TECHNICAL NOTE

Fenchel's Duality Theorem for Nearly Convex Functions [1]

R. I. BOŢ[2], S. M. GRAD[3], AND G. WANKA[4]

Communicated by J-P. Crouzeix

[1]The authors are grateful to the Associated Editor for helpful suggestions and remarks which improved the quality of the paper

[2]Assistant Professor, Faculty of Mathematics, Chemnitz University of Technology, Chemnitz, Germany.

[3]Ph. D. Student, Faculty of Mathematics, Chemnitz University of Technology, Germany.

[4]Professor, Faculty of Mathematics, Chemnitz University of Technology, Chemnitz, Germany.

Abstract. We present an extension of Fenchel's duality theorem by weakening the convexity assumptions to near convexity. These weak hypotheses are automatically fulfilled in the convex case. Moreover we show by a counterexample that a further extension to closely convex function is not possible under these hypotheses.

Key Words. Fenchel duality, conjugate functions, nearly convex functions.

1 Introduction and Preliminaries

Fenchel's duality theorem (cf. Ref. 1) asserts that for $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ a proper convex function and for $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ a proper concave function fulfilling $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(g)) \neq \emptyset$ there is strong duality between the primal problem $\inf_{x\in\mathbb{R}^n} \big[f(x) - g(x)\big]$ and its Fenchel dual $\sup_{u\in\mathbb{R}^n} \big\{g^*(u) - f^*(u)\big\}$. There were attempts to extend it, for instance see Ref. 2.

In this note we give another extension of Fenchel's duality theorem, for a primal problem having as objective the difference between a nearly convex function and a nearly concave one. The nearly convex functions were introduced by Aleman (Ref. 3) as p-convex (see also Ref. 4), while the nearly convex sets are due to Green and Gustin (Ref. 5). As the name "nearly convex" has been used in the literature also for other concepts, we followed the terminology used in some relevant optimization papers (Refs. 6-8). Nearly convex functions generalize the older *midconvex functions* (cf. Ref. 9) (obtained for $p = 1/2$).

When $X \subseteq \mathbb{R}^n$ we use the classical notations $\mathrm{cl}(X)$, $\mathrm{aff}(X)$, $\mathrm{ri}(X)$ and δ_X for its *closure, affine hull, relative interior* and *indicator function*, respectively. For a convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ we consider the following definitions: *effective domain* $\mathrm{dom}(f) = \big\{x \in \mathbb{R}^n : f(x) < +\infty\big\}$, *epigraph* $\mathrm{epi}(f) = \big\{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\big\}$, *$f$ is proper* if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty \ \forall x \in \mathbb{R}^n$, *lower semicontinuous envelope* $\overline{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that $\mathrm{epi}(\overline{f}) = \mathrm{cl}(\mathrm{epi}(f))$ and *conjugate function* $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, $f^*(p) = \sup_{x \in \mathbb{R}^n} \big\{p^T x - f(x)\big\}$.

Similar notions are defined for a concave function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$: $\mathrm{dom}(g) = \big\{x \in \mathbb{R}^n : g(x) > -\infty\big\}$, $\mathrm{hyp}(g) = \big\{(x, r) \in \mathbb{R}^n \times \mathbb{R} : g(x) \geq r\big\}$, *$g$ is proper* if $\mathrm{dom}(g) \neq \emptyset$ and $g(x) < +\infty \ \forall x \in \mathbb{R}^n$, its *upper semicontinuous envelope* $\overline{g}$ and $g^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g^*(p) = \inf_{x \in \mathbb{R}^n} \big[p^T x - g(x)\big]$. We denote these notions in the same way for both convex and concave functions as the meaning arises from the context. For nearly convex functions these notions are considered in the convex sense, while for nearly concave ones they are taken like for concave functions. For $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and its convex conjugate, respectively for $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ and its concave conjugate, there is the Young-Fenchel inequality $f^*(u) + f(x) \geq u^T x \geq g^*(u) + g(x) \ \forall u, x \in \mathbb{R}^n$.

The following result coming from the convex analysis will be used later.

Lemma 1.1. (see Ref. 6) For a convex set $C \subseteq \mathbb{R}^n$ and any set $S \subseteq \mathbb{R}^n$ satisfying

$S \subseteq C$ we have $\mathrm{ri}(C) \subseteq S$ if and only if $\mathrm{ri}(C) = \mathrm{ri}(S)$.

A set $S \subseteq \mathbb{R}^n$ is called *nearly convex* if there is an $\alpha \in (0,1)$ such that for each $x, y \in S$ it follows that $\alpha x + (1-\alpha)y \in S$. Every convex set is nearly convex, while $\mathbb{Q}$ is nearly convex (with $\alpha = 1/2$), but not convex. We give now some results concerning nearly convex sets.

Lemma 1.2. (see Ref. 3) For every nearly convex set $S \subseteq \mathbb{R}^n$ one has

(i) $\mathrm{ri}(S)$ is convex (may be empty),

(ii) $\mathrm{cl}(S)$ is convex,

(iii) for every $x \in \mathrm{cl}(S)$ and $y \in \mathrm{ri}(S)$ we have $tx+(1-t)y \in \mathrm{ri}(S)$ for each $0 \leq t < 1$.

Lemma 1.3. (see Ref. 6) Let $\emptyset \neq S \subseteq \mathbb{R}^n$ be a nearly convex. Then $\mathrm{ri}(S) \neq \emptyset$ if and only if $\mathrm{ri}(\mathrm{cl}(S)) \subseteq S$, in this case $\mathrm{ri}(S) = \mathrm{ri}(\mathrm{cl}(S))$.

Closely related to the notion of a nearly convex set we consider similar notions for functions. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *nearly convex* if there is an $\alpha \in (0,1)$ such that for all $x, y \in \mathrm{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$ we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y).$$

A function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be *nearly concave* when $-f$ is nearly convex.

Nearly convex/concave functions have nearly convex domains and any convex function is also nearly convex, but there are nearly convex functions that are not convex as shown below.

Example 1.1. Let $F : \mathbb{R} \to \mathbb{R}$ be any discontinuous solution of Cauchy's functional equation $F(x+y) = F(x) + F(y)\ \forall x, y \in \mathbb{R}$. For each of these functions, whose existence is guaranteed in Ref. 10, one has $F((x+y)/2) = (F(x)+F(y))/2\ \forall x, y \in \mathbb{R}$, i.e. these functions are nearly convex. None of them is convex because of the absence of continuity.

The following lemma extends to the nearly convex setting a well-known result from the convex analysis.

Lemma 1.4. Let the functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$. Then f is nearly convex if and only if $\mathrm{epi}(f)$ is nearly convex and g is nearly concave if and only if $\mathrm{hyp}(g)$ is nearly convex.

Remark 1.1. Since $\mathrm{epi}(\overline{f}) = \mathrm{cl}(\mathrm{epi}(f))$, by Lemma 1.2 (ii) and Lemma 1.4 one has that the lower semicontinuous envelope $\overline{f}$ of a nearly convex function f is convex.

Theorem 1.1. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper nearly convex function fulfilling $\mathrm{ri}(\mathrm{epi}(f)) \neq \emptyset$. Then

(a) $\mathrm{ri}(\mathrm{epi}(f)) = \mathrm{ri}(\mathrm{epi}(\overline{f}))$ and $\mathrm{ri}(\mathrm{dom}(f)) = \mathrm{ri}(\mathrm{dom}(\overline{f}))$;

(b) $f(x) = \overline{f}(x), \ \forall x \in \mathrm{ri}(\mathrm{dom}(\overline{f}))$,

(c) $\overline{f}$ is proper.

Proof. Since $\mathrm{ri}(\mathrm{epi}(f)) \neq \emptyset$, by Lemma 1.3 it follows $\mathrm{ri}(\mathrm{epi}(\overline{f})) = \mathrm{ri}(\mathrm{cl}(\mathrm{epi}(f)) = \mathrm{ri}(\mathrm{epi}(f))$. Denoting by $\mathrm{Pr} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ the projection operator defined by $\mathrm{Pr}(x, r) = x$, we have by Theorem 6.6 in Ref. 1 that

$$\mathrm{ri}(\mathrm{dom}(\overline{f})) = \mathrm{ri}(\mathrm{Pr}(\mathrm{epi}(\overline{f})) = \mathrm{Pr}(\mathrm{ri}(\mathrm{epi}(\overline{f})) = \mathrm{Pr}(\mathrm{ri}(\mathrm{epi}(f))) \subseteq \mathrm{Pr}(\mathrm{epi}(f)) = \mathrm{dom}(f).$$

On the other hand, as $\mathrm{dom}(f) \subseteq \mathrm{dom}(\overline{f})$ and the latter is a convex set, by Lemma 1.1 follows $\mathrm{ri}(\mathrm{dom}(f)) = \mathrm{ri}(\mathrm{dom}(\overline{f}))$.

Take an x from $\mathrm{ri}(\mathrm{dom}(\bar{f}))$. By Lemma 7.3 in Ref. 1, we have that for all $\varepsilon > 0$, $(x, \bar{f}(x) + \varepsilon) \in \mathrm{ri}(\mathrm{epi}(\bar{f})) \subseteq \mathrm{epi}(f)$, so $f(x) \leq \bar{f}(x) + \varepsilon$. Letting ε tend to 0 it follows $f(x) \leq \bar{f}(x)$. Since the opposite inequality is always true, (ii) follows.

As f is not identical $+\infty$ it follows that $\bar{f}$ is also not identical $+\infty$. Assuming there exists an $x' \in \mathbb{R}^n$ such that $\bar{f}(x') = -\infty$, we would have (cf. Corollary 7.2.1 in Ref. 1) that $\bar{f}(x) = -\infty \ \forall x \in \mathrm{dom}(\bar{f})$. As $\bar{f}(x) = f(x) \ \forall x \in \mathrm{ri}(\mathrm{dom}(f))$ this contradicts the properness of f. Thus $\bar{f}$ is a proper function. $\square$

Remark 1.2. For F a discontinuous solutions of Cauchy's functional equation (cf. Example 1.1) we have that $\mathrm{ri}(\mathrm{dom}(F)) = \mathbb{R}$, but $\mathrm{ri}(\mathrm{epi}(F)) = \emptyset$. Assuming $\mathrm{ri}(\mathrm{epi}(F)) \neq \emptyset$, this would imply $\bar{F}(x) = F(x) \ \forall x \in \mathrm{ri}(\mathrm{dom}(\bar{F}))$. As the latter set coincides with $\mathbb{R}$, one gets $F = \bar{F}$ and so F is convex, which is not the case.

2 Extension of the Fenchel Duality Theorem

For a proper convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a proper concave one $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ *Fenchel duality's theorem* states that if $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(g)) \neq \emptyset$, then

$$\inf_{x \in \mathbb{R}^n} \big[f(x) - g(x)\big] = \max_{u \in \mathbb{R}^n} \big\{g^*(u) - f^*(u)\big\}.$$

We weaken the conditions imposed in Ref. 1 without altering the conclusion by considering f nearly convex and g nearly concave.

Theorem 2.1. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper nearly convex function and let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper nearly concave function. If the following conditions are simultaneously satisfied

(i) $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(g)) \neq \emptyset$,

(ii) $\mathrm{ri}(\mathrm{epi}(f)) \neq \emptyset$,

(iii) $\mathrm{ri}(\mathrm{hyp}(g)) \neq \emptyset$,

then one has

$$\inf_{x \in \mathbb{R}^n} \big[f(x) - g(x)\big] = \max_{u \in \mathbb{R}^n} \big\{g^*(u) - f^*(u)\big\}.$$

Proof. One can notice that the relations (a)-(c) in Theorem 1.1 are fulfilled. Similarly it follows that $\bar{g}$ is a proper concave and upper semicontinuous function such that $\bar{g}(x) = g(x)\ \forall x \in \mathrm{ri}(\mathrm{dom}(\bar{g}))$ and $\mathrm{ri}(\mathrm{dom}(\bar{g})) = \mathrm{ri}(\mathrm{dom}(g))$.

Denote by $v := \inf\big[f(x) - g(x) : x \in \mathbb{R}^n\big] \geq \inf\big[\bar{f}(x) - \bar{g}(x) : x \in \mathbb{R}^n\big]$. Since $\bar{f} - \bar{g}$ is convex, by Corollary 7.3.1 in Ref. 1 we have

$$\begin{aligned} \inf\big[\bar{f}(x) - \bar{g}(x) : x \in \mathbb{R}^n\big] &= \inf\big[\bar{f}(x) - \bar{g}(x) : x \in \mathrm{ri}(\mathrm{dom}(\bar{f} - \bar{g}))\big] \\ &= \inf\big[\bar{f}(x) - \bar{g}(x) : x \in \mathrm{ri}(\mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g}))\big]. \end{aligned}$$

The sets $\operatorname{dom}(\bar{f})$ and $\operatorname{dom}(\bar{g}))$ are convex and the intersection of their relative interior is not empty, since $\operatorname{ri}(\operatorname{dom}(f)) \cap \operatorname{ri}(\operatorname{dom}(g)) = \operatorname{ri}(\operatorname{dom}(\bar{f})) \cap \operatorname{ri}(\operatorname{dom}(\bar{g}))$. By Theorem 6.5 in Ref. 1, the latter set is equal to $\operatorname{ri}(\operatorname{dom}(\bar{f}) \cap \operatorname{dom}(\bar{g}))$. Thus

$$\begin{aligned} \inf \left[\bar{f}(x) - \bar{g}(x) : x \in \mathbb{R}^n\right] &= \inf \left[\bar{f}(x) - \bar{g}(x) : x \in \operatorname{ri}(\operatorname{dom}(\bar{f})) \cap \operatorname{ri}(\operatorname{dom}(\bar{g}))\right] \\ &= \inf \left[f(x) - g(x) : x \in \operatorname{ri}(\operatorname{dom}(f)) \cap \operatorname{ri}(\operatorname{dom}(g))\right] \geq v. \end{aligned}$$

In conclusion,

$$v = \inf \left[f(x) - g(x) : x \in \mathbb{R}^n\right] = \inf \left[\bar{f}(x) - \bar{g}(x) : x \in \mathbb{R}^n\right].$$

Fenchel's duality theorem (Theorem 31.1 in Ref. 1) yields for $\bar{f}$ and $\bar{g}$ that

$$\inf_{x \in \mathbb{R}^n} \left[\bar{f}(x) - \bar{g}(x)\right] = \max_{u \in \mathbb{R}^n} \left\{(\bar{g})^*(u) - (\bar{f})^*(u)\right\}.$$

As $f^* = (\bar{f})^*$ and $g^* = (\bar{g})^*$ (cf. Ref. 1) one has

$$\inf_{x \in \mathbb{R}^n} \left[f(x) - g(x)\right] = \inf_{x \in \mathbb{R}^n} \left[\bar{f}(x) - \bar{g}(x)\right] = \max_{u \in \mathbb{R}^n} \left\{g^*(u) - f^*(u)\right\}. \qquad \square$$

Remark 2.1. The assumptions of near convexity for f and of near concavity for g do not require the same near convexity constant for both of these functions.

Remark 2.2. If f and g are proper, $\bar{f}$ convex, $\bar{g}$ concave and (i) holds, one has

$$\inf_{x \in \mathbb{R}^n} \left[\bar{f}(x) - \bar{g}(x)\right] = \max_{u \in \mathbb{R}^n} \left\{g^*(u) - f^*(u)\right\}. \tag{1}$$

The question whether

$$\inf_{x\in\mathbb{R}^n}\left[f(x)-g(x)\right]=\inf_{x\in\mathbb{R}^n}\left[\bar{f}(x)-\bar{g}(x)\right] \tag{2}$$

is true or not under weaker hypotheses than in Theorem 2.1, like relaxing the near convexity assumptions to close convexity, arises naturally. A function with the closure of the epigraph convex is called *closely convex* (cf. Ref. 7) and analogously one defines *closely concave* functions. The next example shows that (2) may fail when f is closely convex, g is closely concave and the assumptions (i)-(iii) in Theorem 2.1 hold.

Example 2.1. Consider the sets

$$\begin{aligned} X &= \{(x,y)^T\in\mathbb{R}^2 : x\geq 0, y\geq 0, x\in\mathbb{Q}, y\in\mathbb{Q}, x+y<1\} \\ &\cup \{(x,y)^T\in\mathbb{R}^2 : x\geq 0, y\geq 0, 1\leq x+y\leq 2\} \end{aligned}$$

$$\begin{aligned} \text{and } Y &= \{(x,y)^T\in\mathbb{R}^2 : x\geq 0, y\geq 0, x\in\mathbb{R}\backslash\mathbb{Q}, y\in\mathbb{R}\backslash\mathbb{Q}, x+y<1\} \\ &\cup \{(x,y)^T\in\mathbb{R}^2 : x\geq 0, y\geq 0, 1\leq x+y\leq 2\}, \end{aligned}$$

and $f,g:\mathbb{R}^2\to\overline{\mathbb{R}}$,

$$f(x,y)=\begin{cases} x, & \text{if } (x,y)\in X, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } g(x,y)=\begin{cases} -y, & \text{if } (x,y)\in Y, \\ -\infty, & \text{otherwise.} \end{cases}$$

Obviously f and g are proper and $(3/4, 3/4) \in \mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(g))$, $(3/4, 3/4, 1) \in \mathrm{ri}(\mathrm{epi}(f))$ and $(3/4, 3/4, -1) \in \mathrm{ri}(\mathrm{hyp}(g))$, whence hypotheses (i)-(iii) in Theorem 2.1 are valid. X and Y are not nearly convex, thus, as $\mathrm{dom}(f) = X$ and $\mathrm{dom}(g) = Y$, f is not nearly convex and g is not nearly concave. On the other hand we have

$$\mathrm{cl}(\mathrm{epi}(f)) = \{(x, y, r)^T \in \mathbb{R}^3 : x \geq 0, y \geq 0, x + y \leq 2, x \leq r\}$$

$$\text{and } \mathrm{cl}(\mathrm{hyp}(g)) = \{(x, y, r)^T \in \mathbb{R}^3 : x \geq 0, y \geq 0, x + y \leq 2, y \leq -r\}$$

and these sets are convex. Hence f is closely convex and g is closely concave. Therefore, with (i) in Theorem 2.1 fulfilled, (1) is valid. One has

$$\bar{f}(x, y) = \begin{cases} x, & \text{if } (x, y) \in Z, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } \bar{g}(x, y) = \begin{cases} -y, & \text{if } (x, y) \in Z, \\ -\infty, & \text{otherwise,} \end{cases}$$

where $Z = \{(x, y)^T \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 2\}$, thus

$$\inf_{(x,y) \in \mathbb{R}^2} [\bar{f}(x, y) - \bar{g}(x, y)] = \inf\{x + y : x \geq 0, y \geq 0, x + y \leq 2\} = 0.$$

Thus by Fenchel's duality theorem,

$$\inf_{(x,y) \in \mathbb{R}^2} \left[\bar{f}(x, y) - \bar{g}(x, y)\right] = \max_{(u,v) \in \mathbb{R}^2} \left\{g^*(u, v) - f^*(u, v)\right\} = 0.$$

Simple calculations lead to

$$\inf_{(x,y)\in\mathbb{R}^2}[f(x,y)-g(x,y)] = \inf\{x+y : x\geq 0, y\geq 0, 1\leq x+y\leq 2\} = 1.$$

Therefore (2) is obviously violated, thus Fenchel's duality theorem does not hold when its hypotheses are further weakened by taking the functions involved only closely convex, respectively closely concave.

Remark 2.3. If f is proper convex and g is proper concave, (ii) and (iii) are automatically fulfilled and Theorem 2.1 becomes Fenchel's duality theorem.

Giving Theorem 2.1 for $F, G : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$, $F(x,y) = f(x) + \delta_{\{x\in\mathbb{R}^n : Ax=y\}}(x)$ and $G(x,y)=g(y)$, we extend to near convexity Fenchel's duality result for the composition with a linear mapping $A:\mathbb{R}^n\to\mathbb{R}^m$, generalizing Corollary 31.2.1 in Ref. 1.

Theorem 2.2. Let f be proper nearly convex on $\mathbb{R}^n$, let g be proper nearly concave on $\mathbb{R}^m$, and let A be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. If one has

(i) $\exists x' \in \operatorname{ri}(\operatorname{dom}(f))$ such that $Ax' \in \operatorname{ri}(\operatorname{dom}(g))$,

(ii) $\operatorname{ri}(\operatorname{epi}(f)) \neq \emptyset$,

(iii) $\operatorname{ri}(\operatorname{hyp}(g)) \neq \emptyset$, it follows

$$\inf_{x\in\mathbb{R}^n} \big[f(x) - g(Ax)\big] = \max_{v\in\mathbb{R}^m} \big\{g^*(v) - f^*(A^*v)\big\}.$$

Remark 2.4. By Remark 2.3, the assertion of Corollary 31.2.1 in Ref. 1 is valid under the condition $\exists x' \in \mathrm{ri}(\mathrm{dom}(f))$ such that $Ax' \in \mathrm{ri}(\mathrm{dom}(g))$, without any closedness assumption concerning f or g as taken in the mentioned book.

References

1. Rockafellar, R. T., *Convex analysis*, Princeton University Press, Princeton, 1970.

2. Penot, J. P., and Volle, M., *On quasi-convex duality*, Mathematics of Operations Research, Vol. 15, No. 4, pp. 597–625, 1987.

3. Aleman, A., *On some generalizations of convex sets and convex functions*, Mathematica - Revue d'Analyse Numérique et de la Théorie de l'Approximation, Vol. 14, pp. 1–6, 1985.

4. Cobzaş, Ş., and Muntean, I., *Duality relations and characterizations of best approximation for p-convex sets*, Revue d'Analyse Numérique et de Théorie de l'Approximation, Vol. 16, No. 2, pp. 95–108, 1987.

5. Green, J. W., and Gustin, W., *Quasiconvex sets*, Canadian Journal of Mathematics, Vol. 2, pp. 489–507, 1950.

6. Boţ, R. I., Kassay, G., and Wanka, G., *Strong duality for generalized convex optimization problems*, Journal of Optimization Theory and Applications, Vol.

127, No. 1, pp. 45–70, 2005.

7. Breckner, W. W., and Kassay, G., *A systematization of convexity concepts for sets and functions*, Journal of Convex Analysis, Vol. 4, pp. 109–127, 1997.

8. Jeyakumar, V., and Gwinner, J., *Inequality systems and optimization*, Journal of Mathematical Analysis and Applications, Vol. 159, pp. 51–71, 1991.

9. Roberts, A. W., and Varberg, D. E., *Convex functions*, Pure and Applied Mathematics, Vol. 57, Academic Press, New York-London, 1973.

10. Hamel, G., *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung:* $f(x+y)=f(x)+f(y)$, Mathematische Annalen, Vol. 60, pp. 459–462, 1905.