Sequential optimality conditions for composed convex optimization problems

Radu Ioan Boţ [*] Ernö Robert Csetnek [†] Gert Wanka [‡]

Abstract. Using a general approach which provides sequential optimality conditions for a general convex optimization problem, we derive necessary and sufficient optimality conditions for composed convex optimization problems. Further, we give sequential characterizations for a subgradient of the precomposition of a K-increasing lower semicontinuous convex function with a K-convex and K-epi-closed (continuous) function, where K is a nonempty convex cone. We prove that several results from the literature dealing with sequential characterizations of subgradients are obtained as particular cases of our results. We also improve the above mentioned statements.

[*]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: radu.bot@mathematik.tu-chemnitz.de

[†]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: robert.csetnek@mathematik.tu-chemnitz.de

[‡]Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: gert.wanka@mathematik.tu-chemnitz.de

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1 Introduction

Motivated by [9] and [18], we have given in [2] qualification free necessary and sufficient sequential optimality conditions for the general convex optimization problem

$$(P_\phi) \quad \inf_{x \in X} \phi(x, 0),$$

where $\phi : X \times Y \to \overline{\mathbb{R}}$, the so-called *perturbation function*, is proper, convex and lower semicontinuous (see [6] or [19] for more details on the perturbation theory). More precisely, if X is a reflexive Banach space and Y is a Banach space, we proved that an element $a \in \operatorname{dom}(\phi(\cdot, 0))$ is an optimal solution of the problem (P_ϕ) if and only if there exist sequences $(x_n, y_n) \in \operatorname{dom}(\phi)$, $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$ such that $x_n^* \to 0, x_n \to a, y_n \to 0$ $(n \to +\infty)$ and $\phi(x_n, y_n) - \langle y_n^*, y_n \rangle - \phi(a, 0) \to 0$ $(n \to +\infty)$. This sequential characterization is obtained by using the Brøndsted-Rockafellar Theorem ([4], [18]).

It is shown in [2] that the sequential generalizations of the Pshenichnyi-

Rockafellar Lemma ([14], [15]) given by Jeyakumar and Wu (see Theorem 3.3 and Corollary 3.5 in [9]) and a sequential Lagrange multiplier condition given by Thibault (see Theorem 4.1 in [18]), respectively, follow as particular cases of this general approach. Moreover, one can improve these results. Other sequential characterizations can be found in literature in [7], [8], [9], [10], [12], [17].

The aim of this paper is to prove that some other results given in the past on this topic can also be derived from the general case mentioned above and that they can be improved. We start by giving necessary and sufficient optimality conditions for the convex optimization problem with geometric and cone constraints. Using this result, we obtain as a special case sequential characterizations of an optimal solution of the following composed convex optimization problem

$$(P) \quad \inf_{x\in X}[f(x)+g(h(x))],$$

where X is a reflexive Banach space, Y is a reflexive Banach space partially ordered by a nonempty convex cone K, $f : X \to \overline{\mathbb{R}}$ is proper, convex and lower semicontinuous, $h : X \to Y^{\bullet} = Y \cup \{\infty_Y\}$ is proper, K-convex and K-epi-closed, $g : Y^{\bullet} \to \overline{\mathbb{R}}$ is proper, convex, lower semicontinuous, K-increasing on $h(\mathrm{dom}(h))+K$ and $g(\infty_Y) = +\infty$. The sequential characterization of a subgradient of the function $g \circ h$ at $a \in \mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g))$ given by Thibault (see Theorem 3.1 in [18]) follows as a particular case.

If instead of the K-epi-closedness we suppose that $h : X \to Y$ is continuous and $g : Y \to \overline{\mathbb{R}}$ is K-increasing on Y, while Y is not anymore assumed to be reflexive, then for an appropriate choice of the perturbation function ϕ we obtain another sequential characterization of an arbitrary $x^* \in \partial(g \circ h)(a)$, where $a \in h^{-1}(\mathrm{dom}(g))$. For this sequential characterization Thibault considered (see Corollary 3.2 in [18]) that K is a closed convex normal cone and g is K-increasing on $h(X) + K$. We show that if the function g is supposed to be K-increasing on the whole space Y, then this sequential characterization holds even if the cone K is not normal. Moreover, we show that, unlike in [18] in this result we can renounce to the closedness of the cone K.

The paper is organized as follows. In the next section we give some definitions and results from convex analysis that will be used in the paper. In section 3 we deal first with sequential optimality conditions for convex optimization problems with geometric and cone constraints. Further, we prove that some sequential characterizations regarding composed convex optimization problems given in the literature follow as particular cases of our general approach.

2 Preliminaries

Consider two separated locally convex vector spaces X and Y and their topological dual spaces X^* and Y^*, endowed with the weak* topologies $\omega(X^*, X)$ and $\omega(Y^*, Y)$, respectively. Consider also a nonempty convex cone $K \subseteq Y$ and $K^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \ \forall y \in K\}$ its *positive dual cone*, where we denote by $\langle y^*, y\rangle$ the value of the linear continuous functional $y^* \in Y^*$ at

$y \in Y$. On Y we consider the partial order induced by K, "$\leq_K$", defined by $y_1 \leq_K y_2 \Leftrightarrow y_2 - y_1 \in K, y_1, y_2 \in Y$. To Y we attach an abstract maximal element with respect to "$\leq_K$", denoted by ∞_Y and let $Y^\bullet := Y \cup \{\infty_Y\}$. Then for every $y \in Y$ one has $y \leq_K \infty_Y$ and we consider on $Y^\bullet$ the following operations: $y + \infty_Y = \infty_Y + y = \infty_Y$ and $t\infty_Y = \infty_Y$ for all $y \in Y$ and all $t \geq 0$.

The *indicator function* of $C \subseteq X$, denoted by δ_C, is defined as $\delta_C : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$,

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a function $f : X \to \overline{\mathbb{R}}$ we denote by $\operatorname{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\operatorname{dom}(f) \neq \emptyset$ and $f(x) > -\infty \ \forall x \in X$. For $x \in X$ such that $f(x) \in \mathbb{R}$ we consider the (classical) *convex sudifferential* of f at x defined by

$$\partial f(x) = \{x^* \in X^* : f(u) - f(x) \geq \langle x^*, u - x\rangle \ \forall u \in X\}.$$

An arbitrary element $x^* \in \partial f(x)$ (if it exists) is called *subgradient* of the function f at the point $x \in X$. If f is proper then for $a \in \operatorname{dom}(f)$ we have the following relation:

$$\inf_{x \in X} f(x) = f(a) \Leftrightarrow 0 \in \partial f(a).$$

The *Fenchel-Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\} \ \forall x^* \in X^*.$$

We have the so-called *Young-Fenchel inequality*

$$f^*(x^*) + f(x) \geq \langle x^*, x\rangle \ \forall x \in X \ \forall x^* \in X^*.$$

We mention here an important property of conjugate functions: if f is proper, then f is convex and lower semicontinuous if and only if $f^{**} = f$, where f^{**} is the biconjugate of f, defined by $f^{**}(x) = \sup\limits_{x^* \in X^*} \{\langle x^*, x\rangle - f^*(x^*)\} \ \forall x \in X$ (see [6], [19]).

The following characterization of the subdifferential of a proper function f by means of conjugate functions will be useful in the paper (see [6], [19]):

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle,$$

where $x \in \mathrm{dom}(f)$ and $x^* \in X^*$.

A function $g : Y^\bullet \to \overline{\mathbb{R}}$ such that $g(\infty_Y) = +\infty$ is called K-*increasing* on a subset S of Y if for every $s_1, s_2 \in S$ such that $s_1 \leq_K s_2$ one has $g(s_1) \leq g(s_2)$.

Some of the above notions given for functions with extended real values can be formulated also for function having their ranges in infinite dimensional spaces.

For a function $h : X \to Y^\bullet$ we denote by $\mathrm{dom}(h) = \{x \in X : h(x) \in Y\}$ its *domain* and by $\mathrm{epi}_K(h) = \{(x, y) \in X \times Y : h(x) \leq_K y\}$ its K-*epigraph*. We say that h is *proper* if its domain is a nonempty set. The function h is said to be K-*convex* if $h(tx_1 + (1-t)x_2) \leq_K th(x_1) + (1-t)h(x_2) \ \forall x_1, x_2 \in X \ \forall t \in [0, 1]$. Further, for an arbitrary $\lambda \in K^*$ we define the function $\lambda h : X \to \overline{\mathbb{R}}$,

$$(\lambda h)(x) = \begin{cases} \langle \lambda, h(x)\rangle, & \text{if } x \in \mathrm{dom}(h), \\ +\infty, & \text{otherwise.} \end{cases}$$

The function h is said to be K*-epi-closed* if $\mathrm{epi}_K(h)$ is a closed subset of $X \times Y$ ([11]). Also, h is called *star* K*-lower semicontinuous* at $x \in X$ if $\forall \lambda \in K^*$ the function λh is lower semicontinuous at x. The function h is said to be *star* K*-lower semicontinuous* if it is star K-lower semicontinuous at every $x \in X$.

Remark 2.1 (a) Besides the two generalizations of lower semicontinuity defined above for functions taking values in infinite dimensional spaces, there exists in the literature another notion of lower semicontinuity, called K*-lower semicontinuity*, wich has been introduced in [13] and refined in [5]. One can show that K-lower semicontinuity implies star K-lower semicontinuity, which yields K-epi-closedness (see [11]), while the opposite assertions are not valid in general.

The following example of a K-convex function which is K-epi-closed, but not star K-lower semicontinuous was given in [3]: $h : \mathbb{R} \to (\mathbb{R}^2)^\bullet = \mathbb{R}^2 \cup \{\infty_{\mathbb{R}^2}\}$,

$$h(x) = \begin{cases} (\frac{1}{x}, x), & \text{if } x > 0, \\ \infty, & \text{otherwise}, \end{cases}$$

and $K := \mathbb{R}^2_+ = [0, +\infty) \times [0, +\infty)$. One can see that for $\lambda = (0, 1)^T \in (\mathbb{R}^2_+)^* = \mathbb{R}^2_+$ the function λh is not lower semicontinuous.

For more on lower semicontinuity on topological vector spaces we refer the reader to [1], [5], [11], [13], [16].

(b) It is known that when $Y = \mathbb{R}$ and $K = \mathbb{R}_+ = [0, +\infty)$, all the lower semicontinuity notions mentioned above coincide, becoming the classical lower semicontinuity of functions with extended real values.

3 Sequential optimality conditions

In this section we derive several necessary and sufficient sequential optimality conditions for different classes of convex optimization problems.

3.1 The general case

For the rest of the paper, we consider $(X, \|\cdot\|)$ a reflexive Banach space, $(Y, \|\cdot\|)$ a Banach spaces and $(X^*, \|\cdot\|_*)$, $(Y^*, \|\cdot\|_*)$ their topological dual spaces. Let $\{x_n^* : n \in \mathbb{N}\}$ be a sequence in X^* and $x^* \in X^*$. We write $x_n^* \xrightarrow{\omega^*} x^*$ ($x_n^* \xrightarrow{\|\cdot\|_*} x^*$) for the case when x_n^* converges to x^* in the weak* (strong) topology. We make the following convention: if in a certain property we write $x_n^* \to x^*$ $(n \to +\infty)$, we understand that the property holds no matter which of the two topologies (weak* or strong) is used. The following property will be frequently used in the paper:

$$\text{if } x_n^* \to 0 \text{ and } x_n \to a \ (n \to +\infty), \text{ then } \langle x_n^*, x_n \rangle \to 0 \ (n \to +\infty),$$

where $\{x_n : n \in \mathbb{N}\} \subseteq X$, $a \in X$ and $x_n \to a$ $(n \to +\infty)$ means $\|x_n - a\| \to 0$ $(n \to +\infty)$, that is the convergence in the topology induced by the norm on X. On $X \times Y$ we use the norm $\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}$, for $(x, y) \in X \times Y$. Similarly we define the norm on $X^* \times Y^*$.

We give the following sequential optimality condition concerning the general optimization problem

$$(P_\phi) \quad \inf_{x \in X} \phi(x, 0),$$

where $\phi : X \times Y \to \overline{\mathbb{R}}$ is a so-called *perturbation function* (see [6], [19] for more on the perturbation theory).

Theorem 3.1 ([2]) *Let $\phi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex and lower semi-continuous function such that $\inf_{x \in X} \phi(x, 0) < +\infty$. The following statements are equivalent:*

(a) $a \in \mathrm{dom}(\phi(\cdot, 0))$ is an optimal solution of the problem (P_ϕ);

(b) there exist sequences $(x_n, y_n) \in \mathrm{dom}(\phi)$, $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$ such that

$$x_n^* \to 0, x_n \to a, y_n \to 0 \ (n \to +\infty) \ \textit{and}$$

$$\phi(x_n, y_n) - \langle y_n^*, y_n \rangle - \phi(a, 0) \to 0 \ (n \to +\infty).$$

Remark 3.2 Using an idea due to Thibault ([17]) one can derive the following refined version of the above sequential optimality condition.

In the same hypotheses as in Theorem 3.1 the following assertions are equivalent:

(a) $a \in \mathrm{dom}(\phi(\cdot, 0))$ is an optimal solution of the problem (P_ϕ);

(b) there exist sequences $(x_n, y_n) \in \mathrm{dom}(\phi)$, $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$ such that

$$x_n^* \to 0, x_n \to a, y_n \to 0, \langle y_n^*, y_n \rangle \to 0 \ (n \to +\infty) \text{ and}$$

$$\phi(x_n, y_n) - \phi(a, 0) \to 0 \ (n \to +\infty).$$

This refinement can be obtained also from Proposition 1.1 in [12] (see [2] for more details).

3.2 Sequential optimality conditions for convex optimization problems with geometric and cone constraints

In this subsection we derive sequential optimality conditions for a convex optimization problem with geometric and cone constraints

$$(P_K) \inf_{\substack{x \in C \\ g(x) \in -K}} f(x),$$

where C is a closed convex subset of X, K is a nonempty convex cone of Y, $f : X \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function and $g : X \to Y^\bullet$ is proper, K-convex and K-epi-closed. We suppose also that $C \cap g^{-1}(-K) \cap \operatorname{dom}(f) \neq \emptyset$. In the following we derive a sequential form of the Lagrange multiplier condition for (P_K) by applying Theorem 3.1 to the following perturbation function

$$\phi : X \times X \times Y \to \overline{\mathbb{R}},\ \phi(x,p,q) = \begin{cases} f(x+p), & \text{if } x \in C \text{ and } g(x) \leq_K q, \\ +\infty, & \text{otherwise.} \end{cases}$$

One can easily show that ϕ is proper, convex and lower semicontinuous such that $\inf_{x \in X} \phi(x,0,0) < +\infty$. The conjugate of ϕ is $\phi^* : X^* \times X^* \times Y^* \to \overline{\mathbb{R}}$,

$$\phi^*(x^*,p^*,q^*) = \begin{cases} f^*(p^*) + (-q^*g + \delta_C)^*(x^* - p^*), & \text{if } q^* \in -K^*, \\ +\infty, & \text{otherwise,} \end{cases}$$

as a straightforward calculation shows.

Theorem 3.3 *The element $a \in C \cap g^{-1}(-K) \cap \mathrm{dom}(f)$ is an optimal solution of the problem (P_K) if and only if*

$$\begin{cases} \exists (x_n, p_n, q_n) \in C \times \mathrm{dom}(f) \times Y, g(x_n) \leq_K q_n, \exists (u_n^*, v_n^*, q_n^*) \in X^* \times X^* \times K^*, \\ u_n^* \in \partial f(p_n), v_n^* \in \partial (q_n^* g + \delta_C)(x_n), \langle q_n^*, q_n - g(x_n) \rangle = 0 \ \forall n \in \mathbb{N}, \\ u_n^* + v_n^* \to 0, x_n \to a, p_n \to a, q_n \to 0 \ (n \to +\infty) \ \textit{and} \\ f(p_n) - \langle u_n^*, p_n - x_n \rangle + \langle q_n^*, q_n \rangle - f(a) \to 0 \ (n \to +\infty). \end{cases} \tag{1}$$

Proof. According to Theorem 3.1, the element $a \in C \cap g^{-1}(-K) \cap \mathrm{dom}(f)$ solves the problem (P_K) if and only if there exist sequences $(x_n, p_n, q_n) \in \mathrm{dom}(\phi)$, $(x_n^*, p_n^*, -q_n^*) \in \partial \phi(x_n, p_n, q_n)$ such that

$$x_n^* \to 0, x_n \to a, (p_n, q_n) \to (0, 0) \ (n \to +\infty) \text{ and}$$

$$\phi(x_n, p_n, q_n) - \langle (p_n^*, -q_n^*), (p_n, q_n) \rangle - \phi(a, 0, 0) \to 0 \ (n \to +\infty).$$

Since $(x_n, p_n, q_n) \in \mathrm{dom}(\phi)$ we get $x_n \in C, x_n + p_n \in \mathrm{dom}(f)$ and $g(x_n) \leq_K q_n$ $\forall n \in \mathbb{N}$. We have $(x_n^*, p_n^*, -q_n^*) \in \partial \phi(x_n, p_n, q_n)$ if and only if

$$\phi(x_n, p_n, q_n) + \phi^*(x_n^*, p_n^*, -q_n^*) = \langle x_n^*, x_n \rangle + \langle p_n^*, p_n \rangle + \langle -q_n^*, q_n \rangle \Leftrightarrow$$

$$f(x_n + p_n) + f^*(p_n^*) + (q_n^* g + \delta_C)^*(x_n^* - p_n^*) = \langle x_n^*, x_n \rangle + \langle p_n^*, p_n \rangle + \langle -q_n^*, q_n \rangle,$$

where $q_n^* \in K^*$ $\forall n \in \mathbb{N}$. As $q_n - g(x_n) \in K$ we obtain $\langle q_n^*, q_n - g(x_n) \rangle \geq 0$ $\forall n \in \mathbb{N}$. Using this and the Young-Fenchel inequality we get $f(x_n + p_n) + f^*(p_n^*) + (q_n^* g +$

$\delta_C)^*(x_n^*-p_n^*) \geq \langle p_n^*, x_n+p_n\rangle+\langle x_n^*-p_n^*, x_n\rangle-(q_n^*g+\delta_C)(x_n) = \langle x_n^*, x_n\rangle+\langle p_n^*, p_n\rangle+\langle -q_n^*, g(x_n)\rangle \geq \langle x_n^*, x_n\rangle+\langle p_n^*, p_n\rangle+\langle -q_n^*, q_n\rangle$. Hence $(x_n^*, p_n^*, -q_n^*) \in \partial\phi(x_n, p_n, q_n)$ if and only if $p_n^* \in \partial f(x_n+p_n), x_n^*-p_n^* \in \partial(q_n^*g+\delta_C)(x_n)$ and $\langle q_n^*, q_n-g(x_n)\rangle = 0\ \forall n \in \mathbb{N}$. As a consequence, we obtain that $a \in C \cap g^{-1}(-K) \cap \operatorname{dom}(f)$ is an optimal solution of the problem (P_K) if and only if

$$\begin{cases} \exists(x_n, p_n, q_n) \in C \times X \times Y, x_n+p_n \in \operatorname{dom}(f), g(x_n) \leq_K q_n, \\ \exists(x_n^*, p_n^*, q_n^*) \in X^* \times X^* \times K^*, p_n^* \in \partial f(x_n+p_n), x_n^*-p_n^* \in \partial(q_n^*g+\delta_C)(x_n), \\ \langle q_n^*, q_n-g(x_n)\rangle = 0\ \forall n \in \mathbb{N}, x_n^* \to 0, x_n \to a, p_n \to 0, q_n \to 0\ (n \to +\infty) \text{ and} \\ f(x_n+p_n)-\langle p_n^*, p_n\rangle+\langle q_n^*, q_n\rangle-f(a) \to 0\ (n \to +\infty). \end{cases} \tag{2}$$

Introducing the new variables p_n', u_n^* and v_n^* instead of p_n, p_n^* and x_n^* by $p_n' := x_n+p_n, u_n^* := p_n^*$ and $v_n^* := x_n^*-p_n^*$ for all $n \in \mathbb{N}$, one can see that (2) is equivalent to (1) (again denoting p_n' by p_n $\forall n \in \mathbb{N}$), which completes the proof. $\square$

Remark 3.4 Let us notice that for a different choice of the perturbation function ϕ, we have given in [2] another sequential optimality condition for the problem (P_K) in case $g : X \to Y$ is continuous and K is a closed convex cone.

For the special case when $C = X$, we obtain the following sequential characterization of an optimal solution of the optimization problem

$$(P_K') \inf_{g(x) \leq_K 0} f(x).$$

Corollary 3.5 *The element $a \in g^{-1}(-K) \cap \operatorname{dom}(f)$ is an optimal solution of* *the problem* (P'_K) *if and only if*

$$\begin{cases} \exists(x_n, p_n, q_n) \in X \times \mathrm{dom}(f) \times Y, g(x_n) \leq_K q_n, \exists(u_n^*, v_n^*, q_n^*) \in X^* \times X^* \times K^*, \\ u_n^* \in \partial f(p_n), v_n^* \in \partial(q_n^* g)(x_n), \langle q_n^*, q_n - g(x_n)\rangle = 0 \ \forall n \in \mathbb{N}, \\ u_n^* + v_n^* \to 0, x_n \to a, p_n \to a, q_n \to 0 \ (n \to +\infty) \text{ and} \\ f(p_n) - \langle u_n^*, p_n - x_n\rangle + \langle q_n^*, q_n\rangle - f(a) \to 0 \ (n \to +\infty). \end{cases} \tag{3}$$

3.3 Sequential optimality conditions for composed convex optimization problems

The following optimization problem is considered in this subsection

$$(P) \quad \inf_{x \in X} [f(x) + g(h(x))],$$

where $f : X \to \overline{\mathbb{R}}$ is proper, convex and lower semicontinuous, Y is partially ordered by a nonempty convex cone K, $h : X \to Y^\bullet$ is proper, K-convex, $g : Y^\bullet \to \overline{\mathbb{R}}$ is proper, convex, lower semicontinuous and $g(\infty_Y) = +\infty$. We suppose also that $\mathrm{dom}(f) \cap \mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g)) \neq \emptyset$. This subsection is divided in two parts. In subsection 3.3.1 we consider the case h is K-epi-closed and g is K-increasing on $h(\mathrm{dom}(h)) + K$, while in the second part we take h continuous and g K-increasing on Y.

3.3.1 The case h is K-epi-closed

Throughout this subsection we assume that Y is a reflexive Banach space, h is K-epi-closed and g is K-increasing on $h(\mathrm{dom}(h)) + K$. The problem (P) is a convex optimization problem and for characterizing its optimal solutions the following sequential optimality condition can be derived from Corollary 3.5 (see Remark 3.12(b) for a discussion on the several reasons why we apply this method).

Theorem 3.6 *The element $a \in \mathrm{dom}(f) \cap \mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g))$ is an optimal solution of the problem (P) if and only if*

$$\begin{cases} \exists (x_n, p_n, q_n, q_n') \in X \times \mathrm{dom}(f) \times \mathrm{dom}(g) \times Y, h(x_n) \leq_K q_n', \\ \exists (u_n^*, e_n^*, u_n^{*\prime}, q_n^*), q_n^* \in K^*, u_n^* \in \partial f(p_n), q_n^* + e_n^* \in \partial g(q_n), \\ u_n^{*\prime} \in \partial (q_n^* h)(x_n), \langle q_n^*, q_n' - h(x_n) \rangle = 0 \ \forall n \in \mathbb{N}, \\ u_n^* + u_n^{*\prime} \to 0, e_n^* \to 0, x_n \to a, p_n \to a, q_n \to h(a), q_n' \to h(a) \ (n \to +\infty), \\ f(p_n) - \langle u_n^*, p_n - x_n \rangle + \langle q_n^*, h(x_n) - h(a) \rangle - f(a) \to 0 \ (n \to +\infty) \text{ and} \\ g(q_n) - \langle q_n^*, q_n - h(a) \rangle - g(h(a)) \to 0 \ (n \to +\infty). \end{cases} \tag{4}$$

Proof. One can prove that $a \in \mathrm{dom}(f) \cap \mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g))$ is an optimal solution of the problem (P) if and only if $(a, h(a))$ is an optimal solution of the problem

$$(P_K') \quad \inf_{h(x) - y \leq_K 0} (f(x) + g(y)) \Leftrightarrow \inf_{G(x,y) \leq_K 0} F(x, y),$$

where $F : X \times Y \to \overline{\mathbb{R}}, F(x,y) = f(x) + g(y)$ and $G : X \times Y \to Y^\bullet, G(x,y) = h(x) - y \; \forall (x,y) \in X \times Y$. The hypotheses regarding the functions f, g and h imply that F is proper, convex, lower semicontinuous and G is proper, K-convex and K-epi-closed. Applying Corollary 3.5 to the problem (P'_K), which is a problem with cone constraints in $X \times Y$, we get that $a \in \mathrm{dom}(f) \cap \mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g))$ is an optimal solution of the problem (P) if and only if

$$\begin{cases} \exists (x_n, y_n, p_n, q_n, q'_n) : (p_n, q_n) \in \mathrm{dom}(F), G(x_n, y_n) \leq_K q'_n, \\ \exists (u_n^*, v_n^*, u_n^{*\prime}, v_n^{*\prime}, q_n^*) : q_n^* \in K^*, (u_n^*, v_n^*) \in \partial F(p_n, q_n), \\ (u_n^{*\prime}, v_n^{*\prime}) \in \partial (q_n^* G)(x_n, y_n), \langle q_n^*, q'_n - G(x_n, y_n) \rangle = 0 \; \forall n \in \mathbb{N}, \\ (u_n^*, v_n^*) + (u_n^{*\prime}, v_n^{*\prime}) \to (0,0), (x_n, y_n) \to (a, h(a)), (p_n, q_n) \to (a, h(a)), \\ q'_n \to 0 \text{ and } F(p_n, q_n) - \langle (u_n^*, v_n^*), (p_n, q_n) - (x_n, y_n) \rangle + \\ \langle q_n^*, q'_n \rangle - F(a, h(a)) \to 0 \; (n \to +\infty). \end{cases} \tag{5}$$

We have $\mathrm{dom}(F) = \mathrm{dom}(f) \times \mathrm{dom}(g)$, $F^*(x^*, y^*) = f^*(x^*) + g^*(y^*)$ and thus $(x^*, y^*) \in \partial F(x,y) \Leftrightarrow x^* \in \partial f(x)$ and $y^* \in \partial g(y)$, for $(x,y) \in X \times Y$ and $(x^*, y^*) \in X^* \times Y^*$. Further, for $\lambda \in K^*$ we have

$$(\lambda G)^*(x^*, y^*) = \begin{cases} (\lambda h)^*(x^*), & \text{if } y^* + \lambda = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

and $(x^*, y^*) \in \partial(\lambda G)(x,y)$ if and only if $y^* + \lambda = 0$ and $x^* \in \partial(\lambda h)(x)$. Hence $a \in \mathrm{dom}(f) \cap \mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g))$ is an optimal solution of the problem (P) if and only if

$$\begin{cases} \exists (x_n, y_n, p_n, q_n, q'_n) \in X \times Y \times \operatorname{dom}(f) \times \operatorname{dom}(g) \times Y : h(x_n) \leq_K y_n + q'_n, \\ \exists (u_n^*, v_n^*, u_n^{*\prime}, q_n^*) : q_n^* \in K^*, u_n^* \in \partial f(p_n), v_n^* \in \partial g(q_n), u_n^{*\prime} \in \partial(q_n^* h)(x_n), \\ \langle q_n^*, q'_n + y_n - h(x_n)\rangle = 0 \ \forall n \in \mathbb{N}, u_n^* + u_n^{*\prime} \to 0, v_n^* - q_n^* \to 0, x_n \to a, p_n \to a, \\ y_n \to h(a), q_n \to h(a), q'_n \to 0 \ (n \to +\infty) \text{ and } f(p_n) + g(q_n) - \langle u_n^*, p_n - x_n\rangle - \\ \langle v_n^*, q_n - y_n\rangle + \langle q_n^*, q'_n\rangle - f(a) - g(h(a)) \to 0 \ (n \to +\infty). \end{cases} \tag{6}$$

With the following notations: $q''_n := y_n + q'_n$ and $e_n^* := v_n^* - q_n^*$, $\forall n \in \mathbb{N}$, we obtain that (6) is equivalent to

$$\begin{cases} \exists (x_n, y_n, p_n, q_n, q''_n) \in X \times Y \times \operatorname{dom}(f) \times \operatorname{dom}(g) \times Y : h(x_n) \leq_K q''_n, \\ \exists (u_n^*, e_n^*, u_n^{*\prime}, q_n^*) : q_n^* \in K^*, u_n^* \in \partial f(p_n), q_n^* + e_n^* \in \partial g(q_n), u_n^{*\prime} \in \partial(q_n^* h)(x_n), \\ \langle q_n^*, q''_n - h(x_n)\rangle = 0 \ \forall n \in \mathbb{N}, u_n^* + u_n^{*\prime} \to 0, e_n^* \to 0, x_n \to a, p_n \to a, \\ y_n \to h(a), q_n \to h(a), q''_n \to h(a) \ (n \to +\infty) \text{ and } f(p_n) + g(q_n) - \langle u_n^*, p_n - x_n\rangle - \\ \langle q_n^* + e_n^*, q_n - y_n\rangle + \langle q_n^*, q''_n - y_n\rangle - f(a) - g(h(a)) \to 0 \ (n \to +\infty). \end{cases} \tag{7}$$

Since $\langle e_n^*, q_n - y_n\rangle \to 0 (n \to +\infty)$, we obtain that the element $a \in \operatorname{dom}(f) \cap \operatorname{dom}(h) \cap h^{-1}(\operatorname{dom}(g))$ is an optimal solution of the problem (P) if and only if

$$\begin{cases} \exists (x_n, y_n, p_n, q_n, q''_n) \in X \times Y \times \operatorname{dom}(f) \times \operatorname{dom}(g) \times Y : h(x_n) \leq_K q''_n, \\ \exists (u_n^*, e_n^*, u_n^{*\prime}, q_n^*) : q_n^* \in K^*, u_n^* \in \partial f(p_n), q_n^* + e_n^* \in \partial g(q_n), u_n^{*\prime} \in \partial(q_n^* h)(x_n), \\ \langle q_n^*, q''_n - h(x_n)\rangle = 0 \ \forall n \in \mathbb{N}, u_n^* + u_n^{*\prime} \to 0, e_n^* \to 0, x_n \to a, p_n \to a, \\ y_n \to h(a), q_n \to h(a), q''_n \to h(a) \ (n \to +\infty) \text{ and} \\ f(p_n) + g(q_n) - \langle u_n^*, p_n - x_n\rangle - \langle q_n^*, q_n - q''_n\rangle - f(a) - g(h(a)) \to 0 \ (n \to +\infty). \end{cases} \tag{8}$$

Let us notice that in the above condition the sequence $\{y_n : n \in \mathbb{N}\}$ is superfluous, that is the conditions in (8) are equivalent to

$$\begin{cases} \exists(x_n, p_n, q_n, q_n'') \in X \times \operatorname{dom}(f) \times \operatorname{dom}(g) \times Y : h(x_n) \leq_K q_n'', \\ \exists(u_n^*, e_n^*, u_n^{*\prime}, q_n^*) : q_n^* \in K^*, u_n^* \in \partial f(p_n), q_n^* + e_n^* \in \partial g(q_n), u_n^{*\prime} \in \partial(q_n^* h)(x_n), \\ \langle q_n^*, q_n'' - h(x_n)\rangle = 0 \ \forall n \in \mathbb{N}, u_n^* + u_n^{*\prime} \to 0, e_n^* \to 0, x_n \to a, p_n \to a, \\ q_n \to h(a), q_n'' \to h(a) \ (n \to +\infty) \text{ and} \\ f(p_n) + g(q_n) - \langle u_n^*, p_n - x_n\rangle - \langle q_n^*, q_n - q_n''\rangle - f(a) - g(h(a)) \to 0 \ (n \to +\infty). \end{cases} \tag{9}$$

Indeed, the direct implication is obvious, while for the reverse one we take $y_n := h(a) \ \forall n \in \mathbb{N}$.

Let us introduce now the following real sequences: $a_n := f(p_n) + g(q_n) - \langle u_n^*, p_n - x_n\rangle - \langle q_n^*, q_n - q_n''\rangle - f(a) - g(h(a))$, $b_n := g(q_n) - \langle q_n^*, q_n - h(a)\rangle - g(h(a))$ and $c_n := f(p_n) - \langle u_n^*, p_n - x_n\rangle + \langle q_n^*, h(x_n) - h(a)\rangle - f(a) \ \forall n \in \mathbb{N}$. We prove that if the condition

$$\begin{cases} (x_n, p_n, q_n, q_n'') \in X \times \operatorname{dom}(f) \times \operatorname{dom}(g) \times Y, u_n^* \in \partial f(p_n), q_n^* + e_n^* \in \partial g(q_n), \\ u_n^{*\prime} \in \partial(q_n^* h)(x_n), \langle q_n^*, q_n'' - h(x_n)\rangle = 0 \ \forall n \in \mathbb{N} \text{ and} \\ u_n^* + u_n^{*\prime} \to 0, e_n^* \to 0, x_n \to a, q_n \to h(a) \ (n \to +\infty) \end{cases} \tag{10}$$

is satisfied, then we have

$$a_n \to 0 \ (n \to +\infty) \text{ if and only if } b_n \to 0 \text{ and } c_n \to 0 \ (n \to +\infty). \tag{11}$$

Indeed, if (10) is fulfilled, then

$$a_n = b_n + c_n \ , \tag{12}$$

hence the sufficiency of relation (11) is trivial. We point out that for this implication we need only the fulfillment of $\langle q_n^*, q_n'' - h(x_n)\rangle = 0 \ \forall n \in \mathbb{N}$.

Assume now that $a_n \to 0$ $(n \to +\infty)$. Since $u_n^* \in \partial f(p_n)$ we have $f(a) - f(p_n) \geq \langle u_n^*, a - p_n\rangle \ \forall n \in \mathbb{N}$. Moreover, $u_n^{*\prime} \in \partial(q_n^* h)(x_n)$, hence $\langle q_n^*, h(a)\rangle - \langle q_n^*, h(x_n)\rangle \geq \langle u_n^{*\prime}, a - x_n\rangle \ \forall n \in \mathbb{N}$. We obtain that $c_n \leq \langle u_n^*, p_n - a\rangle + \langle u_n^{*\prime}, x_n - a\rangle - \langle u_n^*, p_n - x_n\rangle = \langle u_n^* + u_n^{*\prime}, x_n - a\rangle$. Also, from $q_n^* + e_n^* \in \partial g(q_n)$ we get $g(h(a)) - g(q_n) \geq \langle q_n^* + e_n^*, h(a) - q_n\rangle$ and so

$$b_n \leq \langle q_n^* + e_n^*, q_n - h(a)\rangle - \langle q_n^*, q_n - h(a)\rangle = \langle e_n^*, q_n - h(a)\rangle.$$

On the other hand,

$$b_n = a_n - c_n \geq a_n - \langle u_n^* + u_n^{*\prime}, x_n - a\rangle.$$

Combining the last two inequalities we obtain $b_n \to 0$ $(n \to +\infty)$. From (12) we also get $c_n \to 0$ $(n \to +\infty)$ and hence (11) is fulfilled.

Thus the condition (9) is equivalent to (4) and the proof is complete. □

In the following corollary we give a sequential characterization of the subgradients of the function $g \circ h$ at $a \in \mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g))$.

Corollary 3.7 *For $a \in \mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g))$ we have $x^* \in \partial(g \circ h)(a)$ if* *and only if*

$$\begin{cases} \exists(x_n, q_n, q_n') \in X \times \operatorname{dom}(g) \times Y, h(x_n) \leq_K q_n', \exists(e_n^*, x_n^*, q_n^*), q_n^* \in K^*, \\ q_n^* + e_n^* \in \partial g(q_n), x_n^* \in \partial(q_n^* h)(x_n), \langle q_n^*, q_n' - h(x_n)\rangle = 0 \ \forall n \in \mathbb{N}, \\ x_n \to a, q_n \to h(a), q_n' \to h(a), x_n^* \to x^*, e_n^* \to 0 \ (n \to +\infty), \\ g(q_n) - \langle q_n^*, q_n - h(a)\rangle - g(h(a)) \to 0 \ (n \to +\infty) \ \textit{and} \\ \langle q_n^*, h(x_n) - h(a)\rangle \to 0 \ (n \to +\infty). \end{cases} \tag{13}$$

Proof. We have $x^* \in \partial(g \circ h)(a) \Leftrightarrow 0 \in \partial(-x^* + g \circ h)(a) \Leftrightarrow a$ is an optimal solution of the problem (P) with $f : X \to \mathbb{R}$, $f(x) = \langle -x^*, x\rangle, \forall x \in X$. According to Theorem 3.6, we get that $x^* \in \partial(g \circ h)(a)$ if and only if

$$\begin{cases} \exists(x_n, p_n, q_n, q_n') \in X \times X \times \operatorname{dom}(g) \times Y, h(x_n) \leq_K q_n', \exists(e_n^*, u_n^{*\prime}, q_n^*), q_n^* \in K^*, \\ q_n^* + e_n^* \in \partial g(q_n), u_n^{*\prime} \in \partial(q_n^* h)(x_n), \langle q_n^*, q_n' - h(x_n)\rangle = 0 \ \forall n \in \mathbb{N}, \\ x_n \to a, p_n \to a, q_n \to h(a), q_n' \to h(a), u_n^{*\prime} \to x^*, e_n^* \to 0 \ (n \to +\infty), \\ g(q_n) - \langle q_n^*, q_n - h(a)\rangle - g(h(a)) \to 0 \ (n \to +\infty) \text{ and} \\ \langle q_n^*, h(x_n) - h(a)\rangle \to 0 \ (n \to +\infty), \end{cases} \tag{14}$$

where we used the continuity of the function f and the fact that $\partial f(x) = \{-x^*\} \ \forall x \in X$. The desired conclusion follows easily, since in the condition (14) the sequence p_n is superfluous (we made the notation $x_n^* := u_n^{*\prime} \ \forall n \in \mathbb{N}$). $\square$

Remark 3.8 Corollary 3.7 above is exactly the result given by Thibault in Theorem 3.1 in [18].

3.3.2 The case h is continuous

Consider again the problem

$$(P) \quad \inf_{x \in X} [f(x) + g(h(x))],$$

with the following hypotheses: $f : X \to \overline{\mathbb{R}}$ is proper, convex and lower semicontinuous, Y is partially ordered by a nonempty convex cone K, $h : X \to Y$ is K-convex and continuous, $g : Y \to \overline{\mathbb{R}}$ is proper, convex, lower semicontinuous and K-increasing on Y. We want to mention that, unlike in the previous subsection, the results in this subsection hold even Y is not reflexive. Suppose that $\mathrm{dom}(f) \cap h^{-1}(\mathrm{dom}(g)) \neq \emptyset$ and consider the perturbation function $\phi : X \times Y \to \overline{\mathbb{R}}$,

$$\phi(x,y) = f(x) + g(h(x) + y) \ \forall (x,y) \in X \times Y, \tag{15}$$

which is in this situation proper, convex and lower semicontinuous. The conjugate function $\phi^* : X^* \times Y^* \to \overline{\mathbb{R}}$ has for all $(x^*, y^*) \in X^* \times Y^*$ the following form

$$\phi^*(x^*, y^*) = \begin{cases} (f + y^* h)^*(x^*) + g^*(y^*), & \text{if } y^* \in K^*, \\ +\infty, & \text{otherwise,} \end{cases}$$

where we took into consideration that $g^*(y^*) = +\infty \ \forall y^* \in Y^* \setminus K^*$. By means of the general result Theorem 3.1 applied for this perturbation function we obtain the following sequential optimality conditions for (P).

Theorem 3.9 *The element $a \in \mathrm{dom}(f) \cap h^{-1}(\mathrm{dom}(g))$ is an optimal solution of the problem (P) if and only if*

$$\begin{cases} \exists (x_n, y_n) \in \mathrm{dom}(f) \times \mathrm{dom}(g), \exists (u_n^*, v_n^*, y_n^*) \in X^* \times X^* \times K^*, \\ u_n^* \in \partial f(x_n), v_n^* \in \partial (y_n^* h)(x_n), y_n^* \in \partial g(y_n) \ \forall n \in \mathbb{N}, \\ u_n^* + v_n^* \to 0, x_n \to a, y_n \to h(a) \ (n \to +\infty), \\ f(x_n) + \langle y_n^*, h(x_n) - h(a) \rangle - f(a) \to 0 \ (n \to +\infty) \ \textit{and} \\ g(y_n) - \langle y_n^*, y_n - h(a) \rangle - g(h(a)) \to 0 \ (n \to +\infty). \end{cases} \tag{16}$$

Proof. Applying Theorem 3.1 we obtain that $a \in \mathrm{dom}(f) \cap h^{-1}(\mathrm{dom}(g))$ is an optimal solution of the problem (P) if and only if

$$\begin{cases} \exists (x_n, y_n) \in X \times Y, x_n \in \mathrm{dom}(f), h(x_n) + y_n \in \mathrm{dom}(g), \\ \exists (x_n^*, y_n^*) \in \partial \phi(x_n, y_n) \ \forall n \in \mathbb{N}, x_n^* \to 0, x_n \to a, y_n \to 0 \ (n \to +\infty) \text{ and} \\ \phi(x_n, y_n) - \langle y_n^*, y_n \rangle - \phi(a, 0) \to 0 \ (n \to +\infty). \end{cases} \tag{17}$$

The condition $(x_n^*, y_n^*) \in \partial \phi(x_n, y_n)$ is equivalent to $y_n^* \in K^*$ and $f(x_n) + g(h(x_n) + y_n) + (f + y_n^* h)^*(x_n^*) + g^*(y_n^*) = \langle x_n^*, x_n \rangle + \langle y_n^*, y_n \rangle \ \forall n \in \mathbb{N}$. Using the Young-Fenchel inequality one can see that for all $n \in \mathbb{N}$:

$$f(x_n) + (y_n^* h)(x_n) + (f + y_n^* h)^*(x_n^*) - \langle x_n^*, x_n \rangle \geq 0$$

and

$$g(h(x_n) + y_n) + g^*(y_n^*) - \langle y_n^*, h(x_n) + y_n \rangle \geq 0.$$

Since the sum of the terms in the left-hand side of the inequalities above is equal to zero, both of them must be equal to zero. This is the case if and only if $x_n^* \in$

$\partial(f+y_n^*h)(x_n)$ and $y_n^* \in \partial g(h(x_n)+y_n)\ \forall n \in \mathbb{N}$. Hence $a \in \mathrm{dom}(f) \cap h^{-1}(\mathrm{dom}(g))$ is an optimal solution of (P) if and only if

$$\begin{cases} \exists (x_n, y_n) \in X \times Y, x_n \in \mathrm{dom}(f), h(x_n) + y_n \in \mathrm{dom}(g), \\ \exists (x_n^*, y_n^*) \in X^* \times K^*, x_n^* \in \partial(f + y_n^* h)(x_n), y_n^* \in \partial g(h(x_n) + y_n)\ \forall n \in \mathbb{N}, \\ x_n^* \to 0, x_n \to a, y_n \to 0\ (n \to +\infty) \text{ and} \\ f(x_n) + g(h(x_n) + y_n) - \langle y_n^*, y_n \rangle - f(a) - g(h(a)) \to 0\ (n \to +\infty). \end{cases} \tag{18}$$

The function h being continuous, the following subdifferential sum formula holds:

$$\partial(f + y_n^* h)(x_n) = \partial f(x_n) + \partial(y_n^* h)(x_n)\ \forall n \in \mathbb{N} \tag{19}$$

(see Theorem 2.8.7 in [19]). Thus $x_n^* \in \partial(f + y_n^* h)(x_n)$ if and only if there exist $u_n^* \in \partial f(x_n)$ and $v_n^* \in \partial(y_n^* h)(x_n)$ such that $x_n^* = u_n^* + v_n^*\ \forall n \in \mathbb{N}$. Introducing a new variable by $y_n' := h(x_n) + y_n\ \forall n \in \mathbb{N}$ and employing once more the continuity of the function h we get that (18) is equivalent to

$$\begin{cases} \exists (x_n, y_n') \in \mathrm{dom}(f) \times \mathrm{dom}(g), \exists (u_n^*, v_n^*, y_n^*) \in X^* \times X^* \times K^*, \\ u_n^* \in \partial(f)(x_n), v_n^* \in \partial(y_n^* h)(x_n), y_n^* \in \partial g(y_n')\ \forall n \in \mathbb{N}, \\ u_n^* + v_n^* \to 0, x_n \to a, y_n' \to h(a)\ (n \to +\infty) \text{ and} \\ f(x_n) + g(y_n') - \langle y_n^*, y_n' - h(x_n) \rangle - f(a) - g(h(a)) \to 0\ (n \to +\infty). \end{cases} \tag{20}$$

Let us consider now the following real sequences: $\alpha_n := f(x_n) + g(y_n') - \langle y_n^*, y_n' - h(x_n) \rangle - f(a) - g(h(a))$, $\beta_n := f(x_n) - f(a) + \langle y_n^*, h(x_n) - h(a) \rangle$ and $\gamma_n := g(y_n') - g(h(a)) - \langle y_n^*, y_n' - h(a) \rangle\ \forall n \in \mathbb{N}$. We have $\alpha_n = \beta_n + \gamma_n\ \forall n \in \mathbb{N}$

and if the condition

$$\begin{cases} (x_n, y_n') \in \operatorname{dom}(f) \times \operatorname{dom}(g), (u_n^*, v_n^*, y_n^*) \in X^* \times X^* \times K^*, \\ u_n^* \in \partial(f)(x_n), v_n^* \in \partial(y_n^* h)(x_n), y_n^* \in \partial g(y_n') \ \forall n \in \mathbb{N}, \\ u_n^* + v_n^* \to 0, x_n \to a, \ (n \to +\infty), \end{cases} \tag{21}$$

is satisfied, then

$$\alpha_n \to 0 \ (n \to +\infty) \text{ if and only if } \beta_n \to 0 \text{ and } \gamma_n \to 0 \ (n \to +\infty). \tag{22}$$

We omit the proof of (22), since it can be done in the lines of the one given for the relation (11) in the proof of Theorem 3.6. Hence the condition (20) is equivalent to (16). □

Taking in the previous result $f : X \to \mathbb{R}, f(x) = \langle -x^*, x \rangle \ \forall x \in X$, where $x^* \in X^*$ is fixed, we get the following corollary.

Corollary 3.10 *For $a \in h^{-1}(\operatorname{dom}(g))$ we have $x^* \in \partial(g \circ h)(a)$ if and only if*

$$\begin{cases} \exists (x_n, y_n) \in X \times \operatorname{dom}(g), \exists (v_n^*, y_n^*) \in X^* \times K^*, v_n^* \in \partial(y_n^* h)(x_n), y_n^* \in \partial g(y_n), \\ v_n^* \to x^*, x_n \to a, y_n \to h(a) \ (n \to +\infty), \\ g(y_n) - \langle y_n^*, y_n - h(a) \rangle - g(h(a)) \to 0 \ (n \to +\infty) \ \textit{and} \\ \langle y_n^*, h(x_n) - h(a) \rangle \to 0 \ (n \to +\infty). \end{cases} \tag{23}$$

Remark 3.11 The above sequential characterization of an arbitrary $x^* \in \partial(g \circ h)(a)$ was given by Thibault in case X and Y are both reflexive Banach spaces, K is a closed convex normal cone and g is K-increasing on $h(X)+K$ (see Corollary 3.2 in [18]). We proved that if the function g is K-increasing on Y, then this result holds even if the cone K is not normal and Y is an arbitrary Banach space. Moreover, the closedness condition regarding the cone K, requested by Thibault in [18], is not needed anymore.

Remark 3.12 (a) One can prove that the perturbation function defined at the beginning of the subsection 3.3.2 is lower semicontinuous even in the more general case when h is star K-lower semicontinuous (this follows because of $\phi^{**} = \phi$). This means that it is possible to derive sequential optimality conditions even in this case. Nevertheless, in order to obtain the result given by Thibault (Corollary 3.2 in [18]), we have to suppose that h is continuous, as this fact was used twice in the proof of Theorem 3.9 above. Even if the subdifferential sum formula (19) holds also in the case h is star K-lower semicontinuous and f is continuous (because we take $f = -x^*$ in order to obtain the result of Thibault), we still need the continuity of the function h in order to ensure that the sequence y_n' has the limit $h(a)$ as $n \to +\infty$ (see the equivalence between the conditions (18) and (20) in the proof of Theorem 3.9).

(b) Under the hypotheses mentioned in the beginning of the subsection 3.3.1 one can not prove that the perturbation function ϕ defined in the relation (15) is lower semicontinuous and hence in case h is K-epi-closed, Theorem 3.1 is not applicable for this perturbation function. This is one of the reasons why the first sequential optimality condition for the composed convex optimization problem (P), namely Theorem 3.6, is derived via Corollary 3.5, a result which is given for an optimization problem with cone constraints (of course, Corollary 3.5 is obtained from the general result Theorem 3.1). Another reason is that the condition g is K-increasing on $h(\mathrm{dom}(h)) + K$ (which is the case in subsection 3.3.1) is not sufficient in order to guarantee the convexity of the above mentioned perturbation function. In order to ensure the convexity of this function ϕ, g has to be K-increasing on Y, which is actually the case in subsection 3.3.2.

References

[1] Aït Mansour, M.A., Metrane, A., Théra, M. (2006): *Lower semicontinuous regularization for vector-valued mappings*, Journal of Global Optimization 35, 283-309.

[2] Boţ, R.I., Csetnek, E.R., Wanka, G. (2008): *Sequential optimality conditions in convex programming via perturbation approach*, Journal of Convex Analysis, 15(1).

[3] Boţ, R.I., Grad, S.-M., Wanka, G. (2008): *On strong and total Lagrange duality for convex optimization problems*, Journal of Mathematical Analysis and Applications 337(2), 1315–1325.

[4] Brøndsted, A., Rockafellar, R.T. (1965): *On the subdifferential of convex functions*, Proceedings of the American Mathematical Society 16, 605–611.

[5] Combari, C., Laghdir, M., Thibault, L. (1994): *Sous-différentiels de fonctions convexes composées*, Annales des Sciences Mathématiques du Québec 18(2), 119–148.

[6] Ekeland, I., Temam, R. (1976): *Convex analysis and variational problems*, North-Holland Publishing Company, Amsterdam.

[7] Ioffe, A. (2006): *Three theorems on subdifferentiation of convex integral functionals*, Journal of Convex Analysis 13 (3-4), 759–772.

[8] Jeyakumar, V., Lee, G.M., Dinh, N. (2003): *New sequential Lagrange multiplier conditions characterizing optimality without constraint qualifications for convex programs*, SIAM Journal on Optimization 14(2), 534–547.

[9] Jeyakumar, V., Wu, Z.Y. (2006): *A qualification free sequential Pshenichnyi-Rockafellar Lemma and convex semidefinite programming*, Journal of Convex Analysis 13 (3-4), 773–784.

[10] Jeyakumar, V., Wu, Z.Y., Lee, G.M., Dinh, N. (2006): *Liberating the subgradient optimality conditions from constraint qualifications*, Journal of Global Optimization 36(1), 127–137.

[11] Luc, D.T., (1989): *Theory of vector optimization*, Springer-Verlag, Berlin.

[12] Penot, J.P. (1996): *Subdifferential calculus without qualification assumptions*, Journal of Convex Analysis 3(2), 207–219.

[13] Penot, J.P., Théra, M. (1982): *Semi-continuous mappings in general topology*, Archiv der Mathematik 38, 158-166.

[14] Pshenichnyi, B.N. (1971): *Necessary conditions for an extremum*, Marcel Dekker, New York.

[15] Rockafellar, R.T. (1970): *Convex analysis*, Princeton University Press, Princeton.

[16] Tanaka, T. (1997): *Generalized semicontinuity and existence theorems for cone saddle points*, Applied Mathematics and Optimization 36, 313-322.

[17] Thibault, L. (1994): *A short note on sequential convex subdifferential calculus* (unpublished paper).

[18] Thibault, L. (1997): *Sequential convex subdifferential calculus and sequential Lagrange multipliers*, SIAM Journal on Control and Optimization 35(4), 1434–1444.

[19] Zălinescu, C. (2002): *Convex analysis in general vector spaces*, World Scientific, Singapore.