New regularity conditions for strong and total Fenchel-Lagrange duality in infinite dimensional spaces

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Abstract. We give new regularity conditions for convex optimization problems in separated locally convex spaces. We completely characterize the *stable strong* and *strong Fenchel-Lagrange duality*. Then we give similar statements for the case when a solution of the primal problem is assumed as known, obtaining complete characterizations for the so-called *total* and, respectively, *stable total Fenchel-Lagrange duality*. For particular settings the conditions we consider turn into some constraint qualifications already used by different authors, like Farkas-Minkowski CQ, locally Farkas-Minkowski CQ and basic CQ and we rediscover and improve some recent results in the literature.

Keywords. Conjugate functions, Fenchel-Lagrange dual, constraint qualifications, (locally) Farkas-Minkowski condition, stable strong duality

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1 Introduction

To a convex optimization problem

$$(P) \qquad \inf_{\substack{x \in U, \\ g(x) \in -C}} f(x),$$

one can attach different dual problems. Usually the classical Lagrange dual problem is considered, but the recently introduced Fenchel-Lagrange-type duals gather more and more attention. These duals can be obtained via perturbations or by

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constructing Fenchel dual problems to the infimum problem that appears in the formulation of the Lagrange dual. In the recent paper [1] the authors have dealt with both Lagrange and Fenchel-Lagrange duality, giving weak conditions that ensure *strong* duality and, respectively, completely characterizing the so-called *stable strong* duality.

In another recent paper [2] we went further, completely characterizing via subdifferentials the so-called *total* Lagrange duality, namely the situation when there is strong duality, but moreover a solution of the primal problem is assumed to be known, for all the convex optimization problems for which the objective functions satisfy some weak conditions. There we have also completely characterized via subdifferentials the *stable total* Lagrange duality and via epigraphs the strong Lagrange duality for all the convex optimization problems with the objective functions fulfilling some weak hypotheses.

In this paper we extend the investigations and results to three types of Fenchel-Lagrange-type dual problems one can introduce to (P). New complete characterizations via epigraphs are given for stable strong and strong duality, respectively via subdifferentials for stable total and total duality for each type of Fenchel-Lagrange dual problem we consider. The conditions we use generalize the constraint qualifications called Farkas-Minkowski (FM), locally Farkas-Minkowski (LFM) and the basic constraint qualification (BCQ) treated in works like [8–10, 12, 13, 16, 17, 21]. We also show how our results generalize some recent ones from [5, 8, 10, 21]. The conditions we consider in this paper belong to the recently introduced class of so-called closedness type conditions and they are weaker than the generalized interiority type regularity conditions, as it is proved for instance in [4]. Let us also mention that, unlike most of the papers in locally convex spaces with topological assumptions on the functions, we consider the constraint function C-epi-closed, neither C-lower semicontinuous as introduced in [19], nor star C-lower semicontinuous as in [10, 15]. Thus we work in a more general framework. An example we gave in [2] sustains this option.

The paper is organized as follows. In order to make the paper self-contained, we dedicate Section 2 to the necessary preliminaries. In Section 3 we deal with strong and stable strong Fenchel-Lagrange duality, completely characterizing them via conditions involving epigraphs. Section 4 is dedicated to similar characterizations, this time for total and stable total duality and the equivalent conditions use subdifferentials. Then we give optimality conditions for the problem (P). A short conclusive section closes the paper.

2 Preliminaries

Consider two separated locally convex vector spaces X and Y and their continuous dual spaces X^* and Y^*, endowed with the weak* topologies $w(X^*, X)$ and $w(Y^*, Y)$, respectively. Let the nonempty closed convex cone $C \subseteq Y$ and its dual cone $C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \ \forall y \in Y\}$ be given, where we denote by $\langle y^*, y\rangle = y^*(y)$ the value at y of the continuous linear functional y^*. On Y we consider the partial order induced by C, "$\leq_C$", defined by $z \leq_C y \Leftrightarrow y - z \in C$, $z, y \in Y$. To Y we attach a greatest element with respect to "$\leq_C$" which does not belong to Y denoted by ∞_Y and let $Y^\bullet = Y \cup \{\infty_Y\}$. Then for any $y \in Y^\bullet$ one has $y \leq_C \infty_Y$ and we consider on $Y^\bullet$ the following operations: $y + \infty_Y = \infty_Y + y = \infty_Y$ and $t \cdot \infty_Y = \infty_Y$ for all $y \in Y$ and all $t \geq 0$. Denote also the set of nonnegative real numbers by $\mathbb{R}_+ = [0, +\infty)$.

Given a subset U of X, by $\mathrm{cl}(U)$ we denote its *closure* in the corresponding topology, its *boundary* by $\mathrm{bd}(U)$, while its *indicator* function $\delta_U : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and, respectively, *support* function $\sigma_U : X^* \to \overline{\mathbb{R}}$ are defined as follows

$$\delta_U(x) = \begin{cases} 0, & \text{if } x \in U, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } \sigma_U(x^*) = \sup_{x \in U} \langle x^*, x\rangle.$$

Next we give some notions regarding functions.

For a function $f : X \to \overline{\mathbb{R}}$ we have

- the *domain*: $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$,
- the *epigraph*: $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$,
- the *conjugate regarding the set* $U \subseteq X$: $f_U^* : X^* \to \overline{\mathbb{R}}$ given by $f_U^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in U\}$,
- f is *proper*: $f(x) > -\infty \ \forall x \in X$ and $\mathrm{dom}(f) \neq \emptyset$,
- the *subdifferential* of f at x, where $f(x) \in \mathbb{R}$: $\partial f(x) = \{x^* \in X^* : f(u) - f(x) \geq \langle x^*, u - x\rangle \ \forall u \in X\}$.

One can easily notice that $\delta_U^* = \sigma_U$. When $U = X$ the conjugate regarding the set U is the classical *(Fenchel-Moreau) conjugate* function of f denoted by f^*. For a function and its conjugate regarding some set $U \subseteq X$ the *Young-Fenchel inequality* holds

$$f_U^*(x^*) + f(x) \geq \langle x^*, x\rangle \ \forall x \in U \ \forall x^* \in X^*.$$

Given a proper function $f : X \to \overline{\mathbb{R}}$, for all $x \in \mathrm{dom}(f)$ and $x^* \in X^*$ one has

$$x^* \in \partial f(x) \Leftrightarrow f^*(x^*) + f(x) = \langle x^*, x\rangle.$$

For two proper functions $f, g : X \to \overline{\mathbb{R}}$, we always have $\partial f(x) + \partial g(x) \subseteq \partial(f + g)(x) \ \forall x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. The *infimal convolution* of f and g is defined by

$$f \Box g : X \to \overline{\mathbb{R}}, \ \big(f \Box g\big)(a) = \inf\{f(x) + g(a - x) : x \in X\},$$

and it is said to be *exact* at some $a \in X$ when there is an $x \in X$ such that $\big(f \Box g\big)(a) = f(x) + g(a - x)$. Let us also recall a result from [3], needed later.

Lemma 1. *Let $f, g : X \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions, with the intersection of their domains nonempty. Then*

$$\operatorname{epi}((f+g)^*) = \operatorname{cl}(\operatorname{epi}(f^* \Box g^*)) = \operatorname{cl}(\operatorname{epi}(f^*) + \operatorname{epi}(g^*)).$$

There are notions given for functions with extended real values that can be formulated also for functions having their ranges in infinite dimensional spaces.

For a function $g : X \to Y^\bullet$ one has

- the *domain*: $\operatorname{dom}(g) = \{x \in X : g(x) \in Y\}$,
- g is *proper*: $\operatorname{dom}(g) \neq \emptyset$,
- g is *C-convex*: $g(tx + (1-t)y) \leq_C tg(x) + (1-t)g(y)\ \forall x, y \in X\ \forall t \in [0,1]$,
- for $\lambda \in C^*$, $(\lambda g) : X \to \overline{\mathbb{R}}$, $(\lambda g)(x) = \langle \lambda, g(x) \rangle$ for $x \in \operatorname{dom}(g)$ and $(\lambda g)(x) = +\infty$ otherwise,
- the *C-epigraph* $\operatorname{epi}_C(g) = \{(x, y) \in X \times Y : y \in g(x) + C\}$,
- g is C-epi-*closed* if $\operatorname{epi}_C(g)$ is closed,
- g is *star C-lower-semicontinuous* at $x \in X$: (λg) is lower-semicontinuous at $x\ \forall \lambda \in C^*$,
- for a subset $W \subseteq Y$: $g^{-1}(W) = \{x \in X : \exists z \in W \text{ s.t. } g(x) = z\}$.

Remark 1. There are other extensions of lower semicontinuity for functions taking values in infinite dimensional spaces used in convex optimization, we mention here just the C-lower semicontinuity, introduced in [19] and refined in [6]. Between these types of generalized lower semicontinuity there is the following relation (cf. [15, 18, 19])

C-lower semicontinuity $\Rightarrow$ star C-lower semicontinuity $\Rightarrow$ C-epi-closedness.

The opposite implications do not always hold, see [2] for an example of a C-convex function, which is C-epi-closed, but not star C-lower semicontinuous or [19] for a function which is C-epi-closed but not C-lower semicontinuous (in this example the function is not C-convex). We work here with C-epi-closedness, i.e. in the most general framework.

The following statement was given in [14] and [15] under the assumption of continuity, respectively star C-lower semicontinuity, alongside C-convexity, for the vector function involved. We give it here in a more general way, by considering the function g C-convex and only C-epi-closed.

Lemma 2. *Let $U \subseteq X$ be a nonempty convex closed set and a proper, C-convex and C-*epi-*closed function $g : X \to Y^\bullet$ such that $U \cap g^{-1}(-C) \neq \emptyset$. Then*

$$\operatorname{epi}(\sigma_{U\cap g^{-1}(-C)}) = \operatorname{cl}(\operatorname{epi}(\sigma_U) + \bigcup_{\lambda\in C^*} \operatorname{epi}((\lambda g)^*)) = \operatorname{cl}\Big(\bigcup_{\lambda\in C^*} \operatorname{epi}((\lambda g)^*_U)\Big).$$

Proof. The first equality was proven in Lemma 1 in [2]. For each $\lambda \in C^*$ one has $(\lambda g)^*_U(y) \leq \sigma_U \square (\lambda g)^*(y)\ \forall y \in X^*$, which yields $\operatorname{epi}(\sigma_U) + \operatorname{epi}((\lambda g)^*) \subseteq \operatorname{epi}((\lambda g)^*_U)$, immediately followed by $\operatorname{epi}(\sigma_{U\cap g^{-1}(-C)}) \subseteq \operatorname{cl}(\cup_{\lambda\in C^*} \operatorname{epi}((\lambda g)^*_U))$. On the other hand, $\delta_{U\cap g^{-1}(-C)}(x) \geq \delta_U(x)+(\lambda g)(x)\ \forall x \in X\ \forall \lambda \in C^*$, thus $\delta^*_{U\cap g^{-1}(-C)}$ $(y) \leq (\delta_U+(\lambda g))^*(y)\ \forall y \in X^*\ \forall \lambda \in C^*$, which yields $\operatorname{epi}(\sigma_{U\cap g^{-1}(-C)}) \supseteq \operatorname{epi}((\lambda g)^*_U)$ $\forall \lambda \in C^*$. Consequently, we obtain $\operatorname{epi}(\sigma_{U\cap g^{-1}(-C)}) \supseteq \operatorname{cl}(\cup_{\lambda\in C^*} \operatorname{epi}((\lambda g)^*_U))$, and so we are done. $\square$

Taking $U = X$ this turns into the following characterization.

Corollary 1. *Given the proper, C-convex and C-*epi-*closed function $g : X \to Y^\bullet$ fulfilling $g^{-1}(-C) \neq \emptyset$, there is*

$$\operatorname{epi}(\sigma_{g^{-1}(-C)}) = \operatorname{cl}(\bigcup_{\lambda\in C^*} \operatorname{epi}((\lambda g)^*)).$$

From the general case we get as special cases some results previously given for semi-infinite systems of convex inequalities. This is the reason why we recall some notations used in the literature on semi-infinite programming. Let T be a possibly infinite index set and denote by $\mathbb{R}^T$ the space of all functions $x : T \to \mathbb{R}$, endowed with the product topology and with the operations being the usual pointwise ones. For simplicity, denote $x_t = x(t)\ \forall x \in \mathbb{R}^T\ \forall t \in T$. The dual space of $\mathbb{R}^T$ is $(\mathbb{R}^T)^*$, the *space of generalized finite sequences* $\lambda = (\lambda_t)_{t\in T}$ such that $\lambda_t \in \mathbb{R}\ \forall t \in T$, and with finitely many λ_t different from zero. The positive cone in $\mathbb{R}^T$ is $\mathbb{R}^T_+ = \{x \in \mathbb{R}^T : x_t = x(t) \geq 0\ \forall t \in T\}$, and its dual is the positive cone in $(\mathbb{R}^T)^*$, namely $(\mathbb{R}^T_+)^* = \{\lambda = (\lambda_t)_{t\in T} \in (\mathbb{R}^T)^* : \lambda_t \geq 0\ \forall t \in T\}$.

In order to avoid repetitions we introduce here the framework and the optimization problems we will use later in the paper.

The spaces X and Y are defined like in the beginning of the section, the latter being ordered by the nonempty convex closed cone C. Let U be a nonempty convex closed subset of X. Consider the proper, C-convex and, unless otherwise specified, C-epi-closed function $g : X \to Y^\bullet$ such that the set $\mathcal{A} = U \cap g^{-1}(-C) = \{x \in U : g(x) \in -C\}$ is nonempty. It is clear from the way it is defined that

$\mathcal{A}$ is convex and closed. For a proper convex and lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ which satisfies $\mathcal{A} \cap \operatorname{dom}(f) \neq \emptyset$ consider the optimization problem

$$(P) \qquad \inf_{x \in \mathcal{A}} f(x).$$

For any $p \in X^*$ we also consider the linearly perturbed optimization problem

$$(P_p) \qquad \inf_{x \in \mathcal{A}} [f(x) + \langle p, x \rangle].$$

To (P_p) one can attach the Lagrange dual problem

$$(D_p^L) \qquad \sup_{\lambda \in C^*} \inf_{x \in U} [f(x) + \langle p, x \rangle + (\lambda g)(x)].$$

Let us focus on the inner minimization problem that appears in (D_p^L). For any $\lambda \in C^*$ it can be rewritten as

$$\inf_{x \in X} [f(x) + \langle p, x \rangle + \delta_U(x) + (\lambda g)(x)].$$

To this problem one can attach different Fenchel dual problems, obtaining via (D_p^L) different Fenchel-Lagrange-type dual problems to (P_p). The name Fenchel-Lagrange is given to the following dual problems because they are "combinations" of the classical Fenchel and Lagrange dual problems. Keeping together f and δ_U, respectively (λg) and $\langle p, \cdot \rangle$, one gets the following Fenchel dual problem to the minimization problem given above $\sup_{\beta \in X^*} [-f_U^*(\beta) - (\lambda g)^*(-p - \beta)]$ and the Fenchel-Lagrange-type dual problem obtained in this way to (P_p) is

$$(\widetilde{D}_p) \qquad \sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-f_U^*(\beta) - (\lambda g)^*(-p - \beta)\}.$$

When (λg), δ_U and $\langle p, \cdot \rangle$ remain together, the Fenchel-Lagrange-type dual problem to (P_p) becomes

$$(\bar{D}_p) \qquad \sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{-f^*(\beta) - (\lambda g)_U^*(-p - \beta)\}.$$

Finally, when f and (λg) are separated, while $\langle p, \cdot \rangle$ and δ_U stay together, the following Fenchel-Lagrange-type dual problem to (P_p) is obtained

$$(D_p) \qquad \sup_{\substack{\lambda \in C^*, \\ \beta, \alpha \in X^*}} \{-f^*(\beta) - \sigma_U(-p - \alpha) - (\lambda g)^*(\alpha - \beta)\}.$$

For $p = 0$ we call these problems $(\widetilde{D})$, $(\bar{D})$ and, respectively, (D) and they are Fenchel-Lagrange-type dual problems to (P). These dual problems can be obtained also by means of the perturbation theory. For more on this subject see [4].

Remark 2. When (λg) and f and, respectively, δ_U and $\langle p, \cdot \rangle$, are grouped together one can obtain also the following Fenchel-Lagrange-type dual problem to (P_p)

$$(D'_p) \qquad \sup_{\substack{\lambda \in C^*, \\ \beta \in X^*}} \{ -(f + (\lambda g))^*(\beta) - \sigma_U(-p - \beta) \}.$$

This dual will not be mentioned further in the paper, but one can notice that results similar to the ones obtained for the other three dual problems can be given for (D'_p), too.

For the optimization problem (P) we denote by $v(P)$ its optimal objective value and this notation is extended to all the optimization problems we use in this paper. There is always weak duality for (P_p) and its duals, i.e. the optimal objective value of (P_p) is always greater than or equal to the optimal objective values of each of its duals. For each $p \in X^*$, the optimal objective values of the problems considered above fulfill the following inequalities

$$v(D_p) \leq \begin{array}{c} v(\bar{D}_p) \\ v(\widetilde{D}_p) \end{array} \leq v(D_p^L) \leq v(P_p).$$

Below there are some examples, given in case $p = 0$, where the inequalities involving the optimal objective values of the three Fenchel-Lagrange-type dual problems are strictly fulfilled.

Example 1. (see [20]) Let $X = \mathbb{R}^2$, equipped with the Euclidean norm $\|(\cdot, \cdot)\|$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and the functions $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^2 \to (\mathbb{R}^2)^\bullet = \mathbb{R}^2 \cup \{\infty_{\mathbb{R}^2}\}$. Denote the closed ball centered in the origin with radius $\lambda > 0$ by $B_\lambda = \{(x_1, x_2) \in \mathbb{R}^2 : \|(x_1, x_2)\| \leq \lambda\}$.

(i) Let $U = B_1 - (1, 0)$, $f(x_1, x_2) = x_1 + x_2$ $\forall (x_1, x_2) \in \mathbb{R}^2$ and $g(x_1, x_2) = (x_1, x_2)$ if $(x_1, x_2) \in B_1 + (1, 0)$ and $g(x_1, x_2) = \infty_{\mathbb{R}^2}$ otherwise. For them we have $f^* = \delta_{\{(1,1)\}}$, $\sigma_U(y_1, y_2) = \|(y_1, y_2)\| - y_1$, $f_U^*(y_1, y_2) = \sigma_U(y_1 - 1, y_2 - 1)$, $(\lambda g)^*(y_1, y_2) = \|(y_1 - \lambda_1, y_2 - \lambda_2)\| + y_1 - \lambda_1$ and $(\lambda g)_U^*(y_1, y_2) = 0$ $\forall (y_1, y_2) \in \mathbb{R}^2$, where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2_+$. With these conjugates, we obtain $v(D) = v(\widetilde{D}) = 1 - \sqrt{2} < 0 = v(\bar{D})$.

(ii) Consider the real numbers a and b such that $0 < b \leq a$. Take $U = B_a - (a, 0)$, $f(x_1, x_2) = \delta_{B_b + (b,0)}(x_1, x_2) + x_1 + x_2$ and $g(x_1, x_2) = (0, 0)$ $\forall (x_1, x_2) \in \mathbb{R}^2$. For every $(y_1, y_2) \in \mathbb{R}^2$ we have $f^*(y_1, y_2) = b\|(y_1, y_2) - (1, 1)\| + b(y_1 - 1)$, $\sigma_U(y_1, y_2) = a\|(y_1, y_2)\| - ay_1$, $f_U^*(y_1, y_2) = 0$, and for all $\lambda \in \mathbb{R}^2_+$ $(\lambda g)^* = \delta_{\{(0,0)\}}$ and $(\lambda g)_U^* = \sigma_U$. With these conjugates, we obtain $v(D) = v(\bar{D}) = 1 - \sqrt{2} < 0 = v(\widetilde{D})$.

Remark 3. From these examples we see that in general no order can be established between $v(\bar{D})$ and $v(\widetilde{D})$.

Let us recall that by *strong duality* we understand the situation when the optimal objective values of the primal and dual problem coincide and the dual problem has an optimal solution. When there is strong duality and the primal problem has an optimal solution, too, we say that we have *total duality*. With *stable strong/total duality* we refer to the situation when there is strong/total duality for all the optimization problems obtained by linearly perturbing the objective function of the primal problem. We also write min (max) instead of inf (sup) when the infimum (supremum) is attained.

3 New characterizations for strong Fenchel-Lagrange duality

We introduce first some closedness conditions which completely characterize stable strong Fenchel-Lagrange duality regarding an optimization problem of the type (P). Let us consider the following regularity conditions for f and $\mathcal{A}$

$$(C_1(f,\mathcal{A})) \qquad \operatorname{epi}(f^*) + \operatorname{epi}(\sigma_U) + \underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*) \text{ is closed,}$$

$$(C_2(f,\mathcal{A})) \qquad \operatorname{epi}(f^*) + \underset{\lambda\in C^*}{\cup} \operatorname{epi}(((\lambda g) + \delta_U)^*) \text{ is closed}$$

and

$$(C_3(f,\mathcal{A})) \qquad \operatorname{epi}((f+\delta_U)^*) + \underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*) \text{ is closed.}$$

As one can notice, these regularity conditions require the closedness of some sets constructed with the epigraphs of the conjugates of the functions involved in the primal optimization problem (P). Such conditions are said to be of closedness type. Despite being recently introduced, conditions belonging to this class were treated in papers like [1–5, 8, 10, 14–16], because they proved to be weaker than the classical generalized interior type regularity conditions considered so far in the literature for the same kinds of convex optimization problems.

Because f and $\delta_{\mathcal{A}}$ are proper, convex and lower semicontinuous, one has by Lemma 1 $\operatorname{epi}(f+\delta_{\mathcal{A}})^* = \operatorname{cl}(\operatorname{epi}(f^*)+\operatorname{epi}(\sigma_{\mathcal{A}}))$. By Lemma 2, this is further equal to $\operatorname{cl}(\operatorname{epi}(f^*)+\operatorname{cl}(\operatorname{epi}(\sigma_U)+\cup_{\lambda\in C^*}\operatorname{epi}((\lambda g)^*)))$, which is actually $\operatorname{cl}(\operatorname{epi}(f^*)+\operatorname{epi}(\sigma_U)+\cup_{\lambda\in C^*}\operatorname{epi}\,((\lambda g)^*))$. Thus

$$\operatorname{epi}(f+\delta_{\mathcal{A}})^* = \operatorname{cl}\Big(\operatorname{epi}(f^*) + \operatorname{epi}(\sigma_U) + \underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*)\Big). \tag{1}$$

On the other hand, using Lemma 2 and the considerations in its proof we get

$$\begin{aligned}\operatorname{epi}(f^*) + \operatorname{epi}(\sigma_U) + \underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*) &\subseteq \operatorname{epi}(f^*) + \underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*_U)\\ &\subseteq \operatorname{epi}(f^*) + \operatorname{epi}(\sigma_{\mathcal{A}}) \subseteq \operatorname{epi}(f+\delta_{\mathcal{A}})^*,\end{aligned}$$

which yields via (1)

$$\operatorname{cl}(\operatorname{epi}(f^*)+\operatorname{epi}(\sigma_U)+\underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*))=\operatorname{cl}(\operatorname{epi}(f^*)+\underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*_U))=\operatorname{epi}(f+\delta_{\mathcal{A}})^*,$$

thus the fulfillment of $(C_1(f,\mathcal{A}))$ guarantees the satisfaction of $(C_2(f,\mathcal{A}))$.

We also know that Lemma 1 states that $\operatorname{epi}((f+\delta_U)^*) = \operatorname{cl}(\operatorname{epi}(f^*)+\operatorname{epi}(\sigma_U))$, which yields

$$\operatorname{epi}(f^*) + \operatorname{epi}(\sigma_U) + \underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*) \subseteq \operatorname{epi}((f+\delta_U)^*) + \underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*).$$

Moreover, via (1) one has

$$\begin{aligned}\operatorname{cl}(\operatorname{epi}((f+\delta_U)^*)+\underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*))&=\operatorname{cl}(\operatorname{cl}(\operatorname{epi}(f^*)+\operatorname{epi}(\sigma_U))+\underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*))\\ &= \operatorname{cl}(\operatorname{epi}(f^*) + \operatorname{epi}(\sigma_U) + \underset{\lambda\in C^*}{\cup} \operatorname{epi}((\lambda g)^*)) = \operatorname{epi}(f+\delta_{\mathcal{A}})^*.\end{aligned}$$

It is obvious now that the validity of $(C_1(f,\mathcal{A}))$ implies also the satisfaction of $(C_3(f,\mathcal{A}))$. Moreover, analogously it can be shown that when any of the three conditions we introduced above is valid, so is the following regularity condition used in [2]

$$(C(f,\mathcal{A})) \qquad \underset{\lambda\in C^*}{\cup} \operatorname{epi}((f+(\lambda g)+\delta_U)^*) \text{ is closed.}$$

The condition $(C_1(f,\mathcal{A}))$ was used in [5] in order to characterize the optimal solutions of the problem (P), while the condition $(C_2(f,\mathcal{A}))$ was introduced in [1] for stable Fenchel-Lagrange duality. These regularity conditions completely characterize the stable Fenchel-Lagrange duality for (P) and the mentioned dual problems as follows.

Theorem 1. *The set $\mathcal{A}$ and the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ satisfy condition $(C_1(f,\mathcal{A}))$ if and only if there is stable strong duality for the problems (P) and (D), i.e. one has strong duality for the pair of problems (P_p) and (D_p) for all $p \in X^*$.*

Proof. "$\Rightarrow$" Let $p \in X^*$. It is known that weak duality for the pair of problems (P_p) and (D_p) always holds, namely $v(P_p) \geq v(D_p)$. If $v(P_p) = -\infty$, then we get strong duality for (P_p) and (D_p) via weak duality, otherwise $v(P_p) \in \mathbb{R}$. We have $v(P_p) = -(f+\delta_{\mathcal{A}})^*(-p)$, thus $(-p, -v(P_p)) \in \operatorname{epi}(f+\delta_{\mathcal{A}})^*$.

Because of (1), the satisfaction of $(C_1(f,\mathcal{A}))$ guarantees the existence of some $\bar\lambda \in C^*$, $\bar\beta, \bar\alpha \in X^*$ such that $(\bar\beta, f^*(\bar\beta)) \in \operatorname{epi}(f^*)$, $(\bar\alpha - \bar\beta, (\bar\lambda g)^*(\bar\alpha - \bar\beta)) \in \operatorname{epi}((\bar\lambda g)^*)$ and $(-p-\bar\alpha, \sigma_U(-p-\bar\alpha)) \in \operatorname{epi}(\sigma_U)$, fulfilling moreover

$$f^*(\bar\beta) + (\bar\lambda g)^*(\bar\alpha - \bar\beta) + \sigma_U(-p-\bar\alpha) \leq -v(P_p).$$

By weak duality for (P_p) and (D_p) we obtain

$$v(P_p) = \max_{\substack{\lambda\in C^*,\\ \beta,\alpha\in X^*}} \{-f^*(\beta) + (\lambda g)^*(\alpha - \beta) - \sigma_U(-p-\alpha)\},$$

i.e. strong duality for the pair of problems (P_p) and (D_p). As $p \in X^*$ has been arbitrarily chosen we obtain that there is stable strong duality for the problems (P) and (D).

"$\Leftarrow$" Take some $(p,v) \in \operatorname{epi}(f+\delta_{\mathcal{A}})^*$. Because $v(P_{-p}) = -(f+\delta_{\mathcal{A}})^*(p)$, we have $-v(P_{-p}) \leq v$. There is strong duality for the problems (P_{-p}) and (D_{-p}), i.e. there are some $\bar\lambda \in C^*$, $\bar\beta, \bar\alpha \in X^*$ such that

$$f^*(\bar\beta) + (\bar\lambda g)^*(\bar\alpha - \bar\beta) + \sigma_U(p-\bar\alpha) = -v(P_{-p}) \leq v.$$

Consequently, $(\bar\beta, f^*(\bar\beta)) \in \operatorname{epi}(f^*)$, $(\bar\alpha - \bar\beta, (\bar\lambda g)^*(\bar\alpha - \bar\beta)) \in \operatorname{epi}((\bar\lambda g)^*)$ and $(p-\bar\alpha, v - f^*(\bar\beta) - (\bar\lambda g)^*(\bar\alpha - \bar\beta)) \in \operatorname{epi}(\sigma_U)$. Together, these yield $(p,v) \in \operatorname{epi}(f^*) + \operatorname{epi}((\bar\lambda g)^*) + \operatorname{epi}(\sigma_U)$. Since (p,v) was arbitrarily chosen in $\operatorname{epi}(f+\delta_{\mathcal{A}})^*$, by (1) we obtain the validity of $(C_1(f,\mathcal{A}))$. □

Analogously one can prove also the following two statements.

Theorem 2. (see [1]) *The set $\mathcal{A}$ and the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ satisfy condition $(C_2(f,\mathcal{A}))$ if and only if there is stable strong duality for the problems (P) and $(\bar D)$, i.e. one has strong duality for the pair of problems (P_p) and $(\bar D_p)$ for all $p \in X^*$.*

Theorem 3. *The set $\mathcal{A}$ and the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ satisfy condition $(C_3(f,\mathcal{A}))$ if and only if there is stable strong duality for the problems (P) and $(\widetilde D)$, i.e. one has strong duality for the pair of problems (P_p) and $(\widetilde D_p)$ for all $p \in X^*$.*

Remark 4. In the following we consider f not necessarily lower semicontinuous, respectively g not necessarily C-epi-closed. When f is continuous at some point of U, due to Theorem 2.8.7(*iii*) in [22] and Proposition 2.2 in [3] we get that $(C_1(f,\mathcal{A}))$ is equivalent to $(C_3(f,\mathcal{A}))$, while when f is continuous at some point of $U \cap \operatorname{dom}(g)$ $(C_2(f,\mathcal{A}))$ becomes condition $(C(f,\mathcal{A}))$ used in [2]. On the other hand, when g is continuous at some point of U, $(C_1(f,\mathcal{A}))$ is equivalent to $(C_2(f,\mathcal{A}))$, while when g is continuous at some point of $U \cap \operatorname{dom}(f)$ $(C_3(f,\mathcal{A}))$

becomes actually condition $(C(f,\mathcal{A}))$ in [2].

If we take $f(x)=0\ \forall x\in X$, conditions $(C_1(f,\mathcal{A}))$ and $(C_3(f,\mathcal{A}))$ collapse into

$(C_1(0,\mathcal{A}))$ $\qquad \operatorname{epi}(\sigma_U)+\bigcup\limits_{\lambda\in C^*}\operatorname{epi}((\lambda g)^*)$ is closed in the product topology of $(X^*,w(X^*,X))\times\mathbb{R}$.

In the literature $(C_1(0,\mathcal{A}))$ is known as the *closed cone constraint qualification* $(CCCQ)$. It was introduced in [14] as a weak condition for strong Lagrange duality for problems of type (P), while in [4] it was shown that it guarantees under some additional assumptions (for instance condition (FRC) which will be used later in this paper) also strong Fenchel-Lagrange duality for such problems. Note also that when g is continuous at some point of $\mathcal{A}$, $(CCCQ)$ is equivalent to saying that $\cup_{\lambda\in C^*}\operatorname{epi}((\lambda g)+\delta_U)^*$ is closed, i.e. condition $(C(0,\mathcal{A}))$ in [2]. Meanwhile, it is obvious that when $f(x)=0\ \forall x\in X$, conditions $(C_2(0,\mathcal{A}))$ and $(C(0,\mathcal{A}))$ from [2] coincide.

In the following statement we completely characterize by using the condition $(CCCQ)$ the strong duality for the problem of minimizing a linear continuous functional over $\mathcal{A}$ and its Fenchel-Lagrange-type dual problem. For the case $g:X\to Y$ continuous the following statement rediscovers Theorem 3.2 in [4]. Here it follows directly from Theorem 1.

Corollary 2. *$\mathcal{A}$ fulfills the condition $(CCCQ)$ if and only if for each $p\in X^*$ one has*

$$\inf_{x\in\mathcal{A}}\langle p,x\rangle=\max_{\substack{\lambda\in C^*,\\ \alpha\in X^*}}\{-\sigma_U(-p-\alpha)-(\lambda g)^*(\alpha)\}=\max_{\substack{\lambda\in C^*,\\ \alpha\in X^*}}\{-\sigma_U(-\alpha)-(\lambda g)^*(\alpha-p)\}.$$

Remark 5. Note also that for f everywhere equal to 0 Theorem 2 rediscovers as a special case Corollary 1 in [2].

Similar characterizations are available also for the strong duality for (P) and its Fenchel-Lagrange-type dual (D). Because in its formulation appear only the constraints of (P), we can say that $(CCCQ)$ is a *constraint qualification*. Though, $(CCCQ)$ is not enough to ensure strong duality for (P) and (D) and in order to achieve it one has to introduce some weak assumptions on the objective function f, too. Let us consider the following condition (cf. [5, 8, 10])

(CC) $\qquad \operatorname{epi}(f^*)+\operatorname{epi}(\sigma_{\mathcal{A}})$ is closed.

According to [3], if one removes the assumption of lower semicontinuity from f and takes it continuous at some point of $\mathcal{A}$, then condition (CC) is automatically satisfied.

Theorem 4. *$\mathcal{A}$ fulfills the condition $(CCCQ)$ if and only if for each proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ which satisfies $\mathcal{A} \cap \operatorname{dom}(f) \neq \emptyset$ and (CC) one has strong duality between (P) and (D), i.e.*

$$\inf_{x\in\mathcal{A}} f(x) = \max_{\substack{\lambda\in C^*,\\ \beta,\alpha\in X^*}} \{-f^*(\beta) - \sigma_U(-\alpha) - (\lambda g)^*(\alpha - \beta)\}.$$

Proof. The sufficiency follows from the previous corollary by taking f linear and continuous. To prove the necessity take first some function f which fulfills the hypothesis.

If $v(P) = -\infty$ weak duality for (P) and (D) yields that we are done, otherwise we have $v(P) \in \mathbb{R}$. It is obvious that $(f + \delta_{\mathcal{A}})^*(0) = -v(P)$. Further we have $(0, -v(P)) \in \operatorname{epi}((f + \delta_{\mathcal{A}})^*)$. Because of (CC) and Lemma 1, there is a $\bar{\beta} \in X^*$ such that $f^*(\bar{\beta}) + \sigma_{\mathcal{A}}(-\bar{\beta}) = -v(P)$, which can be rewritten as $\inf_{x\in\mathcal{A}}\langle\bar{\beta}, x\rangle = v(P) + f^*(\bar{\beta})$. By Corollary 2, $(CCCQ)$ yields that there are some $\bar{\lambda} \in C^*$ and $\bar{\alpha} \in X^*$ such that $\inf_{x\in\mathcal{A}}\langle\bar{\beta}, x\rangle = -\sigma_U(-\bar{\alpha}) - (\bar{\lambda}g)^*(\bar{\alpha} - \bar{\beta})$. Consequently, $-f^*(\bar{\beta}) - \sigma_U(-\bar{\alpha}) - (\bar{\lambda}g)^*(\bar{\alpha} - \bar{\beta}) = v(P)$. By weak duality one obtains that the necessity is proven, i.e. the optimal objective values of (P) and (D) coincide and $(\bar{\lambda}, \bar{\beta}, \bar{\alpha})$ solves (D). □

Before closing this section we show that the results we gave so far generalize some characterizations of the strong duality for optimization problems with (possibly) infinitely many real inequality constraints via the so-called Farkas-Minkowski property (FM) of a system of (infinitely many) convex or linear inequalities. Such assertions were treated in papers dealing with semi-infinite programming problems, like [8,10,11]. Note that (FM) is a special case of $(CCCQ)$.

Remark 6. When T is a possibly infinite index set consider the family of functions $g_t : X \to \overline{\mathbb{R}}$, each of them assumed to be proper, convex and continuous at some point of $\{x \in U : g_t(x) \leq 0 \ \forall t \in T\}$. Note that, unlike [8, 10], we do not ask the functions g_t, $t \in T$, to be also lower semicontinuous. Take $C = \mathbb{R}^T_+$, denote by $\infty_{\mathbb{R}^T}$ the element attached to $\mathbb{R}^T$ as the greatest with respect to the order induced by the positive cone, and let $(\mathbb{R}^T)^\bullet = \mathbb{R}^T \cup \{\infty_{\mathbb{R}^T}\}$. Consider the function

$$g : X \to (\mathbb{R}^T)^\bullet,\ g(x) = \begin{cases} (g_t(x))_{t\in T}, & \text{if } x \in \bigcap\limits_{t\in T} \operatorname{dom}(g_t), \\ \infty_{\mathbb{R}^T}, & \text{otherwise.} \end{cases}$$

Note that in this case we do not assume g to be $\mathbb{R}^T_+$-epi-closed. One can easily show that g is proper and $\mathbb{R}^T_+$-convex. In this situation the condition $(CCCQ)$ becomes, due to Theorem 2.8.7(iii) in [22] and Proposition 2.2 in [3], equivalent to $(C(0, \mathcal{A}))$ from [2]. By Remark 4 in [2] the latter is in this particular instance equivalent to saying that $\mathrm{epi}(\sigma_U) + \mathrm{cone}(\cup_{t \in T} \mathrm{epi}(g_t^*))$ is closed, which is actually the so-called condition *Farkas-Minkowski* (FM) in [10].

Remark 7. One can notice that when taking g as in Remark 6, Corollary 2 and Theorem 4 improve Theorem 4.1 in [10] in the sense that the results given there remain true also when removing the lower semicontinuity assumption from the functions g_t, $t \in T$.

4 New characterizations for total Fenchel-Lagrange duality

It is also interesting to study via duality the situation when the primal problem is assumed to have an optimal solution. Using the terminology from [2], in this case we denote strong duality by *total duality*. For a proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ and the set $\mathcal{A}$ we introduce the following regularity conditions at $x \in \mathcal{A} \cap \mathrm{dom}(f)$

$$(GBCQ_1(f,\mathcal{A})) \qquad \partial(f + \delta_{\mathcal{A}})(x) = \partial f(x) + \partial \delta_U(x) + \bigcup_{\substack{\lambda \in C^*,\\ (\lambda g)(x)=0}} \partial(\lambda g)(x),$$

$$(GBCQ_2(f,\mathcal{A})) \qquad \partial(f + \delta_{\mathcal{A}})(x) = \partial f(x) + \bigcup_{\substack{\lambda \in C^*,\\ (\lambda g)(x)=0}} \partial(\delta_U + (\lambda g))(x)$$

and

$$(GBCQ_3(f,\mathcal{A})) \qquad \partial(f + \delta_{\mathcal{A}})(x) = \partial(f + \delta_U)(x) + \bigcup_{\substack{\lambda \in C^*,\\ (\lambda g)(x)=0}} \partial(\lambda g)(x).$$

We say that f and $\mathcal{A}$ satisfy the condition $(GBCQ_i(f,\mathcal{A}))$ when $(GBCQ_i(f,\mathcal{A}))$ is valid for all $x \in \mathcal{A} \cap \mathrm{dom}(f)$, $i = 1, 2, 3$.

Take $\bar{x} \in \mathcal{A} \cap \mathrm{dom}(f)$. Then one has

$$\partial f(\bar{x}) + \partial \delta_U(\bar{x}) + \bigcup_{\substack{\lambda \in C^*,\\ (\lambda g)(\bar{x})=0}} \partial(\lambda g)(\bar{x}) \subseteq \partial f(\bar{x}) + \bigcup_{\substack{\lambda \in C^*,\\ (\lambda g)(\bar{x})=0}} \partial(\delta_U + (\lambda g))(\bar{x}) \tag{2}$$

and

$$\partial f(\bar{x}) + \partial \delta_U(\bar{x}) + \bigcup_{\substack{\lambda \in C^*,\\ (\lambda g)(\bar{x})=0}} \partial(\lambda g)(\bar{x}) \subseteq \partial(f + \delta_U)(\bar{x}) + \bigcup_{\substack{\lambda \in C^*,\\ (\lambda g)(\bar{x})=0}} \partial(\lambda g)(\bar{x}). \tag{3}$$

Let $p \in X^*$ such that $p \in \partial(f + \delta_U)(\bar{x}) + \cup_{\substack{\lambda \in C^*,\\ (\lambda g)(\bar{x})=0}} \partial(\lambda g)(\bar{x})$. Thus, there is a $\bar{\lambda} \in C^*$ such that $(\bar{\lambda} g)(\bar{x}) = 0$ fulfilling $p \in \partial(f + \delta_U)(\bar{x}) + \partial(\bar{\lambda} g)(\bar{x})$. Consequently, there is a $\bar{\beta} \in \partial(f + \delta_U)(\bar{x})$ such that $p - \bar{\beta} \in \partial(\bar{\lambda} g)(\bar{x})$, which means that for all $x \in X$ one has $f(x) + \delta_U(x) - f(\bar{x}) - \delta_U(\bar{x}) \geq \langle \bar{\beta}, x - \bar{x} \rangle$ and

$(\bar\lambda g)(x)-(\bar\lambda g)(\bar x) \geq \langle p-\bar\beta, x-\bar x\rangle$. Summing these inequalities up and noticing that $\delta_{\mathcal{A}}(x) \geq \delta_U(x)+(\bar\lambda g)(x)\ \forall x \in X$, we get $f(x)+\delta_{\mathcal{A}}(x)-f(\bar x)-\delta_{\mathcal{A}}(\bar x) \geq \langle p, x-\bar x\rangle$ $\forall x \in X$. This is nothing but $p \in \partial(f+\delta_{\mathcal{A}})(\bar x)$, therefore the inclusion "$\supseteq$" in the expression of $(GBCQ_3(f,\mathcal{A}))$ at $\bar x$ is secured. Analogously one can prove that "$\supseteq$" in the expression of $(GBCQ_2(f,\mathcal{A}))$ is valid at each $\bar x \in \mathcal{A}\cap \mathrm{dom}(f)$ and, via (2), that "$\supseteq$" in the expression of $(GBCQ_1(f,\mathcal{A}))$ holds at each $\bar x \in \mathcal{A}\cap\mathrm{dom}(f)$, too.

Remark 8. If $(GBCQ_1(f,\mathcal{A}))$ is fulfilled at any $\bar x \in \mathcal{A}\cap\mathrm{dom}(f)$ then

$$\partial(f+\delta_{\mathcal{A}})(\bar x) \subseteq \partial f(\bar x)+\partial\delta_U(\bar x)+\bigcup_{\substack{\lambda\in C^*,\\ (\lambda g)(x)=0}} \partial(\lambda g)(\bar x).$$

Hence, by (2) and (3) also $(GBCQ_2(f,\mathcal{A}))$ and $(GBCQ_3(f,\mathcal{A}))$ must be satisfied at $\bar x$. Moreover, one can verify in an analogous way that the fulfillment of any of these conditions at $\bar x \in \mathcal{A}\cap\mathrm{dom}(f)$ guarantees the validity at $\bar x$ of the following condition used in [2]

$$(GBCQ(f,\mathcal{A})) \qquad \partial(f+\delta_{\mathcal{A}})(\bar x) = \bigcup_{\substack{\lambda\in C^*,\\ (\lambda g)(\bar x)=0}} \partial(f+\delta_U+(\lambda g))(\bar x).$$

We show that these conditions completely characterize the stable total duality for (P) and its Fenchel-Lagrange-type dual problems (D), $(\bar D)$ and $(\widetilde D)$.

Theorem 5. *Let be given the proper convex lower semicontinuous function $f: X \to \overline{\mathbb{R}}$. $\mathcal{A}$ and f fulfill the condition $(GBCQ_1(f,\mathcal{A}))$ at $\bar x \in \mathcal{A}\cap\mathrm{dom}(f)$ if and only if for each $p \in X^*$ for which the infimum over $\mathcal{A}$ of the function $f+\langle p,\cdot\rangle$ is attained at $\bar x$ one has total duality for (P_p) and (D_p), i.e.*

$$f(\bar x)+\langle p,\bar x\rangle = \min_{x\in\mathcal{A}}[f(x)+\langle p,x\rangle] = \max_{\substack{\lambda\in C^*,\\ \alpha,\beta\in X^*}} \{-f^*(\beta)-\sigma_U(-p-\alpha)-(\lambda g)^*(\alpha-\beta)\}. \tag{4}$$

Proof. Let $\bar x \in \mathcal{A}\cap\mathrm{dom}(f)$. For a $p \in X^*$ it can be easily seen that $\bar x$ is an optimal solution of (P_p) if and only if $0 \in \partial(f+\langle p,\cdot\rangle+\delta_{\mathcal{A}})(\bar x)$, which is equivalent to $-p \in \partial(f+\delta_{\mathcal{A}})(\bar x)$.

"$\Rightarrow$" Let $p \in X^*$ such that $\bar x$ solves (P_p). Thus $-p \in \partial(f+\delta_{\mathcal{A}})(\bar x)$. Because the condition $(GBCQ_1(f,\mathcal{A}))$ is valid at $\bar x$, one gets a $\bar\lambda \in C^*$ such that $(\bar\lambda g)(\bar x)=0$ and $-p \in \partial f(\bar x)+\partial\delta_U(\bar x)+\partial(\bar\lambda g))(\bar x)$.

Thus there are some $\bar\beta \in \partial f(\bar x)$ and $\bar\alpha \in \partial(\bar\lambda g)(\bar x)$ such that $-p-\bar\alpha-\bar\beta \in \partial\delta_U(\bar x)$, i.e. $f(\bar x)+f^*(\bar\beta)=\langle\bar\beta,\bar x\rangle$, $(\bar\lambda g)^*(\bar\alpha)+(\bar\lambda g)(\bar x)=\langle\bar\alpha,\bar x\rangle$ and $\sigma_U(-p-\bar\alpha-\bar\beta)+\delta_U(\bar x)=\langle -p-\bar\alpha-\bar\beta,\bar x\rangle$. Note that $\delta_U(\bar x)=(\bar\lambda g)(\bar x)=0$. Summing up the equalities obtained above we get

$$f(\bar x)+f^*(\bar\beta)+(\bar\lambda g)^*(\bar\alpha)+\sigma_U(-p-\bar\alpha-\bar\beta) = -\langle p,\bar x\rangle,$$

which, taking into consideration the way p was chosen, can be rewritten as

$$v(P_p) = f(\bar{x}) + \langle p, \bar{x} \rangle = -f^*(\bar{\beta}) - (\bar{\lambda} g)^*(\bar{\alpha}) - \sigma_U(-p - \bar{\alpha} - \bar{\beta}). \tag{5}$$

Remark that the expression in the right-hand side of (5) is less than or equal to $v(D_p)$, thus we get $v(P_p) \leq v(D_p)$. Because the weak duality for the two problems means that the latter inequality is satisfied also in the reverse direction, we get $v(P_p) = v(D_p)$. Moreover, by (5) we see that $(\bar{\lambda}, \bar{\beta}, \bar{\alpha} - \bar{\beta})$ solves (D_p), therefore (4) holds.

"$\Leftarrow$" Take now $p \in \partial(f + \delta_{\mathcal{A}})(\bar{x})$. By the considerations from the beginning of the proof this means that $\bar{x}$ is an optimal solution to (P_{-p}). By (4) there are some $\bar{\lambda} \in C^*$ and $\bar{\alpha}, \bar{\beta} \in X^*$ such that

$$f(\bar{x}) - \langle p, \bar{x} \rangle = -f^*(\bar{\beta}) - \sigma_U(p - \bar{\alpha}) - (\bar{\lambda} g)^*(\bar{\alpha} - \bar{\beta}). \tag{6}$$

By the Young-Fenchel inequality we have that the expression in the right-hand side of (6) is at most $f(\bar{x}) + (\bar{\lambda} g)(\bar{x}) + \delta_U(\bar{x}) - \langle p, \bar{x} \rangle$. As $\delta_U(\bar{x}) = 0$, this implies $(\bar{\lambda} g)(\bar{x}) \geq 0$. On the other hand, $(\bar{\lambda} g)(\bar{x}) \leq 0$ because of the feasibility of $\bar{x}$ to (P_{-p}). Consequently, $(\bar{\lambda} g)(\bar{x}) = 0$. Using this, we can rewrite (6) in the following way

$$\begin{aligned} &\big(f(\bar{x}) + f^*(\bar{\beta}) - \langle \bar{\beta}, \bar{x} \rangle\big) + \big(\delta_U(\bar{x}) + \sigma_U(p - \bar{\alpha}) - \langle p - \bar{\alpha}, \bar{x} \rangle\big) \\ &\quad + \big((\bar{\lambda} g)(\bar{x}) + (\bar{\lambda} g)^*(\bar{\alpha} - \bar{\beta}) - \langle \bar{\alpha} - \bar{\beta}, \bar{x} \rangle\big) = 0. \end{aligned}$$

Applying the Young-Fenchel inequality for the functions involved above we get that in each case it is fulfilled as equality, thus $\bar{\beta} \in \partial f(\bar{x})$, $\bar{\alpha} - \bar{\beta} \in \partial(\bar{\lambda} g)(\bar{x})$ and $p - \bar{\alpha} \in \partial \delta_U(\bar{x})$. The inclusion "$\subseteq$" in the expression of $(GBCQ_1(f, \mathcal{A}))$ at $\bar{x}$ follows at once, and, since "$\supseteq$" holds, too, we obtain the validity of $(GBCQ_1(f, \mathcal{A}))$. □

In an analogous way one can verify the next two theorems.

Theorem 6. *Let be given the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$. $\mathcal{A}$ and f fulfill the condition $(GBCQ_2(f, \mathcal{A}))$ at $\bar{x} \in \mathcal{A} \cap \operatorname{dom}(f)$ if and only if for each $p \in X^*$ for which the infimum over $\mathcal{A}$ of the function $f + \langle p, \cdot \rangle$ is attained at $\bar{x}$ one has total duality for (P_p) and $(\bar{D}_p)$.*

Theorem 7. *Let be given the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$. $\mathcal{A}$ and f fulfill the condition $(GBCQ_3(f, \mathcal{A}))$ at $\bar{x} \in \mathcal{A} \cap \operatorname{dom}(f)$ if and only if for each $p \in X^*$ for which the infimum over $\mathcal{A}$ of the function $f + \langle p, \cdot \rangle$ is attained at $\bar{x}$ one has total duality for (P_p) and $(\widetilde{D}_p)$.*

The following statements follow naturally.

Theorem 8. *Let be given the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$.*

(i) $\mathcal{A}$ *and* f *fulfill the condition* $(GBCQ_1(f,\mathcal{A}))$ *if and only if for each* $p \in X^*$ *for which the infimum over* $\mathcal{A}$ *of the function* $f + \langle p, \cdot \rangle$ *is attained one has total duality for* (P_p) *and* (D_p).

(ii) $\mathcal{A}$ *and* f *fulfill the condition* $(GBCQ_2(f,\mathcal{A}))$ *if and only if for each* $p \in X^*$ *for which the infimum over* $\mathcal{A}$ *of the function* $f + \langle p, \cdot \rangle$ *is attained one has total duality for* (P_p) *and* $(\bar{D}_p)$.

(iii) $\mathcal{A}$ *and* f *fulfill the condition* $(GBCQ_3(f,\mathcal{A}))$ *if and only if for each* $p \in X^*$ *for which the infimum over* $\mathcal{A}$ *of the function* $f + \langle p, \cdot \rangle$ *is attained one has total duality for* (P_p) *and* $(\widetilde{D}_p)$.

Remark 9. Like in Remark 4, consider now f not necessarily lower semicontinuous, respectively g not necessarily C-epi-closed. When f is continuous at some point of U, $(GBCQ_1(f,\mathcal{A}))$ is equivalent to $(GBCQ_3(f,\mathcal{A}))$, while when f is continuous at some point of $U \cap \operatorname{dom}(g)$ $(GBCQ_2(f,\mathcal{A}))$ becomes condition $(GBCQ(f,\mathcal{A}))$ used in [2]. On the other hand, when g is continuous at some point of U $(GBCQ_1(f,\mathcal{A}))$ is equivalent to $(GBCQ_2(f,\mathcal{A}))$, while when g is continuous at some point of $U \cap \operatorname{dom}(f)$ $(GBCQ_3(f,\mathcal{A}))$ becomes actually condition $(GBCQ(f,\mathcal{A}))$ in [2].

When $f(x) = 0$ for all $x \in X$, $(GBCQ_1(f,\mathcal{A}))$ and $(GBCQ_3\ (f,\mathcal{A}))$ at some $x \in \mathcal{A}$ turn both into a condition which generalizes the classical basic constraint qualification at x

$$(GBCQ_1(0,\mathcal{A})) \qquad \partial\delta_{\mathcal{A}}(x) = \partial\delta_U(x) + \bigcup_{\substack{\lambda \in C^*,\\ (\lambda g)(x)=0}} \partial(\lambda g)(x).$$

Remark 10. Note that $(GBCQ_2(0,\mathcal{A}))$ at $x \in \mathcal{A}$ is actually condition $(GBCQ\ (0,\mathcal{A}))$ at x from [2]. If g is continuous at some point of $\mathcal{A}$, $(GBCQ_1(0,\mathcal{A}))$ coincides with $(GBCQ(0,\mathcal{A}))$.

If the set $\mathcal{A}$ satisfies $(GBCQ_1(0,\mathcal{A}))$ for all $x \in \mathcal{A}$ we say that it fulfills $(GBCQ_1(0,\mathcal{A}))$.

A direct consequence of Theorem 5 and also of Theorem 7 is the next result, where the condition $(GBCQ_1(0,\mathcal{A}))$ at some $\bar{x} \in \mathcal{A}$ completely characterizes the total Fenchel-Lagrange duality for optimization problems consisting in minimizing linear functionals that attain their minimum over $\mathcal{A}$ at $\bar{x}$.

Corollary 3. $\mathcal{A}$ *fulfills the condition* $(GBCQ_1(0,\mathcal{A}))$ *at* $\bar{x} \in \mathcal{A}$ *if and only if for each* $p \in X^*$ *such that* $\langle p, \cdot \rangle$ *attains its minimum over* $\mathcal{A}$ *at* $\bar{x}$ *one has*

$$\langle p, \bar{x} \rangle = \min_{x \in \mathcal{A}} \langle p, x \rangle = \max_{\substack{\lambda \in C^*,\\ \alpha \in X^*}} \{-\sigma_U(-p-\alpha) - (\lambda g)^*(\alpha)\}.$$

Note that when f takes everywhere the value zero Theorem 5 collapses into Theorem 6 in [2]. The next theorem completely characterizes via $(GBCQ_1(0,\mathcal{A}))$ at some $\bar{x} \in \mathcal{A}$ the strong duality for convex optimization problems consisting in minimizing over the set $\mathcal{A}$ of proper convex lower semicontinuous functions $f : X \to \overline{\mathbb{R}}$ which attain their minima over $\mathcal{A}$ at $\bar{x}$ and fulfill the following condition (see [3])

(FRC) $\qquad f^* \Box \delta_{\mathcal{A}}^*$ is a lower semicontinuous function and it is exact at 0,

and their Fenchel-Lagrange-type dual problems.

Remark 11. The condition (FRC) is implied by (CC), consult [3] for details. There one finds an example for which (FRC) holds, unlike (CC). Consequently if one removes the assumption of lower semicontinuity from f and takes it continuous at some point of $\mathcal{A}$, then condition (FRC) is automatically satisfied.

Theorem 9. *$\mathcal{A}$ fulfills the condition $(GBCQ_1(0,\mathcal{A}))$ at $\bar{x} \in \mathcal{A}$ if and only if for each proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ that fulfills $\mathcal{A} \cap \mathrm{dom}(f) \neq \emptyset$, attains its minimum over $\mathcal{A}$ at $\bar{x}$ and satisfies (FRC) one has total duality for (P) and (D), i.e.*

$$f(\bar{x}) = \inf_{x \in \mathcal{A}} f(x) = \max_{\substack{\lambda \in C^*,\\ \alpha,\beta \in X^*}} \{-f^*(\beta) - \sigma_U(-\alpha) - (\lambda g)^*(\alpha - \beta)\}.$$

Proof. Because the sufficiency follows from the preceding theorem by taking f linear, we prove here only the necessity. Take some f as requested in the hypothesis. We have

$$f(\bar{x}) = \inf_{x \in \mathcal{A}} f(x) = -(f + \delta_{\mathcal{A}})^*(0)$$

and (FRC) guarantees (see [4]) that there is some $\bar{\beta} \in X^*$ such that $(f+\delta_{\mathcal{A}})^*(0) = f^*(\bar{\beta}) + \sigma_{\mathcal{A}}(-\bar{\beta})$. Further we get

$$0 = \big(f(\bar{x}) + f^*(\bar{\beta}) - \langle \bar{\beta}, \bar{x} \rangle\big) + \big(\sigma_{\mathcal{A}}(-\bar{\beta}) + \delta_{\mathcal{A}}(\bar{x}) - \langle -\bar{\beta}, \bar{x} \rangle\big) \geq 0,$$

therefore the Young-Fenchel inequality applied for both pairs f and f^*, and $\delta_{\mathcal{A}}$ and $\sigma_{\mathcal{A}}$, respectively, is fulfilled as equality, i.e. $\bar{\beta} \in \partial f(\bar{x})$ and $-\bar{\beta} \in \partial \delta_{\mathcal{A}}(\bar{x})$. As $(GBCQ_1(0,\mathcal{A}))$ is satisfied at $\bar{x}$, there are a $\bar{\lambda} \in C^*$ and a $\bar{\alpha} \in X^*$ such that $(\bar{\lambda} g)(\bar{x}) = 0$, $-\bar{\alpha} \in \partial \delta_U(\bar{x})$ and $\bar{\alpha} - \bar{\beta} \in \partial(\bar{\lambda} g)(\bar{x})$. Thus $\sigma_U(-\bar{\alpha}) = \langle -\bar{\alpha}, \bar{x} \rangle$ and $(\bar{\lambda} g)^*(\bar{\alpha} - \bar{\beta}) = \langle \bar{\alpha} - \bar{\beta}, \bar{x} \rangle$, therefore $(\bar{\lambda} g)^*(\bar{\alpha} - \bar{\beta}) + \sigma_U(-\bar{\alpha}) = -\langle \bar{\beta}, \bar{x} \rangle = -f(\bar{x}) - f^*(\bar{\beta})$, i.e.

$$f(\bar{x}) = -(\bar{\lambda} g)^*(\bar{\alpha} - \bar{\beta}) - \sigma_U(-\bar{\alpha}) - f^*(\bar{\beta}).$$

The right-hand side of the equality above is less than or equal to $v(D)$, while the left-hand side coincides, by hypothesis, with $v(P)$. We get $v(P) \leq v(D)$, but the opposite inequality stands also because of weak duality. Therefore $v(P) = v(D)$ and, because there are some feasible points where the objective functions of these two problems coincide, we have total duality. $\square$

Such statements are valid also for $(GBCQ_1(0, \mathcal{A}))$ as follows.

Theorem 10. *The following statements are equivalent:*

(i) *$\mathcal{A}$ fulfills condition $(GBCQ_1(0, \mathcal{A}))$,*

(ii) *for each $p \in X^*$ that attains its minimum over $\mathcal{A}$ one has*

$$\langle p, \bar{x} \rangle = \inf_{x \in \mathcal{A}} \langle p, x \rangle = \max_{\substack{\lambda \in C^*, \\ \alpha \in X^*}} \{-\sigma_U(-p - \alpha) - (\lambda g)^*(\alpha)\},$$

(iii) *for each proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ that attains its minimum over $\mathcal{A}$ and satisfies (FRC) and $\mathcal{A} \cap \operatorname{dom}(f) \neq \emptyset$ one has*

$$\inf_{x \in \mathcal{A}} f(x) = \max_{\substack{\lambda \in C^*, \\ \alpha, \beta \in X^*}} \{-f^*(\beta) - \sigma_U(-\alpha) - (\lambda g)^*(\alpha - \beta)\}.$$

Remark 12. Using Remark 4.2 in [4], one can show that Theorems 9 and 10 can be proven in a slightly more general context, namely by asking the functions $f^* \square \delta^*_{\mathcal{A}}$ to be lower semicontinuous only at 0 and exact at 0, instead of requiring f to satisfy (FRC).

Remark 13. Let T be a possibly infinite index set and let g be as in Remark 6. In this setting the condition $(GBCQ_1(0, \mathcal{A}))$ at x becomes, due to Theorem 2.8.7(*iii*) in [22], equivalent to $(GBCQ(0, \mathcal{A}))$ from [2]. By Remark 9 in [2] the latter is in this particular instance equivalent to the so-called *locally Farkas-Minkowski* condition at x (cf. [8,9])

$$(LFM) \qquad \partial \delta_U(x) + \operatorname{cone}\Big(\bigcup_{t \in T(x)} \partial g_t(x) \Big) = \partial \delta_{\mathcal{A}}(x),$$

where $T(x) = \{t \in T : g_t(x) = 0\}$, which is known also under the name *basic constraint qualification* (BCQ) at x (cf. [8]). In this case $(GBCQ_1(0, \mathcal{A}))$ becomes exactly the condition (LFM) in [10].

Remark 14. If $g : X \to (\mathbb{R}^m)^\bullet$ and $U = X$, $(GBCQ_1(0, \mathcal{A}))$ is actually the condition (BCQ) considered in [21]. If $m = 1$, i.e. $g : X \to \overline{\mathbb{R}}$, when $C = \mathbb{R}_+$ $(GBCQ_1(0, \mathcal{A}))$ is actually the condition (5) in [21], while when $U = \operatorname{dom}(g)$

and $x \in \mathrm{bd}(\mathcal{A})$, $(GBCQ_1(0,\mathcal{A}))$ at x becomes the condition extended (BCQ) at x in [13]. Considering $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, the convex functions $c_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1,\dots,r$, and $\mathcal{A} = \{x \in \mathbb{R}^n : Ax = b, c_j(x) \leq 0,\ j = 1,...,r\}$, $(GBCQ_1(0,\mathcal{A}))$ becomes exactly the condition (BCQ) in its original formulation due to Hiriart-Urruty and Lemaréchal [12]. For comparisons between other constraint qualifications and different particular instances of (BCQ) we refer to [13,16,17,21].

Remark 15. When T is a possibly infinite index set and $g = (g_t)_{t\in T}$ such that each g_t, $t \in T$, is continuous at some point of $\mathcal{A}$ and $C = (\mathbb{R}^T_+)^*$, Theorem 10 yields, via Remark 13, a result similar to Theorem 5.1 in [10], improving it because the functions g_t, $t \in T$, are no more required to be lower semicontinuous as there and also in the sense that (ii) in the mentioned statement can be generalized by taking f not continuous at some point of $\mathcal{A} \cap \mathrm{dom}(f)$ like in the original paper, but only fulfilling the condition (FRC) or the weaker condition mentioned in Remark 12. Moreover, if T contains only one element, and when $C = \mathbb{R}_+$, Theorem 10 generalizes Proposition 2.5 in [21].

Remark 16. Comparing Theorems 1–3 and 5–7 one sees that $(C_i(f,\mathcal{A}))$ implies $(GBCQ_i(f,\mathcal{A}))$ for $i = 1,\dots,3$. Consequently, $(CCCQ)$ guarantees the fulfillment of $(GBCQ_1(0,\mathcal{A}))$. This observation generalizes Corollary 2 in [8]. See Example 4.1 in [10] for a situation where $(GBCQ_1(0,\mathcal{A}))$ is valid, while $(CCCQ)$ fails.

We conclude this section by giving optimality conditions for the problem (P).

Theorem 11. *If $\mathcal{A}$ fulfills the condition $(GBCQ_1(0,\mathcal{A}))$ and $f : X \to \overline{\mathbb{R}}$ is a proper convex lower semicontinuous function which satisfies (FRC), an $\bar{x} \in \mathcal{A} \cap \mathrm{dom}(f)$ is an optimal solution to (P) if and only if there is some $\bar{\lambda} \in C^*$ such that $(\bar{\lambda}g)(\bar{x}) = 0$ and $0 \in \partial f(\bar{x}) + \partial\delta_U(\bar{x}) + \partial(\bar{\lambda}g)(\bar{x})$.*

Proof. It is known that $f(\bar{x}) = v(P)$ if and only if $0 \in \partial(f + \delta_{\mathcal{A}})(\bar{x})$. Because (FRC) holds, by Theorem 3.2 in [3] this is further equivalent to $0 \in \partial f(\bar{x}) + \partial\delta_{\mathcal{A}}(\bar{x})$. We have also $(GBCQ_1(0,\mathcal{A}))$ fulfilled, thus $\bar{x}$ solves (P) if and only if

$$0 \in \partial f(\bar{x}) + \partial\delta_U(\bar{x}) + \bigcup_{\substack{\lambda\in C^*,\\ (\lambda g)(\bar{x})=0}} \partial(\lambda g)(x).$$

This means that there is some $\bar{\lambda} \in C^*$ fulfilling $(\bar{\lambda}g)(\bar{x}) = 0$ and $\partial(\bar{\lambda}g)(\bar{x}) \neq \emptyset$ such that $0 \in \partial f(\bar{x}) + \partial\delta_U(\bar{x}) + \partial(\bar{\lambda}g)(\bar{x})$. □

Remark 17. The previous statement is valid also when replacing condition (FRC) with the one considered in Remark 12. In the particular setting described in Remark 6, though without assuming the functions g_t, $t \in T$, continuous any-where and taking them only lower semicontinuous, in which case g is C-lower semicontinuous by Proposition 1.8 in [19], so also star C-lower semicontinuous, Theorem 11 rediscovers Theorem 3 in [8] and improves it in the sense that we take the function f to fulfill (FRC) instead of the stronger condition (CC). Note also that if g is continuous, Theorem 11 rediscovers and improves Theorem 4.2 in [5] and Corollary 5.8 in [7]. If moreover f is continuous, Theorem 11 rediscovers Corollary 3.2 in [14].

Using Theorem 9 one can also prove the following optimality conditions statement.

Theorem 12. *Let $\bar{x} \in \mathcal{A} \cap \mathrm{dom}(f)$. $\mathcal{A}$ fulfills the condition $(GBCQ_1(0, \mathcal{A}))$ at $\bar{x}$ if and only if for each proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ which satisfies (FRC) the fact that $\bar{x}$ is an optimal solution to (P) is equivalent with the existence of some $\bar{\lambda} \in C^*$ such that $(\bar{\lambda}g)(\bar{x}) = 0$ and $0 \in \partial f(\bar{x}) + \partial \delta_U(\bar{x}) + \partial(\bar{\lambda}g)(\bar{x})$.*

Remark 18. This statement generalizes and improves by dropping the continuity assumptions Theorem 4 in [8] and Theorem 4.1$(i) \Leftrightarrow (iii)$ in [16].

5 Conclusions and future research

We completely characterize the *strong*, *stable strong*, *total* and *stable total* Fenchel -Lagrange duality for convex optimization problems through equivalent conditions. By *total* duality we understand the situation when there is *strong* duality for the primal and the dual problem, i.e. their optimal objective values coincide and the dual has an optimal solution, but also a solution of the primal problem is assumed to be known. When strong duality, respectively, total duality takes place for all the problems obtained by perturbing with a linear function the objective function of the primal problem, we say that we have *stable* strong/total duality. To a given convex optimization problem we consider three Fenchel-Lagrange-type dual problems and for each of them we introduce conditions which are equivalent to stable strong, respectively to stable total duality. For all convex optimization problems with the objective functions satisfying some additional weak assumptions we completely characterize the strong and total Fenchel-Lagrange duality through some conditions derived from the previously mentioned ones. For convex optimization problems having (in)finitely many convex inequalities as constraints these conditions turn into the so-called conditions Farkas-Minkowski (FM), locally Farkas-Minkowski (LFM) and basic constraint qualification (BCQ). Different results in the literature are also rediscovered as special cases and some of them are improved in their original context.

Given the connections between the conditions we used in this paper and the (locally) Farkas-Minkowski conditions that completely characterize some Farkas-type results for systems of (in)finitely many convex inequalities, one could formulate such statements also for the conditions dealt with here and in [2].

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