).$$

Therefore there exist $(u^*, r) \in epi(f^*)$ and $(v^*, s) \in A^* \times id_{\mathbb{R}}(epi(g^*))$ such that $u^* + v^* = 0$ and $r + s = -v(P_A)$. Further there exists a $y^* \in Y^*$ such that $A^*y^* = v^*$ and $g^*(y^*) \leq s$. Thus $u^* = -A^*y^*$ and

$$v(P_A) = -r - s \leq -f^*(u^*) - g^*(y^*) = -f^*(-A^*y^*) - g^*(y^*) \leq v(D_A),$$

which delivers the desired conclusion. $\square$

Remark 7. Assume that (RC_A) is fulfilled, namely that $epi((f + g \circ A)^*) = epi(f^*) + A^* \times id_{\mathbb{R}}(epi(g^*))$ (cf. (6)). We prove that (FRC_A) is also fulfilled. As we have seen in the proof of Theorem 4.1, the following relations hold (see also the proof of Theorem 2.4)

$$epi((f + g \circ A)^*) = cl(epi(f^* \square A^*g^*)) \supseteq epi(f^* \square A^*g^*)$$

$$\supseteq epi(f^*) + epi(A^*g^*) \supseteq epi(f^*) + A^* \times id_{\mathbb{R}}(epi(g^*)).$$

Because of (RC_A), for all these inclusions equality holds. Thus $epi(f^* \square A^*g^*)$ is closed and $epi(f^* \square A^*g^*) = epi(f^*) + epi(A^*g^*)$. So, the conclusion follows. That (FRC_A) is actually weaker than (RC_A) will be shown in the example below.

In case $X = Y$ and $A = id_X$ is the identity mapping of X, the problems (P_A) and (D_A) become

$$(P) \quad \inf_{x \in X} \{f(x) + g(x)\}$$

and

$$(D) \quad \sup_{y^* \in Y^*} \{-f^*(-y^*) - g^*(y^*)\},$$

respectively. The functions A^*g^* and $A^* \times id_{\mathbb{R}}$ will be nothing else than g^* and the identity mapping of $X^* \times \mathbb{R}$, respectively. The regularity condition (FRC_A) can now be written as

$$(FRC): \quad f^*\Box g^* \text{ is lower semicontinuous and } epi(f^*\Box g^*) \cap (\{0\} \times \mathbb{R}) = (epi(f^*) + epi(g^*)) \cap (\{0\} \times \mathbb{R})$$

or, equivalently,

$$(FRC): f^*\Box g^* \text{ is a lower semicontinuous function and is exact at } 0.$$

The following theorem states the existence of strong duality between (P) and (D), assuming (FRC) fulfilled.

Theorem 4.2 *If (FRC) is fulfilled, then $v(P) = v(D)$ and (D) has an optimal solution.*

By Remark 7, we have that $epi(f^*) + epi(g^*)$ is closed in the product topology of $(X^*, w(X^*, X)) \times \mathbb{R}$ if and only if $epi(f^*\Box g^*)$ is closed and $epi(f^*\Box g^*) = epi(f^*) + epi(g^*)$. This is the same with $f^*\Box g^*$ is lower semicontinuous and is exact at every $p \in X^*$. Therefore, if (RC) holds, then (FRC) is also fulfilled. The following example shows that (FRC) is indeed weaker than (RC) (see also [4]).

Example. Let $X = \mathbb{R}^2$, $C = \{(x_1, x_2)^T : x_1 \geq 0\}$, $D = \{(x_1, x_2)^T : 2x_1 + x_2^2 \leq 0\}$, $f = \delta_C$ and $g = \delta_D$. Obviously, f and g are proper, convex and lower semicontinuous and $dom(f) \cap dom(g) = \{(0, 0)^T\}$. The conjugate functions f^* and g^* are

$$f^*(u_1^*, u_2^*) = \delta_C^*(u_1^*, u_2^*) = \begin{cases} 0, & \text{if } u_1^* \leq 0, u_2^* = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$g^*(v_1^*, v_2^*) = \delta_D^*(v_1^*, v_2^*) = \begin{cases} \frac{(v_2^*)^2}{v_1^*}, & \text{if } v_1^* > 0, \\ 0, & \text{if } v_1^* = v_2^* = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

For every $(x_1^*, x_2^*)^T \in \mathbb{R}^2$, $(f + g)^*(x_1^*, x_2^*) = \sup\limits_{x_1 = x_2 = 0} \{(x_1^*)^T x_1 + (x_2^*)^T x_2\} = 0$ and, on the other hand,

$$\begin{aligned}
f^*\Box g^*(x_1^*, x_2^*) &= \inf_{\substack{u_1^*+v_1^*=x_1^* \\ u_2^*+v_2^*=x_2^*}} \{\delta_C^*(u_1^*, u_2^*) + \delta_D^*(v_1^*, v_2^*)\} \\
&= \inf_{\substack{u_1^*+v_1^*=x_1^* \\ u_2^*+v_2^*=x_2^*}} \begin{cases} \frac{(v_2^*)^2}{v_1^*}, & \text{if } u_1^* \leq 0, u_2^* = 0, v_1^* > 0, \\ 0, & \text{if } u_1^* \leq 0, u_2^* = 0, v_1^* = v_2^* = 0 \end{cases} \\
&= \inf_{\substack{v_1^* \geq x_1^* \\ v_2^* = x_2^*}} \begin{cases} \frac{(v_2^*)^2}{v_1^*}, & \text{if } v_1^* > 0, \\ 0, & \text{if } v_1^* = v_2^* = 0 \end{cases} \\
&= 0.
\end{aligned}$$

Thus $f^*\Box g^*$ is lower semicontinuous on $\mathbb{R}^2$. Moreover, $f^*\Box g^*$ is exact at $(0,0)^T$ (the infimum is attained for $(v_1^*, v_2^*)^T = (0,0)^T$) and so (FRC) is fulfilled.

On the other hand, the function $f^*\Box g^*$ is not exact at every point of $\mathbb{R}^2$. Taking, for instance, $(x_1^*, x_2^*)^T = (1,1)^T$, the infimum in the infimal convolution of $f^*\Box g^*$ at $(1,1)^T$ is not attained. By Proposition 2.2, the sets $epi(f^*) + epi(g^*)$ and $epi((f+g)^*)$ are not equal, which means that, by Theorem 2.1, $epi(f^*) + epi(g^*)$ can not be closed.

Let us also notice that for $(0,0)^T \in dom(f) \cap dom(g)$ it holds $\partial f(0,0) = (-\infty, 0] \times \{0\}$, $\partial g(0,0) = [0, +\infty) \times \{0\}$, $\partial(f+g)(0,0) = \mathbb{R} \times \mathbb{R}$ and so $\partial(f+g)(0,0) \neq \partial f(0,0) + \partial g(0,0)$. Thus the Fenchel duality can hold even though the subdifferential sum rule fails.

We conclude this section by treating the duality for a particular case of (P_A), namely the convex optimization problem over a finite intersection of closed convex sets. The primal problem is defined in the following way

$$(P_C) \quad \inf_{x \in C} f(x),$$

where $C = \bigcap_{i=1}^{m} C_i \neq \emptyset$, $C_1, ..., C_m$ are closed convex subsets of the nontrivial locally convex space X and $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function such that $dom(f) \cap C \neq \emptyset$. The problem (P_C) has been intensively studied in the past in [2], [8], [9] and [17].

For $Y = X^m$, $A : X \to X^m$, $Ax = (x, ..., x)$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$, $g = \delta_{\prod_{i=1}^{m} C_i}$, (P_C) has the following formulation

$$(P_C) \quad \inf_{x \in X} \{f(x) + g(Ax)\}.$$

The Fenchel dual problem to (P_C) can be derived from (D_A) and looks like

$$(D_C) \quad \sup_{y^* \in Y^*} \{-f^*(-A^*y^*) - g^*(y^*)\}$$

or, equivalently,

$$(D_C) \qquad \sup_{(x_1^*,...,x_m^*)\in (X^*)^m} \left\{ -f^*\left(-\sum_{i=1}^m x_i^*\right) - \sum_{i=1}^m \delta_{C_i}^*(x_i^*) \right\},$$

as $A^*(x_1^*,...,x_m^*) = \sum\limits_{i=1}^m x_i^*$ and $g^*(x_1^*,...,x_m^*) = \sum\limits_{i=1}^m \delta_{C_i}^*(x_i^*)$ for $(x_1^*,...,x_m^*) \in (X^*)^m$. Let us deduce now from (FRC_A) a sufficient condition which guarantees the strong duality between (P_C) and (D_C). Obviously, for $x^* \in X^*$,

$$\begin{aligned} A^*g^*(x^*) &= \inf\{g^*(y^*) : A^*y^* = x^*\} = \inf\left\{\sum_{i=1}^m \delta_{C_i}^*(y_i^*) : \sum_{i=1}^m y_i^* = x^*\right\} \\ &= \delta_{C_1}^*\square ...\square \delta_{C_m}^*(x^*) \end{aligned}$$

and, because of the associativity of the infimal convolution, we get $f^*\square A^*g^* = f^*\square\delta_{C_1}^*\square...\square\delta_{C_m}^*$. On the other hand, by (9), $A^*\times id_{\mathbb{R}}(epi(g^*)) = \sum\limits_{i=1}^m epi(\delta_{C_i}^*)$ and, so,

$$epi(f^*\square A^*g^*) \cap (\{0\}\times\mathbb{R}) = (epi(f^*) + A^*\times id_{\mathbb{R}}(epi(g^*))) \cap (\{0\}\times\mathbb{R})$$

becomes

$$epi(f^*\square\delta_{C_1}^*\square...\square\delta_{C_m}^*) \cap (\{0\}\times\mathbb{R}) = \left(epi(f^*) + \sum_{i=1}^m epi(\delta_{C_i}^*)\right) \cap (\{0\}\times\mathbb{R}).$$

Noticing that this last equality holds if and only if $f^*\square\delta_{C_1}^*\square...\square\delta_{C_m}^*$ is exact at 0, we can write (FRC_A) as follows

(FRC_C) : $f^*\square\delta_{C_1}^*\square...\square\delta_{C_m}^*$ is a lower semicontinuous function and is exact at 0.

Theorem 4.3 *If (FRC_C) is fulfilled, then $v(P_C) = v(D_C)$ and (D_C) has an optimal solution.*

Remark 8. Theorem 4.3 generalizes similar results given in the past in the literature in particular spaces and under much stronger regularity conditions (cf. [8], [9], [17]).

5 Converse duality

The aim of this last section is to study the existence of the so-called converse duality for the optimization problems introduced in Section 4, namely the situation when the optimal objective values of the primal and of the dual are equal but the primal has an optimal solution. The approach is based on a fruitful idea used by Bauschke in [2] and later by Ng and Song in [17].

Let X and Y be nontrivial locally convex spaces with X^* and Y^* the continuous dual spaces endowed with the weak* topologies $w^*(X^*, X)$ and $w^*(Y^*, Y)$, respectively, $A : X \to Y$ a linear continuous mapping, $f : X \to \mathbb{R} \cup \{+\infty\}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ proper, convex and lower semicontinuous functions such that $A(dom(f)) \cap dom(g) \neq \emptyset$ and $0 \in dom(f^*) + A^*(dom(g^*))$. For the optimal objective value of the dual (D_A) we have the following expression

$$-v(D_A) = -\sup_{y^* \in Y^*} \{-f^*(-A^*y^*) - g^*(y^*)\} = \inf_{y^* \in Y^*} \{f^*(-A^*y^*) + g^*(y^*)\}.$$

By Theorem 4.1, if an appropriate regularity condition is fulfilled, than the optimal objective value of the infimum problem

$$\inf_{y^* \in Y^*} \{f^*(-A^*y^*) + g^*(y^*)\}$$

is equal to the optimal objective value of its dual and the last one has an optimal solution. In the hypotheses we made we have that the continuous dual of X^* and Y^* are $X^{**} = X$ and $Y^{**} = Y$, respectively that $(A^*)^* = A$, $f^{**}(x) = f(x), \forall x \in X$ and $g^{**}(y) = g(y), \forall y \in Y$. Therefore the Fenchel dual of the infimum problem from above becomes

$$\sup_{x \in X} \{-g(Ax) - f(x)\} = -\inf_{x \in X} \{f(x) + g(Ax)\}$$

and its objective value is equal to $-v(P_A)$. Let us notice that the assumptions $A(dom(f)) \cap dom(g) \neq \emptyset$ and $0 \in dom(f^*) + A^*(dom(g^*))$ provide the following relation

$$-\infty < v(D_A) \leq v(P_A) < +\infty. \tag{10}$$

The regularity condition which guarantees the converse duality can be derived from (FRC_A) and looks like

$$\begin{aligned}(CFRC_A):\quad & g\Box(-A)f \text{ is lower semicontinuous and } epi(g\Box(-A)f) \cap \\ & (\{0\} \times \mathbb{R}) = (epi(g) + (-A) \times id_{\mathbb{R}}(epi(f))) \cap (\{0\} \times \mathbb{R}).\end{aligned}$$

By Theorem 4.1 it follows that, under this regularity condition, $v(P_A) = v(D_A)$ and (P_A) has an optimal solution.

In the following we give an equivalent formulation for $(CFRC_A)$ which turns out to be very close to a well-known result given in literature.

For the beginning we prove that, because of the fact that $v(P_A)$ is finite (cf. (10)), the relation

$$epi(g\Box(-A)f) \cap (\{0\} \times \mathbb{R}) = (epi(g) + (-A) \times id_{\mathbb{R}}(epi(f))) \cap (\{0\} \times \mathbb{R}) \tag{11}$$

is fulfilled if and only if (P_A) has an optimal solution. This means that (11) gives a complete characterization of the existence of optimal solutions for (P_A).

Therefore assume that (11) holds. So,

$$\begin{aligned} g\Box(-A)f(0) &= \inf_{y\in Y}\{g(y)+(-A)f(-y)\} = \inf_{y\in Y}\left\{g(y)+\inf_{\substack{-Ax=-y\\ x\in X}} f(x)\right\} \\ &= \inf_{x\in X}\{f(x)+g(Ax)\} = v(P_A)\in\mathbb{R}. \end{aligned}$$

Because

$$(0,v(P_A))\in(epi(g)+(-A)\times id_{\mathbb{R}}(epi(f)))\cap(\{0\}\times\mathbb{R}),$$

there exist $(x,r)\in epi(g)$, $z\in X$ and $s\in\mathbb{R}$ such that $x=Az$, $f(z)\le s$ and $r+s=v(P_A)$. Then we have $f(z)+g(Az)\le v(P_A)$ and z is an optimal solution for (P_A).

On the other hand, we notice that the inclusion $(epi(g)+(-A)\times id_{\mathbb{R}}(epi(f)))\cap(\{0\}\times\mathbb{R})\subseteq epi(g\Box(-A)f)\cap(\{0\}\times\mathbb{R})$ is always fulfilled. Taking an element $(0,r)$ in $epi(g\Box(-A)f)\cap(\{0\}\times\mathbb{R})$, this is nothing else than $v(P_A)=\inf_{x\in X}\{f(x)+g(Ax)\}\le r$. The primal (P_A) having an optimal solution, there exists a $z\in X$ such that $f(z)+g(Az)\le r$. Thus $(0,r)=(Az,g(Az))+(-Az,r-g(Az))\in(epi(g)+(-A)\times id_{\mathbb{R}}(epi(f)))\cap(\{0\}\times\mathbb{R})$, which concludes the proof.

Next we discuss the other assumption in $(CFRC_A)$, namely that $g\Box(-A)f$ is a lower semicontinuous function. We start by calculating its value on an arbitrary $z\in Y$

$$\begin{aligned} g\Box(-A)f(z) &= \inf_{y\in Y}\{g(y)+(-A)f(z-y)\} = \inf_{y\in Y}\left\{g(y)+\inf_{\substack{-Ax=z-y\\ x\in X}} f(x)\right\} \\ &= \inf_{x\in X}\{f(x)+g(Ax+z)\}. \end{aligned}$$

Let now define the following so-called perturbation function for the problem (P_A) $\Phi_A: X\times Y\to\mathbb{R}\cup\{+\infty\}$, $\Phi_A(x,z)=f(x)+g(Ax+z)$. It fulfills $\Phi_A(x,0)=f(x)+g(Ax),\forall x\in X$ and, so, by means of the perturbation theory (see, also [5], [6], [12]) Φ_A provides the following dual to (P_A)

$$\sup_{y^*\in Y^*}\{-\Phi_A^*(0,y^*)\},$$

which is exactly the Fenchel dual problem

$$(D_A)\quad \sup_{y^*\in Y^*}\{-f^*(-A^*y^*)-g^*(y^*)\}.$$

The function $h_A : Y \to \mathbb{R}\cup\{+\infty\}$, $h_A(z) = \inf\limits_{x\in X} \Phi_A(x,z) = \inf\limits_{x\in X}\{f(x)+g(Ax+z)\}$ is called the infimal value function of (P_A) (cf. [12]). Thus $h_A = g\Box(-A)f$ and $(CFRC_A)$ is nothing else than

$(CFRC_A)$: The infimal value function h_A is lower semicontinuous and (P_A) has an optimal solution.

Thus the fact that $(CFRC_A)$ implies the converse duality for (P_A) and (D_A) is not surprising at all (cf. Proposition III.2.1 in [12]).

In the last part of the section we give the perturbation and the infimal value functions for the problems (P) and (P_C), respectively.

In case $X = Y$, $A = id_X$ is the identity mapping of X and $f, g : X \to \mathbb{R}\cup\{+\infty\}$ are proper, convex and lower semicontinuous functions such that $dom(f) \cap dom(g) \neq \emptyset$ and $0 \in dom(f^*) + dom(g^*)$, the perturbation and infimal value functions for (P) become $\Phi : X\times X \to \mathbb{R}\cup\{+\infty\}$, $\Phi(x,z) = f(x)+g(x+z)$ and $h : X \to \mathbb{R}\cup\{+\infty\}$, $h(z) = \inf\limits_{x\in X}\Phi(x,z) = \inf\limits_{x\in X}\{f(x)+g(x+z)\}$, respectively. The same duality scheme described above leads to the Fenchel dual (D) introduced in Section 4.

For the convex optimization problem over a finite intersection of closed and convex sets we make again, under the hypotheses $dom(f)\cap\bigcap\limits_{i=1}^{m} C_i \neq \emptyset$ and $0 \in dom(f^*) + \sum\limits_{i=1}^{m} dom(\delta^*_{C_i})$, the following particularizations $Y = X^m$, $A : X \to X^m$, $Ax = (x,...,x)$ and $g : Y \to \mathbb{R}\cup\{+\infty\}$, $g = \delta_{\prod\limits_{i=1}^{m} C_i}$. Then the perturbation and the infimal value functions for (P_C) will be $\Phi_C : X\times X^m \to \mathbb{R}\cup\{+\infty\}$,

$$\Phi_C(x,z_1,...,z_m) = \begin{cases} f(x), & \text{if } x \in \bigcap\limits_{i=1}^{m}(C_i - z_i), \\ +\infty, & \text{otherwise,} \end{cases}$$

and $h_C : X^m \to \mathbb{R}\cup\{+\infty\}$,

$$h_C(z_1,...,z_m) = \inf_{x\in X}\Phi_C(x,z_1,...,z_m) = \inf\left\{f(x) : x \in \bigcap_{i=1}^{m}(C_i - z_i)\right\},$$

respectively. One can easily prove that the duality approach in [12] leads to the Fenchel dual problem (D_C). Let us also mention that the existence of converse duality between (P_C) and (D_C) has been discussed under much stronger assumptions by Bauschke in [2] and, respectively, Ng and Song in [17].

6 Conclusions

In this paper we give a new regularity condition for the convex subdifferential sum formula of a convex function with the precomposition of a convex function with a continuous linear mapping. This condition is proved to be weaker than the generalized interior-point regularity conditions given so far in the literature. We employ these insights by giving a sufficient condition which ensures the existence of strong duality between a convex optimization problem with the objective function being the sum of a convex function with the precomposition of a convex function with a continuous linear mapping and its Fenchel dual. Further, we investigate the so-called converse duality, namely the situation when the optimal objective values of the primal and the dual are equal and the primal problem has an optimal solution. As an application, we discuss the strong conical hull intersection property (CHIP) for a finite family of closed convex sets.

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