Duality for location problems with unbounded unit balls

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Abstract. Given an optimization problem with a composite of a convex and componentwise increasing function with a convex vector function as objective function, by means of the conjugacy approach based on the perturbation theory, we determine a dual to it. Necessary and sufficient optimality conditions are derived using strong duality. Furthermore, as special case of this problem, we consider a location problem, where the "distances" are measured by gauges of closed convex sets. We prove that the geometric characterization of the set of optimal solutions for this location problem given by Hinojosa and Puerto in a recently published paper can be obtained via the presented dual problem. Finally, the Weber and the minmax location problems with gauges are given as applications.

Keywords. Convex programming, Location, Conjugate duality, Gauges, Optimality conditions

1 Introduction

Location problems play an important role in a lot of fields of applications, as they appear in many areas such as transportation planning, industrial engineering, telecommunication, computer science, etc. A lot of research has been carried out in location analysis, the results of these problems being to locate some items, to optimize transportation costs, to minimize covered distances etc. Among the large number of papers and books dealing with location analysis we mention [2], [5], [6], [8], [9], [10] and [16].

This paper is based on the work of Y. Hinojosa and J. Puerto [6], in which the authors introduced a location problem, where the "distances" were measured by gauges

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of closed (not necessarily bounded) convex sets. For this problem the authors obtained a characterization of the set of optimal solutions and gave some methods to solve it.

The goal of our paper is to treat the problem introduced in [6] by means of duality. On the other hand, we show how it is possible to derive the optimality conditions for this optimization problem via strong duality.

In order to do this we consider at first a more general optimization problem, and then we particularize the results for the location problems in [6]. The objective function of the original optimization problem we consider is a composite of a convex and componentwise increasing function with a convex vector function. Applying the Fenchel-Rockafellar duality concept based on conjugacy and perturbations (cf. [3]), we construct a dual problem to it and we prove the strong duality. Then, by means of strong duality, we derive the optimality conditions for the primal optimization problem.

Afterwards, we study optimization problems with monotonic gauges as objective functions as particular cases of the general problem treated before. Analogously to the general problem, we construct a dual problem and prove the strong duality, and then we derive the optimality conditions.

In Section 5 we consider the optimization problem treated by Y. Hinojosa and J. Puerto in [6]. The obtained optimality conditions turn out to be the same as in [6]. The sections 6 and 7 are devoted to the specialization of the Weber and the minmax problem with gauges, respectively.

In the past most of the references concerning location problems have considered distances induced by norms, but recently some papers have been published that consider the use of gauges like [4], [12] and [17]. These lead to more general models, for example to model situations where the symmetry property of a norm does not make sense.

2 Notations and preliminary results

In this section we provide some definitions and results that we shall use in the sequel. As usual, $\mathbb{R}^n$ denotes the n-dimensional real space, for $n \in \mathbb{N}$. Throughout this paper all the vectors are considered as column vectors belonging to $\mathbb{R}^n$, unless otherwise specified. An upper index ^{T} transposes a column vector to a raw one and viceversa. The inner product of two vectors $x = \big(x_1, ..., x_n\big)^T$ and $y = \big(y_1, ..., y_n\big)^T$ in the n-dimensional real space is denoted $x^T y = \sum_{i=1}^n x_i y_i$. Now we recall the concepts of gauges and polar sets and we relate them with some other concepts of convex analysis.

Definition 2.1. Let $C \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The function γ_C defined by

$$\gamma_C(x) := \inf \big\{\alpha > 0 : \, x \in \alpha C\big\}$$

is called the gauge of C. The set C is called the unit ball associated with γ_C. As usual, we set $\gamma_C(x) := +\infty$, if there is no $\alpha > 0$ such that $x \in \alpha C$.

Definition 2.2. Let $C \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. The set given by

$$C^0 = \{y \in \mathbb{R}^n : \, y^T x \leq 1, \ \forall x \in C\}$$

is called the polar set of C.

Remark 2.3. C^0 is a closed convex set containing the origin.

Definition 2.4. Let $C \subseteq \mathbb{R}^n$ be a convex set. The function σ_C given by

$$\sigma_C(y) := \sup\{y^T x : \ x \in C\}$$

is called the support function of C.

Theorem 2.5. ([7]) *Let C be a closed convex set containing the origin. Then*

(i) its gauge γ_C is a nonnegative closed sublinear function,

(ii) $\{x \in \mathbb{R}^n : \, \gamma_C(x) \leq r\} = rC$, for all $r > 0$.

Proposition 2.6. ([7]) *Let C be a closed convex set containing the origin. Its gauge γ_C is the support function of the set C^0, namely*

$$\gamma_C(x) = \sigma_{C^0}(x) = \sup\{y^T x : \ y \in C^0\}.$$

Corollary 2.7. ([7]) *Let C be a closed convex set containing the origin. Its support function σ_C is the gauge of C^0 and is denoted by γ_{C^0}, i.e.*

$$\sigma_C(y) = \gamma_{C^0}(y) = \inf\left\{\alpha > 0 : \ y \in \alpha C^0\right\}.$$

Proposition 2.8. *The conjugate function $\gamma_C^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of γ_C verifies*

$$\gamma_C^*(y) = \begin{cases} 0, & if \ y \in C^0, \\ +\infty, & otherwise, \end{cases}$$

where C^0 is the polar set of C.

Proof. By the definition of the conjugate function of $\gamma_C(x)$ we get

$$\begin{aligned} \gamma_C^*(y) &= \sup_{x \in \mathbb{R}^n} \left\{y^T x - \gamma_C(x)\right\} = \sup_{x \in \mathbb{R}^n} \left\{y^T x - \inf\left\{\alpha > 0 : \ x \in \alpha C\right\}\right\} \\ &= \sup_{x \in \mathbb{R}^n} \Big\{y^T x + \sup_{\substack{\alpha > 0, \\ x \in \alpha C}} (-\alpha)\Big\} = \sup_{\substack{\alpha > 0, \\ x \in \alpha C}} \left\{y^T x - \alpha\right\} = \sup_{\substack{\alpha > 0, \\ z \in C}} \left\{y^T (\alpha z) - \alpha\right\} \\ &= \sup_{\alpha > 0} \alpha \Big\{ \sup_{z \in C} \left\{y^T z - 1\right\}\Big\} = \begin{cases} 0, & \text{if } y \in C^0, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

□

Remark 2.9. By Theorem 2.5 and Remark 2.3 the fact that $y \in C^0$ is equivalent to the inequality $\gamma_{C^0}(y) \leq 1$, so, one can write $\gamma_C^*(y) = \begin{cases} 0, & \text{if } \gamma_{C^0}(y) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$

3 The optimization problem with a composed convex function as objective function

Let $(X, \|\cdot\|)$ be a normed space and X^* the topological dual space of X. $\langle x^*, x\rangle$ will denote the value at $x \in X$ of the continuous linear functional $x^* \in X^*$. Further, let $g_i : X \to \mathbb{R}, i = 1, \ldots, m$, be convex and continuous functions and $f : \mathbb{R}^m \to \mathbb{R}$ be a convex and componentwise increasing function, i.e. for $y = (y_1, \ldots, y_m)^T, z = (z_1, \ldots, z_m)^T \in \mathbb{R}^m$,

$$y_i \geq z_i, i = 1, \ldots, m \Rightarrow f(y) \geq f(z).$$

The optimization problem which we consider is the following one

$$(P) \quad \inf_{x \in X} f(g(x)),$$

where $g : X \to \mathbb{R}^m$, $g(x) = (g_1(x), \ldots, g_m(x))^T$.

In this section we find out a dual problem to (P) and prove the existence of weak and strong duality. Moreover, by means of strong duality we derive the optimality conditions for (P).

The approach, we use to find a dual problem to (P), is the so-called Fenchel-Rockafellar approach and it was very well described in [3] and [11]. It offers the possibility to construct different dual problems to a primal optimization problem by perturbing it in different ways (cf. [13], [14] and [15]).

In order to find a dual problem to (P) we consider the following perturbation function $\Psi : \underbrace{X \times \ldots \times X}_{m+1} \times \mathbb{R}^m \to \mathbb{R}$,

$$\Psi(x, q, d) = f((g_1(x + q_1), \ldots, g_m(x + q_m))^T + d),$$

where $q = (q_1, \ldots, q_m) \in X \times \ldots \times X$ and $d \in \mathbb{R}^m$ are the so-called perturbation variables.

Then the dual problem to (P), obtained by using the perturbation function Ψ, is

$$(D) \quad \sup_{\substack{\lambda \in \mathbb{R}^m, p_i \in X^*, \\ i=1,\ldots,m}} \{-\Psi^*(0, p, \lambda)\},$$

where $\Psi^* : \underbrace{X^* \times \ldots \times X^*}_{m+1} \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is the conjugate function of Ψ. Here, $p_i, i = 1, ..., m$, and $\lambda \in \mathbb{R}^m$ are the dual variables.

We recall that for a function $h : Y \to \mathbb{R}$, Y being a Hausdorff locally convex vector space, its conjugate function $h^* : Y^* \to \mathbb{R} \cup \{+\infty\}$ has the form $h^*(y^*) = \sup\limits_{y \in Y}\{\langle y^*, y\rangle - h(y)\}$. Y^* is the topological dual space to Y.

Therefore the conjugate function of Ψ can be calculated by the following formula

$$\Psi^*(x^*, p, \lambda) = \sup_{\substack{q_i \in X, i=1,\dots,m, \\ x \in X, d \in \mathbb{R}^m}} \Big\{ \langle x^*, x\rangle + \sum_{i=1}^m \langle p_i, q_i\rangle + \lambda^T d - f((g_1(x+q_1), \dots, g_m(x+q_m))^T + d)\Big\}.$$

To treat this expression we introduce at first the new variable t instead of d and then the new variables r_i instead of q_i by

$$t = d + (g_1(x+q_1), \dots, g_m(x+q_m))^T \in \mathbb{R}^m$$

and

$$r_i = x + q_i \in X, i = 1, ..., m.$$

This implies

$$\begin{aligned}
\Psi^*(x^*, p, \lambda) &= \sup_{\substack{q_i \in X, i=1,\dots,m, \\ x \in X, t \in \mathbb{R}^m}} \Big\{ \langle x^*, x\rangle + \sum_{i=1}^m \langle p_i, q_i\rangle \\
&\quad + \lambda^T \big(t - (g_1(x+q_1), \dots, g_m(x+q_m))^T\big) - f(t)\Big\} \\
&= \sup_{\substack{r_i \in X, i=1,\dots,m, \\ x \in X}} \Big\{ \langle x^*, x\rangle + \sum_{i=1}^m \langle p_i, r_i - x\rangle \\
&\quad - \lambda^T \big((g_1(r_1), \dots, g_m(r_m))^T\big)\Big\} + \sup_{t \in \mathbb{R}^m}\{\lambda^T t - f(t)\} \\
&= \sum_{i=1}^m \sup_{r_i \in X}\{\langle p_i, r_i\rangle - \lambda_i g_i(r_i)\} + \sup_{x \in X}\left\langle x^* - \sum_{i=1}^m p_i, x\right\rangle \\
&\quad + f^*(\lambda) \\
&= f^*(\lambda) + \sum_{i=1}^m (\lambda_i g_i)^*(p_i) + \sup_{x \in X}\left\langle x^* - \sum_{i=1}^m p_i, x\right\rangle.
\end{aligned}$$

We have now to consider $x^* = 0$ and, so, the dual problem of (P) has the following form

$$(D) \quad \sup_{\substack{\lambda \in \mathbb{R}^m, p_i \in X^*, \\ i=1,\dots,m}} \left\{ -f^*(\lambda) - \sum_{i=1}^m (\lambda_i g_i)^*(p_i) + \inf_{x \in X}\left\langle \sum_{i=1}^m p_i, x\right\rangle\right\}.$$

In the objective function of (D), if $\sum\limits_{i=1}^{m} p_i \neq 0_{X^*}$, there exists $x_0 \in X, x_0 \neq 0_X$, such that $\left\langle \sum\limits_{i=1}^{m} p_i, x_0 \right\rangle < 0$. But, for all $\alpha > 0$ we have

$$\inf_{x \in X} \left\langle \sum_{i=1}^{m} p_i, x \right\rangle < \alpha \cdot \left\langle \sum_{i=1}^{m} p_i, x_0 \right\rangle,$$

and this means that, in this case, $\inf\limits_{x \in X} \left\langle \sum\limits_{i=1}^{m} p_i, x \right\rangle = -\infty$.

In conclusion, in order to have the supremum in (D), we must consider $\sum\limits_{i=1}^{m} p_i = 0$.

By this, the dual problem of (P) is

$$(D) \quad \sup_{\substack{\lambda \in \mathbb{R}^m, p_i \in X^*, \\ i=1,...,m, \sum\limits_{i=1}^{m} p_i = 0}} \left\{ -f^*(\lambda) - \sum_{i=1}^{m} (\lambda_i g_i)^*(p_i) \right\}. \tag{1}$$

Now, let us point out a property of the conjugate of a componentwise increasing function.

Proposition 3.1. *Let be $f : \mathbb{R}^m \to \mathbb{R}$ a componentwise increasing function. Then $f^*(\lambda) = +\infty$ for all $\lambda \in \mathbb{R}^m \setminus \mathbb{R}^m_+$.*

Proof. Let be $\lambda \in \mathbb{R}^m \setminus \mathbb{R}^m_+$. Then there exists at least one $i \in \{1, ..., m\}$ such that $\lambda_i < 0$. But

$$f^*(\lambda) = \sup_{d \in \mathbb{R}^m} \left\{ \lambda^T d - f(d) \right\} \geq \sup_{\substack{d=(0,...,d_i,...,0), \\ d_i \in \mathbb{R}}} \left\{ \lambda_i d_i - f(0, ..., d_i, ..., 0) \right\},$$

this means that

$$\begin{aligned} f^*(\lambda) \geq & \sup_{d_i \in \mathbb{R}} \left\{ \lambda_i d_i - f(0, ..., d_i, ..., 0) \right\} \geq \sup_{d_i < 0} \left\{ \lambda_i d_i - f(0, ..., d_i, ...0) \right\} \\ \geq & \sup_{d_i < 0} \left\{ \lambda_i d_i \right\} - f(0, ..., 0) = +\infty, \end{aligned}$$

i.e. $f^*(\lambda) = +\infty, \ \forall \ \lambda \in \mathbb{R}^m \setminus \mathbb{R}^m_+$. □

By Proposition 3.1, the dual problem of (P) becomes

$$(D) \quad \sup_{\substack{\lambda \in \mathbb{R}^m_+, p_i \in X^*, \\ i=1,...,m, \sum\limits_{i=1}^{m} p_i = 0}} \left\{ -f^*(\lambda) - \sum_{i=1}^{m} (\lambda_i g_i)^*(p_i) \right\}.$$

Let us point out that, by the Fenchel-Rockafellar approach, between (P) and (D) weak duality, i.e. $\inf(P) \geq \sup(D)$, always holds (cf. [3]).

But, we are interested in the existence of strong duality $\inf(P) = \max(D)$. This can be shown by proving that the problem (P) is stable (cf. [3]). Therefore, we show that the stability criterion described in Proposition III.2.3 in [3] is fulfilled. For the beginning we need the following proposition.

Proposition 3.2. *The function* $\Psi : \underbrace{X \times \ldots \times X}_{m+1} \times \mathbb{R}^m \to \mathbb{R}$,

$$\Psi(x, q, d) = f((g_1(x + q_1), \ldots, g_m(x + q_m))^T + d)$$

is convex.

The convexity of Ψ follows from the convexity of the functions f and g and the fact that f is a componentwise increasing function.

Theorem 3.3. (strong duality for (P)) *If* $\inf(P) > -\infty$, *then the dual problem has an optimal solution and strong duality holds, i.e.*

$$\inf(P) = \max(D).$$

Proof. By Proposition 3.2, we have that the perturbation function Ψ is convex. Moreover, $\inf(P)$ is a finite number and the function

$$(q_1, \ldots, q_m, d) \longrightarrow \Psi(0, q_1, \ldots, q_m, d)$$

is finite and continuous in $(\underbrace{0, ..., 0}_{m}, 0_{\mathbb{R}^m}) \in \underbrace{X \times \ldots \times X}_{m} \times \mathbb{R}^m$. This means that the stability criterion in Proposition III.2.3 in [3] is fulfilled, which implies that the problem (P) is stable. Finally, the Propositions IV.2.1 and IV.2.2 in [3] lead to the desired conclusions. □

The structure of the problem (P) looks like a scalarization of a vector optimization problem by means of the monotonic function f. The results concerning duality for the problem (P) could be used to derive duality statements in the multiobjective optimization. But, this is the subject of some of our present research.

The last part of this section is devoted to the presentation of the optimality conditions for the primal problem (P). They are derived, by the use of the equality between the optimal values of the primal and dual problem.

Theorem 3.4. (optimality conditions for (P))

(1) *Let $\bar{x} \in X$ be an optimal solution to (P). Then there exist $\bar{p}_i \in X^*, i = 1, ..., m$, and $\bar{\lambda} \in \mathbb{R}^m_+$, such that $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ is an optimal solution to (D) and the*

following optimality conditions are satisfied

$$(i) \quad f(g(\bar{x})) + f^*(\bar{\lambda}) = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}),$$

$$(ii) \quad \bar{\lambda}_i g_i(\bar{x}) + (\bar{\lambda}_i g_i)^*(\bar{p}_i) = \langle \bar{p}_i, \bar{x} \rangle, i = 1, \ldots, m,$$

$$(iii) \quad \sum_{i=1}^{m} \bar{p}_i = 0.$$

(2) *If $\bar{x} \in X$, $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ is feasible to (D) and (i)-(iii) are fulfilled, then $\bar{x}$ is an optimal solution to (P), $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ is an optimal solution to (D) and strong duality holds*

$$f(g(\bar{x})) = -f^*(\bar{\lambda}) - \sum_{i=1}^{m} (\bar{\lambda}_i g_i)^*(\bar{p}_i).$$

Proof. (1) By Theorem 3.3 follows that there exist $\bar{p}_i \in X^*, i = 1, ..., m$, and $\bar{\lambda} \in \mathbb{R}^m_+$, such that $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ is a solution to (D) and $\inf(P) = \max(D)$. This means that $\sum_{i=1}^{m} \bar{p}_i = 0$ and

$$f(g(\bar{x})) = -f^*(\bar{\lambda}) - \sum_{i=1}^{m} (\bar{\lambda}_i g_i)^*(\bar{p}_i). \tag{2}$$

The last equality is equivalent to

$$0 = f(g(\bar{x})) + f^*(\bar{\lambda}) - \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) + \sum_{i=1}^{m} [\bar{\lambda}_i g_i(\bar{x}) + (\bar{\lambda}_i g_i)^*(\bar{p}_i) - \langle \bar{p}_i, \bar{x} \rangle]. \tag{3}$$

From the definition of the conjugate functions we have that the following so-called Young-inequalities

$$f(g(\bar{x})) + f^*(\bar{\lambda}) \geq \bar{\lambda}^T g(\bar{x}) = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \tag{4}$$

and

$$\bar{\lambda}_i g_i(\bar{x}) + (\bar{\lambda}_i g_i)^*(\bar{p}_i) \geq \langle \bar{p}_i, \bar{x} \rangle, i = 1, \ldots, m, \tag{5}$$

are true. By (4) and (5) all the terms of the sum in (3) must be equal to zero. In conclusion, the equalities in (i) and (ii) must hold.
(2) All the calculations and transformations done within part (1) may be carried out in the inverse direction starting from the conditions (i), (ii) and (iii). Thus the equality (2) results, which is the strong duality and shows that $\bar{x}$ solves (P) and $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ solves (D). □

4 The case of monotonic gauges

In this section we give an application to the problem presented above. Therefore, let $\gamma_C : \mathbb{R}^m \to \mathbb{R}$ be a monotonic gauge of a closed convex set C containing the origin. Recall that γ_C is a monotonic gauge on $\mathbb{R}^m$ (cf. [1]), if $\gamma_C(u) \leq \gamma_C(v)$ for every u and v in $\mathbb{R}^m$ satisfying $|u_i| \leq |v_i|$ for each $i = 1, ..., m$.
As in the Section 3 X is assumed to be a normed space and $g : X \to \mathbb{R}^m$, $g(x) = (g_1(x), \ldots, g_m(x))^T$, where $g_i, i = 1, ..., m$, are convex and continuous functions.

Let us introduce now the following primal problem

$$(P_{\gamma_C}) \quad \inf_{x \in X} \gamma_C^+(g(x)),$$

where $\gamma_C^+ : \mathbb{R}^m \to \mathbb{R}$, $\gamma_C^+(t) := \gamma_C(t^+)$, with $t^+ = (t_1^+, ..., t_m^+)^T$ and $t_i^+ = \max\{0, t_i\}$, $i = 1, ..., m$.

Proposition 4.1. *The function $\gamma_C^+ : \mathbb{R}^m \to \mathbb{R}$ is convex and componentwise increasing.*

Proof. First, let us point out that the function $(\cdot)^+ : \mathbb{R}^m \to \mathbb{R}_+^m$, defined by $t^+ = (t_1^+, ..., t_m^+)^T$, for $t \in \mathbb{R}^m$, is a convex function. This means that, for $u, v \in \mathbb{R}^m$ and $\alpha \in [0, 1]$, it holds

$$(\alpha u + (1 - \alpha)v)^+ \leqq \alpha u^+ + (1 - \alpha)v^+.$$

Here, " $\leqq$ " is the ordering induced on $\mathbb{R}^m$ by the cone of non-negative elements $\mathbb{R}_+^m$. By the positive sublinearity and monotonicity of the gauge γ_C, we have for $u, v \in \mathbb{R}^m$ and $\alpha \in [0, 1]$,

$$\begin{aligned}\gamma_C^+(\alpha u + (1 - \alpha)v) = \gamma_C((\alpha u + (1 - \alpha)v)^+) &\leq \gamma_C(\alpha u^+ + (1 - \alpha)v^+) \\ &\leq \alpha\gamma_C(u^+) + (1 - \alpha)\gamma_C(v^+) = \alpha\gamma_C^+(u) + (1 - \alpha)\gamma_C^+(v).\end{aligned}$$

This means that the function γ_C^+ is convex.

In order to prove that γ_C^+ is componentwise increasing, let $u, v \in \mathbb{R}^m$ be such that $u_i \leq v_i$, $i = 1, ..., m$. It follows $u_i^+ \leq v_i^+$, which implies that $|u_i^+| \leq |v_i^+|$, $i = 1, ..., m$. γ_C being a monotonic gauge, we have $\gamma_C(u^+) \leq \gamma_C(v^+)$, where $u^+ = (u_1^+, ..., u_m^+)^T$, $v^+ = (v_1^+, ..., v_m^+)^T$ or, equivalently, $\gamma_C^+(u) \leq \gamma_C^+(v)$.
Hence the function γ_C^+ is componentwise increasing. □

By the approach described in Section 3, a dual problem to (P_{γ_C}) is

$$(D_{\gamma_C}) \quad \sup_{\substack{\lambda \in \mathbb{R}_+^m, p_i \in X^*, \\ i=1,...,m, \sum\limits_{i=1}^m p_i = 0}} \left\{ -(\gamma_C^+)^*(\lambda) - \sum_{i=1}^m (\lambda_i g_i)^*(p_i) \right\}.$$

Proposition 4.2. *The conjugate function* $(\gamma_C^+)^* : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ *of* γ_C^+ *verifies*

$$(\gamma_C^+)^*(\lambda) = \begin{cases} 0, & if\ \lambda \in \mathbb{R}^m_+ \ and\ \gamma_{C^0}(\lambda) \leq 1, \\ +\infty, & otherwise, \end{cases}$$

where γ_{C^0} *is the gauge of the polar set* C^0.

Proof. For $\lambda \in \mathbb{R}^m \setminus \mathbb{R}^m_+$ the assertion is a consequence of Proposition 3.1 and Proposition 4.1.
Let be $\lambda \in \mathbb{R}^m_+$. For $t \in \mathbb{R}^m$, we have $|t_i| \geq |t_i^+|,\ i = 1, ..., m$, which implies that $\gamma_C(t) \geq \gamma_C(t^+) = \gamma_C^+(t)$ and

$$\gamma_C^*(\lambda) = \sup_{t \in \mathbb{R}^m} \{\lambda^T t - \gamma_C(t)\} \leq \sup_{t \in \mathbb{R}^m} \{\lambda^T t - \gamma_C^+(t)\} = (\gamma_C^+)^*(\lambda). \tag{6}$$

On the other hand, for the conjugate of the gauge γ_C we have the following formula (see Remark 2.9)

$$\gamma_C^*(\lambda) = \sup_{t \in \mathbb{R}^m} \{\lambda^T t - \gamma_C(t)\} = \begin{cases} 0, & \text{if } \gamma_{C^0}(\lambda) \leq 1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{7}$$

If $\gamma_{C^0}(\lambda) > 1$, we have that $+\infty = \gamma_C^*(\lambda) \leq (\gamma_C^+)^*(\lambda)$. From here, $(\gamma_C^+)^*(\lambda) = +\infty$.
Let be now $\gamma_{C^0}(\lambda) \leq 1$. Because $\lambda \geqq 0$, it follows that $\lambda^T t \leq \lambda^T t^+$, for every $t \in \mathbb{R}^m$. Furthermore, by Theorem 2.5, from $\gamma_{C^0}(\lambda) \leq 1$ it follows that $\lambda \in C^0$ and then by Proposition 2.6 we obtain that $\lambda^T t^+ \leq \gamma_C(t^+)$. From these inequalities we obtain for the conjugate function of γ_C^+

$$0 \leq \gamma_C^*(\lambda) \leq (\gamma_C^+)^*(\lambda) = \sup_{t \in \mathbb{R}^m} \{\lambda^T t - \gamma_C(t^+)\} \leq \sup_{t \in \mathbb{R}^m} \{\lambda^T t^+ - \gamma_C(t^+)\} \leq 0.$$

Consequently, there is $(\gamma_C^+)^*(\lambda) = 0$ and the proposition is proved. □

By Proposition 4.2 the dual of (P_{γ_C}) has the following formulation

$$(D_{\gamma_C}) \quad \sup_{\substack{\lambda \in \mathbb{R}^m_+,\, p_i \in X^*,\, i=1,...,m, \\ \sum_{i=1}^m p_i = 0,\, \gamma_{C^0}(\lambda) \leq 1}} \left\{ -\sum_{i=1}^m (\lambda_i g_i)^*(p_i) \right\}.$$

In the objective function of this dual we separate the terms for which $\lambda_i > 0$ from the terms for which $\lambda_i = 0$ and then the dual can be written as

$$(D_{\gamma_C}) \quad \sup_{\substack{p_i \in X^*,\, i=1,...,m,\, \sum_{i=1}^m p_i = 0, \\ \gamma_{C^0}(\lambda) \leq 1,\, I \subseteq \{1,...,m\}, \\ \lambda_i > 0\, (i \in I),\, \lambda_i = 0\, (i \notin I)}} \left\{ -\sum_{i \in I} (\lambda_i g_i)^*(p_i) - \sum_{i \notin I} (0)^*(p_i) \right\}. \tag{8}$$

For $i \notin I$, it holds

$$0^*(p_i) = \sup_{x \in X}\{\langle p_i, x\rangle - 0\} = \sup_{x \in X}\langle p_i, x\rangle = \begin{cases} 0, & \text{if } p_i = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

For $i \in I$ there is $(\lambda_i g_i)^*(p_i) = \lambda_i g_i^*\left(\frac{1}{\lambda_i} p_i\right)$ (cf. [3]). Denoting $p_i := \frac{1}{\lambda_i} p_i$, we obtain

$$(D_{\gamma_C}) \qquad \sup_{(I,\lambda,p) \in Y_{\gamma_C}} \left\{ -\sum_{i \in I} \lambda_i g_i^*(p_i) \right\},$$

with

$$Y_{\gamma_C} = \Big\{(I, \lambda, p) : I \subseteq \{1, ..., m\}, \lambda = (\lambda_1, ..., \lambda_m)^T, p = (p_1, ..., p_m),$$
$$\gamma_{C^0}(\lambda) \leq 1,\ \lambda_i > 0\,(i \in I),\ \lambda_i = 0\,(i \notin I),\ \sum_{i \in I} \lambda_i p_i = 0\Big\}.$$

Because $\inf(P_{\gamma_C})$ is finite being greater or equal than zero and γ_C^+ is a convex and componentwise increasing function, by Theorem 3.3 we can formulate the following strong duality theorem for the problems (P_{γ_C}) and (D_{γ_C}).

Theorem 4.3. (strong duality for (P_{γ_C})) *The dual problem (D_{γ_C}) has an optimal solution and strong duality holds, i.e.*

$$\inf(P_{\gamma_C}) = \max(D_{\gamma_C}).$$

Similarly to the general problem (P) the optimality conditions for (P_{γ_C}) can be derived.

Theorem 4.4. (optimality conditions for (P_{γ_C}))

(1) Let $\bar{x}$ be an optimal solution to (P_{γ_C}). Then there exists an optimal solution $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_{\gamma_C}$ to (D_{γ_C}), such that the following optimality conditions are satisfied

$$\begin{array}{rl} (i) & \bar{I} \subseteq \{1, ..., m\},\ \bar{\lambda}_i > 0\ (i \in \bar{I}),\ \bar{\lambda}_i = 0\ (i \notin \bar{I}), \\ (ii) & \gamma_{C^0}(\bar{\lambda}) \leq 1,\ \sum\limits_{i \in \bar{I}} \bar{\lambda}_i \bar{p}_i = 0, \\ (iii) & \gamma_C^+(g(\bar{x})) = \sum\limits_{i \in \bar{I}} \bar{\lambda}_i g_i(\bar{x}), \\ (iv) & g_i(\bar{x}) + g_i^*(\bar{p}_i) = \langle \bar{p}_i, \bar{x}\rangle,\ i \in \bar{I}. \end{array}$$

(2) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_{\gamma_C}$ and $(i) - (iv)$ are fulfilled, then $\bar{x}$ is an optimal solution to (P_{γ_C}), $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_{\gamma_C}$ is an optimal solution to (D_{γ_C}) and strong duality holds

$$\gamma_C^+(g(\bar{x})) = -\sum_{i \in \bar{I}} \bar{\lambda}_i g_i^*(\bar{p}_i).$$

Proof. (1) By Theorem 4.3 follows that there exists an optimal solution $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_{\gamma_C}$ to (D_{γ_C}) such that $(i) - (ii)$ are fulfilled and

$$\gamma_C^+(g(\bar{x})) = -\sum_{i \in \bar{I}} \bar{\lambda}_i g_i^*(\bar{p}_i).$$

This equality is equivalent to

$$0 = \left[\gamma_C^+(g(\bar{x})) + (\gamma_C^+)^*(\bar{\lambda}) - \sum_{i \in \bar{I}} \bar{\lambda}_i g_i(\bar{x})\right] + \sum_{i \in \bar{I}} \bar{\lambda}_i [g_i(\bar{x}) + g_i^*(\bar{p}_i) - \langle \bar{p}_i, \bar{x} \rangle]. \tag{9}$$

Using Young's inequality we have

$$\gamma_C^+(g(\bar{x})) + (\gamma_C^+)^*(\bar{\lambda}) \geq \bar{\lambda}^T g(\bar{x}) = \sum_{i \in \bar{I}} \bar{\lambda}_i g_i(\bar{x}) \tag{10}$$

and

$$g_i(\bar{x}) + g_i^*(\bar{p}_i) \geq \langle \bar{p}_i, \bar{x} \rangle, \ i \in \bar{I}. \tag{11}$$

All terms of the sum in (9) inside brackets are positive, therefore

$$\gamma_C^+(g(\bar{x})) + (\gamma_C^+)^*(\bar{\lambda}) = \bar{\lambda}^T g(\bar{x}) = \sum_{i \in \bar{I}} \bar{\lambda}_i g_i(\bar{x}) \tag{12}$$

and

$$g_i(\bar{x}) + g_i^*(\bar{p}_i) = \langle \bar{p}_i, \bar{x} \rangle, \ i \in \bar{I}. \tag{13}$$

But, by Proposition 4.2, we have that $(\gamma_C^+)^*(\bar{\lambda}) = 0$, and, so, $\gamma_C^+(g(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i g_i(\bar{x})$.
In conclusion, the relations $(iii) - (iv)$ must also hold.
(2) All the calculations and transformations done within part (1) may be carried out in the inverse direction. □

5 The location model with unbounded unit balls

In this section we consider the problem treated by Hinojosa and Puerto in [6]. This is a single facility location problem, where gauges of closed convex sets are used to model distances.

Throughout this section let $\mathcal{A} := \{a_1, ..., a_m\}$ be a subset of $\mathbb{R}^n$ which represents the set of existing facilities. Each facility $a_i \in \mathcal{A}$ has an associated gauge φ_{a_i}, whose unit ball is a closed convex set C_{a_i} containing the origin. Let $w = \{w_{a_1}, ..., w_{a_m}\}$ be a set of positive weights and let $\gamma_C : \mathbb{R}^m \to \mathbb{R}$ be a monotonic gauge of a closed convex set C containing the origin. The distance from an existing facility $a_i \in \mathcal{A}$ to a new facility $x \in \mathbb{R}^n$ is given by $\varphi_{a_i}(x-a_i)$. By $\varphi^0_{a_i}$ we denote the gauge of the polar set $C^0_{a_i}$.

The location problem studied in [6] is

$$(P_{\gamma_C}(\mathcal{A})) \quad \inf_{x\in\mathbb{R}^n} \gamma_C\big(w_{a_1}\varphi_{a_1}(x-a_1), ..., w_{a_m}\varphi_{a_m}(x-a_m)\big).$$

Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be the vector function defined by $g(x) := (g_1(x), ..., g_m(x))^T$, where $g_i(x) = w_{a_i}\varphi_{a_i}(x-a_i)$ for all $i = 1, ..., m$.

Because

$$\gamma^+_C(g(x)) = \gamma_C(g^+(x)) = \gamma_C(g(x)),\ \forall x \in \mathbb{R}^n,$$

$(P_{\gamma_C}(\mathcal{A}))$ can be written in the equivalent form

$$(P_{\gamma_C}(\mathcal{A})) \quad \inf_{x\in\mathbb{R}^n} \gamma^+_C(g(x)),$$

which is a particular case of the problem studied in the previous section. We mention, that instead of the space X considered in the case of the general optimization problem, we take here analogously to [6], the space $\mathbb{R}^n$. Therefore the dual problem to $(P_{\gamma_C}(\mathcal{A}))$ is

$$(D_{\gamma_C}(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_{\gamma_C}(\mathcal{A})} \left\{-\sum_{i\in I}\lambda_i g^*_i(p_i)\right\},$$

with

$$Y_{\gamma_C}(\mathcal{A}) = \Big\{(I,\lambda,p):\ I \subseteq \{1,...,m\}, \lambda = (\lambda_1,...,\lambda_m)^T, p = (p_1,...,p_m),\\ \gamma_{C^0}(\lambda) \leq 1,\ \lambda_i > 0\,(i\in I),\ \lambda_i = 0\,(i \notin I),\ \sum_{i\in I}\lambda_i p_i = 0\Big\}.$$

As the dual space X^* is also $\mathbb{R}^n$, the dual variable p belongs to $\underbrace{\mathbb{R}^n \times ... \times \mathbb{R}^n}_{m}$. This is also valid for the duals in the rest of the paper.

In our case $g_i(x) = w_{a_i}\varphi_{a_i}(x-a_i),\ i = 1, ..., m$, hence (cf. [3])

$$g^*_i(p_i) = (w_{a_i}\varphi_{a_i}(x-a_i))^*(p_i) = (w_{a_i}\varphi_{a_i})^*(p_i) + p_i^T a_i = w_{a_i}\varphi^*_{a_i}\left(\frac{p_i}{w_{a_i}}\right) + p_i^T a_i.$$

By Remark 2.9, $\varphi^*_{a_i}\left(\frac{p_i}{w_{a_i}}\right) = \begin{cases} 0, & \text{if } \varphi^0_{a_i}\left(\frac{p_i}{w_{a_i}}\right) \leq 1, \\ +\infty, & \text{otherwise,}\end{cases}$ and, denoting $p_i := \frac{p_i}{w_{a_i}},\ i \in I$, the dual problem to $(P_{\gamma_C}(\mathcal{A}))$ becomes

$$(D_{\gamma_C}(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_{\gamma_C}(\mathcal{A})} \left\{-\sum_{i\in I}\lambda_i w_{a_i} p_i^T a_i\right\},$$

with

$$Y_{\gamma_C}(\mathcal{A}) = \Big\{(I,\lambda,p) : I \subseteq \{1,...,m\}, \lambda = (\lambda_1,...,\lambda_m)^T, p = (p_1,...,p_m),\ \varphi^0_{a_i}(p_i) \leq 1,$$

$$i \in I,\ \gamma_{C^0}(\lambda) \leq 1,\ \lambda_i > 0\ (i \in I),\ \lambda_i = 0\ (i \notin I),\ \sum_{i \in I} \lambda_i w_{a_i} p_i = 0 \Big\}.$$

Using the Theorems 4.3 and 4.4 we can present for $(P_{\gamma_C}(\mathcal{A}))$ and $(D_{\gamma_C}(\mathcal{A}))$ the strong duality theorem and the optimality conditions.

Theorem 5.1. (strong duality for $(P_{\gamma_C}(\mathcal{A}))$) *The dual problem $(D_{\gamma_C}(\mathcal{A}))$ has an optimal solution and strong duality holds, i.e.*

$$\inf(P_{\gamma_C}(\mathcal{A})) = \max(D_{\gamma_C}(\mathcal{A})).$$

Theorem 5.2. (optimality conditions for $(P_{\gamma_C}(\mathcal{A}))$)
(1) Let $\bar{x}$ be an optimal solution to $(P_{\gamma_C}(\mathcal{A}))$. Then there exists an optimal solution $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_{\gamma_C}(\mathcal{A})$ to $(D_{\gamma_C}(\mathcal{A}))$, such that the following optimality conditions are satisfied

$$\begin{aligned}
&(i) && \bar{I} \subseteq \{1, ..., m\},\ \bar{\lambda}_i > 0\ (i \in \bar{I}),\ \bar{\lambda}_i = 0\ (i \notin \bar{I}),\\
&(ii) && \gamma_{C^0}(\bar{\lambda}) \leq 1,\ \varphi^0_{a_i}(\bar{p}_i) \leq 1,\ i \in \bar{I},\ \textstyle\sum_{i \in \bar{I}} \bar{\lambda}_i w_{a_i} \bar{p}_i = 0,\\
&(iii) && \gamma_C(w_{a_1}\varphi_{a_1}(\bar{x} - a_1), ..., w_{a_m}\varphi_{a_m}(\bar{x} - a_m)) = \textstyle\sum_{i \in \bar{I}} \bar{\lambda}_i w_{a_i} \varphi_{a_i}(\bar{x} - a_i),\\
&(iv) && \varphi_{a_i}(\bar{x} - a_i) = \bar{p}_i^{\,T}(\bar{x} - a_i),\ i \in \bar{I}.
\end{aligned}$$

(2) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_{\gamma_C}$ and $(i) - (iv)$ are fulfilled, then $\bar{x}$ is an optimal solution to $(P_{\gamma_C}(\mathcal{A}))$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_{\gamma_C}(\mathcal{A})$ is an optimal solution to $(D_{\gamma_C}(\mathcal{A}))$ and strong duality holds

$$\gamma_C(w_{a_1}\varphi_{a_1}(\bar{x} - a_1), ..., w_{a_m}\varphi_{a_m}(\bar{x} - a_m)) = -\sum_{i \in \bar{I}} \bar{\lambda}_i w_{a_i} \bar{p}_i^T a_i.$$

Proof. Theorem 5.2 is a direct consequence of Theorem 4.4 and the fact that $\gamma_C^+(g(x)) = \gamma_C(g(x))$. □

Remark 5.3. The optimality conditions obtained for the optimization problem $(P_{\gamma_C}(\mathcal{A}))$ are the same as the conditions obtained by Y. Hinojosa and J. Puerto in Lemma 7 in [6]. In the paper cited above the authors gave an geometrical description of the set of optimal solutions, but, as one can see, by means of duality one obtains the same characterization of this set.

In the next two sections of this paper we present some particular cases of the problem $(P_{\gamma_C}(\mathcal{A}))$, namely, the Weber problem and the minmax problem with gauges of closed convex sets.

6 The Weber problem with gauges of closed convex sets

The Weber problem with gauges of closed convex sets is

$$(P_w(\mathcal{A})) \quad \inf_{x\in\mathbb{R}^n} \sum_{i=1}^{m} w_{a_i}\varphi_{a_i}(x-a_i),$$

where $\varphi_{a_i},\ i=1,...,m$, are gauges whose unit balls are the closed convex sets $C_{a_i},\ i=1,...,m$, which contain the origin, and $w=\{w_{a_1},...,w_{a_m}\}$ is a set of the positive weights. As one can see, the problem above is equivalent to the following one

$$(P_w(\mathcal{A})) \quad \inf_{x\in\mathbb{R}^n} l_1(g(x)),$$

where $l_1:\mathbb{R}^m\to\mathbb{R},\ l_1(\lambda)=\sum\limits_{i=1}^{m}|\lambda_i|$ and $g:\mathbb{R}^n\to\mathbb{R}^m$ is the vector function defined by $g(x):=(g_1(x),...,g_m(x))^T$, with $g_i(x)=w_{a_i}\varphi_{a_i}(x-a_i)$ for all $i=1,...,m$. One may observe that the function l_1 is a monotonic gauge, actually, a monotonic norm.

Taking $\gamma_C(\lambda):=l_1(\lambda)$ for all $\lambda\in\mathbb{R}^m$, by the results obtained in the previous section, the dual problem to $(P_w(\mathcal{A}))$ becomes

$$(D_w(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_w(\mathcal{A})} \left\{-\sum_{i\in I}\lambda_i w_{a_i} p_i^T a_i\right\},$$

with

$$Y_w(\mathcal{A})=\Big\{(I,\lambda,p): I\subseteq\{1,...,m\}, \lambda=(\lambda_1,...,\lambda_m)^T, p=(p_1,...,p_m),\ \varphi^0_{a_i}(p_i)\le 1,$$

$$i\in I,\ l_1^0(\lambda)\le 1,\ \lambda_i>0\ (i\in I),\ \lambda_i=0\ (i\notin I),\ \sum_{i\in I}\lambda_i w_{a_i}p_i=0\Big\}.$$

Remark 6.1. In case the gauge γ_C of a convex set C is a norm, the gauge of the polar set C^0 actually becomes the dual norm. Because the dual norm of the l_1-norm is $l_1^0(\lambda)=\max\limits_{i=1,...,m}|\lambda_i|$, we obtain the following formulation for the dual problem

$$(D_w(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_w(\mathcal{A})} \left\{-\sum_{i\in I}\lambda_i w_{a_i} p_i^T a_i\right\},$$

with

$$Y_w(\mathcal{A})=\Big\{(I,\lambda,p): I\subseteq\{1,...,m\}, \lambda=(\lambda_1,...,\lambda_m)^T, p=(p_1,...,p_m),\ \varphi^0_{a_i}(p_i)\le 1,$$

$$i \in I, \max_{i\in I} \lambda_i \leq 1, \ \lambda_i > 0 \ (i \in I), \ \lambda_i = 0 \ (i \notin I), \sum_{i\in I} \lambda_i w_{a_i} p_i = 0 \Big\}.$$

Let us give now the strong duality theorem and the optimality conditions for $(P_w(\mathcal{A}))$ and its dual $(D_w(\mathcal{A}))$.

Theorem 6.2. (strong duality for $(P_w(\mathcal{A}))$**)** *The dual problem* $(D_w(\mathcal{A}))$ *has an optimal solution and strong duality holds, i.e.*

$$\inf(P_w(\mathcal{A})) = \max(D_w(\mathcal{A})).$$

Theorem 6.3. (optimality conditions for $(P_w(\mathcal{A}))$**)**
(1) Let $\bar{x}$ *be an optimal solution to* $(P_w(\mathcal{A}))$*. Then there exists an optimal solution* $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_w(\mathcal{A})$ *to* $(D_w(\mathcal{A}))$*, such that the following optimality conditions are satisfied*

$$\begin{array}{rl}
(i) & \bar{I} \subseteq \{1, ..., m\}, \ \bar{\lambda}_i > 0 \ (i \in \bar{I}), \ \bar{\lambda}_i = 0 \ (i \notin \bar{I}), \\
(ii) & \max\limits_{i \in \bar{I}} \bar{\lambda}_i \leq 1, \ \varphi^0_{a_i}(\bar{p}_i) \leq 1, \ i \in \bar{I}, \ \sum\limits_{i \in \bar{I}} \bar{\lambda}_i w_{a_i} \bar{p}_i = 0, \\
(iii) & \sum\limits_{i=1}^{m} w_{a_i} \varphi_{a_i}(\bar{x} - a_i) = \sum\limits_{i \in \bar{I}} \bar{\lambda}_i w_{a_i} \varphi_{a_i}(\bar{x} - a_i), \\
(iv) & \varphi_{a_i}(\bar{x} - a_i) = \bar{p}_i^T(\bar{x} - a_i), \ i \in \bar{I}.
\end{array}$$

(2) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_w(\mathcal{A})$ *and* $(i) - (iv)$ *are fulfilled, then* $\bar{x}$ *is an optimal solution to* $(P_w(\mathcal{A}))$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_w(\mathcal{A})$ *is an optimal solution to* $(D_w(\mathcal{A}))$ *and strong duality holds*

$$\sum_{i=1}^{m} w_{a_i} \varphi_{a_i}(\bar{x} - a_i) = - \sum_{i \in \bar{I}} \bar{\lambda}_i w_{a_i} \bar{p}_i^T a_i.$$

Proof. Theorem 6.3 is a direct consequence of Theorem 5.2. □

7 The minmax problem with gauges of closed convex sets

The optimization problem studied in this last section is the minmax problem with gauges of closed convex sets

$$(P_m(\mathcal{A})) \quad \inf_{x \in \mathbb{R}^n} \max_{i=1,...,m} w_{a_i} \varphi_{a_i}(x - a_i),$$

where $\varphi_{a_i}, \ i = 1, ..., m$, and $w = \{w_{a_1}, ..., w_{a_m}\}$ are considered like in the previous section. One can see that this problem is equivalent to the following one

$$(P_m(\mathcal{A})) \quad \inf_{x \in \mathbb{R}^n} l_\infty(g(x)),$$

where $l_\infty : \mathbb{R}^m \to \mathbb{R}$, $l_\infty(\lambda) = \max\limits_{i=1,...,m} |\lambda_i|$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ is the vector function defined by $g(x) := (g_1(x), ..., g_m(x))^T$, with $g_i(x) = w_{a_i}\varphi_{a_i}(x - a_i)$ for all $i = 1, ..., m$. One may observe that the function l_∞ is also a monotonic norm.

Taking $\gamma_C(\lambda) := l_\infty(\lambda)$ for all $\lambda \in \mathbb{R}^m$, the dual problem to $(P_m(\mathcal{A}))$ becomes

$$(D_m(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_m(\mathcal{A})} \left\{ -\sum_{i\in I} \lambda_i w_{a_i} p_i^T a_i \right\},$$

with

$$Y_m(\mathcal{A}) = \Big\{ (I,\lambda,p) : I \subseteq \{1,...,m\}, \lambda = (\lambda_1,...,\lambda_m)^T, p = (p_1,...,p_m),\ \varphi^0_{a_i}(p_i) \leq 1,$$

$$i \in I,\ l^0_\infty(\lambda) \leq 1,\ \lambda_i > 0\ (i \in I),\ \lambda_i = 0\ (i \notin I), \sum_{i\in I} \lambda_i w_{a_i} p_i = 0 \Big\}.$$

Remark 7.1. Because the dual norm of the l_∞−norm is $l^0_\infty(\lambda) = \sum\limits_{i=1}^{m} |\lambda_i|$ we obtain the following formulation for the dual problem

$$(D_m(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_m(\mathcal{A})} \left\{ -\sum_{i\in I} \lambda_i w_{a_i} p_i^T a_i \right\},$$

with

$$Y_m(\mathcal{A}) = \Big\{ (I,\lambda,p) : I \subseteq \{1,...,m\}, \lambda = (\lambda_1,...,\lambda_m)^T, p = (p_1,...,p_m),\ \varphi^0_{a_i}(p_i) \leq 1,$$

$$i \in I, \sum_{i=1}^{m} \lambda_i \leq 1,\ \lambda_i > 0\ (i \in I),\ \lambda_i = 0\ (i \notin I), \sum_{i\in I} \lambda_i w_{a_i} p_i = 0 \Big\}.$$

Like in the previous section we give now the strong duality theorem and the optimality conditions for $(P_m(\mathcal{A}))$ and its dual $(D_m(\mathcal{A}))$.

Theorem 7.2. (strong duality for $(P_m(\mathcal{A}))$) *The dual problem $(D_m(\mathcal{A}))$ has an optimal solution and strong duality holds, i.e.*

$$\inf(P_m(\mathcal{A})) = \max(D_m(\mathcal{A})).$$

Theorem 7.3. (optimality conditions for $(P_m(\mathcal{A}))$)
(1) Let $\bar{x}$ be an optimal solution to $(P_m(\mathcal{A}))$. Then there exists an optimal solution $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_m(\mathcal{A})$ to $(D_m(\mathcal{A}))$, such that the following optimality conditions are

satisfied

$$\begin{array}{rl} (i) & \bar{I} \subseteq \{1,...,m\},\ \bar{\lambda}_i > 0\ (i \in \bar{I}),\ \bar{\lambda}_i = 0\ (i \notin \bar{I}), \\ (ii) & \sum\limits_{i\in\bar{I}} \bar{\lambda}_i \leq 1,\ \varphi^0_{a_i}(\bar{p}_i) \leq 1,\ i \in \bar{I},\ \sum\limits_{i\in\bar{I}} \bar{\lambda}_i w_{a_i} \bar{p}_i = 0, \\ (iii) & \max\limits_{i=1,...,m} w_{a_i}\varphi_{a_i}(\bar{x}-a_i) = \sum\limits_{i\in\bar{I}} \bar{\lambda}_i w_{a_i}\varphi_{a_i}(\bar{x}-a_i), \\ (iv) & \varphi_{a_i}(\bar{x}-a_i) = \bar{p}_i{}^T(\bar{x}-a_i),\ i \in \bar{I}. \end{array}$$

(2) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_m(\mathcal{A})$ and $(i) - (iv)$ are fulfilled, then $\bar{x}$ is an optimal solution to $(P_m(\mathcal{A}))$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_m(\mathcal{A})$ is an optimal solution to $(D_m(\mathcal{A}))$ and strong duality holds

$$\max_{i=1,...,m} w_{a_i}\varphi_{a_i}(\bar{x}-a_i) = -\sum_{i\in\bar{I}} \bar{\lambda}_i w_{a_i} \bar{p}_i^T a_i.$$

Proof. Theorem 7.3 is a direct consequence of Theorem 5.2. □

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