The conjugate of the pointwise maximum of two convex functions revisited

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Abstract. In this paper we use the tools of the convex analysis in order to give a suitable characterization for the epigraph of the conjugate of the pointwise maximum of two proper, convex and lower semicontinuous functions in a normed space. By using this characterization we obtain, as a natural consequence, the formula for the biconjugate of the pointwise maximum of two functions, provided the so-called Attouch-Brézis regularity condition holds.

Key Words. pointwise maximum, conjugate functions, Attouch-Brézis regularity condition

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1 Introduction and preliminaries

Let X be a nontrivial normed space and X^* its topological dual space. By $\sigma(X^*, X)$ we denote the weak* topology induced by X on X^*, by $\|\cdot\|_{X^*}$ the dual norm of X^* and by $\langle x^*, x\rangle$ the value at $x \in X$ of the continuous linear functional $x^* \in X^*$. For a set $D \subseteq X$ we denote the *closure* and the *convex hull* of D by $\text{cl}(D)$ and $\text{co}(D)$, respectively. We also use the *strong quasi relative interior* of a nonempty convex set D denoted $\text{sqri}(D)$, which contains all the elements $x \in D$ for which the cone generated by $D - x$ is a closed linear subspace. Furthermore, the *indicator function* of a nonempty set $D \subseteq X$ is denoted by δ_D.

Considering now a function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, we denote by $\text{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *effective domain* and by $\text{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We say that $f : X \to \overline{\mathbb{R}}$ is *proper* if $f(x) > -\infty$ for all $x \in X$ and $\text{dom}(f) \neq \emptyset$. The (Fenchel-Moreau) *conjugate function* of f is $f^* : X^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}$. The *lower semicontinuous hull* of f is denoted by $\text{cl}(f)$, while the *biconjugate* of f is the function $f^{**} : X^{**} \to \overline{\mathbb{R}}$ defined by $f^{**}(x^{**}) = (f^*)^*(x^{**})$.

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Definition 1.1 *Let $f, g : X \to \overline{\mathbb{R}}$ be given. The function $f \Box g : X \to \overline{\mathbb{R}}$ defined by*

$$f \Box g(x) = \inf \{f(y) + g(x - y) : y \in X\}$$

is called the infimal convolution function of f and g. We say that $f \Box g$ is exact if for all $x \in X$ there exists some $y \in X$ such that $f \Box g(x) = f(y) + g(x - y)$.

Having two proper functions $f, g : X \to \overline{\mathbb{R}}$, we denote by $f \vee g : X \to \overline{\mathbb{R}}$, $f \vee g(x) = \max\{f(x), g(x)\}$ the pointwise maximum of f and g. In this paper we rediscover first the formula for the conjugate of $f \vee g$. This formula is a classical one in the convex analysis (see, for example, [5] and [7]), but we show how it can be obtained as a nice application of the Lagrange duality theory.

Then we represent $\text{epi}((f \vee g)^*)$ as the closure of the reunion of the epigraphs of the conjugates of all convex combinations $\lambda f + (1 - \lambda)g$, when $\lambda \in (0, 1)$, where the closure can be taken both in $(X^*, \sigma(X^*, X)) \times \mathbb{R}$ and in $(X^*, \|\cdot\|_{X^*}) \times \mathbb{R}$. This formulae turn out to be suitable in order to show that $(f \vee g)^{**} = f^{**} \vee g^{**}$, provided a regularity condition is fulfilled. In this way, on the one hand, we extend and, on the other hand, we give a simpler proof of Theorem 6 in [5].

Throughout this paper we assume that $f, g : X \to \overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions fulfilling $\text{dom}(f) \cap \text{dom}(g) \neq \emptyset$.

For all $x^* \in X^*$ we have

$$-(f \vee g)^*(x^*) = \inf_{\substack{x \in \text{dom}(f) \cap \text{dom}(g), y \in \mathbb{R}, \\ f(x) - y \leq 0, g(x) - y \leq 0}} \{y - \langle x^*, x \rangle\}.$$

Since between the convex optimization problem

$$\inf_{\substack{x \in \text{dom}(f) \cap \text{dom}(g), y \in \mathbb{R}, \\ f(x) - y \leq 0, g(x) - y \leq 0}} \{y - \langle x^*, x \rangle\}$$

and its Lagrange dual

$$\sup_{\lambda \geq 0, \mu \geq 0} \inf_{\substack{x \in \text{dom}(f) \cap \text{dom}(g), \\ y \in \mathbb{R}}} \{y - \langle x^*, x \rangle + \lambda(f(x) - y) + \mu(g(x) - y)\}$$

strong duality holds, we obtain

$$-(f \vee g)^*(x^*) = \max_{\lambda \in [0,1]} \inf_{x \in \text{dom}(f) \cap \text{dom}(g)} [\lambda f(x) + (1 - \lambda)g(x) - \langle x^*, x \rangle].$$

Throughout the paper we write max(min) instead of sup(inf) in order to point out that the supremum(infimum) is attained.

With the conventions $0f := \delta_{\text{dom}(f)}$ and $0g := \delta_{\text{dom}(g)}$ the conjugate of $f \vee g$ turns out to be (see also [5], [7])

$$\begin{aligned} (f \vee g)^*(x^*) &= \min_{\lambda \in [0,1]} \sup_{x \in X} \left[\langle x^*, x \rangle - \lambda f(x) - (1 - \lambda)g(x)\right] \\ &= \min_{\lambda \in [0,1]} \big(\lambda f + (1 - \lambda)g\big)^*(x^*). \end{aligned} \quad (1)$$

Obviously, (1) leads to the following formula for the epigraph of $(f \vee g)^*$

$$\mathrm{epi}((f \vee g)^*) = \bigcup_{\lambda \in [0,1]} \mathrm{epi}\left((\lambda f + (1-\lambda) g)^*\right)$$

$$= \bigcup_{\lambda \in (0,1)} \mathrm{epi}\left((\lambda f + (1-\lambda) g)^*\right) \bigcup \mathrm{epi}\left((f + \delta_{\mathrm{dom}(g)})^*\right) \bigcup \mathrm{epi}\left((g + \delta_{\mathrm{dom}(f)})^*\right). \quad (2)$$

2 An alternative formulation for $\mathrm{epi}((f \vee g)^*)$

In this section we assume first that the dual space X^* is endowed with the weak* topology $\sigma(X^*, X)$ and give, by using some tools of the convex analysis, a new alternative formulation for $\mathrm{epi}((f \vee g)^*)$, in case $f, g : X \to \overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions with $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$.

Let us start by remarking that for all $x \in X$

$$f \vee g(x) = \Big(\sup_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g) \Big)(x).$$

As for all $\lambda \in (0,1)$ the function $x \to \lambda f(x) + (1-\lambda) g(x)$ is proper, convex and lower semicontinuous, it must be equal to its biconjugate and so we have (the last equality follows from the definition of the conjugate function)

$$f \vee g = \sup_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g) =$$

$$\sup_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g)^{**} = \left[\inf_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g)^* \right]^*. \quad (3)$$

Proposition 2.1 *The function $x^* \to \inf_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g)^*(x^*)$, $x^* \in X^*$ is proper and convex.*

Proof. As $(\lambda f + (1-\lambda) g)^*$ is proper for all $\lambda \in (0,1)$ it follows that $x^* \to \inf_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g)^*(x^*)$ cannot be identical $+\infty$. If there exists an $x^* \in X^*$ such that $\inf_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g)^*(x^*) = -\infty$, then $f \vee g$ must be identical $+\infty$ and this contradicts $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$.

The properness being proved we show next the convexity. For this we consider the function $\Phi : X^* \times \mathbb{R} \to \overline{\mathbb{R}}$ defined by

$$\Phi(x^*, \lambda) = \begin{cases} (\lambda f + (1-\lambda) g)^*(x^*), & \text{if } x^* \in X^*, \lambda \in (0,1), \\ +\infty, & \text{otherwise.} \end{cases}$$

Since the function

$$(x^*, \lambda) \to (\lambda f + (1-\lambda) g)^*(x^*) = \sup_{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)} \{\langle x^*, x \rangle + \lambda (g(x) - f(x)) - g(x)\}$$

is convex on $X^* \times (0,1)$, being the pointwise supremum of a family of affine functions, it follows that Φ is also convex. By a well-known result from the convex analysis, the convexity of the *infimal value function* of Φ

$$x^* \to \inf_{\lambda \in \mathbb{R}} \Phi(x^*, \lambda) = \inf_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g)^*(x^*)$$

follows immediately and this concludes the proof. $\square$

By a similar argument like in the proof above one can easily see that $\mathrm{cl}_{(X^*,\sigma(X^*,X))\times\mathbb{R}}\left(\inf_{\lambda\in(0,1)}(\lambda f+(1-\lambda)g)^*\right)$ is also a proper function, being in the same time convex and lower semicontinuous. Thus, in the view of the Fenchel-Moreau theorem relation, (3) implies that

$$(f\vee g)^* = \mathrm{cl}_{(X^*,\sigma(X^*,X))\times\mathbb{R}}\left[\inf_{\lambda\in(0,1)}(\lambda f+(1-\lambda)g)^*\right]$$

or, equivalently,

$$\mathrm{epi}((f\vee g)^*) = \mathrm{cl}_{(X^*,\sigma(X^*,X))\times\mathbb{R}}\left(\mathrm{epi}\left(\inf_{\lambda\in(0,1)}(\lambda f+(1-\lambda)g)^*\right)\right).$$

The following proposition leads to a first formulation for $\mathrm{epi}((f\vee g)^*)$.

Proposition 2.2 *Let τ be a vector topology on X^*. Then one has*

$$\bigcup_{\lambda\in(0,1)}\mathrm{epi}\left((\lambda f+(1-\lambda)g)^*\right)\subseteq \mathrm{epi}\left(\inf_{\lambda\in(0,1)}(\lambda f+(1-\lambda)g)^*\right)$$

$$\subseteq \mathrm{cl}_{(X^*,\tau)\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)}\mathrm{epi}\left((\lambda f+(1-\lambda)g)^*\right)\right).$$

Proof. As the first inclusion is obvious, we prove just the second one. For this we consider $(x^*,r)\in\mathrm{epi}\left(\inf_{\lambda\in(0,1)}(\lambda f+(1-\lambda)g)^*\right)$, $\mathcal{V}(x^*)$ an arbitrary open neighborhood of x^* in τ and $\varepsilon>0$. As $\inf_{\lambda\in(0,1)}(\lambda f+(1-\lambda)g)^*\le r$, there exists an $\lambda_\varepsilon\in(0,1)$ such that $(\lambda_\varepsilon f+(1-\lambda_\varepsilon)g)^*(x^*)<r+\varepsilon/2$. Thus $(x^*,r+\varepsilon/2)\in\left(\bigcup_{\lambda\in(0,1)}\mathrm{epi}\left((\lambda f+(1-\lambda)g)^*\right)\right)$. Since $(x^*,r+\varepsilon/2)$ belongs also to $\mathcal{V}(x^*)\times(r-\varepsilon,r+\varepsilon)$, it follows that the intersection of this arbitrary neighborhood of (x^*,r) with $\bigcup_{\lambda\in(0,1)}\mathrm{epi}\left((\lambda f+(1-\lambda)g)^*\right)$ is non-empty. In conclusion (x^*,r) must belong to $\mathrm{cl}_{(X^*,\tau)\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)}\mathrm{epi}\left((\lambda f+(1-\lambda)g)^*\right)\right)$.$\square$

Taking into account the result in Proposition 2.2 we obtain the following formula for the epigraph of $(f\vee g)^*$

$$\mathrm{epi}((f\vee g)^*) = \mathrm{cl}_{(X^*,\sigma(X^*,X))\times\mathbb{R}}\left(\mathrm{epi}\left(\inf_{\lambda\in(0,1)}(\lambda f+(1-\lambda)g)^*\right)\right)$$

$$= \mathrm{cl}_{(X^*,\sigma(X^*,X))\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)}\mathrm{epi}\left((\lambda f+(1-\lambda)g)^*\right)\right).$$

Next we take into consideration also the strong (norm) topology on X^*. Obviously, one has that

$$\mathrm{cl}_{(X^*,\|\cdot\|_{X^*})\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)}\mathrm{epi}\left((\lambda f+(1-\lambda)g)^*\right)\right)$$

$$\subseteq \operatorname{cl}_{(X^*,\sigma(X^*,X))\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)} \operatorname{epi}\left((\lambda f+(1-\lambda)g)^*\right)\right) = \operatorname{epi}((f\vee g)^*).$$

We prove the following auxiliary result.

Proposition 2.3 *The following inclusion always holds*

$$\operatorname{epi}\left((f+\delta_{\operatorname{dom}(g)})^*\right) \subseteq \operatorname{cl}_{(X^*,\|\cdot\|_{X^*})\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)} \operatorname{epi}\left((\lambda f+(1-\lambda)g)^*\right)\right).$$

Proof. Let $(x^*,r)\in \operatorname{epi}\left((f+\delta_{\operatorname{dom}(g)})^*\right)$ or, equivalently, $(f+\delta_{\operatorname{dom}(g)})^*(x^*)\leq r$. Because g is proper, convex and lower semicontinuous it follows that g^* is a proper function and therefore there exists a $y^*\in X^*$ such that $g^*(y^*)\in\mathbb{R}$.

For all $n\geq 1$ we denote $\lambda_n:=1/n$ and $\mu_n:=(n-1)/n$ and get

$$(\lambda_n g+\mu_n f)^*\left(\lambda_n y^*+\mu_n x^*\right) = \sup_{x\in\operatorname{dom}(f)\cap\operatorname{dom}(g)} \{\langle \lambda_n y^*+\mu_n x^*, x\rangle$$

$$-\lambda_n g(x)-\mu_n f(x)\} \leq \lambda_n \sup_{x\in\operatorname{dom}(f)\cap\operatorname{dom}(g)}\{\langle y^*,x\rangle - g(x)\}+$$

$$\mu_n \sup_{x\in\operatorname{dom}(f)\cap\operatorname{dom}(g)}\{\langle x^*,x\rangle - f(x)\} \leq \lambda_n \sup_{x\in X}\{\langle y^*,x\rangle - g(x)\}+$$

$$\mu_n \sup_{x\in X}\{\langle x^*,x\rangle - (f+\delta_{\operatorname{dom}(g)})(x)\} = \lambda_n g^*(y^*)+\mu_n (f+\delta_{\operatorname{dom}(g)})^*(x^*)\leq$$

$$r+\lambda_n\left(g^*(y^*)-(f+\delta_{\operatorname{dom}(g)})^*(x^*)\right).$$

Thus for all $n\geq 1$ it holds

$$\left(\lambda_n y^*+\mu_n x^*, r+\lambda_n\left(g^*(y^*)-(f+\delta_{\operatorname{dom}(g)})^*(x^*)\right)\right)\in$$

$$\operatorname{epi}\left((\lambda_n g+\mu_n f)^*\right) \subseteq \bigcup_{\lambda\in(0,1)} \operatorname{epi}\left((\lambda f+(1-\lambda)g)^*\right),$$

which implies that $(x^*,r)\in \operatorname{cl}_{(X^*,\|\cdot\|_{X^*})\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)} \operatorname{epi}\left((\lambda f+(1-\lambda)g)^*\right)\right)$. □

Because of the symmetry of the functions f and g, by Proposition 2.3, we also have

$$\operatorname{epi}\left((g+\delta_{\operatorname{dom}(f)})^*\right) \subseteq \operatorname{cl}_{(X^*,\|\cdot\|_{X^*})\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)} \operatorname{epi}\left((\lambda f+(1-\lambda)g)^*\right)\right)$$

and so (2) implies that

$$\operatorname{epi}((f\vee g)^*) \subseteq \operatorname{cl}_{(X^*,\|\cdot\|_{X^*})\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)} \operatorname{epi}\left((\lambda f+(1-\lambda)g)^*\right)\right).$$

Thus

$$\begin{aligned}
\operatorname{epi}((f \vee g)^*) &= \operatorname{cl}_{(X^*,\sigma(X^*,X))\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)} \operatorname{epi}\left((\lambda f + (1-\lambda)g)^*\right)\right) \\
&= \operatorname{cl}_{(X^*,\|\cdot\|_{X^*})\times\mathbb{R}}\left(\bigcup_{\lambda\in(0,1)} \operatorname{epi}\left((\lambda f + (1-\lambda)g)^*\right)\right). \qquad (4)
\end{aligned}$$

Using again Proposition 2.2 it follows that

$$\operatorname{epi}((f \vee g)^*) = \operatorname{cl}_{(X^*,\|\cdot\|_{X^*})\times\mathbb{R}}\left(\operatorname{epi}\left(\inf_{\lambda\in(0,1)} (\lambda f + (1-\lambda)g)^*\right)\right),$$

which means that

$$(f \vee g)^* = \operatorname{cl}_{(X^*,\|\cdot\|_{X^*})\times\mathbb{R}}\left[\inf_{\lambda\in(0,1)} (\lambda f + (1-\lambda)g)^*\right]. \qquad (5)$$

In the last part of this section we turn back to the formula for $\operatorname{epi}((f \vee g)^*)$ given in (2) and show what it becomes, provided the Attouch-Brézis regularity condition is fulfilled. Recall that f and g satisfy the Attouch-Brézis regularity condition (cf. [1]) if

$$(AB) \;\; X \text{ is a Banach space and } 0 \in \operatorname{sqri}(\operatorname{dom}(f) - \operatorname{dom}(g)).$$

In case f and g are proper, convex and lower semicontinuous functions such that $\operatorname{dom}(f) \cap \operatorname{dom}(g) \neq \emptyset$ and (AB) is fulfilled, then $(f+g)^* = f^* \square g^*$ and $f^* \square g^*$ is exact (see Theorem 1.1 in [1]). One can notice, by means of Corollary 4 in [6], that this is also the case even if X is a Fréchet space.

Let us come now to two propositions, which we not prove here, since the proof of the first one can be found in [3], while the proof of the second one is elementary.

Proposition 2.4 *Let* $f, g : X \to \overline{\mathbb{R}}$ *be proper functions such that* $\operatorname{dom}(f) \cap \operatorname{dom}(g) \neq \emptyset$. *Then the following statements are equivalent:*

(i) $\operatorname{epi}((f+g)^*) = \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$*;*

(ii) $(f+g)^* = f^* \square g^*$ *and* $f^* \square g^*$ *is exact.*

Proposition 2.5 *Let* $f : X \to \overline{\mathbb{R}}$ *be a proper function and* $\lambda > 0$. *Then* $\operatorname{epi}((\lambda f)^*) = \lambda \operatorname{epi}(f^*)$.

Remark 2.1. If f and g are proper, convex and lower semicontinuous functions such that $\operatorname{dom}(f) \cap \operatorname{dom}(g) \neq \emptyset$, then the statements in Proposition 2.4 are nothing else than assuming that $\operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ is a closed set in $(X^*, \sigma(X^*, X)) \times \mathbb{R}$. This property remains true even if X is a separated locally convex space (see [3], [4]).

Thus, assuming that for f and g the Attouch-Brézis regularity condition is fulfilled, one has

$$\mathrm{epi}((\lambda f + \mu g)^*) = \lambda\,\mathrm{epi}(f^*) + \mu\,\mathrm{epi}(g^*), \forall \lambda, \mu > 0.$$

Unfortunately, one cannot apply Theorem 1.1 in [1] for proving that $\mathrm{epi}((f + \delta_{\mathrm{dom}(g)})^*) = \mathrm{epi}(f^*) + \mathrm{epi}(\delta^*_{\mathrm{dom}(g)})$, as $\delta_{\mathrm{dom}(g)}$ is not necessarily lower semicontinuous. Nevertheless, this follows from Theorem 2.8.7 (v) in [7], since f and $\delta_{\mathrm{dom}(g)}$ are both li-convex functions. Indeed, f is li-convex being convex and lower semicontinuous, while $\delta_{\mathrm{dom}(g)}$ is li-convex being the marginal function of $\delta_{\mathrm{epi}(g)}$ (one can apply Proposition 2.2.18 $(iii) \Rightarrow (i)$ in [7], as $\delta_{\mathrm{epi}(g)}$ is ideally convex and for all $x \in X$ it holds $\delta_{\mathrm{dom}(g)}(x) = \inf_{r\in\mathbb{R}} \delta_{\mathrm{epi}(g)}(x, r)$). Analogously, we get $\mathrm{epi}((g + \delta_{\mathrm{dom}(f)})^*) = \mathrm{epi}(g^*) + \mathrm{epi}(\delta^*_{\mathrm{dom}(f)})$ and so relation (2) becomes

$$\mathrm{epi}((f \vee g)^*) = \bigcup_{\lambda\in(0,1)} \big(\lambda\,\mathrm{epi}(f^*) + (1-\lambda)\,\mathrm{epi}(g^*)\big)$$

$$\bigcup \big(\,\mathrm{epi}(f^*) + \mathrm{epi}(\delta^*_{\mathrm{dom}(g)})\big) \bigcup \big(\,\mathrm{epi}(g^*) + \mathrm{epi}(\delta^*_{\mathrm{dom}(f)})\big). \tag{6}$$

Remark 2.2. Let us notice that if the Attouch-Brézis regularity condition is fulfilled, then the conjugate of $f \vee g$ at $x^* \in X^*$ looks like

$$(f \vee g)^*(x^*) = \min \Bigg\{ \inf_{\substack{\lambda\in(0,1), y^*, z^*\in X^*,\\ \lambda y^* + (1-\lambda) z^* = x^*}} [\lambda f^*(y^*) + (1-\lambda) g^*(z^*)],$$

$$\min_{\substack{y^*, z^*\in X^*,\\ y^*+z^*=x^*}} [f^*(y^*) + \delta^*_{\mathrm{dom}(g)}(z^*)], \min_{\substack{y^*, z^*\in X^*,\\ y^*+z^*=x^*}} [g^*(y^*) + \delta^*_{\mathrm{dom}(f)}(z^*)] \Bigg\}. \tag{7}$$

In Remark 3 in [5] Fitzpatrick and Simons give an example which shows that the equality

$$(f \vee g)^*(x^*) = \min_{\substack{\lambda\in[0,1], y^*, z^*\in X^*,\\ \lambda y^* + (1-\lambda) z^* = x^*}} [\lambda f^*(y^*) + (1-\lambda) g^*(z^*)]$$

is not true for all $x^* \in X^*$.

3 Rediscovering the formula for the biconjugate of the pointwise maximum

In the following we prove, by using relation (4), that the functions $(f \vee g)^{**}$ and $f^{**} \vee g^{**}$ are identical on X^{**}, provided the Attouch-Brézis regularity condition (AB) holds. We actually propose a more simple proof than the one given to this statement in [5].

Theorem 3.1 *Assume that $f, g : X \to \overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions such that the Attouch-Brézis regularity condition (AB) is fulfilled. Then for all $x^{**} \in X^{**}$ $(f \vee g)^{**}(x^{**}) = f^{**} \vee g^{**}(x^{**})$.*

Proof. By the properties of the conjugate functions, because of $f(x) \leq f \vee g(x)$ and $g(x) \leq f \vee g(x)$ for all $x \in X$, we get $f^{**}(x^{**}) \leq (f \vee g)^{**}(x^{**})$ and $g^{**}(x^{**}) \leq (f \vee g)^{**}(x^{**})$ for all $x^{**} \in X^{**}$. From here, $f^{**} \vee g^{**}(x^{**}) \leq (f \vee g)^{**}(x^{**})$ for all $x^{**} \in X^{**}$.

In order to prove the reverse inequality, let be $x^{**} \in X^{**}$ such that $f^{**} \vee g^{**}(x^{**}) < +\infty$. Furthermore, consider an arbitrary $w^* \in \operatorname{dom}((f \vee g)^*)$. Thus, by (4),

$$(w^*, (f \vee g)^*(w^*)) \in \operatorname{epi}(f \vee g)^* = \operatorname{cl}_{(X^*, \|\cdot\|_{X^*}) \times \mathbb{R}} \left(\bigcup_{\lambda \in (0,1)} \operatorname{epi}\left((\lambda f + (1-\lambda) g)^*\right) \right)$$

$$= \operatorname{cl}_{(X^*, \|\cdot\|_{X^*}) \times \mathbb{R}} \left(\bigcup_{\lambda \in (0,1)} \left(\lambda \operatorname{epi}(f^*) + (1-\lambda) \operatorname{epi}(g^*)\right) \right).$$

Then there exist for all $n \geq 1$ $\lambda_n \in (0,1)$ and $(w_n^*, r_n) \in \lambda_n \operatorname{epi}(f^*) + (1 - \lambda_n) \operatorname{epi}(g^*)$ such that $\lim_{n \to +\infty} \|w_n^* - w^*\|_{X^*} = 0$ and $\lim_{n \to +\infty} r_n = (f \vee g)^*(w^*)$. Further, there exist for all $n \geq 1$ $(u_n^*, s_n) \in \operatorname{epi}(f^*)$ and $(v_n^*, t_n) \in \operatorname{epi}(g^*)$ such that $w_n^* = \lambda_n u_n^* + (1 - \lambda_n) v_n^*$ and $r_n = \lambda_n s_n + (1 - \lambda_n) t_n \geq \lambda_n f^*(u_n^*) + (1 - \lambda_n) g^*(v_n^*)$. Applying the Young-Fenchel inequality we get for all $n \geq 1$

$$r_n \geq \lambda_n \{\langle x^{**}, u_n^* \rangle - f^{**}(x^{**})\} + (1 - \lambda_n)\{\langle x^{**}, v_n^* \rangle - g^{**}(x^{**})\}$$

$$= \langle x^{**}, \lambda_n u_n^* + (1 - \lambda_n) v_n^* \rangle - \lambda_n f^{**}(x^{**}) - (1 - \lambda_n) g^{**}(x^{**})$$

$$\geq \langle x^{**}, w_n^* \rangle - f^{**} \vee g^{**}(x^{**}).$$

This implies that for all $n \geq 1$ $f^{**} \vee g^{**}(x^{**}) \geq \langle x^{**}, w_n^* \rangle - r_n$ and letting now n converge towards $+\infty$, we obtain $f^{**} \vee g^{**}(x^{**}) \geq \langle x^{**}, w^* \rangle - (f \vee g)^*(w^*)$. Since $w^* \in \operatorname{dom}((f \vee g)^*)$ was arbitrary chosen, we get

$$f^{**} \vee g^{**}(x^{**}) \geq \sup_{w^* \in \operatorname{dom}((f \vee g)^*)} \{\langle x^{**}, w^* \rangle - (f \vee g)^*(w^*)\} = (f \vee g)^{**}(x^{**})$$

and this delivers the desired conclusion. □

Remark 3.1. C. Zălinescu suggested for Theorem 3.1 the following proof based on relation (5). By using some properties of the conjugate functions one has

$$(f \vee g)^{**} = \left(\operatorname{cl}_{(X^*, \|\cdot\|_{X^*}) \times \mathbb{R}} \left(\inf_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g)^* \right) \right)^* =$$

$$\left(\inf_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g)^* \right)^* = \sup_{\lambda \in (0,1)} (\lambda f + (1-\lambda) g)^{**}.$$

Further, using relation (0.2) in [5], Theorem 2.3.1 (v) in [7] and the fact that f^{**} and g^{**} are proper, it holds

$$\sup_{\lambda\in(0,1)}(\lambda f+(1-\lambda)g)^{**}=\sup_{\lambda\in(0,1)}[(\lambda f)^{**}+((1-\lambda)g)^{**}]=$$

$$\sup_{\lambda\in(0,1)}[\lambda f^{**}+(1-\lambda)g^{**}]=f^{**}\vee g^{**}.$$

Remark 3.2. Recently in [2] it was shown that if $\operatorname{co}\big(\operatorname{epi}(f^*)\cup\operatorname{epi}(g^*)\big)$ is closed in $(X^*,\sigma(X^*,X))\times\mathbb{R}$, then $(f\vee g)^{**}=f^{**}\vee g^{**}$ on X^{**}, provided that $f,g:X\to\overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions and X is a separated locally convex space.

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