A new condition for maximal monotonicity via representative functions

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Abstract. In this paper we give a weaker sufficient condition for the maximal monotonicity of the operator $S + A^*TA$, where $S : X \rightrightarrows X^*$, $T : Y \rightrightarrows Y^*$ are two maximal monotone operators, $A : X \to Y$ is a linear continuous mapping and X, Y are reflexive Banach spaces. We prove that our condition is weaker than the generalized interior-point conditions given so far in the literature. This condition is formulated using the representative functions of the operators involved. In particular, we rediscover some sufficient conditions given in the past using the so-called Fitzpatrick function for the maximal monotonicity of the sum of two maximal monotone operators and for the precomposition of a maximal monotone operator with a linear operator, respectively.

Key Words. maximal monotone operator, representative function, Fitzpatrick function, subdifferential

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1 Introduction

The maximal monotonicity of operators defined on a Banach space has been intensively studied since the beginning of this theory in the 60's. We mention here the papers of Browder ([8]) and Rockafellar ([21], [22]) who made the first important steps in this field. Obviously, the sum of two maximal monotone operators is monotone, but not always maximal. The challenge is to give as weak as possible sufficient conditions in order to assure the maximality of the sum.

The literature is quite rich in such conditions, given in the framework of reflexive Banach spaces (see [1], [4], [6], [7], [11], [15], [17], [22], [24], [26], [28]). A

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comprehensive study on this problem may be found in [23], where many sufficient conditions are compared and classified. This book and the lecture notes ([20]) due to Phelps are important references for the theory of maximal monotone operators.

A generalization of this problem is to find conditions to ensure the maximality of the operator A^*TA, where T is a maximal monotone operator and A is a linear and continuous mapping. In this case the number of conditions given in the literature is relatively small ([4], [6], [11], [15], [17], [26]). Excepting the condition given in [6], which is formulated using the conjugate of the *Fitzpatrick function* of T (notion introduced in [12]), the other ones are generalized interior-point conditions.

The most general case treated in the literature so far is referring to the maximality of the operator $S + A^*TA$, where $S : X \rightrightarrows X^*$, $T : Y \rightrightarrows Y^*$ are two maximal monotone operators, $A : X \to Y$ is a linear continuous mapping and X, Y are reflexive Banach spaces. In this paper we give a weak sufficient condition that guarantees the maximality of this operator. It is stated in terms of the representatives of the operators involved. The notion of a *representative function* of a monotone operator has been introduced in [12] and it is intensively studied in [4], [10], [14], [17], [19]. The most prominent example of representative functions of a monotone operator is the *Fitzpatrick function* ([12]; for other examples see [4], [10], [16], [17]). The sufficient condition we give in this paper turns out to be weaker than the generalized interior-point conditions given by Pennanen in [15] and by Penot and Zălinescu in [19]. The proof uses an idea due to Borwein ([4]) and shows once more the usefulness of convex analysis in treating the problem of maximality of monotone operators. Other links between monotone operators and convex analysis may be found in [2], [6], [7], [10], [13], [14], [16], [17], [23], [24], [25]. As particular cases we rediscover the sufficient conditions given in [6] and [7] using Fitzpatrick functions for the maximal monotonicity of the precomposition of a maximal monotone operator with a linear operator, respectively, for the sum of two maximal monotone operators. The conditions in [6] and [7] are the weakest known so far in the literature for the maximal monotonicity in reflexive Banach spaces. We close the paper by particularizing our condition for the case of the subdifferential mapping of a proper, convex and lower semicontinuous function.

The paper is organized as follows. In the next section we present some definitions and preliminary results from the theory of convex analysis and monotone operators that will be used later. In section 3 we give the announced constraint qualification which ensures the maximality of the operator $S + A^*TA$, proving that it is weaker than the ones given in [15] and [19]. In section 4 we treat some particular cases of the main result. A short concluding section and a list of references close the paper.

2 Preliminaries

In this section we give the necessary notions and results in order to make the paper as self-contained as possible.

2.1 Elements of convex analysis

The main result of the paper (Theorem 2) is given in a reflexive Banach space. However, as some preliminary results are valid in a more general setting, we start by considering a non-trivial locally convex space X and its continuous dual space X^*, endowed with an arbitrary locally convex topology τ giving X as dual. Significant choices for τ are the weak* topology $\omega(X^*, X)$ on X^* or the norm topology of X^* when X is a reflexive Banach space. We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. For a subset C of X we denote by $\mathrm{cl}(C)$, $\mathrm{aff}(C)$, $\mathrm{ri}(C)$ and ${}^{ic}C$ its *closure*, *affine hull*, *relative interior* and *intrinsic relative algebraic interior*, respectively. In fact, $\mathrm{ri}(C)$ is the topological interior of C with respect to $\mathrm{cl}(\mathrm{aff}(C))$. Note that if C is a convex set, then an element $x \in X$ belongs to ${}^{ic}C$ if and only if $\bigcup_{\lambda>0} \lambda(C - x)$ is a closed linear subspace of X (see also [28]). The *indicator function* of C, denoted by δ_C, is defined as $\delta_C : X \to \overline{\mathbb{R}}$,

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

We consider also the *first projection*, the function $\mathrm{pr}_1 : X \times Y \to X$, for Y some non-trivial locally convex space, $\mathrm{pr}_1(x, y) = x, \forall (x, y) \in X \times Y$.

For a function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we denote by $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty, \forall x \in X$. For $x \in X$ such that $f(x) \in \mathbb{R}$ we define the *subdifferential* of f at x by $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle, \forall y \in X\}$. The *Fenchel-Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}, \forall x^* \in X^*.$$

We have the so called Young-Fenchel inequality

$$f^*(x^*) + f(x) \geq \langle x^*, x\rangle, \forall x \in X, \forall x^* \in X^*.$$

Consider also the *identity function* on X, $\mathrm{id}_X : X \to X, \mathrm{id}_X(x) = x, \forall x \in X$. Given two functions, $f : M_1 \to M_2$ and $g : N_1 \to N_2$, where M_1, M_2, N_1, N_2 are nonempty sets, we define the function $f \times g : M_1 \times N_1 \to M_2 \times N_2$ by $f \times g(m, n) = (f(m), g(n)), \forall (m, n) \in M_1 \times N_1$. Given a linear continuous mapping $A : X \to Y$, we denote by $\mathrm{Im}(A)$ its *image-set*, $\mathrm{Im}(A) = \{Ax : x \in X\}$ and by A^* its *adjoint* *operator*, $A^* : Y^* \to X^*$ given by $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle, \forall y^* \in Y^*, \forall x \in X$. For the proper function $f : X \to \overline{\mathbb{R}}$ we also define the *infimal function of f through A* as $Af : Y \to \overline{\mathbb{R}}, Af(y) = \inf\{f(x) : x \in X, Ax = y\}, \forall y \in Y$. When an infimum or a supremum is attained we write min, respectively max instead of inf, respectively sup.

Definition 1. A set $M \subseteq X$ is said to be *closed regarding the subspace $Z \subseteq X$* if $M \cap Z = \mathrm{cl}(M) \cap Z$.

It is worth noting that a closed set is closed regarding any subspace.

The following result has been proved in [6] and will play later an important role.

Proposition 1. ([6]) *Let X, Y and U be non-trivial locally convex spaces, $A : X \to Y$ a linear continuous mapping and $f : Y \to \overline{\mathbb{R}}$ a proper, convex and lower semi-continuous function such that $f \circ A$ is proper on X. Consider moreover the linear mapping $M : U \to X^*$. The following statements are equivalent:*

(a) $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*))$ *is closed regarding the subspace* $\mathrm{Im}(M) \times \mathbb{R}$ *in the product topology of* $(X^*, \tau) \times \mathbb{R}$,

(b) $(f \circ A)^*(Mu) = \min\{f^*(y^*) : A^*y^* = Mu\}, \forall u \in U$.

Theorem 1. ([5]) *Let X and Y be non-trivial locally convex spaces, $A : X \to Y$ a linear continuous mapping, $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ proper, convex and lower semi-continuous functions such that $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$. Then*

(i) $\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))$ *is closed in the product topology* $(X^*, \tau) \times \mathbb{R}$ *if and only if*

$$(f + g \circ A)^*(x^*) = \min\{f^*(x^* - A^*y^*) + g^*(y^*) : y^* \in Y^*\}, \forall x^* \in X^*.$$

(ii) If $\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))$ *is closed in the product topology* $(X^*, \tau) \times \mathbb{R}$, *then* $\forall x \in \mathrm{dom}(f) \cap A^{-1}(\mathrm{dom}(g))$ *one has*

$$\partial(f + g \circ A)(x) = \partial f(x) + A^*\partial g(Ax).$$

Remark 1. In [5] the authors proved that the condition

$\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*))$ is closed in the product topology $(X^*, \tau) \times \mathbb{R}$

is equivalent to

$$\mathrm{epi}(f + g \circ A)^* = \mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*)).$$

The following corollary can be obtained by taking in Theorem 1 $X = Y$ and $A = \mathrm{id}_X$.

Corollary 1. ([5]) *Let X be a non-trivial locally convex space and $f, g : X \to \overline{\mathbb{R}}$ proper, convex and lower semi-continuous functions such that* $\mathrm{dom}(f) \bigcap \mathrm{dom}(g) \neq \emptyset$. *Then*

(i) $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ *is closed in the product topology* $(X^*, \tau) \times \mathbb{R}$ *if and only if*

$$(f+g)^*(x^*) = \min\{f^*(x^* - y^*) + g^*(y^*) : y^* \in X^*\}, \forall x^* \in X^*.$$

(ii) If $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ *is closed in the product topology* $(X^*, \tau) \times \mathbb{R}$, *then* $\forall x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ *one has*

$$\partial(f+g)(x) = \partial f(x) + \partial g(x).$$

Remark 2. The condition

$$\mathrm{epi}(f^*) + \mathrm{epi}(g^*) \text{ is closed in the product topology } (X^*, \tau) \times \mathbb{R}$$

is equivalent to

$$\mathrm{epi}(f+g)^* = \mathrm{epi}(f^*) + \mathrm{epi}(g^*).$$

2.2 Monotone operators

In the following we recall some notations and results concerning monotone operators. Consider further X a Banach space equipped with the norm $\|\cdot\|$, while the norm on X^* is $\|\cdot\|_*$.

Definition 2. The multifunction $S : X \rightrightarrows X^*$ is called a *monotone operator* provided that for every $x, y \in X$ one has

$$\langle y^* - x^*, y - x\rangle \geq 0, \forall x^* \in S(x), \forall y^* \in S(y).$$

For the multifunction $S : X \rightrightarrows X^*$ we have

- its *effective domain* $D(S) = \{x \in X : S(x) \neq \emptyset\}$,
- its *range* $R(S) = \cup\{S(x) : x \in X\}$,
- its *graph* $G(S) = \{(x, x^*) : x \in X, x^* \in S(x)\}$.

Definition 3. A monotone operator $S : X \rightrightarrows X^*$ is said to be *maximal* when its graph is not properly included in the graph of any other monotone operator on the same space.

A classical example of a maximal monotone operator is the subdifferential of a proper, convex and lower semicontinuous function on X ([21]). We introduce also the *duality map* $J : X \rightrightarrows X^*$ defined as follows

$$J(x) = \partial\Big(\frac{1}{2}\|x\|^2\Big) = \Big\{x^* \in X^* : \|x\|^2 = \|x^*\|_*^2 = \langle x^*, x\rangle\Big\}, \forall x \in X.$$

Using the duality map one can give a necessary and sufficient condition for the maximality of a monotone operator, as follows.

Proposition 2. ([23]) *A monotone operator S on a reflexive Banach space X is maximal if and only if the mapping $S(x+\cdot)+J(\cdot)$ is surjective, for all $x \in X$.*

Consider that X is a reflexive Banach space. For a function $f : X \times X^* \to \overline{\mathbb{R}}$, we denote by $f^\top$ the *transpose* of f, namely the function $f^\top : X^* \times X, f^\top(x^*, x) = f(x, x^*)$. In a similar way one can define the transpose of a function $f : X^* \times X \to \overline{\mathbb{R}}$. The *pairing* function on $X \times X^*$ is denoted by c, $c(x, x^*) = \langle x^*, x\rangle$. Also we identify the dual of $X \times X^*$ with $X^* \times X$ by the pairing

$$\langle (y^*, y), (x, x^*)\rangle = \langle y^*, x\rangle + \langle x^*, y\rangle.$$

Having a maximal monotone operator $S : X \rightrightarrows X^*$ we associate to it the functions $c_S := c + \delta_{G(S)}$ and $\psi_S := \overline{\text{co}} c_S$, that is the *closed convex hull* of c_S. They have been intensively studied by Burachik and Svaiter in [10] and by Penot in [16], [17]. We consider also the *Fitzpatrick function* of S, defined by

$$\varphi_S : X \times X^* \to \overline{\mathbb{R}}, \varphi_S(x, x^*) = \sup\{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in S(y)\}.$$

Introduced by Fitzpatrick ([12]), it proved to be very important in treating the problem of maximality of the sum of maximal monotone operators, via convex analysis. The most important results concerning these functions are given in the following proposition.

Proposition 3. ([17], [19], [24]) *Let S be a maximal monotone operator on a reflexive Banach space X. Then*

(a) φ_S is convex and lower semicontinuous;

(b) for each pair $(x, x^*) \in X \times X^*$ we have

$$\varphi_S^*(x^*, x) \geq \varphi_S(x, x^*) \geq \langle x^*, x\rangle.$$

Moreover, $\varphi_S^*(x^*, x) = \varphi_S(x, x^*) = \langle x^*, x\rangle$ if and only if $(x, x^*) \in G(S)$;

(c) $\varphi_S = c_S^{*\top}$, $\psi_S = \varphi_S^{*\top}$ and $\psi_S \geq \varphi_S \geq c$.

Definition 4. Let $S : X \rightrightarrows X^*$ be a maximal monotone operator. We call a *representative function* of S a lower semicontinuous convex function $f_S : X \times X^* \to \overline{\mathbb{R}}$ fulfilling

$$f_S \geq c \text{ and } G(S) = \{(x, x^*) \in X \times X^* : f_S(x, x^*) = \langle x^*, x \rangle\}.$$

We observe that a representative function of a maximal monotone operator is proper. The next result is a direct consequence of Proposition 1 and Proposition 4 in [19].

Lemma 1. *Let $f_S : X \times X^* \to \overline{\mathbb{R}}$ be a representative function of the maximal monotone operator S. Then $f_S^* \geq c^\top$ and*

$$G(S) = \{(x, x^*) \in X \times X^* : f_S^*(x^*, x) = \langle x^*, x \rangle\}.$$

By Proposition 6 in [19], the lower semicontinuous convex function $f : X \times X^* \to \overline{\mathbb{R}}$ is a representative function of the maximal monotone operator S if and only if $\varphi_S \leq f \leq \psi_S$. In particular, φ_S and ψ_S are representative functions of S. It follows that $f^{*\top}$ is a representative function of S when f is also a representative one (see also [27]).

One can see that if $f : X \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, then a representative function of the maximal monotone operator $\partial f : X \rightrightarrows X^*$ is the function $(x, x^*) \mapsto f(x) + f^*(x^*)$. This follows by the Young-Fenchel inequality and from the definition of the subdifferential of f.

According to Example 3 in [18], if f is a sublinear lower semicontinuous function, then the operator $\partial f : X \rightrightarrows X^*$ has a unique representative function, namely the function $(x, x^*) \mapsto f(x) + f^*(x^*)$. This result is true even if X is non-reflexive (see Theorem 3.1 in [9]).

If X is a Hilbert space, then there exists a unique representative function of the maximal monotone operator $\partial(\delta_C) : X \rightrightarrows X$, where C is a nonempty closed convex set in X. Indeed, by Example 3.1 in [3], the Fitzpatrick function of $\partial(\delta_C)$ is $\varphi_{\partial(\delta_C)}(x, x^*) = \delta_C(x) + \delta_C^*(x^*)$. This implies by Proposition 3(c) that $\psi_{\partial(\delta_C)} = \varphi_{\partial(\delta_C)}^{*\top} = \varphi_{\partial(\delta_C)}$. As $f_{\partial(\delta_C)}$ is a representative function of $\partial(\delta_C)$ if and only if $\varphi_{\partial(\delta_C)} \leq f_{\partial(\delta_C)} \leq \psi_{\partial(\delta_C)}$, we get that the unique representative function is $(x, x^*) \mapsto \delta_C(x) + \delta_C^*(x^*)$.

3 Maximal monotonicity of the operator $S + A^*TA$

Throughout this section X and Y are reflexive Banach spaces. Consider $A : X \to Y$ a linear continuous mapping, $S : X \rightrightarrows X^*$ and $T : Y \rightrightarrows Y^*$ two maximal monotone operators with representative functions f_S, respectively f_T, such that

$A(\mathrm{pr}_1(\mathrm{dom}(f_S)))\cap\mathrm{pr}_1(\mathrm{dom}(f_T)) \neq \emptyset$. The operator $S+A^*TA : X \rightrightarrows X^*$, defined by $(S+A^*TA)(x) = S(x)+(A^*\circ T\circ A)(x), \forall x \in X$, is monotone, but not always maximal. Next we prove, using an idea due to Borwein ([4]), that this operator is maximal, provided that the following constraint qualification is fulfilled,

(CQ) $\{(x^* + A^*y^*, x, y, r) : f_S^*(x^*, x) + f_T^*(y^*, y) \leq r\}$ is closed regarding the subspace $X^* \times \Delta_X^A \times \mathbb{R}$,

where $\Delta_X^A = \{(x, Ax) : x \in X\}$.

Theorem 2. *If (CQ) is fulfilled then $S + A^*TA$ is a maximal monotone operator.*

Proof. Let us consider $z \in X$ and $z^* \in X^*$ some fixed elements. According to Proposition 2, the main idea is to prove that there exists $\overline{x} \in X$ such that $z^* \in (S + A^*TA)(\overline{x} + z) + J(\overline{x})$. Consider the functions $F, G : X \times X^* \to \overline{\mathbb{R}}$ defined by

$$F(x, x^*) = \inf_{\substack{(u^*,y^*)\in X^*\times Y^*\\ u^*+A^*y^*=x^*+z^*}} \{f_S(x + z, u^*) + f_T\big(A(x + z), y^*\big) - \langle u^*, z\rangle - \langle y^*, Az\rangle\}$$

and

$$G(x, x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 - \langle z^*, x\rangle, \forall (x, x^*) \in X \times X^*.$$

In the following we compute the conjugates of F and G. For $(\omega^*, \omega) \in X^* \times X$ we have

$$F^*(\omega^*, \omega) = \sup_{(x,x^*)\in X\times X^*} \Big\{\langle\omega^*, x\rangle + \langle x^*, \omega\rangle - \inf_{\substack{(u^*,y^*)\in X^*\times Y^*\\ u^*+A^*y^*=x^*+z^*}} \{f_S(x + z, u^*) + f_T\big(A(x+z), y^*\big) - \langle u^*, z\rangle - \langle y^*, Az\rangle\}\Big\} = \sup_{x\in X,(u^*,y^*)\in X^*\times Y^*} \big\{\langle\omega^*, x\rangle + \langle u^* + A^*y^* - z^*, \omega\rangle - f_S(x + z, u^*) - f_T\big(A(x + z), y^*\big) + \langle u^*, z\rangle + \langle y^*, Az\rangle\big\}$$
$$= \sup_{\substack{x\in X,(u^*,y^*)\in X^*\times Y^*\\ u=x+z\in X}} \big\{\langle\omega^*, u - z\rangle + \langle u^* + A^*y^* - z^*, \omega\rangle - f_S(u, u^*) - f_T\big(Au, y^*\big) + \langle u^*, z\rangle + \langle y^*, Az\rangle\big\} = \sup_{\substack{u\in X\\ (u^*,y^*)\in X^*\times Y^*}} \big\{\langle\omega^*, u\rangle + \langle u^*, \omega+z\rangle + \langle y^*, A(\omega+z)\rangle - f_S(u, u^*) - f_T\big(Au, y^*\big)\big\} - \langle\omega^*, z\rangle - \langle z^*, \omega\rangle.$$

Considering the functions $h : X \times Y \times X^* \times Y^* \to \overline{\mathbb{R}}, h(x, y, x^*, y^*) = f_S(x, x^*) + f_T(y, y^*), B : X \times X^* \times Y^* \to X \times Y \times X^* \times Y^*, B(x, x^*, y^*) = (x, Ax, x^*, y^*)$ and $M : X \times X^* \to X^* \times X \times Y, M(\omega, \omega^*) = (\omega^*, \omega, A\omega)$, we have that

$$F^*(\omega^*, \omega) = (h \circ B)^*\big(M(\omega + z, \omega^*)\big) - \langle\omega^*, z\rangle - \langle z^*, \omega\rangle.$$

One can see by direct computation that the Fenchel conjugate of h is $h^* : X^* \times Y^* \times X \times Y \to \overline{\mathbb{R}}$, $h^*(x^*, y^*, x, y) = f_S^*(x^*, x) + f_T^*(y^*, y)$ and the adjoint operator of the linear continuous mapping B is $B^* : X^* \times Y^* \times X \times Y \to X^* \times X \times Y, B^*(p^*, q^*, a, b) = (p^* + A^*q^*, a, b)$. Hence $B^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(h^*)) = \big\{(B^*(x^*, y^*, x, y), r) : h^*(x^*, y^*, x, y) \le r\big\} = \big\{(x^* + A^*y^*, x, y, r) : f_S^*(x^*, x) + f_T^*(y^*, y) \le r\big\}$ and because $\mathrm{Im}(M) \times \mathbb{R} = X^* \times \Delta_X^A \times \mathbb{R}$, the constraint qualification (CQ) is nothing else than $B^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(h^*))$ is closed regarding the subspace $\mathrm{Im}(M) \times \mathbb{R}$. By Proposition 1 we have that $(h \circ B)^*\big(M(\omega + z, \omega^*)\big) = \min\{h^*(a^*, b^*, a, b) : B^*(a^*, b^*, a, b) = M(\omega + z, \omega^*)\}$ (the properness of the function $h \circ B$ is assured by the assumption $A(\mathrm{pr}_1(\mathrm{dom}(f_S))) \cap \mathrm{pr}_1(\mathrm{dom}(f_T)) \neq \emptyset$). This shows that

$$F^*(\omega^*, \omega) = \min_{(a^*+A^*b^*, a, b) = (\omega^*, \omega+z, A(\omega+z))} \big\{f_S^*(a^*, a) + f_T^*(b^*, b)\big\} - \langle \omega^*, z\rangle - \langle z^*, \omega\rangle.$$

For the conjugate of G, we have

$$\begin{aligned} G^*(\omega^*, \omega) &= \sup_{\substack{x \in X, \\ x^* \in X^*}} \Big\{\langle \omega^*, x\rangle + \langle x^*, \omega\rangle - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|x^*\|_*^2 + \langle z^*, x\rangle\Big\} \\ &= \frac{1}{2}\|\omega^* + z^*\|_*^2 + \frac{1}{2}\|\omega\|^2. \end{aligned}$$

For every $(x, x^*) \in X \times X^*$ and $(u^*, y^*) \in X^* \times Y^*$ such that $u^* + A^*y^* = x^* + z^*$, we obtain by the definition of the representative function that

$$f_S(x + z, u^*) + f_T\big(A(x + z), y^*\big) - \langle u^*, z\rangle - \langle y^*, Az\rangle + G(x, x^*) \ge$$

$$\langle u^*, x + z\rangle + \langle y^*, A(x + z)\rangle - \langle u^*, z\rangle - \langle y^*, Az\rangle + G(x, x^*) =$$

$$\langle u^*, x\rangle + \langle A^*y^*, x\rangle + G(x, x^*) = \langle u^*, x\rangle + \langle x^* + z^* - u^*, x\rangle + G(x, x^*) =$$

$$\langle x^* + z^*, x\rangle + G(x, x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 + \langle x^*, x\rangle \ge 0.$$

This implies $F(x, x^*) + G(x, x^*) \ge 0, \forall (x, x^*) \in X \times X^*$, that is

$$\inf_{(x,x^*) \in X \times X^*} \{F(x, x^*) + G(x, x^*)\} \ge 0.$$

One can see that the functions F and G are convex and the latter is continuous. Thus, by Fenchel's duality theorem ([28]) there exists a pair $(\overline{x}^*, \overline{x})$ such that

$$\begin{aligned} \inf_{(x,x^*) \in X \times X^*} \{F(x, x^*) + G(x, x^*)\} &= \max_{(x^*,x) \in X^* \times X} \{-F^*(x^*, x) - G^*(-x^*, -x)\} \\ &= -F^*(\overline{x}^*, \overline{x}) - G^*(-\overline{x}^*, -\overline{x}). \end{aligned}$$

Using the last two relations we get $F^*(\overline{x}^*,\overline{x})+G^*(-\overline{x}^*,-\overline{x})\leq 0$. So there exists $(\overline{a}^*,\overline{a},\overline{b}^*,\overline{b})\in X^*\times X\times Y^*\times Y$ such that $(\overline{a}^*+A^*\overline{b}^*,\overline{a},\overline{b})=\big(\overline{x}^*,\overline{x}+z,A(\overline{x}+z)\big)$ and

$$f_S^*(\overline{a}^*,\overline{a})+f_T^*(\overline{b}^*,\overline{b})-\langle\overline{x}^*,z\rangle-\langle z^*,\overline{x}\rangle+\frac{1}{2}\|-\overline{x}^*+z^*\|_*^2+\frac{1}{2}\|-\overline{x}\|^2\leq 0.$$

Taking into account that $\overline{b}=A(\overline{x}+z)=A\overline{a}$, $\overline{x}=\overline{a}-z$ and $\overline{x}^*=\overline{a}^*+A^*\overline{b}^*$ we obtain

$$\begin{aligned}
0 \ \geq\ & \Big(f_S^*(\overline{a}^*,\overline{a})-\langle\overline{a}^*,\overline{a}\rangle\Big)+\Big(f_T^*(\overline{b}^*,\overline{b})-\langle\overline{b}^*,\overline{b}\rangle\Big)+\langle\overline{a}^*,\overline{a}\rangle+\langle\overline{b}^*,\overline{b}\rangle\\
- \ & \langle\overline{x}^*,z\rangle-\langle z^*,\overline{x}\rangle+\frac{1}{2}\|-\overline{x}^*+z^*\|_*^2+\frac{1}{2}\|-\overline{x}\|^2=\Big(f_S^*(\overline{a}^*,\overline{a})-\langle\overline{a}^*,\overline{a}\rangle\Big)\\
+ \ & \Big(f_T^*(\overline{b}^*,\overline{b})-\langle\overline{b}^*,\overline{b}\rangle\Big)+\langle\overline{a}^*,\overline{a}\rangle+\langle\overline{b}^*,\overline{b}\rangle-\langle\overline{a}^*+A^*\overline{b}^*,z\rangle-\langle z^*,\overline{a}-z\rangle\\
+ \ & \frac{1}{2}\|-\overline{x}^*+z^*\|_*^2+\frac{1}{2}\|-\overline{x}\|^2=\Big(f_S^*(\overline{a}^*,\overline{a})-\langle\overline{a}^*,\overline{a}\rangle\Big)\\
+ \ & \Big(f_T^*(\overline{b}^*,\overline{b})-\langle\overline{b}^*,\overline{b}\rangle\Big)+\langle\overline{a}^*,\overline{a}\rangle+\langle\overline{b}^*,A\overline{a}\rangle-\langle\overline{a}^*,z\rangle-\langle\overline{b}^*,Az\rangle-\langle z^*,\overline{a}\rangle\\
+ \ & \langle z^*,z\rangle+\frac{1}{2}\|-\overline{x}^*+z^*\|_*^2+\frac{1}{2}\|-\overline{x}\|^2=\Big(f_S^*(\overline{a}^*,\overline{a})-\langle\overline{a}^*,\overline{a}\rangle\Big)\\
+ \ & \Big(f_T^*(\overline{b}^*,\overline{b})-\langle\overline{b}^*,\overline{b}\rangle\Big)+\frac{1}{2}\|-\overline{a}^*-A^*\overline{b}^*+z^*\|_*^2+\frac{1}{2}\|-\overline{a}+z\|^2\\
+ \ & \langle-\overline{a}^*-A^*\overline{b}^*+z^*,-\overline{a}+z\rangle\geq 0,
\end{aligned}$$

where the last inequality follows from Lemma 1. Hence the inequalities above must be fulfilled as equalities, that is

$$f_S^*(\overline{a}^*,\overline{a})=\langle\overline{a}^*,\overline{a}\rangle,\ \ f_T^*(\overline{b}^*,\overline{b})=\langle\overline{b}^*,\overline{b}\rangle$$

and

$$\frac{1}{2}\|-\overline{a}^*-A^*\overline{b}^*+z^*\|_*^2+\frac{1}{2}\|-\overline{a}+z\|^2+\langle-\overline{a}^*-A^*\overline{b}^*+z^*,-\overline{a}+z\rangle=0.$$

Using Lemma 1 and the definition of the duality map J, the last three equalities are equivalent to $\overline{a}^*\in S(\overline{a})$, $\overline{b}^*\in T(\overline{b})$ and, respectively, $z^*-\overline{a}^*-A^*\overline{b}^*\in J(\overline{a}-z)$. Employing once more the relation $(\overline{a}^*+A^*\overline{b}^*,\overline{a},\overline{b})=\big(\overline{x}^*,\overline{x}+z,A(\overline{x}+z)\big)$, we get $z^*-\overline{x}^*\in J(\overline{x})$ and $\overline{x}^*=\overline{a}^*+A^*\overline{b}^*\in S(\overline{a})+A^*T(A\overline{a})=(S+A^*TA)(\overline{a})=(S+A^*TA)(\overline{x}+z)$. Finally, we have

$$z^*=\overline{x}^*+(z^*-\overline{x}^*)\in(S+A^*TA)(\overline{x}+z)+J(\overline{x}).$$

As z and z^* have been arbitrary chosen, Proposition 2 yields the conclusion. $\square$

Remark 3. In the literature one cannot find many conditions for the maximal monotonicity of the operator $S+A^*TA$. We can mention here the one given by Pennanen ([15])

$$(CQ^P) \qquad\qquad 0 \in \operatorname{ri}\big(A(D(S)) - D(T)\big),$$

and, respectively, the one given by Penot and Zălinescu ([19])

(CQ^{PZ}) $\qquad 0 \in {}^{ic}\big(A(\mathrm{pr}_1(\mathrm{dom}(f_S))) - \mathrm{pr}_1(\mathrm{dom}(f_T))\big)$, where f_S and f_T are representative functions of S and T, respectively.

According to Corollary 14 in [19], the conditions (CQ^P) and (CQ^{PZ}) are equivalent. We show that our condition is implied by the aforementioned ones.

We assume that (CQ^{PZ}) is fulfilled. Consider the following functions: $s : X \times X^* \times Y^* \to \overline{\mathbb{R}}, s(x, x^* y^*) = f_S(x, x^*), t : Y \times X^* \times Y^* \to \overline{\mathbb{R}}, t(y, x^*, y^*) = f_T(y, y^*)$ and $C : X \times X^* \times Y^* \to Y \times X^* \times Y^*, C(x, x^*, y^*) = (Ax, x^*, y^*)$. One can see by direct computation that

$$s^* : X^* \times X \times Y \to \overline{\mathbb{R}}, s^*(x^*, x, y) = \begin{cases} f_S^*(x^*, x), & \text{if } y = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$t^* : Y^* \times X \times Y \to \overline{\mathbb{R}}, t^*(y^*, x, y) = \begin{cases} f_T^*(y^*, y), & \text{if } x = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and $C^* : Y^* \times X \times Y \to X^* \times X \times Y, C^*(y^*, x, y) = (A^* y^*, x, y)$. As

$$\bigcup_{\lambda>0} \lambda\big[A(\mathrm{pr}_1(\mathrm{dom}(f_S))) - \mathrm{pr}_1(\mathrm{dom}(f_T))\big] \times X^* \times Y^* = \bigcup_{\lambda>0} \lambda\big[C(\mathrm{dom}(s)) - \mathrm{dom}(t)\big],$$

it follows that $\bigcup\limits_{\lambda>0} \lambda\big[C(\mathrm{dom}(s)) - \mathrm{dom}(t)\big]$ is a closed linear subspace in $Y \times X^* \times Y^*$. By Theorem 2.8.3 in [28] it follows that for all $(x^*, x, y) \in X^* \times X \times Y, (s + t \circ C)^*(x^*, x, y) = \min\big\{s^*\big((x^*, x, y) - C^*(u^*, v, z)\big) + t^*(u^*, v, z) : (u^*, v, z) \in Y^* \times X \times Y\big\}$. On the other hand, by Theorem 1, the last relation holds if and only if $\mathrm{epi}(s^*) + C^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(t^*))$ is closed in $X^* \times X \times Y \times \mathbb{R}$. As $\mathrm{epi}(s^*) = \{(x^*, x, 0, r) : f_S^*(x^*, x) \leq r\}$ and $C^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(t^*)) = \{(A^* y^*, 0, y, r) : f_T^*(y^*, y) \leq r\}$, we have in conclusion that $\{(x^* + A^* y^*, x, y, r) : f_S^*(x^*, x) + f_T^*(y^*, y) \leq r\}$ is closed and hence closed regarding the subspace $X^* \times \Delta_X^A \times \mathbb{R}$, that is (CQ) is fulfilled.

The fact that (CQ) is indeed weaker than (CQ^P) and (CQ^{PZ}) is showed by an example in the next section.

4 Particular cases

In this section we treat some particular cases of the main result given in the previous section. We obtain sufficient conditions for the maximality of the operators A^*TA and $S + T$. Then we give an example which proves that these conditions are indeed weaker than the others given so far in the literature. The subdifferential case is also considered in this section.

4.1 Maximal monotonicity of the operator A^*TA

An important special case of Theorem 2 follows by taking $S : X \rightrightarrows X^*$, $S(x) = \{0\}, \forall x \in X$. In this case $G(S) = X \times \{0\}$ and $S + A^*TA = A^*TA$. We show that the operator S has a unique representative function. To this end, we compute first the Fitzpatrick function φ_S. We have

$$\varphi_S(x, x^*) = \sup\{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in S(y)\}$$

$$= \sup\{\langle x^*, y\rangle : y \in X\} = \begin{cases} 0, & \text{if } x^* = 0, \\ +\infty, & \text{otherwise} \end{cases} = \delta_{X\times\{0\}}.$$

Let f_S be a representative function of S. By Definition 4 we get $\{(x, x^*) : f_S(x, x^*) = \langle x^*, x\rangle\} = G(S) = X \times \{0\}$, hence $f_S(x, 0) = 0, \forall x \in X$. Let $x \in X$ and $x^* \in X^*$, $x^* \neq 0$ be arbitrary elements. Taking into account that $f_S(x, x^*) \geq \varphi_S(x, x^*) = +\infty$, it follows that $f_S = \delta_{X\times\{0\}}$.

As $f_S^{*\top}$ is again a representative function of S, we obtain that $f_S^{*\top} = \delta_{X\times\{0\}}$, that is $f_S^* = \delta_{\{0\}\times X}$. The condition $A(\mathrm{pr}_1(\mathrm{dom}(f_S))) \cap \mathrm{pr}_1(\mathrm{dom}(f_T)) \neq \emptyset$ becomes $A^{-1}(\mathrm{pr}_1(\mathrm{dom}(f_T))) \neq \emptyset$ and the constraint qualification (CQ) can be written as

(CQ_A^X) $\{(A^*y^*, x, y, r) : x \in X, y \in Y, y^* \in Y^*, f_T^*(y^*, y) \leq r\}$ is closed regarding the subspace $X^* \times \Delta_X^A \times \mathbb{R}$.

We prove that (CQ_A^X) is equivalent to the following condition

(CQ_A) $\{(A^*y^*, y, r) : f_T^*(y^*, y) \leq r\}$ is closed regarding the subspace $X^* \times \mathrm{Im}(A) \times \mathbb{R}$.

The conditions (CQ_A^X) and (CQ_A) are nothing else than

$$\{(A^*y^*, x, y, r) : x \in X, y \in Y, y^* \in Y^*, f_T^*(y^*, y) \leq r\} \cap X^* \times \Delta_X^A \times \mathbb{R} =$$

$$\mathrm{cl}\{(A^*y^*, x, y, r) : x \in X, y \in Y, y^* \in Y^*, f_T^*(y^*, y) \leq r\} \cap X^* \times \Delta_X^A \times \mathbb{R}$$

and, respectively,

$$\{(A^*y^*, y, r) : f_T^*(y^*, y) \leq r\} \cap X^* \times \mathrm{Im}(A) \times \mathbb{R} =$$

$$\mathrm{cl}\{(A^*y^*, y, r) : f_T^*(y^*, y) \leq r\} \cap X^* \times \mathrm{Im}(A) \times \mathbb{R}.$$

"$(CQ_A^X) \Rightarrow (CQ_A)$" Take an element $(z^*, Az, s) \in \mathrm{cl}\{(A^*y^*, y, r) : f_T^*(y^*, y) \leq r\} \cap X^* \times \mathrm{Im}(A) \times \mathbb{R}$. Then there exist some sequences $(y_n^*)_{n\in\mathbb{N}} \subseteq Y^*$, $(y_n)_{n\in\mathbb{N}} \subseteq Y$ and $(r_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $A^*y_n^* \to z^*, y_n \to Az, r_n \to s$ $(n \to +\infty)$ and

$f_T^*(y_n^*, y_n) \leq r_n, \forall n \in \mathbb{N}$. Thus $(A^*y_n^*, z, y_n, r_n) \to (z^*, z, Az, s)$ $(n \to +\infty)$. Using that $(A^*y_n^*, z, y_n, r_n) \in \{(A^*y^*, x, y, r) : x \in X, y \in Y, y^* \in Y^*, f_T^*(y^*, y) \leq r\}, \forall n \in \mathbb{N}$, we obtain $(z^*, z, Az, s) \in \operatorname{cl}\{(A^*y^*, x, y, r) : x \in X, y \in Y, y^* \in Y^*, f_T^*(y^*, y) \leq r\} \cap X^* \times \Delta_X^A \times \mathbb{R} = \{(A^*y^*, x, y, r) : x \in X, y \in Y, y^* \in Y^*, f_T^*(y^*, y) \leq r\} \cap X^* \times \Delta_X^A \times \mathbb{R}$. This implies that $z^* = A^*y^*$ and $f_T^*(y^*, Az) \leq s$, so $(z^*, Az, s) \in \{(A^*y^*, y, r) : f_T^*(y^*, y) \leq r\} \cap X^* \times \operatorname{Im}(A) \times \mathbb{R}$. Hence the inclusion $\operatorname{cl}\{(A^*y^*, y, r) : f_T^*(y^*, y) \leq r\} \cap X^* \times \operatorname{Im}(A) \times \mathbb{R} \subseteq \{(A^*y^*, y, r) : f_T^*(y^*, y) \leq r\} \cap X^* \times \operatorname{Im}(A) \times \mathbb{R}$ is true, and because the reverse inclusion is trivial, we get that (CQ_A) is fulfilled.

"$(CQ_A) \Rightarrow (CQ_A^X)$" Let $(z^*, z, Az, s) \in \operatorname{cl}\{(A^*y^*, x, y, r) : x \in X, y \in Y, y^* \in Y^*, f_T^*(y^*, y) \leq r\} \cap X^* \times \Delta_X^A \times \mathbb{R}$ be fixed. Then there exist some sequences $(y_n^*)_{n\in\mathbb{N}} \subseteq Y^*, (x_n)_{n\in\mathbb{N}} \subseteq X, (y_n)_{n\in\mathbb{N}} \subseteq Y$ and $(r_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that $A^*y_n^* \to z^*, x_n \to z, y_n \to Az, r_n \to s$ $(n \to +\infty)$ and $f_T^*(y_n^*, y_n) \leq r_n, \forall n \in \mathbb{N}$. We have $(A^*y_n^*, y_n, r_n) \to (z^*, Az, s)$ $(n \to +\infty)$. Using that $(A^*y_n^*, y_n, r_n) \in \{(A^*y^*, y, r) : f_T^*(y^*, y) \leq r\}, \forall n \in \mathbb{N}$, we obtain $(z^*, Az, s) \in \operatorname{cl}\{(A^*y^*, y, r) : f_T^*(y^*, y) \leq r\} \cap X^* \times \operatorname{Im}(A) \times \mathbb{R} = \{(A^*y^*, y, r) : f_T^*(y^*, y) \leq r\} \cap X^* \times \operatorname{Im}(A) \times \mathbb{R}$. We get $z^* = A^*y^*$ and $f_T^*(y^*, Az) \leq s$, so $(z^*, z, Az, s) \in \{(A^*y^*, x, y, r) : x \in X, y \in Y, y^* \in Y^*, f_T^*(y^*, y) \leq r\} \cap X^* \times \Delta_X^A \times \mathbb{R}$. Thus (CQ_A^X) is fulfilled.

Corollary 2. *If (CQ_A) is fulfilled, then A^*TA is a maximal monotone operator.*

Let us notice that the interior-point condition (CQ^{PZ}) in Remark 3 becomes in this case (see [19])

$$(CQ_A^{PZ})\ 0 \in {}^{ic}(\operatorname{pr}_1(\operatorname{dom}(f_T)) - \operatorname{Im}(A)).$$

Obviously, (CQ_A) is implied by CQ_A^{PZ}; this fact was proved in [6] for the case f_T is the Fitzpatrick function of T. The constraint qualification (CQ_A) and Corollary 2 extends the corresponding results given in the paper [6].

4.2 Maximal monotonicity of the operator $S + T$

A second important special case of Theorem 2 is obtained by taking $Y = X$ and $A = \operatorname{id}_X$. Then $A^* = \operatorname{id}_{X^*}$, $S, T : X \rightrightarrows X^*$ and $S + A^*TA = S + T$. The condition $A(\operatorname{pr}_1(\operatorname{dom}(f_S))) \cap \operatorname{pr}_1(\operatorname{dom}(f_T)) \neq \emptyset$ becomes $\operatorname{pr}_1(\operatorname{dom}(f_S)) \cap \operatorname{pr}_1(\operatorname{dom}(f_T)) \neq \emptyset$ and the constraint qualification (CQ) is in this case

(CQ_+) $\{(x^* + y^*, x, y, r) : f_S^*(x^*, x) + f_T^*(y^*, y) \leq r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$,

where $\Delta_X = \{(x, x) : x \in X\}$.

Corollary 3. *If (CQ_+) is fulfilled, then $S+T$ is a maximal monotone operator.*

The interior point-condition (CQ^{PZ}) in Remark 3 becomes in this case (see [19])

$$(CQ_+^{PZ})\ 0 \in {}^{ic}(\mathrm{pr}_1(\mathrm{dom}(f_S)) - \mathrm{pr}_1 \mathrm{dom}(f_T)).$$

Obviously, (CQ_+^{PZ}) implies (CQ_+). For $f_S = \varphi_S$ and $f_T = \varphi_T$ in (CQ_+) one obtains the constraint qualification given in [6] and [7] for the maximal monotonicity of the sum of two maximal monotone operators.

Another possibility to get (CQ_+) is by deriving it from (CQ_A) when $Y = X \times X$, $A : X \to X \times X$, $Ax = (x, x)$ and $(S,T) : X \times X \rightrightarrows X^* \times X^*$, $(S,T)(x,y) = (S(x), T(y))$. In case S and T are maximal monotone operators, (S,T) is also a maximal monotone operator and it holds $A^*(S,T)A(x) = S(x) + T(x), \forall x \in X$. After some minor calculations one can see that (CQ_A) becomes in this case (CQ_+).

4.3 An example

The example we give in this subsection presents two maximal monotone operators for which (CQ) is fulfilled and (CQ^{PZ}) is not fulfilled. It shows that the constraint qualifications we have introduced in this paper in the general case, but also both for the precomposition of a maximal monotone operator with a linear continuous mapping and for the sum of two maximal monotone operators, are indeed weaker than the other ones given in the literature. Let us notice that since the functions f and g are not both polyhedral, the subdifferential sum formula $\partial(f+g) = \partial f + \partial g$ (and, as consequence, the maximal monotonicity of $\partial f + \partial g$) does not follow automatically.

Example 1. Take $X = \mathbb{R}^2$, equipped with the Euclidean norm $\|\cdot\|_2$, $f, g : \mathbb{R}^2 \to \overline{\mathbb{R}}$, $f = \|\cdot\|_2 + \delta_{\mathbb{R}^2_+}$, $g = \delta_{-\mathbb{R}^2_+}$ and $S = \partial f$, $T = \partial g$. Since f is proper, lower semicontinuous and sublinear and g is the indicator function of a non-empty closed convex set, the operators S and T have unique representative functions, namely

$$f_S(x, x^*) = f(x) + f^*(x^*), \forall (x, x^*) \in \mathbb{R}^2 \times \mathbb{R}^2$$

and, respectively,

$$f_T(x, x^*) = g(x) + g^*(x^*), \forall (x, x^*) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

One can see that $g^* = \delta_{\mathbb{R}^2_+}$ and $f^* = \delta_{\overline{B}(0,1) - \mathbb{R}^2_+}$, where $\overline{B}(0,1)$ is the closed unit ball in $\mathbb{R}^2$. We note that for the set involved in (CQ_+) one has

$$\{(x^* + y^*, x, y, r) : f(x) + f^*(x^*) + g(y) + g^*(y^*) \leq r\} =$$

$$\mathbb{R}^2 \times \{(x, y, r) : x \in \mathbb{R}^2_+, y \in -\mathbb{R}^2_+, \|x\|_2 \leq r\}.$$

As this set is closed, we conclude that (CQ_+) is fulfilled.

Since $\mathrm{pr}_1(\mathrm{dom}(f_S)) = \mathbb{R}^2_+$ and $\mathrm{pr}_1(\mathrm{dom}(f_T)) = -\mathbb{R}^2_+$, the condition (CQ^{PZ}_+) becomes: $\mathbb{R}^2_+$ is a closed linear subspace in $\mathbb{R}^2$, which shows that this condition fails in this case.

4.4 The subdifferential case

Let us consider $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ two proper, convex and lower semicontinuous functions such that $A(\mathrm{dom}(f)) \bigcap \mathrm{dom}(g) \neq \emptyset$. Consider also the maximal monotone operators $\partial f : X \rightrightarrows X^*$ and $\partial g : Y \rightrightarrows Y^*$ and their representative functions $(x, x^*) \mapsto f(x) + f^*(x^*)$, respectively, $(y, y^*) \mapsto g(y) + g^*(y^*)$. The condition (CQ) becomes

(CQ^∂) $\quad\{(x^* + A^*y^*, x, y, r) : f(x) + f^*(x^*) + g(y) + g^*(y^*) \leq r\}$ is closed regarding the subspace $X^* \times \Delta^A_X \times \mathbb{R}$.

By Theorem 2 we obtain the following result concerning the subdifferential sum formula of a convex function with the precomposition of another convex function with a continuous linear mapping.

Corollary 4. *If (CQ^∂) is fulfilled, then $\partial f + A^*\partial gA$ is a maximal monotone operator. Hence*

$$\partial(f + g \circ A) = \partial f + A^*\partial gA.$$

The second assertion of the previous corollary is a direct consequence of the fact that $\partial(f + g \circ A) \supseteq \partial f + A^*\partial gA$, which is always the case. The operator $\partial(f + g \circ A)$ being monotone, the equality must hold.

Next we prove that the condition (CQ^∂) is equivalent to

$$\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*)) \text{ is closed in } X^* \times \mathbb{R}. \tag{1}$$

To this end we make the following notations

$$U := \{(x^* + A^*y^*, x, y, r) : f(x) + f^*(x^*) + g(y) + g^*(y^*) \leq r\}$$

and $V := X^* \times \Delta^A_X \times \mathbb{R}$. Hence (CQ^∂) can be written as $\mathrm{cl}(U) \cap V = U \cap V$.

"$(CQ^\partial) \Rightarrow (1)$" Take an element $(u^*, r) \in \mathrm{cl}(\mathrm{epi}(f^*) + A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*)))$. There exist some sequences $(x^*_n)_{n\in\mathbb{N}} \subseteq X^*, (y^*_n)_{n\in\mathbb{N}} \subseteq Y^*, (r_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ and $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that

$$\begin{cases} x^*_n + A^*y^*_n \to u^*(n \to +\infty) \\ r_n + t_n \to r(n \to +\infty) \\ f^*(x^*_n) \leq r_n, \forall n \in \mathbb{N} \\ g^*(y^*_n) \leq t_n \forall n \in \mathbb{N}. \end{cases}$$

By the hypothesis, there exists $x_0 \in \text{dom}(f) \bigcap A^{-1}(\text{dom}(g)) \neq \emptyset$. We get

$$f(x_0) + g(Ax_0) + f^*(x_n^*) + g^*(y_n^*) \leq r_n + t_n + f(x_0) + g(Ax_0), \forall n \in \mathbb{N},$$

so $(x_n^* + A^*y_n^*, x_0, Ax_0, r_n + t_n + f(x_0) + g(Ax_0)) \in U, \forall n \in \mathbb{N}$, which implies that $(u^*, x_0, Ax_0, r + f(x_0) + g(Ax_0)) \in \text{cl}(U) \cap V = U \cap V$. It follows that there exist $x^* \in X^*$ and $y^* \in Y^*$ such that $u^* = x^* + A^*y^*$ and $f(x_0) + f^*(x^*) + g(Ax_0) + g^*(y^*) \leq r + f(x_0) + g(Ax_0)$, that is $f^*(x^*) + g^*(y^*) \leq r$. The element (u^*, r) can now be written in the following way

$$(u^*, r) = (x^*, r - g^*(y^*)) + (A^*y^*, g^*(y^*)) \in \text{epi}(f^*) + A^* \times \text{id}_{\mathbb{R}}(\text{epi}(g^*)),$$

and so $\text{cl}(\text{epi}(f^*) + A^* \times \text{id}_{\mathbb{R}}(\text{epi}(g^*))) \subseteq \text{epi}(f^*) + A^* \times \text{id}_{\mathbb{R}}(\text{epi}(g^*))$. As the reverse inclusion is obvious, the relation (1) is fulfilled.

"(1) $\Rightarrow$ (CQ^∂)" Let $(z^*, z, Az, r) \in \text{cl}(U) \cap V$ be fixed. There exist some sequences $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*, (y_n^*)_{n\in\mathbb{N}} \subseteq Y^*, (x_n)_{n\in\mathbb{N}} \subseteq X, (y_n)_{n\in\mathbb{N}} \subseteq Y$ and $(r_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ such that

$$\begin{cases} x_n^* + A^*y_n^* \to z^* (n \to +\infty) \\ x_n \to z, y_n \to Az, r_n \to r (n \to +\infty) \\ f(x_n) + f^*(x_n^*) + g(y_n) + g^*(y_n^*) \leq r_n, \forall n \in \mathbb{N}. \end{cases}$$

A simple computation using the definition of the conjugate function shows that the following inequality is true

$$(f + g \circ A)^*(x_n^* + A^*y_n^*) \leq f^*(x_n^*) + g^*(y_n^*), \forall n \in \mathbb{N}.$$

Combining the last two inequalities we get

$$f(x_n) + g(y_n) + (f + g \circ A)^*(x_n^* + A^*y_n^*) \leq r_n, \forall n \in \mathbb{N}.$$

The lower semicontinuity of the functions involved yields the inequality

$$f(z) + g(Az) + (f + g \circ A)^*(z^*) \leq r.$$

This shows (see Remark 1) that $(z^*, r - f(z) - g(Az)) \in \text{epi}(f + g \circ A)^* = \text{epi}(f^*) + A^* \times \text{id}_{\mathbb{R}}(\text{epi}(g^*))$. Hence there exist $x^* \in X^*, y^* \in Y^*, r_1, r_2 \in \mathbb{R}$ such that

$$\begin{cases} (z^*, r - f(z) - g(Az)) = (x^*, r_1) + (A^*y^*, r_2) \\ f^*(x^*) \leq r_1, g^*(y^*) \leq r_2. \end{cases}$$

The last relations imply $f^*(x^*) + g^*(y^*) \leq r_1 + r_2 = r - f(z) - g(Az) \Leftrightarrow f(z) + f^*(x^*) + g(Az) + g^*(y^*) \leq r$. Finally, $(z^*, z, Az, r) = (x^* + A^*y^*, z, Az, r)$ and since by the last inequality we obtain $(z^*, z, Az, r) \in U \cap V$, the inclusion $\text{cl}(U) \cap V \subseteq U \cap V$ is true. The reverse inclusion is always true, hence (CQ^∂) is fulfilled.

In conclusion, we rediscover a sufficient condition for the subdifferential sum formula of a convex function with the precomposition of another convex function with a continuous linear mapping, given in [5] in the framework of locally convex spaces. For the special cases of the precomposition with a linear continuous operator and for the sum we get the following two results.

Corollary 5. *If (CQ_A^∂) is fulfilled, then $\partial(g \circ A)$ is a maximal monotone operator. Hence*

$$\partial(g \circ A) = A^*\partial g A,$$

where

(CQ_A^∂) $\quad\{(A^*y^*, y, r) : g(y) + g^*(y^*) \leq r\}$ *is closed regarding the subspace* $X^* \times \mathrm{Im}(A) \times \mathbb{R}$.

The condition (CQ_A^∂) is equivalent to

$$A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(g^*)) \text{ is closed in } X^* \times \mathbb{R}. \tag{2}$$

Corollary 6. *If (CQ_+^∂) is fulfilled, then $\partial f + \partial g$ is a maximal monotone operator. Hence*

$$\partial(f + g) = \partial f + \partial g,$$

where

(CQ_+^∂) $\{(x^* + y^*, x, y, r) : f(x) + f^*(x^*) + g(y) + g^*(y^*) \leq r\}$ *is closed regarding the subspace* $X^* \times \Delta_X \times \mathbb{R}$.

It turns out that (CQ_+^∂) is equivalent to

$$\mathrm{epi}(f^*) + \mathrm{epi}(g^*) \text{ is closed in } X^* \times \mathbb{R}. \tag{3}$$

5 Some open questions

As we can see, there exist two types of conditions for the maximality of monotone operators, given so far in the literature: interior point-conditions and closedness conditions.

The new conditions introduced in this paper are obviously of closedness type. As the reader can easily observe, we have in fact a family of sufficient conditions, which can be obtained by considering different representative functions. Let us denote by $CQ(f_S, f_T)$ the constraint qualification we get by taking the representatives f_S and f_T of the operators S and T, respectively. It could be an interesting research subject to find out what kind of relations exist between the different conditions. One should take here into consideration the sufficient conditions that ensure the maximal monotonicity of $S+T$ written by using the Fitzpatrick function and its conjugate:

$CQ_+(\varphi_S, \varphi_T)$ $\{(x^*+y^*, x, y, r) : \varphi_S^*(x^*, x) + \varphi_T^*(y^*, y) \le r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$,

$CQ_+(\varphi_S, \varphi_T^*)$ $\{(x^*+y^*, x, y, r) : \varphi_S^*(x^*, x) + \varphi_T(y, y^*) \le r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$,

$CQ_+(\varphi_S^*, \varphi_T^*)$ $\{(x^*+y^*, x, y, r) : \varphi_S(x, x^*) + \varphi_T(y, y^*) \le r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$.

If they are not comparable, it would be important to provide some relevant examples proving this fact.

As one can see, the interior point-condition (CQ^{PZ}) in [19] is written also in terms of the representative functions of the operators involved. Nevertheless, it makes no sense to raise a similar problem as above, since for every pair of representative functions (f_S, f_T), the condition (CQ^{PZ}) is equivalent to (CQ^P) and this is written in terms of the operators S and T.

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