Duality for multiobjective fractional programming problems

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Abstract

In this paper we present some duality assertions to a non-convex multiobjective fractional optimization problem. To the primal problem we attach an intermediate multiobjective convex optimization problem, using an approach due to Dinkelbach ([6]), for which we construct then a dual problem. This is expressed in terms of the conjugates of the numerator and denominator of the components of the primal objective function as well as the functions describing the set of constraints. The weak, strong and converse duality statements for the intermediate problems allow us to give dual characterizations for the efficient solutions of the initial fractional problem.

1 Introduction

The duality theory for convex multiobjective optimization problems is a field of the optimization theory which has intensively developed during the last decades. Among the duality concepts one can meet in the literature we mention here those of Wolfe ([20]), Weir-Mond ([19]), Nakayama ([11]), Jahn ([9]) and Wanka and Boţ ([17]). The last one is proposing a new conjugate dual based on the perturbation approach described in [7]. The papers [3] and [4] are completely studying the relations between all these duality concepts.

This paper considers duality for a special class of optimization problems called multiobjective fractional optimization problems, i.e. problems with

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multiple objective functions that are quotients of convex and concave functions. These kind of problems form indeed a separate class since they are in general not convex.

The case of fractional optimization has been investigated by Charnes and Cooper ([5]) for linear fractional objectives. Dinkelbach ([6]) gave the connection between a fractional and a certain parametrized program. Schaible ([16]) introduced a transformation that made it tractable to work with fractional problems. Bector ([1]) presented a dual for fractional optimization problems and Schaible ([15]) calculated the corresponding Lagrange dual. Kaul and Lyall ([10]) and Bector, Chandra and Singh ([2]) formulated duals and duality assertions for multiobjective fractional problems, but under some differentiability assumptions. Ohlendorf and Tammer ([12]) proposed a Fenchel-type dual for a fractional vector optimization problem.

The aim of this paper is to extend the approach in [17] to a non-convex multiobjective fractional programming problem. By using the parametrization approach of Dinkelbach ([6]) we attach to the primal problem an intermediate multiobjective convex optimization problem. Then we scalarize the primal intermediate problem and consider the conjugate dual for it introduced in [18]. Inspired by the dual of the scalarized problem we construct a multiobjective dual for the intermediate vector one. Weak, strong and converse duality are proved. These statements for the intermediate problems allow us to give dual characterizations for the efficient solutions of the initial fractional problem.

In a forthcoming paper we discuss the relations between the proposed intermediate multiobjective dual and other multiobjectve dual problems introduced so far in the literature (see for instance [12]).

The paper is structured as follows.

In Section 2 we formulate the multiobjective fractional primal problem (P) and - making use of the parametrization of Dinkelbach ([6]) - an intermediate convex problem $(P_\mu), \mu \in \mathbb{R}^m$, which is equivalent to the original in some sense. Furthermore we recall different efficiency definitions that will be necessary in the following.

In Section 3 we present the Fenchel-Lagrange dual for the corresponding real-valued problem that one gets by applying a linear scalarizing functional to the objective function of the parametrized primal problem. A strong duality theorem and necessary optimality conditions are established.

In Section 4 we derive the dual (D_μ) for the multiobjective problem $(P_\mu), \mu \in \mathbb{R}^m$. This dual is presented in a compact way with three conjugate functions in each component of the objective function. For this dual problem weak, strong and converse duality theorems are formulated and proved. These statements are then used in order to give dual characterizations for the efficient solutions of the initial fractional problem.

Finally, we describe the way in which the converse duality result can be applied, by treating a particular multiobjective fractional optimization problem.

2 The primal problem and its parametrization

2.1 The primal problem

Before introducing the primal problem, let us present the following definition of an ordering relation induced on $\mathbb{R}^k$ by the ordering cone $\mathbb{R}^k_+$.

Definition 2.1. For $y, z \in \mathbb{R}^k$ we denote $y \leqq z$ if $z - y \in \mathbb{R}^k_+ = \{u = (u_1, \dots, u_k)^T \in \mathbb{R}^k : u_i \geq 0, i = 1, \dots, k\}$.

Using the above definition the following multiobjective fractional primal problem (P) can be introduced

$$(P) \qquad \begin{array}{l} \text{v-}\min\limits_{x \in \mathcal{A}} \ \Phi(x) \\ \mathcal{A} = \left\{ x \in \mathbb{R}^n : \ g(x) = \left[g_1(x), \cdots, g_k(x) \right]^T \leqq 0 \right\}, \end{array}$$

where $\mathcal{A}$ is assumed to be non-empty, $\forall x \in \mathbb{R}^n$, $\Phi(x) = \left[\Phi_1(x), \cdots, \Phi_m(x)\right]^T = \left[\frac{f_1(x)}{h_1(x)}, \cdots, \frac{f_m(x)}{h_m(x)}\right]^T$, $f_i : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ are convex and proper functions, $(-h_i) : \mathbb{R}^n \to \mathbb{R}$ are convex functions fulfilling $h_i(x) > 0$, $\forall x \in \mathcal{A}$, $i = 1, \dots, m$, $g_j : \mathbb{R}^n \to \mathbb{R}$ are real-valued convex functions, $j = 1, \dots, k$ and $\bigcap\limits_{i=1}^{m} \operatorname{ri}(\operatorname{dom} f_i) \neq \emptyset$.

Note that $\mathcal{A}$ is convex, but nevertheless (P) is in general a non-convex problem.

In order to point out the optimal solutions of the problem (P), let us introduce the following definitions of efficiency and proper efficiency.

Definition 2.2 (Efficiency for problem (P))**.** An element $\bar{x} \in \mathcal{A}$ is said to be *efficient (or minimal) for* (P) if

$$\{\Phi(\bar{x}) - \mathbb{R}^m_+\} \cap \Phi(\mathcal{A}) = \{\Phi(\bar{x})\},$$

or, equivalently, if there is no $x \in \mathcal{A}$ such that

$$\Phi(x) \leqq \Phi(\bar{x})$$

and

$$\Phi(x) \neq \Phi(\bar{x}).$$

Definition 2.3 (Proper efficiency for problem (P))**.** A point $\bar{x} \in \mathcal{A}$ is said to be *properly efficient for* (P) if there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \operatorname{int}(\mathbb{R}^m_+)$ such that

$$\sum_{i=1}^{m} \lambda_i \Phi_i(\bar{x}) \leq \sum_{i=1}^{m} \lambda_i \Phi_i(x), \ \forall x \in \mathcal{A}.$$

Let us notice that any properly efficient solution turns out to be an efficient one, too.

2.2 Parametrization according to Dinkelbach

In order to investigate the duality for the non-convex problem (P) we consider the following parametrized optimization problem

$$(P_\mu) \qquad \text{v-}\min_{x \in \mathcal{A}} \ \Phi^{(\mu)}(x),$$

where

$$\Phi^{(\mu)}(x) = \begin{bmatrix} \Phi_1^{(\mu)}(x) \\ \vdots \\ \Phi_m^{(\mu)}(x) \end{bmatrix} = \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} - \begin{bmatrix} \mu_1 \cdot h_1(x) \\ \vdots \\ \mu_m \cdot h_m(x) \end{bmatrix}$$

and $\mu = (\mu_1, ..., \mu_m)^T \in \mathbb{R}^m$. Note that $\Phi_i^{(\mu)}$ are proper and convex if $\mu_i \geq 0, i = 1, \ldots, m$.

Efficiency and proper efficiency for (P_μ) are defined in an analogous manner as done above for (P).

For the single objective case Dinkelbach ([6]) has proved the following result, which is actually the starting point of our approach.

Theorem 2.1 ([6])**.** *Let $\mathcal{A} \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ be real-valued functions with $h(x) > 0, \forall x \in \mathcal{A}$. Then*

$$q_0 := \frac{f(x_0)}{h(x_0)} = \min \left\{ \frac{f(x)}{h(x)} : x \in \mathcal{A} \right\}$$

if and only if

$$f(x_0) - q_0\, h(x_0) = \min_{x \in \mathcal{A}} \{ f(x) - q_0\, h(x) \} = 0.$$

Kaul and Lyall ([10]) and Bector, Chandra and Singh ([2]) stated the connections between the efficient elements of (P) and (P_μ).

Theorem 2.2 ([2], [10])**.** *A point $\bar{x} \in \mathcal{A}$ is efficient for problem (P) if and only if $\bar{x}$ is efficient for problem $(P_{\bar{\mu}})$, where $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m)^T$ and $\bar{\mu}_i := \frac{f_i(\bar{x})}{h_i(\bar{x})}$, $i = 1, \ldots, m$.*

In order to prove a theorem concerning the relationship between properly efficient elements of (P) and (P_μ) we need the definition presented below.

Definition 2.4 (Proper efficiency in the sense of Geoffrion [8])**.** A point $\bar{x} \in \mathcal{A}$ is said to be *properly efficient in the sense of Geoffrion for* (P) if it is efficient and if there is some real number $M > 0$ such that for each $i = 1, \ldots, m$ and each $x \in \mathcal{A}$ satisfying $\Phi_i(x) < \Phi_i(\bar{x})$ there exists at least one $j \in \{1, \ldots, m\}$ such that $\Phi_j(\bar{x}) < \Phi_j(x)$ and

$$\frac{\Phi_i(\bar{x}) - \Phi_i(x)}{\Phi_j(x) - \Phi_j(\bar{x})} \leq M.$$

Proper efficiency in the sense of Geoffrion for problem (P_μ) is defined in an analogous way, with $\Phi^{(\mu)}$ instead of Φ.

Theorem 2.3. *Let be $\bar{x} \in \mathcal{A}$ and assume that $\bar{\mu}_i := \frac{f_i(\bar{x})}{h_i(\bar{x})} \geq 0, i = 1, \ldots, m$. The point $\bar{x}$ is properly efficient in the sense of Geoffrion for problem (P) if and only if $\bar{x}$ is properly efficient (in the sense of Definition 2.3) for problem $(P_{\bar{\mu}})$, where $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m)^T$.*

Proof. By Lemma 1 in [10] follows that $\bar{x}$ is properly efficient in the sense of Geoffrion for problem (P) if and only if $\bar{x}$ is properly efficient in the sense of Geoffrion for problem $(P_{\bar{\mu}})$. Since $\Phi^{(\bar{\mu})}(\mathcal{A})$ is $\mathbb{R}^m_+$-convex ($\Phi^{(\bar{\mu})}(\mathcal{A}) + \mathbb{R}^m_+$ is convex), the set of the properly efficient elements in the sense of Geoffrion for problem $(P_{\bar{\mu}})$ coincides with the set of the properly efficient elements in the sense of Definition 2.3 (cf. Theorem 3.1.4, Theorem 3.4.1 and Theorem 3.4.2 in [14]). That $\Phi^{(\bar{\mu})}(\mathcal{A}) + \mathbb{R}^m_+$ is convex, can be easily understood by the following calculations.

Let $x_1, x_2 \in \Phi^{(\bar{\mu})}(\mathcal{A}) + \mathbb{R}^m_+$, i.e. $\exists\ a_1, a_2 \in \mathcal{A}$ and $k_1, k_2 \in \mathbb{R}^m_+$ such that $x_1 = \Phi^{(\bar{\mu})}(a_1) + k_1$ and $x_2 = \Phi^{(\bar{\mu})}(a_2) + k_2$. Let be $\alpha \in (0, 1)$. Then it holds

$$\begin{aligned} \alpha x_1 + (1-\alpha)x_2 &= \alpha\Phi^{(\bar{\mu})}(a_1) + \alpha k_1 + (1-\alpha)\Phi^{(\bar{\mu})}(a_2) + (1-\alpha)k_2 \\ &\in \alpha\Phi^{(\bar{\mu})}(\mathcal{A}) + \alpha k_1 + (1-\alpha)\Phi^{(\bar{\mu})}(\mathcal{A}) + (1-\alpha)k_2 \\ &\subseteq \Phi^{(\bar{\mu})}(\mathcal{A}) + \mathbb{R}^m_+. \end{aligned}$$

In conclusion we get the equivalence that was to be shown. □

Now let us consider the corresponding scalarized problem

$$(P_{\mu,\lambda}) \qquad \inf_{x \in \mathcal{A}} \Phi^{(\mu,\lambda)}(x),$$

where

$$\Phi^{(\mu,\lambda)}(x) = \sum_{i=1}^{m} \lambda_i \cdot \Phi_i^{(\mu)}(x) = \sum_{i=1}^{m} \lambda_i \cdot (f_i(x) - \mu_i \cdot h_i(x))$$

and $\lambda = (\lambda_1, ..., \lambda_m)^T \in \mathbb{R}^m, \mu = (\mu_1, ..., \mu_m)^T \in \mathbb{R}^m$ are such that $\lambda_i > 0, \mu_i \geq 0, i = 1, \ldots, m$. Obviously, $(P_{\mu,\lambda})$ is a convex optimization problem and we denote its optimal objective value by $v(P_{\mu,\lambda})$.

3 The Fenchel-Lagrange dual for the scalarized problem

In this section we construct a conjugate dual problem for the scalar optimization problem introduced above, called the *Fenchel-Lagrange dual* problem. For the primal problem $(P_{\mu,\lambda})$ this can be written as follows (cf. [17] and [18])

$$(\mathrm{DG}) \qquad \sup_{\substack{p \in \mathbb{R}^n, \\ q \in \mathbb{R}^k_+}} \left\{ -\tilde{f}^*(p) - (q^T g)^*(-p) \right\},$$

where

$$\tilde{f} : \mathbb{R}^n \to \overline{\mathbb{R}}, \tilde{f}(x) = \sum_{i=1}^{m} \lambda_i \cdot (f_i(x) - \mu_i \cdot h_i(x)).$$

Here $\tilde{f}^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, $\tilde{f}^*(p) = \sup\limits_{x \in \mathbb{R}^n} \{p^T x - f(x)\}$ is the conjugate function of $\tilde{f}$. Note that for $\lambda_i > 0$ and $\mu_i \geq 0$, the functions $\lambda_i(f_i - \mu_i h_i), i = 1, \ldots, m$, are convex.

In order to be able to give an appropriate formulation for the dual we need the following lemma.

Lemma 3.1 (Theorem 16.4, [13]). *Let* $\varphi_i : \mathbb{R}^n \to \overline{\mathbb{R}}, i = 1, \ldots, m,$ *be proper convex functions with* $\bigcap\limits_{i=1}^{m} \mathrm{ri}(\mathrm{dom}\, \varphi_i) \neq \emptyset$. *Then*

$$\left(\sum_{i=1}^{m} \varphi_i \right)^* (p) = \inf \left\{ \sum_{i=1}^{m} \varphi_i^*(p_i) : \sum_{i=1}^{m} p_i = p \right\},$$

and the infimum is attained for each $p \in \mathbb{R}^n$.

Because $\bigcap\limits_{i=1}^{m} \text{ri}(\text{dom}\, f_i) \neq \emptyset$ and as the functions h_i are real-valued for $i = 1, \ldots, m$, we can apply Lemma 3.1 and get the following formulation for the dual problem

$$(D_{\mu,\lambda}) \sup_{\substack{q \in \mathbb{R}^k_+, \\ u_i, v_i \in \mathbb{R}^n, \\ i=1,\ldots,m}} \left\{ -\sum_{i=1}^{m} \lambda_i [f_i^*(u_i) + (-\mu_i h_i)^*(v_i)] \right.$$
$$\left. -(q^T g)^* \left(-\sum_{i=1}^{m} \lambda_i (u_i + v_i) \right) \right\}. \quad (1)$$

For the strong duality theorem and the optimality conditions we need a so-called constraint qualification. In order to formulate it let us consider the sets $L = \{j \in \{1, ..., k\} : g_j \text{ is affine}\}$ and $N = \{1, ..., k\} \setminus L$.

Constraint qualification (CQ)

There exists an element $x' \in \bigcap\limits_{i=1}^{m} \text{ri}(\text{dom}\, f_i)$ such that $g_j(x') < 0$, $j \in N$, and $g_j(x') \leq 0, j \in L$.

Theorem 3.1 (Strong Duality, [18])**.** *If $v(P_{\mu,\lambda})$ is finite and the constraint qualification (CQ) is fulfilled, then the problem $(D_{\mu,\lambda})$ has an optimal solution and it holds*

$$v(P_{\mu,\lambda}) = v(D_{\mu,\lambda}).$$

Theorem 3.2 (Optimality conditions)**.** *Let (CQ) be fulfilled and $\bar{x}$ be an optimal solution of $(P_{\mu,\lambda})$. Then there exists $(\bar{u}_1, ..., \bar{u}_m, \bar{v}_1, ..., \bar{v}_m, \bar{q})$, optimal solution of $(D_{\mu,\lambda})$, such that the following optimality conditions hold*

(1) $f_i(\bar{x}) + f_i^*(\bar{u}_i) = \bar{u}_i^T \bar{x}; \ , i = 1, \ldots, m,$

(2) $-\mu_i h_i(\bar{x}) + (-\mu_i h_i)^*(\bar{v}_i) = \bar{v}_i^T \bar{x} \ \ , i = 1, \ldots, m,$

(3) $\bar{q}^T g(\bar{x}) = 0,$

(4) $-(\bar{q}^T g)^* \left(-\sum\limits_{i=1}^{m} \lambda_i (\bar{u}_i + \bar{v}_i) \right) = \left(\sum\limits_{i=1}^{m} \lambda_i (\bar{u}_i + \bar{v}_i) \right)^T \bar{x}.$

Proof. Let $\bar{x}$ be an optimal solution of $(P_{\mu,\lambda})$. According to Theorem 3.1, there exists an optimal solution $(\bar{u}, \bar{v}, \bar{q})$ of $(D_{\mu,\lambda})$ such that strong duality holds,

$$\sum_{i=1}^{m} \lambda_i (f_i(\bar{x}) - \mu_i h_i(\bar{x})) = -\sum_{i=1}^{m} \lambda_i f_i^*(\bar{u}_i) - \sum_{i=1}^{m} \lambda_i (-\mu_i h_i)^*(\bar{v}_i) - (\bar{q}^T g)^* \left(-\sum_{i=1}^{m} \lambda_i (\bar{u}_i + \bar{v}_i) \right).$$

It follows

$$0 = \sum_{i=1}^{m} \lambda_i \Big(f_i(\bar{x}) - \mu_i h_i(\bar{x}) + f_i^*(\bar{u}_i) + (-\mu_i h_i)^*(\bar{v}_i) \Big) + (\bar{q}^T g)^* \left(-\sum_{i=1}^{m} \lambda_i (\bar{u}_i + \bar{v}_i) \right).$$

This can be reformulated as

$$0 = \sum_{i=1}^{m} \lambda_i \Big[(f_i(\bar{x}) + f_i^*(\bar{u}_i) - \bar{u}_i^T \bar{x}) + (-\mu_i h_i(\bar{x}) + (-\mu_i h_i)^*(\bar{v}_i) - \bar{v}_i^T \bar{x}) \Big] + \left(\sum_{i=1}^{m} \lambda_i (\bar{u}_i + \bar{v}_i) \right)^T \bar{x} + \bar{q}^T g(\bar{x}) - \inf_{x \in \mathbb{R}^n} \left[\left(\sum_{i=1}^{m} \lambda_i (\bar{u}_i + \bar{v}_i) \right)^T x + \bar{q}^T g(x) \right] - \bar{q}^T g(\bar{x}).$$

Because of $\lambda_i > 0, i = 1, \ldots, m,$ $g(\bar{x}) \leqq 0$, $\bar{q} \geqq 0$, by applying the Young-Fenchel inequality, the right-hand side of the previous relation is greater than or equal to 0. Consequently, it must be equal to 0 and the conditions $(1) - (4)$ follow immediately. □

4 The multiobjective Fenchel-Lagrange dual

Before introducing a multiobjective dual problem to (P_μ), let us notice that $(D_{\mu,\lambda})$ can be written equivalently as

$$\sup_{(u,v,q) \in B_{\mu,\lambda}} \left\{ \sum_{i=1}^{m} \lambda_i \left[-f_i^*(u_i) - (-\mu_i h_i)^*(v_i) - \left(\frac{1}{m\lambda_i} \cdot q^T g \right)^* \left(-\frac{1}{m\lambda_i} \sum_{j=1}^{m} \lambda_j (u_j + v_j) \right) \right] \right\},$$

where

$$\mathcal{B}_{\mu,\lambda} = \{(u,v,q): \ u=(u_1,\ldots,u_m), u_i \in \mathbb{R}^n, i=1,\ldots,m, \\ v=(v_1,\ldots,v_m), v_i \in \mathbb{R}^n, i=1,\ldots,m, q \in \mathbb{R}^k, q \geqq 0\}.$$

By setting $q_i := \frac{1}{m\lambda_i} \cdot q$ for $i=1,\ldots,m$, we get $\sum_{i=1}^m \lambda_i q_i = q \geqq 0$ and this motivates the following dual to the problem (P_μ)

$$(D_\mu) \qquad \text{v-}\max_{(u,v,q,\lambda,t)\in\mathcal{B}_\mu} \Psi^{(\mu)}(u,v,q,\lambda,t),$$

where

$$\Psi^{(\mu)}(u,v,q,\lambda,t) = \left[\Psi_1^{(\mu)}(u,v,q,\lambda,t),\cdots,\Psi_m^{(\mu)}(u,v,q,\lambda,t)\right]^T,$$
$$\Psi_i^{(\mu)}(u,v,q,\lambda,t) = -\, f_i^*(u_i) - (-\mu_i h_i)^*(v_i) \\ - (q_i^T g)^* \left(-\frac{1}{m\lambda_i}\sum_{j=1}^m \lambda_j(u_j+v_j)\right) + t_i,$$

the set of constraints is defined as

$$\mathcal{B}_\mu = \left\{(u,v,q,\lambda,t): \ \lambda \in \operatorname{int}(\mathbb{R}^m_+), \ \sum_{i=1}^m \lambda_i q_i \geqq 0, \ \sum_{i=1}^m \lambda_i t_i = 0\right\}$$

and the dual variables are $u=(u_1,\ldots,u_m), u_i \in \mathbb{R}^n$, $v=(v_1,\ldots,v_m), v_i \in \mathbb{R}^n$, $q=(q_1,\ldots,q_m), q_i \in \mathbb{R}^k, i=1,\ldots,m$, $\lambda=(\lambda_1,\ldots,\lambda_m)^T \in \operatorname{int}(\mathbb{R}^m_+)$ and $t=(t_1,\ldots,t_m)^T \in \mathbb{R}^m$.

The efficient elements of (D_μ) are defined in an analogous manner as for (P).

Definition 4.1 (Efficiency for problem (D_μ))**.** An element $(\bar{u},\bar{v},\bar{q},\bar{\lambda},\bar{t}) \in \mathcal{B}_\mu$ is said to be *efficient (or maximal) for* (D_μ) if

$$\{\Psi^{(\mu)}(\bar{u},\bar{v},\bar{q},\bar{\lambda},\bar{t}) + \mathbb{R}^m_+\} \cap \Psi^{(\mu)}(\mathcal{B}_\mu) = \{\Psi^{(\mu)}(\bar{u},\bar{v},\bar{q},\bar{\lambda},\bar{t})\}.$$

Theorem 4.1 (Weak duality)**.** *There is no* $(u,v,q,\lambda,t) \in \mathcal{B}_\mu$ *and no* $x \in \mathcal{A}$ *such that*

$$\Psi^{(\mu)}(u,v,q,\lambda,t) \geqq \Phi^{(\mu)}(x),$$

and

$$\Psi^{(\mu)}(u,v,q,\lambda,t) \neq \Phi^{(\mu)}(x).$$

Proof. Assume there is $x \in \mathcal{A}$ and $(u, v, q, \lambda, t) \in \mathcal{B}_\mu$ such that

$$\Phi_i^{(\mu)}(x) \leq \Psi_i^{(\mu)}(u, v, q, \lambda, t), \ \forall\, i \in \{1, \ldots, m\}$$

and $\Phi_j^{(\mu)}(x) < \Psi_j^{(\mu)}(u, v, q, \lambda, t)$ for at least one $j \in \{1, \ldots, m\}$. Because $\lambda \in \operatorname{int}(\mathbb{R}_+^m)$, we get

$$\sum_{i=1}^m \lambda_i \Phi_i^{(\mu)}(x) < \sum_{i=1}^m \lambda_i \Psi_i^{(\mu)}(u, v, q, \lambda, t). \tag{2}$$

By the Young-Fenchel inequality it holds

$$-(q_i^T g)^* \left(-\frac{1}{m\lambda_i} \sum_{j=1}^m \lambda_j (u_j + v_j) \right) \leq q_i^T g(x) - \left(-\frac{1}{m\lambda_i} \sum_{j=1}^m \lambda_j (u_j + v_j) \right)^T x,$$

for $i = 1, \ldots, m$. Applying the definition of conjugate functions we get

$$-\left(\Phi_i^{(\mu)}\right)^*(p_i) = -(f_i + (-\mu_i h_i))^*(p_i) \geq -f_i^*(u_i) - (-\mu_i h_i)^*(v_i),$$

for arbitrary $u_i, v_i \in \mathbb{R}^n$ with $u_i + v_i = p_i$, $i = 1, \ldots, m$. So we get the following estimate

$$\Psi_i^{(\mu)}(u, v, q, \lambda, t) \leq -(\Phi_i^{(\mu)})^*(p_i) + t_i + q_i^T g(x) + \left(\frac{1}{m\lambda_i} \sum_{j=1}^m \lambda_j (u_j + v_j) \right)^T x,$$

for $i = 1, \ldots, m$. For $\lambda \in \operatorname{int}(\mathbb{R}_+^m)$ it follows

$$\begin{aligned}
\sum_{i=1}^m \lambda_i \Psi_i^{(\mu)}(u, v, q, \lambda, t) \leq & -\sum_{i=1}^m \lambda_i (\Phi_i^{(\mu)})^*(p_i) + \sum_{i=1}^m \lambda_i (p_i^T x) \\
& + \sum_{i=1}^m \lambda_i t_i + \left(\sum_{i=1}^m \lambda_i q_i \right)^T g(x) \\
\leq & \sum_{i=1}^m \lambda_i \Phi_i^{(\mu)}(x),
\end{aligned}$$

which is a direct conclusion of the Young-Fenchel inequality for Φ_i, $i = 1, \ldots, m$, and of the fact that $\sum_{i=1}^m \lambda_i t_i = 0$ as well as $\left(\sum_{i=1}^m \lambda_i q_i \right)^T g(x) \leq 0$. But this contradicts relation (2). □

Theorem 4.2 (Strong duality)**.** *Let (CQ) be fulfilled and $\bar{x}$ be a properly efficient element of (P_μ). Then there exists an efficient solution $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\mu$ of (D_μ) and strong duality holds, i.e.*

$$\Phi^{(\mu)}(\bar{x}) = \Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}).$$

Proof. Because $\bar{x}$ is a properly efficient solution of (P_μ), there exists $\bar{\lambda} \in \operatorname{int}(\mathbb{R}^m_+)$ such that $\bar{x}$ solves

$$(P_{\mu,\bar{\lambda}}) \qquad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \bar{\lambda}_i (f_i(x) - \mu_i h_i(x)).$$

Since (CQ) is fulfilled, by Theorem 3.2 there exists an optimal solution $(\tilde{u}, \tilde{v}, \tilde{q}) \in \mathcal{B}_{\mu,\bar{\lambda}}$ of problem $(D_{\mu,\bar{\lambda}})$ such that the optimality conditions (1)-(4) in Theorem 3.2 are fulfilled.

Now a feasible solution $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})$ to problem (D_μ) will be constructed. We are setting $\bar{u} := \tilde{u}, \bar{v} := \tilde{v}, \bar{\lambda}$ like above and

$$\bar{q}_i := \frac{1}{m\bar{\lambda}_i}\tilde{q}, \ \forall i = 1, \dots, m,$$

$$\bar{t}_i := (\bar{u}_i + \bar{v}_i)^T \bar{x} + (\bar{q}_i^T g)^* \left(-\frac{1}{m\bar{\lambda}_i} \sum_{j=1}^{m} \bar{\lambda}_j (\bar{u}_j + \bar{v}_j) \right), \ \forall i = 1, \dots, m.$$

We see that

$$\begin{aligned}
\sum_{i=1}^{m} \bar{\lambda}_i \bar{t}_i &= \left(\sum_{i=1}^{m} \bar{\lambda}_i (\bar{u}_i + \bar{v}_i) \right)^T \bar{x} + \sum_{i=1}^{m} \bar{\lambda}_i (\bar{q}_i^T g)^* \left(-\frac{1}{m\bar{\lambda}_i} \sum_{j=1}^{m} \bar{\lambda}_j (\bar{u}_j + \bar{v}_j) \right) \\
&= \left(\sum_{i=1}^{m} \bar{\lambda}_i (\bar{u}_i + \bar{v}_i) \right)^T \bar{x} + \sum_{i=1}^{m} \bar{\lambda}_i \frac{1}{m\bar{\lambda}_i} (\tilde{q}^T g)^* \left(-\sum_{j=1}^{m} \bar{\lambda}_j (\bar{u}_j + \bar{v}_j) \right) \\
&= \left(\sum_{i=1}^{m} \bar{\lambda}_i (\bar{u}_i + \bar{v}_i) \right)^T \bar{x} + (\tilde{q}^T g)^* \left(-\sum_{i=1}^{m} \bar{\lambda}_i (\bar{u}_i + \bar{v}_i) \right) \\
&= 0 \text{ (by Theorem 3.2, (4))}.
\end{aligned}$$

Moreover we have

$$\sum_{i=1}^{m} \bar{\lambda}_i \bar{q}_i = \sum_{i=1}^{m} \bar{\lambda}_i \frac{1}{m\bar{\lambda}_i} \tilde{q} = \tilde{q} \geqq 0.$$

This means that $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_\mu$.

Furthermore for $\bar{x}$ and $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})$ the equality of the objective values holds, since

$$\begin{aligned}
\Psi_i^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) = & -f_i^*(\bar{u}_i) - (-\mu_i h_i)^*(\bar{v}_i) \\
& - \left(\bar{q}_i^T g\right)^* \left(-\frac{1}{m\bar{\lambda}_i} \sum_{j=1}^m \bar{\lambda}_j (\bar{u}_j + \bar{v}_j)\right) \\
& + (\bar{u}_i + \bar{v}_i)^T \bar{x} + (\bar{q}_i^T g)^* \left(-\frac{1}{m\bar{\lambda}_i} \sum_{j=1}^m \bar{\lambda}_j (\bar{u}_j + \bar{v}_j)\right) \\
= & -f_i^*(\bar{u}_i) - (-\mu_i h_i)^*(\bar{v}_i) + (\bar{u}_i + \bar{v}_i)^T \bar{x},
\end{aligned}$$

for $i = 1, \ldots, m$. Because of Theorem 3.2 (1) and (2), the right-hand side equals $f_i(\bar{x}) - \mu_i h_i(\bar{x})$, for $i = 1, \ldots, m$, and it follows

$$\Psi_i^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) = \Phi_i^{(\mu)}(\bar{x}), \qquad i = 1, \ldots, m.$$

According to Theorem 4.1 we have weak duality, which means that the element $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})$ is an efficient solution for (D_μ) and strong duality between (P_μ) and (D_μ) holds. □

In order to establish a converse duality theorem (following the steps of Wanka and Boţ [17]), we need the following notations and lemmas presented below.

For $\lambda \in \operatorname{int}(\mathbb{R}^m_+)$ let us denote

$$\mathcal{B}_\lambda := \left\{(u, v, q, t) : \sum_{i=1}^m \lambda_i q_i \geqq 0, \sum_{i=1}^m \lambda_i t_i = 0\right\},$$

where $u = (u_1, \ldots, u_m)$, $v = (v_1, \ldots, v_m)$, $q = (q_1, \ldots, q_m)$, $t = (t_1, \ldots, t_m)^T$, $u_i \in \mathbb{R}^m$, $v_i \in \mathbb{R}^m$, $q_i \in \mathbb{R}^k$, $t_i \in \mathbb{R}$, $i = 1, \ldots, m$. Furthermore, let us define

$$\begin{aligned}
M := \Big\{ a \in \mathbb{R}^m : \exists \lambda \in \operatorname{int}(\mathbb{R}^m_+),\ \exists\, (u, v, q, t) \in \mathcal{B}_\lambda \\
\text{such that } \sum_{i=1}^m \lambda_i a_i = \sum_{i=1}^m \lambda_i \Psi_i^{(\mu)}(u, v, q, \lambda, t) \Big\}.
\end{aligned}$$

The following two lemmas have been proved in a some different context by Wanka and Boţ (see Proposition 1 and 2 in [17]).

Lemma 4.1. *It holds* $\Psi^{(\mu)}(\mathcal{B}_\mu) \cap \mathbb{R}^m = M$.

Proof. From the definition it follows directly $\Psi^{(\mu)}(\mathcal{B}_\mu) \cap \mathbb{R}^m \subseteq M$. What remains to show is the inverse inclusion.
Let $a \in M$. Thus there exist $\lambda \in \text{int}(\mathbb{R}^m_+)$ and $(u, v, q, t) \in \mathcal{B}_\lambda$ such that

$$\begin{aligned}\sum_{i=1}^m \lambda_i a_i =& \sum_{i=1}^m \lambda_i \Psi_i^{(\mu)}(u, v, q, \lambda, t)\\ =& -\sum_{i=1}^m \lambda_i f_i^*(u_i) - \sum_{i=1}^m \lambda_i(-\mu_i h_i)^*(v_i)\\ &- \sum_{i=1}^m \lambda_i (q_i^T g)^* \left(-\frac{1}{m\lambda_i}\sum_{j=1}^m \lambda_j(u_j + v_j)\right) + \sum_{i=1}^m \lambda_i t_i.\end{aligned}$$

Defining

$$\bar{t}_i := a_i + f_i^*(u_i) + (-\mu_i h_i)^*(v_i) + (q_i^T g)^* \left(-\frac{1}{m\lambda_i}\sum_{j=1}^m \lambda_j(u_j + v_j)\right),$$

for $i = 1, \ldots, m$, we see that $\sum_{i=1}^m \lambda_i \bar{t}_i = 0$, which means $(u, v, q, \lambda, \bar{t}) \in \mathcal{B}_\mu$. Thus

$$a_i = -f_i^*(u_i) - (-\mu_i h_i)^*(v_i) - (q_i^T g)^* \left(-\frac{1}{m\lambda_i}\sum_{j=1}^m \lambda_j(u_j + v_j)\right) + \bar{t}_i.$$

In conclusion $a = \Psi^{(\mu)}(u, v, q, \lambda, \bar{t}) \in \Psi^{(\mu)}(\mathcal{B}_\mu)$ and we have the inclusion $M \subseteq \Psi^{(\mu)}(\mathcal{B}_\mu) \cap \mathbb{R}^m$, which finishes the proof. □

Lemma 4.2. *An element $\bar{a} \in \mathbb{R}^m$ is efficient in M if and only if for every $a \in M$ with corresponding $\lambda^a \in \text{int}\,\mathbb{R}^m_+$ and $(u^a, v^a, q^a, t^a) \in \mathcal{B}_{\lambda^a}$ it holds*

$$\sum_{i=1}^m \lambda_i^a \bar{a}_i \geq \sum_{i=1}^m \lambda_i^a a_i.$$

Proof. "$\Leftarrow$":
Assume that $\bar{a}$ is not efficient in M, namely there exists $a \in M$ such that $a \in \bar{a} + \mathbb{R}^m_+ \backslash \{0\}$. For the corresponding $\lambda^a \in \text{int}(\mathbb{R}^m_+)$ it holds

$$\sum_{i=1}^m \lambda_i^a \bar{a}_i < \sum_{i=1}^m \lambda_i^a a_i,$$

which contradicts the assertion.

"⇒":
Let $\bar{a}$ be maximal in M and take an arbitrary $a \in M$ with corresponding $\lambda^a \in \operatorname{int} \mathbb{R}^m_+$ and $(u^a, v^a, q^a, t^a) \in \mathcal{B}_{\lambda^a}$. Further, let be an arbitrary $b \in \mathbb{R}^m$, $b \in \bar{a} + \mathbb{R}^m_+ \backslash \{0\}$. Assume

$$\sum_{i=1}^m \lambda_i^a a_i \geq \sum_{i=1}^m \lambda_i^a b_i.$$

If the above relation is fulfilled with equality, i.e. $\sum\limits_{i=1}^m \lambda_i^a a_i = \sum\limits_{i=1}^m \lambda_i^a b_i$, we must have that $b \in M$ and this contradicts the maximality of $\bar{a}$ in M.
If $\sum\limits_{i=1}^m \lambda_i^a a_i > \sum\limits_{i=1}^m \lambda_i^a b_i$, we can choose $c_i \in \mathbb{R}$ such that $c_i > a_i$ and $c_i > b_i, i = 1, \ldots, m$. It follows

$$c := \sum_{i=1}^m \lambda_i^a c_i > \sum_{i=1}^m \lambda_i^a a_i > b := \sum_{i=1}^m \lambda_i^a b_i,$$

which implies that there exists an $r \in (0,1)$ such that $\sum\limits_{i=1}^m \lambda_i^a a_i = (1-r)b + rc$ or, equivalently, $\sum\limits_{i=1}^m \lambda_i^a a_i = \sum\limits_{i=1}^m \lambda_i^a [(1-r)b_i + rc_i]$, consequently $(1-r)b + rc \in M$.

On the other hand,

$$(1-r)b + rc = r(c-b) + b \in \mathbb{R}^m_+ \backslash \{0\} + (\bar{a} + \mathbb{R}^m_+ \backslash \{0\}).$$

This contradicts the maximality of $\bar{a}$ in M.
Summarizing, it holds for all $b \in \bar{a} + \mathbb{R}^m_+ \backslash \{0\}$

$$\sum_{i=1}^m \lambda_i^a a_i < \sum_{i=1}^m \lambda_i^a b_i.$$

Let b converge to $\bar{a}$ and so one gets

$$\sum_{i=1}^m \lambda_i^a \bar{a}_i = \inf \left\{ \sum_{i=1}^m \lambda_i^a b_i : b \in \bar{a} + \mathbb{R}^m_+ \backslash \{0\} \right\} \geq \sum_{i=1}^m \lambda_i^a a_i.$$

□

Next we deal with the converse duality theorem. Therefore we need the following condition.

Definition 4.2. Let be $\lambda \in \operatorname{int}(\mathbb{R}^m_+)$. The condition $(C_{\mu,\lambda})$ is fulfilled when from

$$\inf_{x\in\mathcal{A}} \sum_{i=1}^{m} \lambda_i \Phi_i^{(\mu)}(x) > -\infty$$

it follows that there exists $x_\lambda \in \mathcal{A}$ such that

$$\inf_{x\in\mathcal{A}} \sum_{i=1}^{m} \lambda_i \Phi_i^{(\mu)}(x) = \sum_{i=1}^{m} \lambda_i \Phi_i^{(\mu)}(x_\lambda).$$

This means that $(P_{\mu,\lambda})$ has a solution if $\inf(P_{\mu,\lambda}) > -\infty$.

Now the converse duality theorem for (P_μ) can be formulated extending Theorem 5 in [17].

Theorem 4.3. *Let (CQ) be fulfilled and assume that $(C_{\mu,\lambda})$ holds for all $\lambda \in \operatorname{int}(\mathbb{R}^m_+)$.*

(1) Let $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})$ be an efficient solution of (D_μ). Then

(a) $\Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \operatorname{cl}(\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+)$;

(b) it exists a properly efficient solution $\bar{x}_{\bar{\lambda}} \in \mathcal{A}$ of (P_μ) such that

$$\sum_{i=1}^{m} \bar{\lambda}_i [\Phi_i^{(\mu)}(\bar{x}_{\bar{\lambda}}) - \Psi_i^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})] = 0.$$

(2) If, additionally, $\Phi^{(\mu)}(\mathcal{A})$ is $\mathbb{R}^m_+$-closed ($\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+$ is closed), then there exists a properly efficient solution $\bar{x} \in \mathcal{A}$ of (P_μ) such that

$$\sum_{i=1}^{m} \bar{\lambda}_i \Phi_i^{(\mu)}(\bar{x}_{\bar{\lambda}}) = \sum_{i=1}^{m} \bar{\lambda}_i \Phi_i^{(\mu)}(\bar{x}),$$

and

$$\Phi^{(\mu)}(\bar{x}) = \Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}).$$

Proof.

(1) Let us denote $\bar{a} := \Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})$. Since $\bar{a}$ is maximal in $\Psi^{(\mu)}(\mathcal{B}_\mu)$, we have that $\bar{a} \in \Psi^{(\mu)}(\mathcal{B}_\mu) \cap \mathbb{R}^m = M$.

Assume that $\bar{a} \notin \operatorname{cl}(\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+)$. Then there exists $\lambda^1 \in \mathbb{R}^m \backslash \{0\}$ and an $\alpha \in \mathbb{R}$ such that

$$\sum_{i=1}^{m} \lambda_i^1 \bar{a}_i < \alpha \leq \sum_{i=1}^{m} \lambda_i^1 d_i, \quad \forall\, d \in \operatorname{cl}(\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+).$$

It is obvious that $\lambda^1 \in \mathbb{R}^m_+ \backslash \{0\}$.

As $\bar{a} \in M$, there exist a corresponding $\lambda^{\bar{a}} \in \text{int}(\mathbb{R}^m_+)$ and an element $(u^{\bar{a}}, v^{\bar{a}}, q^{\bar{a}}, t^{\bar{a}}) \in \mathcal{B}_{\lambda^{\bar{a}}}$ such that $\sum\limits_{i=1}^m \lambda_i^{\bar{a}} \bar{a}_i = \sum\limits_{i=1}^m \lambda_i^{\bar{a}} \Psi_i^{(\mu)}(u^{\bar{a}}, v^{\bar{a}}, q^{\bar{a}}, t^{\bar{a}})$. Because of the weak duality theorem, there is

$$\sum_{i=1}^m \lambda_i^{\bar{a}} \bar{a}_i = \sum_{i=1}^m \lambda_i^{\bar{a}} \Psi_i^{(\mu)}(u^{\bar{a}}, v^{\bar{a}}, q^{\bar{a}}, t^{\bar{a}}) \leq \sum_{i=1}^m \lambda_i^{\bar{a}} d_i, \ \forall\, d \in \text{cl}(\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+).$$

Choosing a fixed $s \in (0,1)$ and setting $\lambda^* := s\ \lambda^1 + (1 - s)\lambda^{\bar{a}} \in \text{int}(\mathbb{R}^m_+)$, we get

$$\sum_{i=1}^m \lambda_i^* \bar{a}_i < \sum_{i=1}^m \lambda_i^* d_i, \quad \forall\, d \in \text{cl}(\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+),$$

which implies that $\forall x \in \mathcal{A}$

$$\sum_{i=1}^m \lambda_i^* \bar{a}_i < \sum_{i=1}^m \lambda_i^* \Phi_i^{(\mu)}(x). \tag{3}$$

Since $(C_{\mu,\lambda})$ is fulfilled, Definition 4.2 and inequality (3) ensure the existence of an optimal solution $x_{\lambda^*} \in \mathcal{A}$ of problem (P_{μ,λ^*}) which is even properly efficient for (P_μ). (CQ) is fulfilled as well, so according to Theorem 4.2 there exists an efficient solution for problem (D_μ), say $(u_{\lambda^*}, v_{\lambda^*}, q_{\lambda^*}, \lambda^*, t_{\lambda^*})$, such that

$$\Phi^{(\mu)}(x_{\lambda^*}) = \Psi^{(\mu)}(u_{\lambda^*}, v_{\lambda^*}, q_{\lambda^*}, \lambda^*, t_{\lambda^*}) \in \Psi^{(\mu)}(\mathcal{B}_\mu) \cap \mathbb{R}^m = M.$$

Due to the fact that $\bar{a}$ is efficient in M, we get from Lemma 4.2

$$\sum_{i=1}^m \lambda_i^* \bar{a}_i \geq \sum_{i=1}^m \lambda_i^* \Phi_i^{(\mu)}(x_{\lambda^*}),$$

which contradicts inequality (3). Thus

$$\bar{a} = \Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \text{cl}(\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+).$$

Taking into account again the weak duality we have for each $x \in \mathcal{A}$

$$\sum_{i=1}^m \bar{\lambda}_i \Phi_i^{(\mu)}(x) \geq \sum_{i=1}^m \bar{\lambda}_i \bar{a}_i,$$

and so $(C_{\mu,\bar{\lambda}})$ ensures that the optimal objective value of $(P_{\mu,\bar{\lambda}})$ is attained, i.e. there exists a properly efficient solution $\bar{x}_{\bar{\lambda}} \in \mathcal{A}$ of (P_μ). For $\bar{x}_{\bar{\lambda}}$ the following inequalities hold

$$\begin{aligned}\sum_{i=1}^m \bar{\lambda}_i \bar{a}_i \leq \sum_{i=1}^m \bar{\lambda}_i \Phi_i^{(\mu)}(\bar{x}_{\bar{\lambda}}) &= \inf_{x\in\mathcal{A}} \sum_{i=1}^m \bar{\lambda}_i \Phi_i^{(\mu)}(x) \\ &\leq \sum_{i=1}^m \bar{\lambda}_i \big(\Phi_i^{(\mu)}(x^n) + k_i^n\big), \quad \forall\, n \geq 1\end{aligned}$$

where $(x^n)_{n\geq 1} \subseteq \mathcal{A}$ and $(k^n)_{n\geq 1} \subseteq \mathbb{R}^m_+$ are sequences with the property that $\Phi^{(\mu)}(x^n) + k^n \to \bar{a}$ as $n \to \infty$. Their existence follows from $\bar{a} \in \operatorname{cl}(\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+)$.
If $n \to \infty$, it follows that

$$\sum_{i=1}^m \bar{\lambda}_i \Phi_i^{(\mu)}(\bar{x}_{\bar{\lambda}}) = \sum_{i=1}^m \bar{\lambda}_i \bar{a}_i = \sum_{i=1}^m \bar{\lambda}_i \Psi_i^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}).$$

(2) If $\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+$ is closed, we have from the first part of the proof that $\bar{a} \in \operatorname{cl}(\Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+) = \Phi^{(\mu)}(\mathcal{A}) + \mathbb{R}^m_+$. Again, because of the weak duality theorem (Theorem 4.1), we have $\bar{a} \in \Phi^{(\mu)}(\mathcal{A})$, meaning that there is an $\bar{x} \in \mathcal{A}$ such that $\Phi^{(\mu)}(\bar{x}) = \bar{a} = \Psi^{(\mu)}(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})$.
Thus $\bar{x}$ is properly efficient and it holds

$$\sum_{i=1}^m \bar{\lambda}_i \Phi_i^{(\mu)}(\bar{x}_{\bar{\lambda}}) = \sum_{i=1}^m \bar{\lambda}_i \Phi_i^{(\mu)}(\bar{x}).$$

□

The last two theorems give some dual characterizations for the efficient solutions of the primal multiobjective problem (P). They follow from Theorem 2.3 together with the strong duality and converse duality assertions Theorem 4.2 and Theorem 4.3, respectively.

Theorem 4.4. *Let (CQ) be fulfilled and $\bar{x} \in \mathcal{A}$ be properly efficient in the sense of Geoffrion for problem (P) with $\bar{\mu}_i := \frac{f_i(\bar{x})}{h_i(\bar{x})} \geq 0$, $i = 1, \ldots, m$. Let be $\bar{\mu} := (\bar{\mu}_1, \ldots, \bar{\mu}_m)^T$. Then $\bar{x}$ is properly efficient for $(P_{\bar{\mu}})$, there exists $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t}) \in \mathcal{B}_{\bar{\mu}}$ that is efficient for $(D_{\bar{\mu}})$ and strong duality between $(P_{\bar{\mu}})$ and $(D_{\bar{\mu}})$ holds.*

Theorem 4.5. *Let (CQ) be fulfilled and $\bar{\mu} \in \mathbb{R}^m_+$ be such that the set $\Phi^{\bar{\mu}}(\mathcal{A})$ is $\mathbb{R}^m_+$-closed. Moreover, assume that $(C_{\bar{\mu},\lambda})$ holds for all $\lambda \in \operatorname{int}(\mathbb{R}^m_+)$. Let $(\bar{u}, \bar{v}, \bar{q}, \bar{\lambda}, \bar{t})$ be an efficient solution for $(D_{\bar{\mu}})$. Then there exists a properly efficient solution $\bar{x} \in \mathcal{A}$ for $(P_{\bar{\mu}})$ and strong duality between $(P_{\bar{\mu}})$ and $(D_{\bar{\mu}})$ holds. If $\Phi(\bar{x}) = \bar{\mu}$, then $\bar{x}$ is properly efficient in the sense of Geoffrion for (P).*

5 An example

We study the applicability of the converse duality result by considering a theoretical example. Let the primal problem be defined as follows

$$(P) \qquad \text{v-}\min_{x \in \mathcal{A}} \Phi(x),$$

where

$$\Phi(x) = \left[\frac{5x+6}{2x+3}, \frac{x+1}{5x+2}\right]^T$$

and

$$\mathcal{A} = \{x \in \mathbb{R} : g(x) := -x \leq 0\} = [0, \infty).$$

The corresponding parametric problem (P_μ) will be

$$(P_\mu) \qquad \text{v-}\min_{x \in \mathcal{A}} \Phi^{(\mu)}(x),$$

where $\mu = (\mu_1, \mu_2)^T$ and

$$\Phi^{(\mu)}(x) = \begin{bmatrix} 5x + 6 - 2\mu_1 x - 3\mu_1 \\ x + 1 - 5\mu_2 x - 2\mu_2 \end{bmatrix} = \begin{bmatrix} (5 - 2\mu_1)\, x + 6 - 3\mu_1 \\ (1 - 5\mu_2)\, x + 1 - 2\mu_2 \end{bmatrix}, x \in \mathbb{R}^n.$$

Then the dual (D_μ) looks like

$$(D_\mu) \qquad \text{v-}\max_{(u,v,q,\lambda,t) \in \mathcal{B}_\mu} \Psi^{(\mu)}(u, v, q, \lambda, t) = \text{v-}\max_{(u,v,q,\lambda,t) \in \mathcal{B}_\mu} \begin{bmatrix} \Psi_1^{(\mu)}(u, v, q, \lambda, t) \\ \Psi_2^{(\mu)}(u, v, q, \lambda, t) \end{bmatrix},$$

where

$$\Psi_i^{(\mu)}(u, v, q, \lambda, t_i) = -f_i^*(u_i) - (-\mu_i h_i)^*(v_i)$$
$$-(q_i g)^* \left(-\frac{1}{2\lambda_i} \sum_{j=1}^{2} \lambda_j (u_j + v_j) \right) + t_i,$$

for $i = 1, 2$. The conjugate functions of f_i and h_i, $i = 1, 2$, can be easily calculated

$$f_1^*(u_1) = \begin{cases} -6, & \text{if } u_1 = 5, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$f_2^*(u_2) = \begin{cases} -1, & \text{if } u_2 = 1, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$(-\mu_1 h_1)^*(v_1) = \begin{cases} 3\mu_1, & \text{if } v_1 = -2\mu_1, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$(-\mu_2 h_2)^*(v_2) = \begin{cases} 2\mu_2, & \text{if } v_2 = -5\mu_2, \\ +\infty, & \text{otherwise}. \end{cases}$$

Thus the two objective functions of the dual problem are greater than $-\infty$ only if $u = (5, 1)$ and $v = (-2\mu_1, -5\mu_2)$. Furthermore,

$$(q_i g)^* \left(-\frac{1}{2\lambda_i} \sum_{j=1}^{2} \lambda_j (u_j + v_j) \right) = \begin{cases} 0, & \text{if } q_i = \frac{1}{2\lambda_i} \sum\limits_{j=1}^{2} \lambda_j (u_j + v_j), \\ \infty, & \text{otherwise}, \end{cases}$$

for $i = 1, 2$. Therefore the dual problem (D_μ) becomes

$$\text{v-} \max_{(q,\lambda,t) \in \mathcal{B}_\mu} \Psi^{(\mu)}(q, \lambda, t),$$

where

$$\Psi_1^{(\mu)}(q, \lambda, t) = 6 - 3\mu_1 + t_1,$$
$$\Psi_2^{(\mu)}(q, \lambda, t) = 1 - 2\mu_2 + t_2,$$

and

$$\mathcal{B}_\mu = \Big\{ (q, \lambda, t) : \lambda_i > 0, q_i \geq 0, i = 1, 2, \sum_{i=1}^{2} \lambda_i t_i = 0,$$
$$2\lambda_1 q_1 = 2\lambda_2 q_2 = \lambda_1 (5 - 2\mu_1) + \lambda_2 (1 - 5\mu_2) \Big\}.$$

Choosing $\bar{\mu} = (2, \frac{1}{2})^T$, we see that $(\bar{q}, \bar{\lambda}, \bar{t})$ is feasible for $(D_{\bar{\mu}})$ with $\bar{q} = (\frac{1}{5}, \frac{1}{2})^T$, $\bar{\lambda} = (5, 2)^T$ and $\bar{t} = (0, 0)^T$. Furthermore, the objective value $\Psi^{(\bar{\mu})}(\bar{q}, \bar{\lambda}, \bar{t}) = [0, 0]^T$ can not be improved (in the sense of efficiency) without violating the

constraints, i.e. $(\bar{q}, \bar{\lambda}, \bar{t})$ is efficient for $(D_{\bar{\mu}})$.
For $\bar{\mu} = (2, \frac{1}{2})^T$ the primal objective function $\Phi^{(\bar{\mu})}$ can be written as

$$\Phi^{(\bar{\mu})}(x) = \begin{bmatrix} x \\ -\frac{3}{2}x \end{bmatrix}.$$

We see that $\bar{x} = 0 \in \mathcal{A}$ is the only element satisfying $\Phi^{(\bar{\mu})}(x) = [0, 0]^T$. Obviously, the condition (CQ) is fulfilled, $(C_{\bar{\mu},\lambda})$ holds for all $\lambda \in \operatorname{int}(\mathbb{R}^2_+)$ and $\Phi^{(\bar{\mu})}(\mathcal{A})$ is $\mathbb{R}^2_+$-closed. By Theorem 4.5 it follows that $\bar{x} = 0$ is properly efficient for $(P_{\bar{\mu}})$.
Furthermore, it holds

$$\Phi(0) = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} = \bar{\mu}$$

and the same theorem ensures that $\bar{x}$ is properly efficient in the sense of Geoffrion for (P).

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