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On the relations between different duals assigned to composed optimization problems

Gert Wanka[1], **Radu Ioan Boţ**[2], **Emese Vargyas**[3]

[1] Faculty of Mathematics, Chemnitz University of Technology, 09107 Chemnitz, Germany (e-mail: `wanka@mathematik.tu-chemnitz.de`)
[2] Faculty of Mathematics, Chemnitz University of Technology, 09107 Chemnitz, Germany (e-mail: `bot@mathematik.tu-chemnitz.de`)
[3] Institute of Mathematics and Its Didactics, University of Flensburg, 24943 Flensburg, Germany (e-mail: `vargyas@uni-flensburg.de`)

Abstract For an optimization problem with a composed objective function and composed constraint functions we determine, by means of the conjugacy approach based on the perturbation theory, some dual problems to it. The relations between the optimal objective values of these duals are studied. Moreover sufficient conditions are given in order to achieve equality between the optimal objective values of the duals and strong duality between the primal and the dual problems, respectively. Finally, some special cases of this problem are presented.

Key words Convex programming, Conjugate duality, Perturbation theory, Composed functions

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1 Introduction

Because a lot of classical optimization problems can be reformulated as composed optimization problems, the last ones represent some of the main topics in the theory of optimization and arise in many areas of science and engineering applications. For instance, when finding a feasible point of the system of inequalities $F_i(x) \leq 0,\ i = 1, ..., m$, by minimizing the norm $\|F(x)\|$, where $F = (F_1, ..., F_m)^T : \mathbb{R}^n \to \mathbb{R}^m$ is a vector function, or when solving the Weber problem with infimal distances by minimizing $\sum\limits_{i=1}^{m} w_i d(x, A_i)$, where $d(x, A_i) = \inf\limits_{a_i \in A_i} \gamma_i(x - a_i)$, $\mathcal{A} = \{A_1, ..., A_m\}$ is a family of convex sets, γ_i are the gauges of the sets A_i and w_i are positive weights, $i = 1, ..., m$. All these examples can be seen as particular cases of a composed optimization problem. Among the large number of papers dealing with composed optimization problems, let us mention here [2], [3], [4], [6], [7], [8], [9] and [14]. Most of these have considered additionally assumptions for the objective function and for the constraint functions, respectively, in order to give necessary and sufficient optimality conditions and to study the duality (see [6], [7] and [9]). The problem treated by us leads to more general models, namely where the differentiability of the functions involved does not have to be assumed.

The aim of this paper is to construct some dual problems to an optimization problem, of which objective function and constraint functions are composed functions. In order to do this we apply the Fenchel-Rockafellar duality concept based on conjugacy and perturbations (cf. [5]). Using special perturbation functions we construct three different dual problems in analogy to the well-known Lagrange and Fenchel dual problems (denoted by (D_L) and (D_F) respectively), and a "combination" of the above two that we call the Fenchel-Lagrange dual problem (denoted by (D_{FL})). Then we study the relations between the optimal objective values of the duals and give some sufficient conditions in order to achieve equality between these values and strong duality between the primal and the dual problems, respectively. Finally, some special cases of this problem are presented.

The paper is organized as follows. In section 2 we introduce some definitions and preliminary results that will be used throughout this paper. Section 3 is devoted to the construction of the three dual problems. In section 4 we examine the relations between the optimal objective values of the duals. In the first part of this section we study these relations in the general case and then under different convexity assumptions and regularity conditions. In the last part we provide some sufficient conditions in order to have strong duality. Finally, in section 5 we give some applications of the preceding results. It will turn out that the dual problems obtained by the authors in [2], [12] and [13] can be obtained from the duals introduced in this paper as special cases.

2 Preliminaries

In this section we introduce some definitions we use throughout this paper and prove some elementary results. Let X be a nonempty subset of $\mathbb{R}^m$. We denote by $ri(X)$ the relative interior of the set X, by $x^T y = \sum\limits_{i=1}^{m} x_i y_i$ the inner product of the vectors $x, y \in \mathbb{R}^m$ and by $\mathbb{R}^m_+$ the non-negative orthant of $\mathbb{R}^m$. For the function $f : \mathbb{R}^m \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, the set defined by $dom(f) = \{x \in \mathbb{R}^m : f(x) < +\infty\}$ denotes the effective domain of f. We say that f is proper if $dom(f) \neq \emptyset$ and $f(x) \neq -\infty$ for all $x \in \mathbb{R}^m$.

Definition 1 *When X is a nonempty subset of $\mathbb{R}^m$ and $f : X \to \mathbb{R}$, let be the so-called conjugate relative to the set X defined by*

$$f_X^* : \mathbb{R}^m \to \overline{\mathbb{R}},\ f_X^*(x^*) = \sup_{x \in X} \left\{ {x^*}^T x - f(x) \right\}.$$

Considering the extension of $f : X \to \mathbb{R}$ to the whole space,

$$\tilde{f} : \mathbb{R}^m \to \overline{\mathbb{R}},\ \tilde{f}(x) = \begin{cases} f(x), \text{ if } x \in X, \\ +\infty, \text{ otherwise}, \end{cases}$$

one can see that the conjugate of f relative to the set X is identical to the classical conjugate of $\tilde{f}$ (the Fenchel-Moreau conjugate)

$$\tilde{f}^* : \mathbb{R}^m \to \overline{\mathbb{R}},\ \tilde{f}^*(x^*) = \sup_{x \in \mathbb{R}^m} \{ {x^*}^T x - \tilde{f}(x) \}.$$

Definition 2 *The function $f : \mathbb{R}^m \to \mathbb{R}$ is called componentwise increasing, if for $x = (x_1, ..., x_m)^T,\ y = (y_1, ..., y_m)^T \in \mathbb{R}^m$ such that $x_i \geq y_i,\ i = 1, ..., m$, follows that $f(x) \geq f(y)$.*

Throughout this paper we consider a nonempty set $X \subseteq \mathbb{R}^n$ and the functions $F = (F_1, ..., F_m)^T : X \to \mathbb{R}^m$, $G = (G_1, ..., G_l)^T : X \to \mathbb{R}^l$, $f : \mathbb{R}^m \to \mathbb{R}$ and $g = (g_1, ..., g_k)^T : \mathbb{R}^l \to \mathbb{R}^k$. Additionally, we extend F and G to $\tilde{F} = (\tilde{F}_1, ..., \tilde{F}_m)^T$ and $\tilde{G} = (\tilde{G}_1, ..., \tilde{G}_l)^T$, respectively, with

$$\tilde{F}_i : \mathbb{R}^n \to \overline{\mathbb{R}},\ \tilde{F}_i(x) = \begin{cases} F_i(x), \text{ if } x \in X, \\ +\infty, \quad \text{otherwise}, \end{cases} \quad i = 1, ..., m,$$

and

$$\tilde{G}_j : \mathbb{R}^n \to \overline{\mathbb{R}},\ \tilde{G}_j(x) = \begin{cases} G_j(x), \text{ if } x \in X, \\ +\infty, \quad \text{otherwise}, \end{cases} \quad j = 1, ..., l.$$

As a consequence we have now to make for the functions f and $g_i,\ i = 1, ..., k$, the following conventions

$$\begin{aligned} f(y) = +\infty,\ \text{if } y = (y_1, ..., y_m)^T \text{ with } y_i \in \mathbb{R} \cup \{+\infty\}, \\ i = 1, ..., m,\ \text{and } \exists\ j \in \{1, ..., m\} \text{ such that } y_j = +\infty, \end{aligned} \tag{1}$$

and, for $i = 1, ..., k$,

$$\begin{aligned} g_i(z) = +\infty,\ \text{if } z = (z_1, ..., z_l)^T \text{ with } z_i \in \mathbb{R} \cup \{+\infty\}, \\ i = 1, ..., l,\ \text{and } \exists\ j \in \{1, ..., l\} \text{ such that } z_j = +\infty. \end{aligned} \tag{2}$$

Proposition 1 *If $f : \mathbb{R}^m \to \mathbb{R}$ is a componentwise increasing function, then $f^*(q) = +\infty$ for all $q \in \mathbb{R}^m \setminus \mathbb{R}^m_+$.*

Proof Let be $q \in \mathbb{R}^m \setminus \mathbb{R}^m_+$. Then there exists at least one $i \in \{1, ..., m\}$ such that $q_i < 0$. But

$$\begin{aligned} f^*(q) &= \sup_{d \in \mathbb{R}^m} \left\{ q^T d - f(d) \right\} \geq \sup_{\substack{d = (0, ..., d_i, ..., 0)^T, \\ d_i \in \mathbb{R}}} \left\{ q_i d_i - f(0, ..., d_i, ..., 0) \right\} \\ &= \sup_{d_i \in \mathbb{R}} \left\{ q_i d_i - f(0, ..., d_i, ..., 0) \right\} \geq \sup_{d_i < 0} \left\{ q_i d_i - f(0, ..., d_i, ...0) \right\} \\ &\geq \sup_{d_i < 0} \left\{ q_i d_i \right\} - f(0, ..., 0) = +\infty. \end{aligned}$$

Therefore $f^*(q) = +\infty,\ \forall q \in \mathbb{R}^m \setminus \mathbb{R}^m_+$. $\square$

Proposition 2 *Assume that X is a nonempty convex subset of $\mathbb{R}^n$, $F_i : X \to \mathbb{R}$, $i = 1, ..., m$, are convex functions and $f : \mathbb{R}^m \to \mathbb{R}$ is a convex and componentwise increasing function. Then $f \circ \tilde{F} : \mathbb{R}^n \to \overline{\mathbb{R}}$ is convex.*

Proof We have to prove that for all $x, y \in \mathbb{R}^n$ and for all λ with $0 \leq \lambda \leq 1$,

$$\left(f \circ \tilde{F}\right)(\lambda x + (1-\lambda)y) \leq \lambda \left(f \circ \tilde{F}\right)(x) + (1-\lambda)\left(f \circ \tilde{F}\right)(y). \tag{3}$$

If $x,\ y \in X$ we have that

$$\begin{aligned}\left(f \circ \tilde{F}\right)\big(\lambda x + (1-\lambda)y\big) &= f\big(F\big(\lambda x + (1-\lambda)y\big)\big) \leq f\big(\lambda F(x) + (1-\lambda)F(y)\big) \\ &\leq \lambda\big(f \circ F\big)(x) + (1-\lambda)\big(f \circ F\big)(y) \\ &= \lambda\left(f \circ \tilde{F}\right)(x) + (1-\lambda)\left(f \circ \tilde{F}\right)(y).\end{aligned}$$

If $x \notin X$ or $y \notin X$, or both, we have $\left(f \circ \tilde{F}\right)(x) = +\infty$ or $\left(f \circ \tilde{F}\right)(y) = +\infty$, or both, respectively. So, the inequality (3) holds again. □

Proposition 3 *Assume that X is a nonempty convex subset of $\mathbb{R}^n$, $G_j : X \to \mathbb{R}$, $j = 1, ..., l$, are convex functions and $g_i : \mathbb{R}^l \to \mathbb{R}$, $i = 1, ..., k$, are convex and componentwise increasing functions. Then $g_i \circ \tilde{G} : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i = 1, ..., k$, are convex.*

Proof The proof is analogous to the proof of Proposition 2 . □

In the last part of this section we present some results which will play an important role in the sequel.

Theorem 1 *(cf. Theorem 16.4 in [11]) Let $f_1, ..., f_n : \mathbb{R}^m \to \overline{\mathbb{R}}$ be proper convex functions. If the sets $ri(dom(f_i))$, $i = 1, ..., n$, have a point in common, then*

$$\left(\sum_{i=1}^{n} f_i\right)^*(p) = \inf\left\{\sum_{i=1}^{n} f_i^*(p_i) : \sum_{i=1}^{n} p_i = p\right\},$$

where for each $p \in \mathbb{R}^m$ the infimum is attained.

Theorem 2 *(cf. [10]) Let $F = (F_1, ..., F_m)^T$ with $F_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, ..., m$, be convex functions and $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a convex and componentwise increasing function. If the set $F\left(\bigcap\limits_{i=1}^{m} dom(F_i)\right)$ contains an interior point of $dom(f)$, then it holds*

$$(f \circ F)^*(p) = \inf_{\lambda \in \mathbb{R}^m_+}\left\{f^*(\lambda) + \left(\sum_{i=1}^{m} \lambda_i F_i\right)^*(p)\right\},$$

where for each $p \in \mathbb{R}^n$ the infimum is attained.

3 The composed optimization problem and its duals

In this section we introduce the primal composed optimization problem and construct different dual problems to it.

Let this primal problem be

$$(P) \qquad \inf_{x \in \mathcal{A}} f(F(x)),$$

where

$$\mathcal{A} = \left\{ x \in X : g(G(x)) \underset{\mathbb{R}^k_+}{\leqq} 0 \right\}.$$

As usual, $g(G(x)) \underset{\mathbb{R}^k_+}{\leqq} 0$ means that $g_i(G(x)) \leq 0$ for all $i = 1, ..., k$. In the following we suppose that the feasible set $\mathcal{A}$ is nonempty. The optimal objective value of (P) is denoted by $v(P)$.

The aim of this section is to construct different dual problems to (P). To do this, we use an approach based on the theory of conjugate functions (see [5]). In order to reproduce it, let us consider first a general optimization problem without constraints

$$(PG) \qquad \inf_{x \in \mathbb{R}^n} k(x),$$

with k a mapping from $\mathbb{R}^n$ into $\overline{\mathbb{R}}$. We embed this problem in a family of perturbed problems

$$(PG_p) \qquad \inf_{x \in \mathbb{R}^n} \Phi(x, p),$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^s \to \overline{\mathbb{R}}$ is the so-called perturbation function and has the property that

$$\Phi(x, 0) = k(x), \ \forall x \in \mathbb{R}^n. \tag{4}$$

Here, $\mathbb{R}^s$ is the space of the perturbation variables. The conjugate function of the perturbation function Φ is

$$\Phi^*(x^*, p^*) = \sup_{x \in \mathbb{R}^n,\, p \in \mathbb{R}^s} \left\{ {x^*}^T x + {p^*}^T p - \Phi(x, p) \right\}.$$

Then the problem

$$(DG) \qquad \sup_{p^* \in \mathbb{R}^s} \{ -\Phi^*(0, p^*) \}$$

defines a dual problem of (PG) and its optimal objective value is denoted by $v(DG)$. This approach has the important property that between the primal and the dual problem weak duality holds. The next proposition states this fact.

Proposition 4 *(cf. Proposition 1.1 in [5]) The relation*

$$-\infty \leq v(DG) \leq v(PG) \leq +\infty$$

always holds.

In order to apply the approach described above to the composed optimization problem we introduce the function $k : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$k(x) = \begin{cases} f(\tilde{F}(x)), \text{ if } g(\tilde{G}(x)) \underset{\mathbb{R}^k_+}{\leqq} 0, \\ +\infty, \qquad \text{otherwise,} \end{cases}$$

and thus (P) can be written as an optimization problem without constraints

$$(P) \qquad \inf_{x \in \mathbb{R}^n} k(x).$$

In the following we construct three different perturbation functions and the corresponding dual problems to (P).

3.1 The Lagrange dual problem

At first let us consider the perturbation function $\Phi_L : \mathbb{R}^n \times \mathbb{R}^k \to \overline{\mathbb{R}}$ defined by

$$\Phi_L(x, q) = \begin{cases} f(\tilde{F}(x)), \text{ if } g(\tilde{G}(x)) \underset{\mathbb{R}^k_+}{\leqq} q, \\ +\infty, \qquad \text{otherwise,} \end{cases}$$

with the perturbation variable $q \in \mathbb{R}^k$. It is obvious that relation (4) is fulfilled. For the conjugate of Φ_L we have

$$\begin{aligned} \Phi_L^*(x^*, q^*) &= \sup_{x \in \mathbb{R}^n,\, q \in \mathbb{R}^k} \left\{ {x^*}^T x + {q^*}^T q - \Phi_L(x, q) \right\} \\ &= \sup_{\substack{x \in \mathbb{R}^n,\, q \in \mathbb{R}^k, \\ g(\tilde{G}(x)) \underset{\mathbb{R}^k_+}{\leqq} q}} \left\{ {x^*}^T x + {q^*}^T q - f(\tilde{F}(x)) \right\} \\ &= \sup_{\substack{x \in X,\, q \in \mathbb{R}^k, \\ g(G(x)) \underset{\mathbb{R}^k_+}{\leqq} q}} \left\{ {x^*}^T x + {q^*}^T q - f(F(x)) \right\}. \end{aligned}$$

In order to calculate this expression we introduce the variable a instead of q, by $a := q - g(G(x)) \in \mathbb{R}^k_+$. This implies

$$\begin{aligned} \Phi_L^*(x^*, q^*) &= \sup_{x \in X,\, a \in \mathbb{R}^k_+} \left\{ {x^*}^T x + {q^*}^T g(G(x)) + {q^*}^T a - f(F(x)) \right\} \\ &= \sup_{x \in X} \left\{ {x^*}^T x + {q^*}^T g(G(x)) - f(F(x)) \right\} + \sup_{a \in \mathbb{R}^k_+} \left\{ {q^*}^T a \right\} \\ &= \begin{cases} \sup\limits_{x \in X} \left\{ {x^*}^T x + {q^*}^T g(G(x)) - f(F(x)) \right\}, \text{ if } q^* \in -\mathbb{R}^k_+, \\ +\infty, \qquad \text{otherwise.} \end{cases} \end{aligned}$$

The dual of (P) obtained by the perturbation function Φ_L is

$$(D_L) \qquad \sup_{q^* \in \mathbb{R}^k} \{-\Phi_L^*(0, q^*)\}.$$

Because of

$$\sup_{q^* \in -\mathbb{R}^k_+} \Big\{ -\sup_{x \in X} \Big\{ {q^*}^T g(G(x)) - f(F(x)) \Big\} \Big\} =$$

$$\sup_{q^* \in -\mathbb{R}^k_+} \inf_{x \in X} \Big\{ -{q^*}^T g(G(x)) + f(F(x)) \Big\}$$

and denoting $t := -q^* \in \mathbb{R}^k_+$, the dual becomes

$$(D_L) \qquad \sup_{t \in \mathbb{R}^k_+} \inf_{x \in X} \{ f(F(x)) + t^T g(G(x)) \}.$$

The problem (D_L) is actually the well-known *Lagrange dual problem.* Its optimal objective value is denoted by $v(D_L)$ and Proposition 4 implies $v(D_L) \leq v(P)$.

3.2 The Fenchel dual problem

Let us consider the perturbation function $\Phi_F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ given by

$$\Phi_F(x, p, q) = \begin{cases} f(\tilde{F}(x+p) + q), \text{ if } g(\tilde{G}(x)) \underset{\mathbb{R}^k_+}{\leqq} 0, \\ +\infty, \qquad\qquad\qquad \text{otherwise,} \end{cases}$$

with the perturbation variables $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$. The relation (4) is also fulfilled and it holds

$$\begin{aligned} \Phi_F^*(x^*, p^*, q^*) &= \sup_{x,\, p \in \mathbb{R}^n,\, q \in \mathbb{R}^m} \Big\{ {x^*}^T x + {p^*}^T p + {q^*}^T q - \Phi_F(x, p, q) \Big\} \\ &= \sup_{\substack{x,\, p \in \mathbb{R}^n,\, q \in \mathbb{R}^m, \\ g(\tilde{G}(x)) \underset{\mathbb{R}^k_+}{\leqq} 0}} \Big\{ {x^*}^T x + {p^*}^T p + {q^*}^T q - f(\tilde{F}(x+p) + q) \Big\}. \end{aligned}$$

Introducing the new variables $r := x + p \in \mathbb{R}^n$ and $a := \tilde{F}(x+p) + q \in \mathbb{R}^m$, we obtain

$$\Phi_F^*(x^*, p^*, q^*) = \sup_{\substack{x \in X,\, r \in \mathbb{R}^n,\, a \in \mathbb{R}^m, \\ g(G(x)) \underset{\mathbb{R}^k_+}{\leqq} 0}} \Big\{ {x^*}^T x + {p^*}^T r - {p^*}^T x + {q^*}^T a - {q^*}^T \tilde{F}(r) -$$

$$f(a) \Big\} = \sup_{a \in \mathbb{R}^m} \Big\{ {q^*}^T a - f(a) \Big\} + \sup_{r \in \mathbb{R}^n} \Big\{ {p^*}^T r - {q^*}^T \tilde{F}(r) \Big\} +$$

$$\sup_{x\in\mathcal{A}}\left\{(x^*-p^*)^Tx\right\} = f^*(q^*) + \left(\sum_{i=1}^m q_i^*\tilde{F}_i\right)^* (p^*) + \sup_{x\in\mathcal{A}}\left\{(x^*-p^*)^Tx\right\}.$$

Because of $\left(\sum\limits_{i=1}^m q_i^*\tilde{F}_i\right)^*(p^*) = \left(\sum\limits_{i=1}^m q_i^*F_i\right)_X^*(p^*)$ and, denoting $p := p^*,\ q := q^*$, the dual problem of (P)

$$(D_F) \qquad \sup_{p^*\in\mathbb{R}^n,\, q^*\in\mathbb{R}^m} \{-\Phi_F^*(0,p^*,q^*)\}$$

can be written as

$$(D_F) \qquad \sup_{p\in\mathbb{R}^n,\, q\in\mathbb{R}^m} \left\{-f^*(q) - \left(\sum_{i=1}^m q_iF_i\right)_X^* (p) + \inf_{x\in\mathcal{A}} p^Tx\right\}.$$

Let us call (D_F) the *Fenchel dual problem* and denote its optimal objective value by $v(D_F)$. Proposition 4 implies that $v(D_F) \le v(P)$.

3.3 The Fenchel-Lagrange dual problem

A further dual problem can be obtained by considering a perturbation function which is a combination of the functions Φ_L and Φ_F. Let this be defined by $\Phi_{FL} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^k \to \overline{\mathbb{R}}$,

$$\Phi_{FL}(x,p,q,p',q',t) = \begin{cases} f(\tilde{F}(x+p)+q), \text{ if } g(\tilde{G}(x+p')+q') \underset{\mathbb{R}^k_+}{\leqq} t, \\ +\infty, \qquad\qquad \text{otherwise,} \end{cases}$$

with the perturbation variables $p,\ p' \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q' \in \mathbb{R}^l$ and $t \in \mathbb{R}^k$. Φ_{FL} satisfies relation (4) and thus a dual problem to (P) can be defined by

$$(D_{FL}) \qquad \sup_{\substack{p^*,\,p'^*\in\mathbb{R}^n,\, q^*\in\mathbb{R}^m,\\ q'^*\in\mathbb{R}^l,\, t\in\mathbb{R}^k}} \{-\Phi_{FL}^*(0,p^*,q^*,p'^*,q'^*,t^*)\}.$$

For the conjugate of Φ_{FL} we have

$$\Phi_{FL}^*(x^*,p^*,q^*,p'^*,q'^*,t^*) = \sup_{\substack{x,\,p,\,p'\in\mathbb{R}^n,\, q\in\mathbb{R}^m,\\ q'\in\mathbb{R}^l,\, t\in\mathbb{R}^k}} \Big\{{x^*}^Tx + {p^*}^Tp + {q^*}^Tq + {p'^*}^Tp' +$$
$${q'^*}^Tq' + {t^*}^Tt - \Phi_{FL}(x,p,q,p',q',t)\Big\} = \sup_{\substack{x,\,p,\,p'\in\mathbb{R}^n,\, q\in\mathbb{R}^m,\, q'\in\mathbb{R}^l,\\ t\in\mathbb{R}^k,\, g(\tilde{G}(x+p')+q') \underset{\mathbb{R}^k_+}{\leqq} t}} \Big\{{x^*}^Tx + {p^*}^Tp +$$
$${q^*}^Tq + {p'^*}^Tp' + {q'^*}^Tq' + {t^*}^Tt - f(\tilde{F}(x+p)+q)\Big\}.$$

Introducing the new variables $r := x+p \in \mathbb{R}^n$, $r' := x+p' \in \mathbb{R}^n$, $a := \tilde{F}(x+p)+q \in \mathbb{R}^m$, $b := \tilde{G}(x+p')+q' \in \mathbb{R}^l$ and $c := t - g(\tilde{G}(x+p')+q') \in \mathbb{R}^k_+$, we have

$$\Phi^*_{FL}(x^*,p^*,q^*,p'^*,q'^*,t^*) = \sup_{\substack{x,\,r,\,r'\in\mathbb{R}^n,\,a\in\mathbb{R}^m,\\ b\in\mathbb{R}^l,\,c\in\mathbb{R}^k_+}} \Big\{ {x^*}^T x + {p^*}^T r - {p^*}^T x + {q^*}^T a -$$

$${q^*}^T \tilde{F}(r) + {p'^*}^T r' - {p'^*}^T x + {q'^*}^T b - {q'^*}^T \tilde{G}(r') + {t^*}^T c + {t^*}^T g(b) - f(a) \Big\} =$$

$$\sup_{a\in\mathbb{R}^m} \Big\{ {q^*}^T a - f(a) \Big\} + \sup_{b\in\mathbb{R}^l} \Big\{ {q'^*}^T b + {t^*}^T g(b) \Big\} + \sup_{r\in\mathbb{R}^n} \Big\{ {p^*}^T r - {q^*}^T \tilde{F}(r) \Big\}$$

$$+ \sup_{r'\in\mathbb{R}^n} \Big\{ {p'^*}^T r' - {q'^*}^T \tilde{G}(r') \Big\} + \sup_{x\in\mathbb{R}^n} \Big\{ (x^* - p^* - p'^*)^T x \Big\} + \sup_{c\in\mathbb{R}^k_+} \Big\{ {t^*}^T c \Big\}.$$

Because of

$$\sup_{x\in\mathbb{R}^n} \big\{ -(p^* + p'^*)^T x \big\} = \begin{cases} 0, & \text{if } p^* + p'^* = 0, \\ +\infty, & \text{otherwise}, \end{cases}$$

and

$$\sup_{c\in\mathbb{R}^k_+} \Big\{ {t^*}^T c \Big\} = \begin{cases} 0, & \text{if } t^* \in -\mathbb{R}^k_+, \\ +\infty, & \text{otherwise}, \end{cases}$$

follows that

$$\Phi^*_{FL}(0,p^*,q^*,p'^*,q'^*,t^*) =$$

$$\begin{cases} f^*(q^*) + \left(-\sum\limits_{i=1}^k t_i^* g_i \right)^* (q'^*) + \left(\sum\limits_{i=1}^m q_i^* \tilde{F}_i \right)^* (p^*) + \left(\sum\limits_{i=1}^l q_i'^* \tilde{G}_i \right)^* (p'^*), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } p^* + p'^* = 0 \text{ and } t^* \in -\mathbb{R}^k_+, \\ +\infty, \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise}. \end{cases}$$

Taking into consideration that $\left(\sum\limits_{i=1}^m q_i^* \tilde{F}_i \right)^* (p^*) = \left(\sum\limits_{i=1}^m q_i^* F_i \right)^*_X (p^*)$, $\left(\sum\limits_{i=1}^l q_i'^* \tilde{G}_i \right)^* (p'^*) = \left(\sum\limits_{i=1}^l q_i'^* G_i \right)^*_X (p'^*)$ and denoting $p := p^* = -p'^*$, $q := q^*$, $q' := q'^*$ and $t := -t^*$, the dual becomes

(D_{FL})

$$\sup_{\substack{p\in\mathbb{R}^n,\,q\in\mathbb{R}^m,\\ q'\in\mathbb{R}^l,\,t\in\mathbb{R}^k_+}} \left\{ -f^*(q) - \left(\sum_{i=1}^k t_i g_i \right)^* (q') - \left(\sum_{i=1}^m q_i F_i \right)^*_X (p) - \left(\sum_{i=1}^l q_i' G_i \right)^*_X (-p) \right\}.$$

Let us call (D_{FL}) the *Fenchel-Lagrange dual problem.* By Proposition 4, the weak duality $v(D_{FL}) \leq v(P)$ is also true, where $v(D_{FL})$ is the optimal objective value of (D_{FL}).

4 The relations between the optimal objective values of the dual problems

In the previous section we have seen that the optimal objective values $v(D_L)$, $v(D_F)$ and $v(D_{FL})$ of the dual problems (D_L), (D_F) and (D_{FL}), respectively, are less than or equal to the optimal objective value $v(P)$ of the primal problem (P). Henceforth we are going to investigate the relations between the optimal objective values of the three dual problems.

4.1 The general case

For the beginning we remain in the most general case, namely, without any special assumptions concerning the set X or the functions f, F, g and G.

Proposition 5 *The inequality $v(D_{FL}) \leq v(D_L)$ holds.*

Proof Let $p \in \mathbb{R}^n$, $q \in \mathbb{R}^m$, $q' \in \mathbb{R}^l$ and $t \in \mathbb{R}^k_+$ be fixed. By the definition of the conjugate function we have

$$-f^*(q) = \inf_{y \in \mathbb{R}^m} \left\{ f(y) - q^T y \right\} \leq \inf_{x \in X} \left\{ f(F(x)) - q^T F(x) \right\},$$

$$-\left(\sum_{i=1}^k t_i g_i \right)^* (q') = \inf_{z \in \mathbb{R}^l} \left\{ \left(\sum_{i=1}^k t_i g_i \right)(z) - q'^T z \right\}$$

$$\leq \inf_{x \in X} \left\{ \left(\sum_{i=1}^k t_i g_i \right)(G(x)) - q'^T G(x) \right\},$$

$$-\left(\sum_{i=1}^m q_i F_i \right)^*_X (p) = \inf_{x \in X} \left\{ q^T F(x) - p^T x \right\}$$

and

$$-\left(\sum_{i=1}^l q'_i G_i \right)^*_X (-p) = \inf_{x \in X} \left\{ q'^T G(x) + p^T x \right\}.$$

Adding the inequalities from above we obtain that

$$-f^*(q) - \left(\sum_{i=1}^k t_i g_i \right)^* (q') - \left(\sum_{i=1}^m q_i F_i \right)^*_X (p) - \left(\sum_{i=1}^l q'_i G_i \right)^*_X (-p) \leq$$

$$\inf_{x \in X} \left\{ f(F(x)) + \left(\sum_{i=1}^k t_i g_i \right)(G(x)) \right\} = \inf_{x \in X} \left\{ f(F(x)) + t^T g(G(x)) \right\}.$$

By taking now the supremum over $p \in \mathbb{R}^n$, $q \in \mathbb{R}^m$, $q' \in \mathbb{R}^l$ and $t \in \mathbb{R}^k_+$, we obtain

$$\sup_{\substack{p \in \mathbb{R}^n, q \in \mathbb{R}^m, \\ q' \in \mathbb{R}^l, t \in \mathbb{R}^k_+}} \left\{ -f^*(q) - \left(\sum_{i=1}^k t_i g_i \right)^* (q') - \left(\sum_{i=1}^m q_i F_i \right)^*_X (p) - \left(\sum_{i=1}^l q'_i G_i \right)^*_X (-p) \right\}$$

$$\leq \sup_{t\in\mathbb{R}^k_+} \inf_{x\in X} \left\{ f(F(x)) + t^T g(G(x)) \right\}.$$

This implies that $v(D_{FL}) \leq v(D_L)$. □

Proposition 6 *The inequality $v(D_{FL}) \leq v(D_F)$ holds.*

Proof Let $p \in \mathbb{R}^n,\ q \in \mathbb{R}^m$ and $q' \in \mathbb{R}^l$ be fixed. For each $t \in \mathbb{R}^k_+$ we have

$$-\left(\sum_{i=1}^k t_i g_i\right)^*(q') - \left(\sum_{i=1}^l q'_i G_i\right)^*_X(-p) = -\sup_{z\in\mathbb{R}^l}\left\{ q'^T z - \left(\sum_{i=1}^k t_i g_i\right)(z)\right\}$$

$$-\sup_{x\in X}\left\{-p^T x - \left(\sum_{i=1}^l q'_i G_i\right)(x)\right\} \leq \inf_{x\in X}\left\{\left(\sum_{i=1}^k t_i g_i\right)(G(x)) - q'^T G(x)\right\}$$

$$+\inf_{x\in X}\left\{\left(\sum_{i=1}^l q'_i G_i\right)(x) + p^T x\right\} \leq \inf_{x\in X}\left\{t^T g(G(x)) + p^T x\right\}$$

$$\leq \inf_{x\in\mathcal{A}}\left\{t^T g(G(x)) + p^T x\right\} \leq \inf_{x\in\mathcal{A}}\left\{p^T x\right\}. \tag{5}$$

The last two inequalities in (5) hold because $\mathcal{A} \subseteq X$ and $t^T g(G(x)) \leq 0$ for all $x \in \mathcal{A}$. By adding first $-f^*(q) + \left(\sum\limits_{i=1}^m q_i F_i\right)^*_X (p)$ to both sides of (5) and by taking then the supremum over $p \in \mathbb{R}^n,\ q \in \mathbb{R}^m,\ q' \in \mathbb{R}^l$ and $t \in \mathbb{R}^k_+$, we obtain

$$\sup_{\substack{p\in\mathbb{R}^n,\, q\in\mathbb{R}^m,\\ q'\in\mathbb{R}^l,\, t\in\mathbb{R}^k_+}}\left\{-f^*(q) - \left(\sum_{i=1}^k t_i g_i\right)^*(q') - \left(\sum_{i=1}^m q_i F_i\right)^*_X(p) - \left(\sum_{i=1}^l q'_i G_i\right)^*_X(-p)\right\}$$
$$\leq \sup_{p\in\mathbb{R}^n,\, q\in\mathbb{R}^m}\left\{-f^*(q) - \left(\sum_{i=1}^m q_i F_i\right)^*_X(p) + \inf_{x\in\mathcal{A}} p^T x\right\},$$

which is nothing else than $v(D_{FL}) \leq v(D_F)$. □

Remark 1 Considering similar counterexamples like Wanka and Boţ in [12], it can be shown that the inequalities in Proposition 5 and Proposition 6 can also be strict. Moreover, in general, an ordering between $v(D_L)$ and $v(D_F)$ cannot be established.

4.2 The equivalence of the dual problems (D_L) and (D_{FL})

In this subsection we assume that X is a convex subset, $F_i : X \to \mathbb{R},\ i = 1, ..., m,\ G_j : X \to \mathbb{R},\ j = 1, ..., l$, are convex functions and $f : \mathbb{R}^m \to \mathbb{R},\ g_i : \mathbb{R}^l \to \mathbb{R},\ i = 1, ..., k$, are convex and componentwise increasing functions. Under these hypotheses we prove that the optimal objective values of the

Lagrange and the Fenchel-Lagrange dual problems are equal. According to Proposition 1, the dual (D_{FL}) becomes

$$(D_{FL}) \quad \sup_{\substack{p\in\mathbb{R}^n,\, q\in\mathbb{R}^m_+,\\ q'\in\mathbb{R}^l_+,\, t\in\mathbb{R}^k_+}} \left\{ -f^*(q) - \left(\sum_{i=1}^k t_i g_i\right)^*(q') - \left(\sum_{i=1}^m q_i F_i\right)^*_X(p) - \left(\sum_{i=1}^l q'_i G_i\right)^*_X(-p)\right\}.$$

Theorem 3 *Assume that $X \subseteq \mathbb{R}^n$ is a nonempty convex subset, $F_i : X \to \mathbb{R}$, $i = 1, ..., m$, $G_j : X \to \mathbb{R}$, $j = 1, ..., l$, are convex functions and $f : \mathbb{R}^m \to \mathbb{R}$, $g_i : \mathbb{R}^l \to \mathbb{R}$, $i = 1, ..., k$, are convex and componentwise increasing functions. Then it holds*

$$v(D_L) = v(D_{FL}).$$

Proof Let be $t \in \mathbb{R}^k_+$. By using the extended functions introduced in section 2, the Lagrange dual can be written as

$$\inf_{x\in X} \left\{f(F(x)) + t^T g(G(x))\right\} = \inf_{x\in\mathbb{R}^n} \left\{ \left(f\circ\tilde{F}\right)(x) + t^T\left(g\circ\tilde{G}\right)(x)\right\} =$$

$$\inf_{x\in\mathbb{R}^n} \left\{ \left(f\circ\tilde{F}\right)(x) + \left(\sum_{i=1}^k t_i g_i\circ\tilde{G}\right)(x)\right\} = -\left(f\circ\tilde{F} + \sum_{i=1}^k t_i g_i\circ\tilde{G}\right)^*(0).$$

Because of $ri\left(dom\left(f\circ\tilde{F}\right)\right)\bigcap ri\left(dom\left(\sum_{i=1}^k t_i g_i\circ\tilde{G}\right)\right) = ri(X) \neq \emptyset$ and $f\circ\tilde{F}$, $\sum_{i=1}^k t_i g_i\circ\tilde{G}$ are convex functions (cf. Proposition 2 and Proposition 3), Theorem 1 implies that

$$-\left(f\circ\tilde{F} + \sum_{i=1}^k t_i g_i\circ\tilde{G}\right)^*(0) = -\inf_{p\in\mathbb{R}^n}\left\{\left(f\circ\tilde{F}\right)^*(p) + \left(\sum_{i=1}^k t_i g_i\circ\tilde{G}\right)^*(-p)\right\}. \tag{6}$$

Since we have that $\tilde{F}\left(\bigcap_{i=1}^m dom\left(\tilde{F}_i\right)\right)\bigcap int\big(dom(f)\big) = F(X)\cap\mathbb{R}^m \neq \emptyset$ and $\tilde{G}\left(\bigcap_{j=1}^l dom\left(\tilde{G}_j\right)\right)\bigcap int\left(dom\left(\sum_{i=1}^k t_i g_i\right)\right) = G(X)\cap\mathbb{R}^l \neq \emptyset$, by Theorem 2 follows

$$\left(f\circ\tilde{F}\right)^*(p) = \inf_{q\in\mathbb{R}^m_+}\left\{f^*(q) + \left(\sum_{i=1}^m q_i\tilde{F}_i\right)^*(p)\right\} \tag{7}$$

and

$$\left(\sum_{i=1}^k t_i g_i\circ\tilde{G}\right)^*(-p) = \inf_{q'\in\mathbb{R}^l_+}\left\{\left(\sum_{i=1}^k t_i g_i\right)^*(q') + \left(\sum_{i=1}^k q'_i\tilde{G}_i\right)^*(-p)\right\}. \tag{8}$$

Finally, the relations (6), (7) and (8) give

$$
\begin{aligned}
&\inf_{x\in X}\left\{f(F(x))+t^T g(G(x))\right\} = \\
&-\inf_{\substack{p\in\mathbb{R}^n, q\in\mathbb{R}^m_+,\\ q'\in\mathbb{R}^l_+}}\left\{f^*(q)+\left(\sum_{i=1}^m q_i\tilde{F}_i\right)^*(p)+\left(\sum_{i=1}^k t_i g_i\right)^*(q')+\left(\sum_{i=1}^k q'_i\tilde{G}_i\right)^*(-p)\right\} = \\
&\sup_{\substack{p\in\mathbb{R}^n, q\in\mathbb{R}^m_+,\\ q'\in\mathbb{R}^l_+}}\left\{-f^*(q)-\left(\sum_{i=1}^k t_i g_i\right)^*(q')-\left(\sum_{i=1}^m q_i\tilde{F}_i\right)^*(p)-\left(\sum_{i=1}^k q'_i\tilde{G}_i\right)^*(-p)\right\} = \\
&\sup_{\substack{p\in\mathbb{R}^n, q\in\mathbb{R}^m_+,\\ q'\in\mathbb{R}^l_+}}\left\{-f^*(q)-\left(\sum_{i=1}^k t_i g_i\right)^*(q')-\left(\sum_{i=1}^m q_i F_i\right)^*_X(p)-\left(\sum_{i=1}^k q'_i G_i\right)^*_X(-p)\right\},
\end{aligned}
$$

which implies that $v(D_L)=v(D_{FL})$. □

4.3 The equivalence of the dual problems (D_F) *and* (D_{FL})

The aim of this section is to investigate some necessary conditions in order to ensure the equality between the optimal objective values of the duals (D_F) and (D_{FL}).

To this end we consider a generalized Slater-type constraint qualification. First, let us divide the index set $\{1,...,k\}$ into two subsets,

$$
L := \left\{ i\in\{1,...,k\} \,\middle|\, \begin{array}{l} g_i\circ G : X\to\mathbb{R} \text{ is the restriction to } X \text{ of an} \\ \text{affine function } \widetilde{g_i\circ G} : \mathbb{R}^n\to\mathbb{R} \end{array} \right\}
$$

and $N := \{1,...,k\}\setminus L$. The constraint qualification follows

$$
(CQ) \qquad \exists\, x'\in\ ri(X) : \begin{cases} g_i(G(x'))\leq 0,\ i\in L, \\ g_i(G(x'))<0,\ i\in N, \end{cases}
$$

where $ri(X)$ denotes the relative interior of the set X.

Assume that the constraint qualification (CQ) is fulfilled and, moreover, that X is a convex set, $G_j : X\to\mathbb{R},\ j=1,...,l$, are convex functions and that $g_i : \mathbb{R}^l\to\mathbb{R},\ i=1,...,k$, are convex and componentwise increasing functions. This will imply the equality of the optimal objective values of (D_F) and (D_{FL}). Let us mention that under these hypotheses (D_{FL}) becomes (cf. Proposition 1)

$$
(D_{FL}) \quad \sup_{\substack{p\in\mathbb{R}^n,\, q\in\mathbb{R}^m,\\ q'\in\mathbb{R}^l_+,\, t\in\mathbb{R}^k_+}}\left\{-f^*(q)-\left(\sum_{i=1}^k t_i g_i\right)^*(q')-\left(\sum_{i=1}^m q_i F_i\right)^*_X(p)-\left(\sum_{i=1}^l q'_i G_i\right)^*_X(-p)\right\}.
$$

Theorem 4 *Assume that $X \subseteq \mathbb{R}^n$ is a nonempty convex subset, $G_j : X \to \mathbb{R},\ j = 1, ..., l$, are convex functions, $g_i : \mathbb{R}^l \to \mathbb{R},\ i = 1, ..., k$, are convex and componentwise increasing functions and the constraint qualification (CQ) is fulfilled. Then it holds*

$$v(D_F) = v(D_{FL}).$$

Proof Let be $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$ fixed. If $\inf\limits_{x\in\mathcal{A}} p^T x = -\infty$, then by Proposition 6, $v(D_F) = v(D_{FL}) = -\infty$.
Further let $\inf\limits_{x\in\mathcal{A}} p^T x$ be finite. By Theorem 28.2 in [11] one has

$$\inf_{x\in\mathcal{A}} p^T x = \sup_{t\in\mathbb{R}^k_+} \inf_{x\in X} \left\{ p^T x + t^T g(G(x)) \right\}.$$

Applying again Theorem 1 it follows that

$$\inf_{x\in X} \left\{ p^T x + \left(\sum_{i=1}^k t_i g_i \circ G \right)(x) \right\} = \inf_{x\in\mathbb{R}^n} \left\{ p^T x + \left(\sum_{i=1}^k t_i g_i \circ \tilde{G} \right)(x) \right\} =$$

$$-\left(p^T(\cdot) + \sum_{i=1}^k t_i g_i \circ \tilde{G} \right)^* (0) = -\inf_{u\in\mathbb{R}^n} \left\{ (p^T(\cdot))^*(u) + \left(\sum_{i=1}^k t_i g_i \circ \tilde{G} \right)^* (-u) \right\}.$$

On the other hand, Theorem 2 yields

$$\left(\sum_{i=1}^k t_i g_i \circ \tilde{G} \right)^* (-u) = \inf_{q'\in\mathbb{R}^l_+} \left\{ \left(\sum_{i=1}^k t_i g_i \right)^* (q') + \left(\sum_{i=1}^l q'_i \tilde{G}_i \right)^* (-u) \right\},$$

therefore

$$\inf_{x\in\mathcal{A}} p^T x = \sup_{\substack{u\in\mathbb{R}^n,\, q'\in\mathbb{R}^l_+,\\ t\in\mathbb{R}^k_+}} \left\{ -(p^T(\cdot))^*(u) - \left(\sum_{i=1}^k t_i g_i \right)^* (q') - \left(\sum_{i=1}^l q'_i \tilde{G}_i \right)^* (-u) \right\}.$$

Since

$$(p^T(\cdot))^*(u) = \begin{cases} 0, & \text{if } u = p, \\ +\infty, & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} \inf_{x\in\mathcal{A}} p^T x &= \sup_{q'\in\mathbb{R}^l_+,\, t\in\mathbb{R}^k_+} \left\{ -\left(\sum_{i=1}^k t_i g_i \right)^* (q') - \left(\sum_{i=1}^l q'_i \tilde{G}_i \right)^* (-p) \right\} \\ &= \sup_{q'\in\mathbb{R}^l_+,\, t\in\mathbb{R}^k_+} \left\{ -\left(\sum_{i=1}^k t_i g_i \right)^* (q') - \left(\sum_{i=1}^l q'_i G_i \right)^*_X (-p) \right\}. \end{aligned} \tag{9}$$

By adding $-f^*(q) - \left(\sum\limits_{i=1}^m q_i F_i \right)^*_X (p)$ to both sides of relation (9) and, by taking the supremum over $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$, one obtains

$$\sup_{p\in\mathbb{R}^n,\, q\in\mathbb{R}^m} \left\{ -f^*(q) - \left(\sum_{i=1}^m q_i F_i \right)^*_X (p) + \inf_{x\in\mathcal{A}} p^T x \right\} =$$

$$\sup_{\substack{p\in\mathbb{R}^n,\, q\in\mathbb{R}^m,\\ q'\in\mathbb{R}^l_+,\, t\in\mathbb{R}^k_+}} \left\{ -f^*(q) - \left(\sum_{i=1}^{k} t_i g_i\right)^*(q') - \left(\sum_{i=1}^{m} q_i F_i\right)^*_X(p) - \left(\sum_{i=1}^{l} q'_i G_i\right)^*_X(-p) \right\},$$

which is nothing else than $v(D_F) = v(D_{FL})$. □

4.4 Strong duality for (D_L), (D_F) and (D_{FL})

In the previous subsections we have presented some conditions which ensure the equality between the optimal objective values of the Lagrange and the Fenchel-Lagrange and of the Fenchel and the Fenchel-Lagrange dual problems, respectively. Combining the hypotheses of the theorems 3 and 4 it follows the equality of the optimal objective values of these three duals. Under the same conditions it can be proved that the optimal objective values of the duals are also equal to $v(P)$. In case $v(P)$ is finite this result leads to the strong duality.

Theorem 5 *Assume that $X \subseteq \mathbb{R}^n$ is a nonempty convex subset, $F_i : X \to \mathbb{R}$, $i = 1, ..., m$, $G_j : X \to \mathbb{R}$, $j = 1, ..., l$, are convex functions, $f : \mathbb{R}^m \to \mathbb{R}$, $g_i : \mathbb{R}^l \to \mathbb{R}$, $i = 1, ..., k$, are convex and componentwise increasing functions and the constraint qualification (CQ) is fulfilled. Then it holds*

$$v(P) = v(D_L) = v(D_F) = v(D_{FL}).$$

Provided that $v(P) > -\infty$, the strong duality holds, i.e. the optimal objective value of the primal and the dual problems coincide and the duals have optimal solutions.

Proof By Theorem 3 and Theorem 4 we obtain

$$v(D_L) = v(D_F) = v(D_{FL}). \tag{10}$$

Because $\mathcal{A} = \left\{ x \in X : g(G(x)) \underset{\mathbb{R}^k_+}{\leqq} 0 \right\} \neq \emptyset$, it holds $v(P) \in [-\infty, +\infty)$. If $v(P) = -\infty$, then the weak duality together with (10) lead to

$$v(D_L) = v(D_F) = v(D_{FL}) = -\infty = v(P).$$

Suppose now that $-\infty < v(P) < +\infty$. Because the constraint qualification (CQ) is fulfilled, Theorem 28.2 in [11] states the existence of a $\bar{t} \in \mathbb{R}^k_+$ such that the Lagrange duality holds, namely

$$\begin{aligned} v(P) &= \max_{t\in\mathbb{R}^k_+} \inf_{x\in X} \left\{ f(F(x)) + t^T g(G(x)) \right\} \\ &= \inf_{x\in X} \left\{ f(F(x)) + \bar{t}^T g(G(x)) \right\} = v(D_L). \end{aligned} \tag{11}$$

There is

$$v(P) = v(D_L) = v(D_F) = v(D_{FL}), \tag{12}$$

and $\bar{t} \in \mathbb{R}^k_+$ is an optimal solution to the Lagrange dual (D_L).
As in the proof of Theorem 3 we obtain, using that the infima in the relations (6), (7) and (8) are attained, $\bar{p} \in \mathbb{R}^n,\ \bar{q} \in \mathbb{R}^m_+$ and $\bar{q'} \in \mathbb{R}^l_+$ such that

$$v(P) = \inf_{x\in X} \left\{ f(F(x)) + \bar{t}^T g(G(x)) \right\} =$$
$$\sup_{\substack{p\in\mathbb{R}^n,\, q\in\mathbb{R}^m_+,\\ q'\in\mathbb{R}^l_+}} \left\{ -f^*(q) - \left(\sum_{i=1}^k \bar{t}_i g_i\right)^*(q') - \left(\sum_{i=1}^m q_i F_i\right)^*_X(p) - \left(\sum_{i=1}^k q'_i G_i\right)^*_X(-p) \right\} =$$
$$-f^*(\bar{q}) - \left(\sum_{i=1}^k \bar{t}_i g_i\right)^*(\bar{q'}) - \left(\sum_{i=1}^m \bar{q}_i F_i\right)^*_X(\bar{p}) - \left(\sum_{i=1}^k \bar{q}'_i G_i\right)^*_X(-\bar{p}) = v(D_{FL}).$$

Therefore $(\bar{p},\, \bar{q},\, \bar{q'},\, \bar{t})$ is an optimal solution to (D_{FL}).
It remains to show that $(\bar{p},\, \bar{q})$ is actually an optimal solution to the Fenchel dual (D_F). The relations (5) and (12) imply that

$$\begin{aligned} v(D_{FL}) &= -f^*(\bar{q}) - \left(\sum_{i=1}^k \bar{t}_i g_i\right)^*(\bar{q'}) - \left(\sum_{i=1}^m \bar{q}_i F_i\right)^*_X(\bar{p}) - \left(\sum_{i=1}^k \bar{q'}_i G_i\right)^*_X(-\bar{p}) \\ &\le -f^*(\bar{q}) - \left(\sum_{i=1}^m \bar{q}_i F_i\right)^*_X(\bar{p}) + \inf_{x\in\mathcal{A}} \bar{p}^T x \\ &\le \sup_{\substack{p\in\mathbb{R}^n,\\ q\in\mathbb{R}^m}} \left\{ -f^*(q) - \left(\sum_{i=1}^m q_i F_i\right)^*_X(p) + \inf_{x\in\mathcal{A}} p^T x \right\} = v(D_F) \le v(P), \end{aligned}$$

and so, because of $v(P) = v(D_{FL}) = v(D_F)$ there is

$$\begin{aligned} v(P) &= -f^*(\bar{q}) - \left(\sum_{i=1}^k \bar{t}_i g_i\right)^*(\bar{q'}) - \left(\sum_{i=1}^m \bar{q}_i F_i\right)^*_X(\bar{p}) - \left(\sum_{i=1}^k \bar{q'}_i G_i\right)^*_X(-\bar{p}) \\ &= -f^*(\bar{q}) - \left(\sum_{i=1}^m \bar{q}_i F_i\right)^*_X(\bar{p}) + \inf_{x\in\mathcal{A}} \bar{p}^T x = v(D_F), \end{aligned}$$

which states that $(\bar{p},\, \bar{q})$ is an optimal solution to (D_F). □

5 Special cases

In the last section of this paper we investigate some special cases of the original problem (P) and show that the duality concepts introduced in this paper generalize some results obtained in the past.

5.1 The ordinary optimization problem with inequality constraints and its duals

Let $X \subseteq \mathbb{R}^n$ be a nonempty set, $F : X \to \mathbb{R}$, $G = (G_1, ..., G_k)^T,\ G_i : X \to \mathbb{R},\ i = 1, ..., k$, and the following optimization problem with inequality constraints

$$(P') \qquad \inf_{x\in\mathcal{A}} F(x),$$

where
$$\mathcal{A} = \left\{ x \in X : G(x) \underset{\mathbb{R}^k_+}{\leqq} 0 \right\}.$$

One may observe that (P') is a particular case of the original problem (P) by considering $m = 1,\ F_1 = F$, $f : \mathbb{R} \to \mathbb{R}$ and $g = (g_1, ..., g_k)^T : \mathbb{R}^k \to \mathbb{R}^k$ such that $f(x) = x$ for all $x \in \mathbb{R}$ and $g_i(y) = y_i$ for all $y \in \mathbb{R}^k$ and $i = 1, ..., k$. Let us notice that f and $g_i,\ i = 1, ..., k$, are convex and componentwise increasing functions. In what follows we derive from the duals introduced for (P) corresponding dual problems for (P').

Because of
$$f^*(q) = \sup_{x \in \mathbb{R}} \{\langle q, x\rangle - f(x)\} = \sup_{x \in \mathbb{R}} \{(q-1)x\} = \begin{cases} 0, & \text{if } q = 1, \\ +\infty, & \text{otherwise,} \end{cases}$$
$$\begin{aligned} \left(\sum_{i=1}^k t_i g_i\right)^* (q') &= \sup_{y \in \mathbb{R}^k} \left\{ q'^T y - \sum_{i=1}^k t_i g_i(y) \right\} = \sup_{y \in \mathbb{R}^k} \left\{ q'^T y - \sum_{i=1}^k t_i y_i \right\} \\ &= \sup_{y \in \mathbb{R}^k} \left\{ (q' - t)^T y \right\} = \begin{cases} 0, & \text{if } q' = t, \\ +\infty, & \text{otherwise,} \end{cases} \end{aligned}$$
and
$$\begin{aligned} \left(\sum_{i=1}^l q'_i G_i\right)^*_X (-p) &= \left(\sum_{i=1}^k t_i G_i\right)^*_X (-p) = \sup_{x \in X} \left\{ -p^T x - t^T G(x) \right\} \\ &= -\inf_{x \in X} \left\{ p^T x + t^T G(x) \right\}, \end{aligned}$$
the three dual problems turn out to be
$$(D'_L) \qquad \sup_{t \in \mathbb{R}^k_+} \inf_{x \in X} \left\{ F(x) + t^T G(x) \right\},$$
$$(D'_F) \qquad \sup_{p \in \mathbb{R}^n} \left\{ -F^*_X(p) + \inf_{x \in \mathcal{A}} p^T x \right\},$$
and
$$(D'_{FL}) \quad \sup_{p \in \mathbb{R}^n,\, t \in \mathbb{R}^k_+} \left\{ -F^*_X(p) + \inf_{x \in X} \left\{ p^T x + t^T G(x) \right\} \right\},$$
respectively.

Let us notice that the constraint qualification (CQ) becomes in this case
$$(CQ') \qquad \exists\, x' \in\ ri(X) : \begin{cases} G_i(x') \leq 0, \ i \in L, \\ G_i(x') < 0, \ i \in N, \end{cases}$$
where
$$L := \left\{ i \in \{1, ..., k\} \middle| \begin{array}{l} G_i : X \to \mathbb{R} \text{ is the restriction to } X \text{ of an} \\ \text{affine function } \widetilde{G_i} : \mathbb{R}^n \to \mathbb{R} \end{array} \right\}$$
and $N := \{1, ..., k\} \setminus L$.

In this special case we get from the theorems obtained in section 4 the following results.

Theorem 6 *Assume that $X \subseteq \mathbb{R}^n$ is a nonempty convex subset and $F : X \to \mathbb{R},\ G_j : X \to \mathbb{R},\ j = 1, ..., k,$ are convex functions. Then it holds*

$$v(D'_L) = v(D'_{FL}).$$

Theorem 7 *Assume that $X \subseteq \mathbb{R}^n$ is a nonempty convex subset, $G_j : X \to \mathbb{R},\ j = 1, ..., k,$ are convex functions and the constraint qualification (CQ') is fulfilled. Then it holds*

$$v(D'_F) = v(D'_{FL}).$$

Theorem 8 *Assume that $X \subseteq \mathbb{R}^n$ is a nonempty convex subset, $F : X \to \mathbb{R},\ G_j : X \to \mathbb{R},\ j = 1, ..., k,$ are convex functions and the constraint qualification (CQ') is fulfilled. Then it holds*

$$v(P') = v(D'_L) = v(D'_F) = v(D'_{FL}).$$

Provided that $v(P') > -\infty$, the strong duality holds, i.e. the optimal objective value of the primal and the dual problems coincide and the duals have optimal solutions.

Remark 2 The three duals derived for (P') as well as the theorems enunciated above have been obtained by Wanka and Boţ in [12]. Therefore the results presented in this paper generalize some of the previous work of the authors.

5.2 The composed optimization problem without inequality constraints

Consider X a nonempty subset of $\mathbb{R}^n$, $F = (F_1, ..., F_m)^T,\ F_i : X \to \mathbb{R},\ i = 1, ..., m$, and the optimization problem without inequality constraints

$$(P'') \qquad \inf_{x \in X} f(F(x)).$$

One may see that (P'') can be obtained from the original problem, namely, by taking in (P) the functions $g_i : \mathbb{R}^l \to \mathbb{R},\ i = 1, ..., k$, such that $g_i(y) = 0,\ i = 1, ..., k$, for all $y \in \mathbb{R}^l$.
Because of

$$\left(\sum_{i=1}^{k} t_i g_i\right)^* (q') = (0)^* (q') = \sup_{y \in \mathbb{R}^l} \left\{y^T q'\right\} = \begin{cases} 0, & \text{if } q' = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and $0^*_X(-p) = -\inf_{x \in X} p^T x$, the Fenchel-Lagrange dual problem becomes

$$(D''_{FL}) \qquad \sup_{p \in \mathbb{R}^n,\, q \in \mathbb{R}^m} \left\{ -f^*(q) - \left(\sum_{i=1}^{m} q_i F_i\right)^*_X (p) + \inf_{x \in X} p^T x \right\}.$$

The strong duality theorem follows.

Theorem 9 *Assume that* $X \subseteq \mathbb{R}^n$ *is a nonempty convex subset,* $F = (F_1, ..., F_m)^T$, $F_i : X \to \mathbb{R}$, $i = 1, ..., m$, *are convex functions and* $f : \mathbb{R}^m \to \mathbb{R}$ *is a convex and componentwise increasing function. Then it holds*

$$v(P'') = v(D''_{FL}).$$

Provided that $v(P'') > -\infty$, *the strong duality holds, i.e. the optimal objective value of the primal and the dual problems coincide and the duals have optimal solutions.*

Remark 3 The dual (D''_{FL}) and Theorem 9 generalize the results obtained by Boţ and Wanka in [2] in the special case when $X = \mathbb{R}^n$.

Further we particularize problem (P'') by taking instead of f the function $\gamma_C^+ : \mathbb{R}^m \to \mathbb{R}$, where $\gamma_C^+(t) := \gamma_C(t^+)$, with $t^+ = (t_1^+, ..., t_m^+)^T$ and $t_i^+ = \max\{0, t_i\}$, $i = 1, ..., m$, and $\gamma_C : \mathbb{R}^m \to \mathbb{R}$ is a monotonic gauge of a closed convex set C containing the origin defined by $\gamma_C(x) := \inf\{\alpha > 0 : x \in \alpha C\}$. Recall that γ_C is a monotonic gauge on $\mathbb{R}^m$ (cf. [1]), if $\gamma_C(u) \leq \gamma_C(v)$ for every u and v in $\mathbb{R}^m$ satisfying $|u_i| \leq |v_i|$ for each $i = 1, ..., m$. Therefore γ_C^+ is a convex and componentwise increasing function.

The optimization problem

$$(P^{\gamma_C}) \qquad \inf_{x \in \mathcal{A}} \gamma_C^+(F(x))$$

has as Fenchel-Lagrange dual the following problem

$$(D_{FL}^{\gamma_C}) \qquad \sup_{p \in \mathbb{R}^n,\, q \in \mathbb{R}^m} \left\{ -(\gamma_C^+)^*(q) - \left(\sum_{i=1}^m q_i F_i \right)_X^* (p) + \inf_{x \in X} p^T x \right\}.$$

On the other hand, by Proposition 4.2 in [13], the conjugate function $(\gamma_C^+)^* : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ of γ_C^+ verifies

$$(\gamma_C^+)^*(q) = \begin{cases} 0, & \text{if } q \in \mathbb{R}_+^m \text{ and } \gamma_{C^0}(q) \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

where γ_{C^0} is the gauge of the polar set C^0. Then the Fenchel-Lagrange dual problem looks like

$$(D_{FL}^{\gamma_C}) \qquad \sup_{\substack{p \in \mathbb{R}^n,\, q \in \mathbb{R}_+^m, \\ \gamma_{C^0}(q) \leq 1}} \left\{ - \left(\sum_{i=1}^m q_i F_i \right)_X^* (p) + \inf_{x \in X} p^T x \right\},$$

which is nothing else than the dual problem obtained by the authors in [13] as a theoretical framework for some location problems.

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