Revisiting Some Duality Theorems via the Quasirelative Interior in Convex Optimization [1]

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Abstract. In this paper we deal with regularity conditions, formulated by making use of the quasirelative interior and/or of the quasi interior of the sets involved, which guarantee strong duality for a convex optimization problem with cone (and equality) constraints and its Lagrange dual. We discuss also some results recently on this topic, which are proved to have either superfluous or contradictory assumptions. Several examples illustrate the theoretical considerations.

Key Words. Lagrange duality, separation theorems, regularity conditions, quasirelative interior, quasi interior.

1. Introduction

The literature on regularity conditions in infinite dimensional spaces which guarantee the existence of strong duality between a convex optimization problem with cone (and equality) constraints and its Lagrange dual problem is very vast. Besides the classical interior, in the formulation of these regularity conditions, different *generalized interior notions* were used, like the *core*, the *intrinsic core* or the *strong quasirelative interior* (see Ref. 1 and the excellent book Ref. 2 where a comprehensive list of regularity conditions is presented). We also want to mention here the class of *closedness type* conditions intensively studied in the last time (see, for example, Ref. 3).

Nevertheless, in many theoretical and practical infinite-dimensional convex optimization problems, the interior conditions are useless since for instance, the interior of the set involved in the regularity condition is empty. This is the case, for example, when dealing with the positive cones l^p_+ and $L^p_+(T,\mu)$ of the spaces l^p and $L^p(T,\mu)$, respectively, where (T,μ) is a σ-finite measure space and $p \in [1,+\infty)$. For these two cones even the strong quasirelative interior (which is the weakest generalized interior notion from the aforementioned ones) is empty. In order two overcome such a situation Borwein and Lewis introduced in Ref. 4 the notion of *quasirelative interior* of a convex set, which is a further generalization of the above mentioned interior notions. They also proved that the quasirelative interiors of l^p_+ and

$L^p_+(T,\mu)$ are nonempty.

The number of papers dealing with regularity conditions for convex optimization problem with cone (and equality) constraints in infinite dimensional spaces, formulated by using the quasirelative interior, is not very large. An important contribution in this field is the paper of Jeyakumar and Wolkowicz Ref. 5, even if it has the drawback that the cone defining the constraints is assumed to have a nonempty interior. But in the last time we noticed an increasing number of papers on this topic which try to overcome this fact, like Ref. 6-8.

In our paper we discuss and improve the duality results given in the aforementioned papers. In Ref. 7 the authors consider in the primal problem along with the cone constraints also equality constraints. We show that their so-called Assumption S, given besides other hypotheses as a sufficient condition, is actually equivalent to strong duality. We also point out by means of an example a mistake in the proof of Theorem 3.1 in Ref. 7.

In Ref. 8 along with other regularity conditions a particularization of the Assumption S is considered for an optimization problem with cone constraints in infinite dimensional spaces. Since this condition, actually assumed to be a sufficient condition, is equivalent to strong duality all the other assumptions become superfluous. Concerning the strong duality theorem in Ref. 6 we prove that this result has contradictory assumptions. We also give a valuable strong duality theorem, the regularity condition of which being expressed by using the *quasirelative interior* and/or the *quasi interior* of the sets involved. We illustrate the theoretical considerations by some examples.

The paper is structured as follows. In the next section we give some definitions and preliminary results. Section 3 is devoted to the existing results in the literature dealing with Lagrange duality for the problem with cone (and equality) constraints, the regularity conditions of which are expressed by means of quasirelative interior. In the last section we give a strong duality theorem for an optimization problem and its Lagrange dual under a weak regularity condition also expressed by using the quasirelative interior of the sets involved.

2. Preliminaries

Consider X a real normed space and X^* its topological dual space. We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. For a subset C of X we denote by $\operatorname{co} C$, $\operatorname{cl} C$ and $\operatorname{int} C$ its *convex hull*, *closure* and *interior*, respectively. The set $\operatorname{cone} C := \bigcup_{\lambda \geq 0} \lambda C$ is the *cone generated by* C. If C is convex, one can prove that $\operatorname{cone} \operatorname{co}(C \cup \{O_X\}) = \operatorname{cone} C$. The *normal cone* of C at $x \in C$ is defined as $N_C(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0,\ \forall y \in C\}$. The set

$$T_C(x) = \big\{y \in X : y = \lim_{n\to\infty} \lambda_n(x_n - x), \lambda_n > 0, x_n \in C, \forall n \in \mathbb{N},\ \lim_{n\to\infty} x_n = x\big\}$$

is called the *contingent cone* to C at $x \in X$. In general, we have the following inclusion: $T_C(x) \subseteq \operatorname{cl}\operatorname{cone}(C - x)$. If the set C is convex, then $T_C(x) = \operatorname{cl}\operatorname{cone}(C - x)$ (cf. Ref. 9).

Before coming to the generalized interior notions we consider in this paper, let us make the following notations: $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_{++} = (0, +\infty)$, $\mathbb{R}_- = -\mathbb{R}_+$ and $\mathbb{R}_{--} = -\mathbb{R}_{++}$.

Definition 2.1. Let C be a convex subset of X. The *quasi interior* of C is the set

$$\operatorname{qi} C = \{x \in C : \operatorname{cl} \operatorname{cone}(C - x) = X\}.$$

We have the following characterization of the quasi interior of a convex set.

Proposition 2.1. (Ref. 8) *Let C be a convex subset of X and $x \in C$. Then $x \in \operatorname{qi} C$ if and only if $N_C(x) = \{O_{X^*}\}$.*

The following notion is a refinement of the quasi interior and is due to Borwein and Lewis (Ref. 4).

Definition 2.2. (Ref. 4) Let C be a convex subset of X. The *quasirelative interior* of C is the set

$$\operatorname{qri} C = \{x \in C : \operatorname{cl} \operatorname{cone}(C - x) \text{ is a linear subspace of } X\}.$$

Proposition 2.2. (Ref. 4) *Let C be a convex subset of X and $x \in C$. Then $x \in \operatorname{qri} C$ if and only if $N_C(x)$ is a linear subspace of X^*.*

It follows from the definitions above that $\operatorname{qi} C \subseteq \operatorname{qri} C$ and $\operatorname{qri}\{x\} = \{x\}$, $\forall x \in X$. Also, if $\operatorname{qi} C \neq \emptyset$, then $\operatorname{qi} C = \operatorname{qri} C$ (cf. Ref. 10). If X is a finite dimensional space, then $\operatorname{qi} C = \operatorname{int} C$ (cf. Ref. 10) and $\operatorname{qri} C = \operatorname{ri} C$ (cf. Ref. 4), where $\operatorname{ri} C$ is the relative interior of C. In the following proposition we give some useful properties of the quasirelative interior.

Proposition 2.3. (Ref. 11, Ref. 4) *Let us consider C and D two convex subsets of X, $x \in X$ and $\alpha \in \mathbb{R}$. Then:*

(i) $\operatorname{qri} C + \operatorname{qri} D \subseteq \operatorname{qri}(C + D)$;

(ii) $\operatorname{qri}(C \times D) = \operatorname{qri} C \times \operatorname{qri} D$;

(iii) $\operatorname{qri}(C - x) = \operatorname{qri} C - x$;

(iv) $\operatorname{qri}(\alpha C) = \alpha \operatorname{qri} C$;

(v) $t \operatorname{qri} C + (1-t)C \subseteq \operatorname{qri} C$, $\forall t \in (0, 1]$, *hence* $\operatorname{qri} C$ *is a convex set;*

(vi) *if C is an affine set then* $\operatorname{qri} C = C$*;*

(vii) $\operatorname{qri}(\operatorname{qri} C) = \operatorname{qri} C$.

If $\operatorname{qri} C \neq \emptyset$ *then:*

(viii) $\operatorname{cl} \operatorname{qri} C = \operatorname{cl} C$*;*

(ix) $\operatorname{cl} \operatorname{cone} \operatorname{qri} C = \operatorname{cl} \operatorname{cone} C$.

Proof. For the proof of (i)-(viii) we refer to Ref. 11 and Ref. 4 for more details.

(ix) The inclusion $\operatorname{cl} \operatorname{cone} \operatorname{qri} C \subseteq \operatorname{cl} \operatorname{cone} C$ is always true. We prove that $\operatorname{cone} C \subseteq \operatorname{cl} \operatorname{cone} \operatorname{qri} C$. Consider $x \in \operatorname{cone} C$ arbitrary. There exist $\lambda \geq 0$ and $c \in C$ such that $x = \lambda c$. Take $x_0 \in \operatorname{qri} C$. Using the property (v), we obtain $tx_0 + (1-t)c \in \operatorname{qri} C$, $\forall t \in (0, 1]$, so $\lambda t x_0 + (1-t)x = \lambda(tx_0 + (1-t)c) \in \operatorname{cone} \operatorname{qri} C$ $\forall t \in (0, 1]$. Passing to the limit as $t \searrow 0$ we get $x \in \operatorname{cl} \operatorname{cone} \operatorname{qri} C$ and the conclusion follows. $\square$

We come now to a lemma which will prove to be useful in the following.

Lemma 2.1. *Let A and B be nonempty convex subsets of X such that $A \cap B \neq \emptyset$. If $O_X \in \text{qi}(A - A)$ and $B \cap \text{qri}\, A \neq \emptyset$, then $O_X \in \text{qi}(A - B)$.*

Proof. Take $x \in B \cap \text{qri}\, A$ and let $x^* \in N_{A-B}(O_X)$ be arbitrary. We get $\langle x^*, a - b\rangle \leq 0, \forall a \in A, \forall b \in B$. Then

$$\langle x^*, a - x\rangle \leq 0, \forall a \in A \tag{1}$$

that is $x^* \in N_A(x)$. As $x \in \text{qri}\, A$, $N_A(x)$ is a linear subspace of X^*, hence $-x^* \in N_A(x)$, which is nothing else than

$$\langle x^*, x - a\rangle \leq 0, \forall a \in A. \tag{2}$$

The relations (1) and (2) give us $\langle x^*, a' - a''\rangle \leq 0, \forall a', a'' \in A$, so $x^* \in N_{A-A}(O_X)$. Since $O_X \in \text{qi}(A - A)$ we have $N_{A-A}(O_X) = \{O_{X^*}\}$ (cf. Proposition 2.1) and we get $x^* = O_{X^*}$. As x^* was arbitrary chosen we obtain $N_{A-B}(O_X) = \{O_{X^*}\}$ and, using again Proposition 2.1, the conclusion follows. □

Let us give now some separation theorems in terms of the quasirelative interior.

Theorem 2.1. (Refs. 7-8) *Let C be a convex subset of X and $x_0 \in C \setminus \text{qri}\, C$. Then there exists $x^* \in X^*, x^* \neq O_{X^*}$ such that*

$$\langle x^*, x\rangle \leq \langle x^*, x_0\rangle, \ \forall x \in C.$$

Viceversa, if there exists $x^* \in X^*$, $x^* \neq O_{X^*}$ *such that*

$$\langle x^*, x\rangle \leq \langle x^*, x_0\rangle, \ \forall x \in C$$

and

$$\mathrm{cl}(T_C(x_0) - T_C(x_0)) = X,$$

then $x_0 \in C \setminus \mathrm{qri}\, C$.

Remark 2.1 The condition $\mathrm{cl}(T_C(x_0) - T_C(x_0)) = X$ in the above theorem can be reformulated as follows: $\mathrm{cl}\,\mathrm{cone}(C - C) = X$ or, equivalently, $O_X \in \mathrm{qi}(C - C)$. Indeed, we have

$$\mathrm{cl}[\mathrm{cl}\,\mathrm{cone}(C - x_0) - \mathrm{cl}\,\mathrm{cone}(C - x_0)] = X \Leftrightarrow \mathrm{cl}[\mathrm{cone}(C - x_0) - \mathrm{cone}(C - x_0)] = X$$

$$\Leftrightarrow \mathrm{cl}\,\mathrm{cone}(C - C) = X \Leftrightarrow O_X \in \mathrm{qi}(C - C),$$

where we used the following properties: $\mathrm{cl}(\mathrm{cl}\, E + \mathrm{cl}\, F) = \mathrm{cl}(E + F)$, for arbitrary sets E, F in X and $\mathrm{cone}\, A - \mathrm{cone}\, A = \mathrm{cone}(A - A)$, if A is a convex subset of X such that $0 \in A$.

The condition $x_0 \in C$ in Theorem 2.1 is essential (see Ref. 8). However, if x_0 is an arbitrary element in X, we can give also a separation theorem using the following result due to Cammaroto and Di Bella (Theorem 2.1 in Ref. 6).

Theorem 2.2. (Ref. 6) *Let* S *and* T *be nonempty convex subsets of* X *with* $\mathrm{qri}\, S \neq \emptyset$, $\mathrm{qri}\, T \neq \emptyset$ *and such that* $\mathrm{cl}\,\mathrm{cone}(\mathrm{qri}\, S - \mathrm{qri}\, T)$ *is not a linear subspace of* X*. Then, there exists*

$x^* \in X^*$, $x^* \neq O_{X^*}$, such that $\langle x^*, s\rangle \leq \langle x^*, t\rangle$ for all $s \in S$, $t \in T$.

The following result is a direct consequence of Theorem 2.2.

Corollary 2.1. *Let C be a convex subset of X such that $\operatorname{qri} C \neq \emptyset$, and $\operatorname{cl}\operatorname{cone}(C - x_0)$ is not a linear subspace of X, where $x_0 \in X$. Then there exists $x^* \in X^*, x^* \neq O_{X^*}$ such that $\langle x^*, x\rangle \leq \langle x^*, x_0\rangle,\ \forall x \in C$.*

Proof. We apply Theorem 2.2 with $S := C$ and $T := \{x_0\}$. Then we use Proposition 2.3 (iii) and (ix) to obtain the conclusion. □

3. Revisiting Some Strong Duality Results from the Literature

In this section we revisit some Lagrange duality results for the optimization problem with cone (and equality) constraints recently given in the literature and stated in terms of the quasi interior and quasirelative interior, respectively. We prove that their assumptions are either superfluous or contradictory, respectively.

3.1. The Problem with Cone and Equality Constraints

Consider the following primal optimization problem

$$(\mathrm{P^e}) \qquad \inf_{x \in R} f(x),$$

where

$$R = \{x \in S : g(x) \in -C, h(x) = O_Z\}$$

is assumed to be nonempty, X is a real linear topological space, S a nonempty subset of X

and Y and Z are real normed spaces, Y being also partially ordered by a convex cone C. Let $h : X \to Z$ be an affine-linear mapping and $f : S \to \mathbb{R}$ and $g : S \to Y$ be two functions such that the function $(f, g) : S \to \mathbb{R} \times Y$, defined by $(f, g)(x) = (f(x), g(x)), \forall x \in S$, is convex-like with respect to the cone $\mathbb{R}_+ \times C \subseteq \mathbb{R} \times Y$, that is, the set $(f, g)(S) + \mathbb{R}_+ \times C$ is convex. Let us notice that this property implies that the sets $f(S) + [0, \infty)$ and $g(S) + C$ are convex (the reverse implication does not hold).

The Lagrange dual problem of $(\mathrm{P^e})$ looks like

$$(\mathrm{D_L^e}) \qquad \sup_{\substack{\lambda \in C^* \\ \mu \in Z^*}} \inf_{x \in S} [f(x) + \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle],$$

where $C^* = \{x^* \in X^* : \langle x^*, x \rangle \geq 0, \forall x \in C\}$ is the *dual cone* of C and Z^* is the topological dual space of Z. Let us denote by $v(\mathrm{P^e})$ and $v(\mathrm{D_L^e})$ the optimal objective values of the primal and the dual problem, respectively. Weak duality always holds, that is $v(\mathrm{D_L^e}) \leq v(\mathrm{P^e})$. We say that strong duality holds, if $v(\mathrm{P^e}) = v(\mathrm{D_L^e})$ and $(\mathrm{D_L^e})$ has an optimal solution.

Daniele and Giuffrè give in Ref. 7 a strong duality theorem for $(\mathrm{P^e})$ and its Lagrange dual problem by using a "separation assumption" and some regularity conditions expressed by means of the quasirelative interior. In the above mentioned paper is said that Assumption S is fulfilled at $x_0 \in R$ if

$$T_{\widetilde{M}}(f(x_0), O_Y, O_Z) \cap (\mathbb{R}_{--} \times O_Y \times O_Z) = \emptyset,$$

where

$$\widetilde{M} := \{(f(x) + \alpha, g(x) + y, h(x)) : x \in S \setminus R, \alpha \geq 0, y \in C\}.$$

Let us come now to the strong duality theorem stated by Daniele and Giuffrè (see Theorem 3.1 in Ref. 7).

Let X be a real normed space and S be a convex subset of X; let $(Y, \|\cdot\|_Y)$ a partially ordered real normed space with convex ordering cone C and let $(Z, \|\cdot\|_Z)$ a real normed space. Let $f : S \to \mathbb{R}$ be a given functional and let $g : S \to Y$, $h : X \to Z$ be given mappings such that the function $(f, g) : S \to \mathbb{R} \times Y$ is convex-like with respect to the cone $\mathbb{R}_+ \times C$ of $\mathbb{R} \times Y$ and h is an affine-linear mapping. Let the set $R := \{x \in S : g(x) \in -C, h(x) = O_Z\}$ be nonempty and let us assume that $\operatorname{qri} C \neq \emptyset$, $\operatorname{cl}(C - C) = Y$, $\operatorname{cl} h(S - S) = Z$ and there exists $\hat{x} \in S$ with $g(\hat{x}) \in -\operatorname{qri} C$ and $h(\hat{x}) = O_Z$. If Assumption S is fulfilled at the extremal solution $x_0 \in R$ to problem $(\mathrm{P^e})$, then also problem $(\mathrm{D^e_L})$ is solvable and, if $\lambda_0 \in C^*$, $\mu_0 \in Z^*$ are the extremal points of problem $(\mathrm{D^e_L})$, it results that

$$\langle \lambda_0, g(x_0) \rangle = 0$$

and the extrema of the two problems are equal.

We show in the following that the fulfillment of Assumption S at an optimal solution $x_0 \in R$ of the primal problem is actually equivalent to the existence of strong duality between $(\mathrm{P^e})$ and $(\mathrm{D^e_L})$, namely that the optimal objective values of the two problems are equal and the dual $(\mathrm{D}^{\mathrm{e}}_{\mathrm{L}})$ has an optimal solution $(\bar{u}, \bar{v}) \in C^* \times Z^*$. The latter are the so-called Lagrange multipliers. This means that the other conditions asked in the above theorem, $\mathrm{cl}(C-C) = Y$, $\mathrm{cl}\, h(S-S) = Z$, $\mathrm{qri}\, C \neq \emptyset$, $\exists\, \hat{x} \in S$ such that $g(\hat{x}) \in -\,\mathrm{qri}\, C$, are not needed when Assumption S is requested.

We work under the assumption that $x_0 \in R$ is an optimal solution of the primal problem $(\mathrm{P}^{\mathrm{e}})$ and give first some preparatory results. Let us introduce the following set:

$$\begin{aligned}\mathcal{E}_{v(\mathrm{P}^{\mathrm{e}})} &= \{(f(x_0) - f(x) - \alpha, -g(x) - y, -h(x)) : x \in S, \alpha \geq 0, y \in C\} \\ &= (f(x_0), O_Y, O_Z) - (f, g, h)(S) - \mathbb{R}_+ \times C \times O_Z\end{aligned}$$

which is similar to the *conic extension* introduced by Giannessi in the theory of image space analysis (Ref. 12).

In the hypotheses we work the set $\mathcal{E}_{v(\mathrm{P}^{\mathrm{e}})}$ is convex. Also, one can easily see that the primal problem $(\mathrm{P}^{\mathrm{e}})$ has an optimal solution if and only if $(0, O_Y, O_Z) \in \mathcal{E}_{v(\mathrm{P}^{\mathrm{e}})}$.

We start by giving the following equivalent formulation for the fulfillment of Assumption S at an optimal solution.

Proposition 3.1. *Assume that x_0 is an optimal solution of the problem $(\mathrm{P}^{\mathrm{e}})$. Then the following equivalence holds:*

$$T_{\widetilde{M}}(f(x_0), O_Y, O_Z) \cap (\mathbb{R}_{--} \times O_Y \times O_Z) = \emptyset \Leftrightarrow$$

$$T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z) \cap (\mathbb{R}_{++} \times O_Y \times O_Z) = \emptyset.$$

Proof. Assume first that $T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z) \cap (\mathbb{R}_{++} \times O_Y \times O_Z) = \emptyset$. Since

$$T_{\widetilde{M}}(f(x_0), O_Y, O_Z) \subseteq \operatorname{cl}\operatorname{cone}(\widetilde{M} - (f(x_0), O_Y, O_Z))$$

$$\subseteq -\operatorname{cl}\operatorname{cone}\mathcal{E}_{v(\mathrm{P^e})} = -T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z),$$

we get that $T_{\widetilde{M}}(f(x_0), O_Y, O_Z) \cap (\mathbb{R}_{--} \times O_Y \times O_Z) = \emptyset$.

Conversely, suppose that $T_{\widetilde{M}}(f(x_0), O_Y, O_Z) \cap (\mathbb{R}_{--} \times O_Y \times O_Z) = \emptyset$ and assume that

$$T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z) \cap (\mathbb{R}_{++} \times O_Y \times O_Z) \neq \emptyset,$$

namely that $\exists\ (t, O_Y, O_Z) \in T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z)$ with $t > 0$. Thus there exist $(x_n)_{n\in\mathbb{N}} \subseteq S$, $(\beta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_{++}$, $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_{+}$ and $(y_n)_{n\in\mathbb{N}} \subseteq C$ such that

$$\lim_{n\to\infty}(f(x_n) + \alpha_n) = f(x_0),\ \lim_{n\to\infty}(g(x_n) + y_n) = O_Y,\ \lim_{n\to\infty} h(x_n) = O_Z \tag{3}$$

and

$$\lim_{n\to\infty} \beta_n(f(x_0) - f(x_n) - \alpha_n) = t,\ \lim_{n\to\infty} \beta_n(g(x_n) + y_n) = O_Y,\ \lim_{n\to\infty} \beta_n h(x_n) = O_Z. \tag{4}$$

Since $t > 0$ and $\beta_n > 0$, $\forall n \in \mathbb{N}$, there exists $\bar{n} \in \mathbb{N}$ such that $f(x_n) < f(x_0), \forall n \geq \bar{n}$. But, as x_0 is an optimal solution of the problem ($\mathrm{P^e}$), we must have $x_n \in S \setminus R$, $\forall n \geq \bar{n}$. In this way we find a sequence $(x_n)_{n\geq\bar{n}} \in S \setminus R$ which satisfies the conditions (3) and (4) and this means that $(-t, O_y, O_Z) \in T_{\widetilde{M}}(f(x_0), O_Y, O_Z)$. This leads to the desired contradiction. $\square$

Before stating the announced result we prove another auxiliary lemma.

Lemma 3.1. *For $a \in \mathbb{R} \times Y^* \times Z^* \setminus \{(0, O_{Y^*}, O_{Z^*})\}$, let $H = \{t \in \mathbb{R} \times Y \times Z : \langle a, t \rangle = 0\}$ be a hyperplane in $\mathbb{R} \times Y \times Z$. The following statements are equivalent:*

(i) *H separates the sets $(f(x_0), O_Y, O_Z) - (f, g, h)(S)$ and $\mathbb{R}_{++} \times C \times O_Z$;*

(ii) *H separates the sets $\mathcal{E}_{v(\mathrm{P^e})}$ and $\mathbb{R}_{++} \times C \times O_Z$;*

(iii) *H separates $T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z)$ and $\mathbb{R}_{++} \times C \times O_Z$;*

(iv) *H separates $T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z)$ and $\mathbb{R}_{++} \times O_Y \times O_Z$.*

Proof. (i)$\Rightarrow$ (ii) Assume that

$$\langle a, t \rangle \geq 0, \ \forall\, t \in \mathbb{R}_{++} \times C \times O_Z \text{ and } \langle a, t \rangle \leq 0, \ \forall\, t \in (f(x_0), O_Y, O_Z) - (f, g, h)(S).$$

This means that $\mathbb{R}_{++} \times C \times O_Z \subseteq H^+$ and $f(x_0), O_Y, O_Z) - (f, g, h)(S) \subseteq H^-$, where by H^+ and H^- we denote the half-spaces $\{t \in \mathbb{R} \times Y \times Z : \langle a, t \rangle \geq 0\}$ and $\{t \in \mathbb{R} \times Y \times Z : \langle a, t \rangle \leq 0\}$, respectively. We prove that actually $\mathcal{E}_{\mathrm{v}(\mathrm{P^e})} \subseteq H^-$.

To this end we suppose that $\exists \hat{t} \in \mathcal{E}_{v(\mathrm{P^e})}$ such that $\langle a, \hat{t} \rangle > 0$. Since $\hat{t} \in \mathcal{E}_{v(\mathrm{P^e})} = (f(x_0), O_Y, O_Z) - (f, g, h)(S) - \mathbb{R}_+ \times C \times O_Z$ we get the existence of $t^1 \in (f(x_0), O_Y, O_Z) - (f, g, h)(S)$ and $t^2 \in \mathbb{R}_+ \times C \times O_Z$ such that $\hat{t} = t^1 - t^2$. From $\langle a, \hat{t} \rangle > 0$ we have

$$0 \leq \langle a, t^2 \rangle < \langle a, t^1 \rangle \leq 0,$$

where the third inequality comes from $(f(x_0), O_Y, O_Z) - (f, g, h)(S) \subseteq H^-$ and the first one from the fact that $\mathbb{R}_+ \times C \times O_Z \subseteq H^+$, which is an easy consequence of $\mathbb{R}_{++} \times C \times O_Z \subseteq H^+$. We get a contradiction and this proves the first implication.

(ii)$\Rightarrow$(iii) Assuming now that $\mathbb{R}_{++} \times C \times O_Z \subseteq H^+$ and $\mathcal{E}_{v(\mathrm{P}^e)} \subseteq H^-$, it follows that $T_{\mathcal{E}_{v(\mathrm{P}^e)}}(0, O_Y, O_Z) = \mathrm{cl}\,\mathrm{cone}\,\mathcal{E}_{v(\mathrm{P}^e)} \subseteq \mathrm{cl}\,\mathrm{cone}\,H^- = H^-$ and this gives (iii).

(iii)$\Rightarrow$(iv) Follows automatically using that $\mathbb{R}_{++} \times O_Y \times O_Z \subseteq \mathbb{R}_{++} \times C \times O_Z$.

(iv)$\Rightarrow$(i) We assume that

$$\langle a, t\rangle \geq 0,\ \forall t \in \mathbb{R}_{++} \times O_Y \times O_Z \text{ and } \langle a, t\rangle \leq 0,\ \forall t \in T_{\mathcal{E}_{v(\mathrm{P}^e)}}(0, O_Y, O_Z)$$

and prove the inclusion $\mathbb{R}_{++} \times C \times O_Z \subseteq H^+$. If this is not the case, then there exists $\hat{t} \in \mathbb{R}_{++} \times C \times O_Z$ such that $\langle a, \hat{t}\rangle < 0$.

Consider an element $\bar{t} \in \mathcal{E}_{v(\mathrm{P}^e)}$. Then $\forall \alpha \geq 0$, we have $\bar{t} - \alpha\hat{t} \in \mathcal{E}_{v(\mathrm{P}^e)} \subseteq T_{\mathcal{E}_{v(\mathrm{P}^e)}}(0, O_Y, O_Z)$. Further, it holds

$$\lim_{\alpha \to +\infty} \langle a, \bar{t} - \alpha\hat{t}\rangle = +\infty,$$

but this is a contradiction to $\langle a, t\rangle \leq 0,\ \forall t \in T_{\mathcal{E}_{v(\mathrm{P}^e)}}(0, O_Y, O_Z)$. This means that $\mathbb{R}_{++} \times C \times O_Z \subseteq H^+$. Since $(f(x_0), O_Y, O_Z) - (f, g, h)(S) \subseteq \mathcal{E}_{v(\mathrm{P}^e)} \subseteq T_{\mathcal{E}_{v(\mathrm{P}^e)}}(0, O_Y, O_Z) \subseteq H^-$, the conclusion follows obviously. $\square$

We come now to the main theorem of this section.

Theorem 3.1. *Suppose that $x_0 \in R$ is an optimal solution of the primal problem* (P^e)*.*

Then

$$T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z) \cap (\mathbb{R}_{++} \times O_Y \times O_Z) = \emptyset$$

holds if and only if $v(\mathrm{P^e}) = v(\mathrm{D^e_L})$ *and* $\exists \lambda_0 \in C^*, \exists \mu_0 \in Z^*$ *such that* (λ_0, μ_0) *is an optimal solution of the dual. In this situation we have* $\langle \lambda_0, g(x_0) \rangle = 0$.

Proof. ($\Rightarrow$) Suppose that $T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z) \cap (\mathbb{R}_{++} \times O_Y \times O_Z) = \emptyset$. This implies that $\exists h \in (\mathbb{R}_{++} \times O_Y \times O_Z) \setminus T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z)$. Since $T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z)$ is a closed and convex set, by a separation theorem (see Theorem 3.4 in Ref. 13) we get $a = (\theta, \lambda, \mu) \neq (0, O_{Y^*}, O_{Z^*})$ such that

$$\langle a, h \rangle > 0 \geq \langle a, t \rangle, \ \forall t \in T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z).$$

Since $\mathcal{E}_{v(\mathrm{P^e})} \subseteq T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z)$ it follows $\langle a, t \rangle \leq 0, \ \forall t \in \mathcal{E}_{v(\mathrm{P^e})}$.

Further we prove that $\langle a, t \rangle \geq 0, \ \forall t \in \mathbb{R}_{++} \times C \times O_Z$. To this aim, assume that there exists $\hat{t} \in \mathbb{R}_{++} \times C \times O_Z$ such that $\langle a, \hat{t} \rangle < 0$. Let be $\bar{t} \in \mathcal{E}_{v(\mathrm{P^e})}$ fixed. Then we have that $\bar{t} - \alpha \hat{t} \in \mathcal{E}_{v(\mathrm{P^e})}, \forall \alpha \geq 0$. But the fact that

$$\lim_{\alpha \to +\infty} \langle a, \bar{t} - \alpha \hat{t} \rangle = +\infty$$

leads to contradiction. Thus, by Lemma 3.1 ((iii)$\Rightarrow$(i)), we obtain that

$$\langle a, t \rangle \geq 0, \forall t \in \mathbb{R}_{++} \times C \times O_Z \text{ and } \langle a, t \rangle \leq 0, \forall t \in (f(x_0), O_Y, O_Z) - (f, g, h)(S).$$

This means that the hyperplane $H = \{t \in \mathbb{R} \times Y \times Z : \langle a, t \rangle = 0\}$ separates the sets

$(f(x_0), O_Y, O_Z) - (f, g, h)(S)$ and $\mathbb{R}_{++} \times C \times O_Z$. From the above inequalities we get

$$\theta r + \langle \lambda, y \rangle \geq 0, \forall r > 0, \forall y \in C \tag{5}$$

and

$$\theta(f(x_0) - f(x)) + \langle \lambda, -g(x) - y \rangle + \langle \mu, -h(x) \rangle \leq 0, \forall x \in S, \forall y \in C. \tag{6}$$

Relation (5) implies $\lambda \in C^*$ and $\theta \geq 0$.

Now let us assume that $\theta = 0$. This would mean that $\langle a, f \rangle = 0$, $\forall f \in \mathbb{R}_{++} \times O_Y \times O_Z$, but this is a contradiction to the fact that there exists $h \in \mathbb{R}_{++} \times O_Y \times O_Z$ such that $\langle a, h \rangle > 0$. Therefore we have necessarily $\theta > 0$.

From (6) we obtain

$$f(x) - f(x_0) + \langle \lambda_0, g(x) + y \rangle + \langle \mu_0, h(x) \rangle \geq 0, \ \forall x \in S, \forall y \in C,$$

where $\lambda_0 = (1/\theta)\lambda \in C^*$ and $\mu_0 = (1/\theta)\mu \in Z^*$. Taking $y = O_Y$ and $x = x_0$ in the above inequality, we have $\langle \lambda_0, g(x_0) \rangle \geq 0$ and since $g(x_0) \in -C$ and $\lambda_0 \in C^*$ we get $\langle \lambda_0, g(x_0) \rangle = 0$. Thus

$$v(\mathrm{P^e}) = f(x_0) = \inf_{x \in S} [f(x) + \langle \lambda_0, g(x) \rangle + \langle \mu_0, h(x) \rangle] = v(\mathrm{D^e_L})$$

and (λ_0, μ_0) is an optimal solution of the dual.

($\Leftarrow$) Suppose that $\exists \lambda_0 \in C^*, \exists \mu_0 \in Z^*$ such that (λ_0, μ_0) is an optimal solution of the dual $(\mathrm{D^e_L})$ and $v(\mathrm{P^e}) = v(\mathrm{D^e_L})$. That $\langle \lambda_0, g(x_0) \rangle = 0$ is an easy consequence of this fact. In this way we obtain

$$f(x) - f(x_0) + \langle \lambda_0, g(x)\rangle + \langle \mu_0, h(x)\rangle \geq 0,\ \forall x \in S$$

and so the hyperplane $H = \{(r, y, z) \in \mathbb{R} \times Y \times Z : r + \langle \lambda_0, y\rangle + \langle \mu_0, z\rangle = 0\}$ separates the sets $(f(x_0), O_Y, O_Z) - (f, g, h)(S)$ and $\mathbb{R}_{++} \times C \times O_Z$, namely $(f(x_0), O_Y, O_Z) - (f, g, h)(S) \subseteq H^-$ and $\mathbb{R}_{++} \times C \times O_Z \subseteq H^+$. By Lemma 3.1 ($(i) \Rightarrow (iv)$) we get that

$$\mathbb{R}_{++} \times O_Y \times O_Z \subseteq H^+ \text{ and } T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z) \subseteq H^-.$$

On the other hand, assume that $T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z) \cap (\mathbb{R}_{++} \times O_Y \times O_Z) \neq \emptyset$, i.e. there exists $(\tilde{t}, O_y, O_Z) \in T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z)$, with $\tilde{t} > 0$. The set $T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z)$ being a cone, it follows that $\mathbb{R}_{++} \times O_Y \times O_Z \subseteq T_{\mathcal{E}_{v(\mathrm{P^e})}}(0, O_Y, O_Z)$. Then we obtain $\mathbb{R}_{++} \times O_Y \times O_Z \subseteq H^-$ and so $\mathbb{R}_{++} \times O_Y \times O_Z \subseteq H$, which is a contradiction. □

The following result is a direct consequence of Proposition 3.1 and Theorem 3.1.

Corollary 3.1. *Suppose that x_0 is an optimal solution of the primal problem* $(\mathrm{P^e})$. *Then, Assumption S is fulfilled at x_0 if and only if $v(\mathrm{P^e}) = v(\mathrm{D^e_L})$ and $\exists \lambda_0 \in C^*, \exists \mu_0 \in Z^*$ such that (λ_0, μ_0) is an optimal solution of the dual. In this situation we have* $\langle \lambda_0, g(x_0)\rangle = 0$.

Remark 3.1. We have proved that Assumption S is not only a sufficient condition for the existence of strong duality, but actually an equivalent formulation of this. Daniele and Giuffrè use in their main result in Ref. 7 Assumption S in order to ensure the existence of a separation between $(f, g, h)(S) + \mathbb{R}_+ \times C \times 0_Z$ and $(f(x_0), O_Y, O_Z)$, where $x_0 \in R$ is an optimal solution for ($\mathrm{P^e}$). What they get is that there exists $(\theta,\lambda,\mu)\in\mathbb{R}_+\times C^*\times Z^*$, $(\theta,\lambda,\mu)\neq(0,O_{Y^*},O_{Z^*})$ such that (cf. (3) in Ref. 7)

$$\theta(f(x)+\alpha)+\langle\lambda,g(x)+y\rangle+\langle\mu,h(x)\rangle\geq\theta f(x_0),\forall x\in S,\forall\alpha\geq 0,\forall y\in C.$$

The remaining assumptions are used for guaranteeing the "nonverticality" of the separating hyperplane, namely that $\theta\neq 0$. As one can notice, in the proof of this fact occurred a mistake, namely in the second relation after inequality (8) in Ref. 7. The following example shows that even one has separation between the aforementioned sets, the regularity conditions assumed in Theorem 3.1 in Ref. 7 do not ensure that $\theta\neq 0$.

Example 3.1. Let be $X=\mathbb{R}^2$, $Y=Z=\mathbb{R}$, $C=\mathbb{R}_+$, $S=\mathbb{R}_+\times\mathbb{R}$, $f:S\to\mathbb{R}$, $f(x,y)=-\sqrt{x}$, $g:S\to\mathbb{R}$, $g(x,y)=-x-y$ and $h:\mathbb{R}^2\to\mathbb{R}$, $h(x,y)=x$. One can see that f and g are convex functions and h is affine-linear. More than that, the feasible set $R=\{(x,y)\in S:g(x,y)\leq 0,h(x,y)=0\}$ is nothing else than $\{0\}\times\mathbb{R}$. We also have that $\operatorname{qri}C=\mathbb{R}_{++}$, $\operatorname{cl}(C-C)=\mathbb{R}=Y$, $S-S=\mathbb{R}^2$ and $\operatorname{cl}h(S-S)=\mathbb{R}=Z$. The element $\hat{x}=(0,1)\in S$ fulfills $g(\hat{x})=-1\in-\operatorname{qri}C$ and $h(\hat{x})=0$. The optimal objective value of ($\mathrm{P^e}$) is $v(\mathrm{P^e})=f(x_0)=0$, where $x_0=(0,0)$ is an optimal solution.

Let be $(\theta,\lambda,\mu)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}$, $(\theta,\lambda,\mu)\neq(0,0,0)$ fulfilling

$$\theta(f(x,y)+\alpha)+\lambda(g(x,y)+c)+\mu h(x,y)\geq 0,\forall(x,y)\in S,\forall\alpha\geq 0,\forall c\in\mathbb{R}_+.\tag{7}$$

As one can see in the following, the regularity conditions in Theorem 3.1 in Ref. 7, which, as we have shown, are fulfilled, are not guaranteeing that $\theta > 0$. Indeed, in (7) we must have $\lambda = 0$ and thus it follows that

$$-\theta\sqrt{x} + \mu x \geq 0, \quad \forall x \geq 0,$$

which is true only for $\theta = 0$.

3.2. The Problem with Cone Constraints

In the following we deal with the convex optimization problem

$$\text{(P)} \qquad \inf_{x \in T} f(x),$$

where the feasible set $T = \{x \in S : g(x) \in -C\}$, expressed here only by means of cone constraints, is assumed to be nonempty. The spaces X and Y, the sets S and C and the functions f and g are considered like in the previous subsection. The Lagrange dual problem associated to (P) is having the following formulation

$$(\mathrm{D_L}) \qquad \sup_{\lambda \in C^*} \inf_{x \in S} [f(x) + \langle \lambda, g(x) \rangle].$$

In Ref. 8 Daniele, Giuffrè, Idone and Maugeri say that, for problem (P), Assumption S is fulfilled at $x_0 \in T$ if

$$T_{\widetilde{M}}(f(x_0), O_Y) \cap (\mathbb{R}_{--} \times O_Y) = \emptyset,$$

where

$$\widetilde{M} := \{(f(x) + \alpha, g(x) + y) : x \in S \setminus T, \alpha \geq 0, y \in C\}.$$

They also formulate the following strong duality theorem (Theorem 4 in Ref. 8).

Let X be a real linear topological space and S a nonempty subset of X; let $(Y, \|\cdot\|)$ be a real normed space partially ordered by a convex cone C. Let $f : S \to \mathbb{R}$ and $g : S \to Y$ be two functions such that the function $(f,g) : S \to \mathbb{R} \times Y$ defined above is convex-like with respect to the cone $\mathbb{R}_+ \times C$ of $\mathbb{R} \times Y$. Let the set $T = \{x \in S : g(x) \in -C\}$ be nonempty and let us assume that $\operatorname{qri} C \neq \emptyset$, $\operatorname{cl}(C - C) = Y$ *and there exists $\overline{x} \in S$ with $g(\overline{x}) \in -\operatorname{qri} C$. Then if problem* (P) *is solvable and Assumption S is fulfilled at the extremal solution $x_0 \in T$ to problem* (P)*, also problem* $(\mathrm{D_L})$ *is solvable, the extremal values of both problems are equal and it results*

$$\langle \overline{\lambda}, g(x_0) \rangle = 0,$$

where $\overline{\lambda} \in C^*$ is the extremal point of problem $(\mathrm{D_L})$.

Making the same considerations like in the previous subsection (it is enough to take $Z = 0$ and $h(x) = 0 \; \forall x \in X$) one can easily deduce from Corollary 3.1 that Assumption S is fulfilled if and only if between (P) and $(\mathrm{D_L})$ strong duality holds. This means that the other assumptions in the result presented above are superfluous.

In Remark 6 in Ref. 8 the authors prove that for a concrete optimization problem which is a reformulation of the elastic-torsion problem Assumption S is also necessary. As we have seen above this is true not just in that very special case, but even in general.

Let us also mention that under the convexity assumptions stated for (P) only assuming that $\operatorname{qri} C \neq \emptyset$, $\operatorname{cl}(C - C) = Y$ and the existence of $\overline{x} \in S$ with $g(\overline{x}) \in -\operatorname{qri} C$ is not enough for having strong duality between (P) and ($\mathrm{D_L}$). This follows also from the following example, which was given by Daniele and Giuffrè in Ref. 7.

Example 3.2. Let be $X = S = Y = l^2$, the Hilbert space consisting of all sequences $x = (x_n)_{n\in\mathbb{N}}$ such that $\sum\limits_{n=1}^{\infty} x_n^2 < \infty$ and $C = l^2_+ = \{x = (x_n)_{n\in\mathbb{N}} \in l^2 : x_n \geq 0, \forall n \in \mathbb{N}\}$, the positive cone of l^2. Take $f : l^2 \to \mathbb{R}$, $f(x) = \langle c, x\rangle$, where $c = (c_n)_{n\in\mathbb{N}}, c_n = \frac{1}{n}$, $\forall n \in \mathbb{N}$ and $g : l^2 \to l^2, g(x) = -Ax$, where $(Ax)_n = \frac{1}{2^n} x_n$, $\forall n \in \mathbb{N}$. Then $T = \{x \in l^2 : Ax \in l^2_+\} = l^2_+$. It holds $\operatorname{cl}(l^2_+ - l^2_+) = l^2$ and $\operatorname{qri} l^2_+ = \{x = (x_n)_{n\in\mathbb{N}} \in l^2 : x_n > 0,\ \forall n \in \mathbb{N}\} \neq \emptyset$ (cf. Ref. 4) and one can easily find an $\overline{x} \in l^2$ with $g(\overline{x}) \in -\operatorname{qri} l^2_+$. We also have that

$$v(\mathrm{P}) = \inf_{x\in T} \langle c, x\rangle = 0$$

and $x = O_{l^2}$ is an optimal solution of the primal problem. On the other hand, for $\lambda \in C^* = l^2_+$, it holds

$$\inf_{x\in S}[f(x) + \langle \lambda, g(x)\rangle] = \inf_{x\in l^2}[\langle c, x\rangle + \langle \lambda, g(x)\rangle]$$

$$\inf_{x=(x_n)_{n\in\mathbb{N}}\in l^2} \left(\sum_{n=1}^{\infty} \frac{1}{n} x_n - \sum_{n=1}^{\infty} \lambda_n \frac{1}{2^n} x_n\right) = \inf_{x=(x_n)_{n\in\mathbb{N}}\in l^2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{\lambda_n}{2^n}\right) x_n$$

$$= \begin{cases} 0, & \text{if } \lambda_n = \frac{2^n}{n}, \forall n \in \mathbb{N}, \\ -\infty, & \text{otherwise.} \end{cases}$$

Since $(\frac{2^n}{n})_{n\in\mathbb{N}}$ does not belong to l^2, we obtain $v(\mathrm{D_L}) = -\infty$, hence the optimal objective values of the two problems do not coincide.

This means that along the regularity conditions one needs to make some supplementary assumptions for ensuring strong duality. Since Assumption S is a necessary and sufficient condition, it could be of interest to give weak regularity conditions expressed via the quasirelative interior which prove to be (only) sufficient for having strong duality between (P) and its Lagrange dual. That will be done in the next section.

Before coming to these considerations, let us also mention the recent paper Ref. 6 due to Cammaroto and Di Bella where the following duality result stated in terms of the quasirelative interior is proposed (Theorem 2.2 in Ref. 6).

Let X be a real linear topological space and let S be a nonempty subset of X; let $(Y, \|\cdot\|)$ be a real normed space partially ordered by a convex cone C; let $f : S \to \mathbb{R}$ and $g : S \to Y$ be two functions such that the function $(f, g) : S \to \mathbb{R} \times Y$ defined above is convex-like with respect to the cone $\mathbb{R}_+ \times C$ of $\mathbb{R} \times Y$, $\operatorname{qri}(g(S) + C) \neq \emptyset$ and $\operatorname{cl}\operatorname{cone}[\operatorname{qri}(g(S) + C)]$ is not a linear subspace of Y. Let the set $T = \{x \in S : g(x) \in -C\}$ be nonempty. In addition, suppose that $\operatorname{qri} C \neq \emptyset$ and $\operatorname{cl}(C - C) = Y$. If the problem (P) *is solvable and there exists $\overline{x} \in S$ with $g(\overline{x}) \in -\operatorname{qri} C$, then the problem* $(\mathrm{D_L})$ *is also solvable and the extrema of the problems are equal.*

The following lemma proves that this result cannot be used, since the hypotheses of this theorem are in contradiction.

Lemma 3.2. *Suppose that* $\operatorname{cl}(C-C)=Y$ *and* $\exists \overline{x} \in S$ *such that* $g(\overline{x}) \in -\operatorname{qri} C$. *Then the following are true:*

(a) $O_Y \in \operatorname{qi}(g(S)+C)$*;*

(b) $\operatorname{cl}\operatorname{cone}[\operatorname{qri}(g(S)+C)]$ *is a linear subspace of* Y*.*

Proof. (a) We apply Lemma 2.1 with $A := -C$ and $B := g(S)+C$. The sets A and B are convex and we have $O_Y \in A \cap B$. The condition $\operatorname{cl}(C-C)=Y$ implies $O_Y \in \operatorname{qi}(A-A)$, while the Slater-type condition gives us $g(\overline{x}) \in B \cap \operatorname{qri} A$. Hence, by Lemma 2.1 we obtain $O_Y \in \operatorname{qi}(A-B)$, that is $O_Y \in \operatorname{qi}(-g(S)-C)$, which is nothing else than $O_Y \in \operatorname{qi}(g(S)+C)$.

(b) From (a) it follows that $O_Y \in \operatorname{qri}(g(S)+C)$. Applying Proposition 2.3 (vii) we get $O_Y \in \operatorname{qri}(\operatorname{qri}(g(S)+C))$, which means that $\operatorname{cl}\operatorname{cone}[\operatorname{qri}(g(S)+C)]$ is a linear subspace of Y.□

4. A Valuable Strong Duality Theorem

In the following we give a strong duality theorem for (P) and its Lagrange dual ($\mathrm{D_L}$) under a weak regularity condition expressed by using the quasirelative interior of the sets involved. Different to the similar attempts in Ref. 6 and Ref. 8, we do not assume that the primal problem has an optimal solution. This situation will be treated in a corollary which will follow our main result.

Since in case $v(\mathrm{P}) = -\infty$, strong duality obviously holds, for the rest of the paper we consider that $v(\mathrm{P}) \in \mathbb{R}$.

The conic extension for (P) looks now like

$$\begin{aligned}\mathcal{E}_{v(\mathrm{P})} &= \{(v(\mathrm{P}) - f(x) - \alpha, -g(x) - y) : x \in S, \alpha \geq 0, y \in C\} \\ &= (v(\mathrm{P}), O_Y) - (f, g)(S) - \mathbb{R}_+ \times C,\end{aligned}$$

and is also in this case a convex set fulfilling $(0, O_Y) \in \mathcal{E}_{v(\mathrm{P})}$ if and only if the primal problem (P) has an optimal solution.

We prove first some preliminary results.

Lemma 4.1. *The following statements are true:*

(a) *if* $g(s_0) + y_0 \in \operatorname{qri}(g(S) + C)$ *then* $(v(\mathrm{P}) - f(s_0) - t, -g(s_0) - y_0) \in \operatorname{qri} \mathcal{E}_{v(\mathrm{P})}$, $\forall t > 0$;

(b) *if* $(r_0, y_0) \in \operatorname{qri} \mathcal{E}_{v(\mathrm{P})}$ *then* $-y_0 \in \operatorname{qri}(g(S) + C)$;

(c) $\operatorname{qri} \mathcal{E}_{v(\mathrm{P})} \neq \emptyset$ *if and only if* $\operatorname{qri}(g(S) + C) \neq \emptyset$.

Proof. (a) Let us suppose that $g(s_0) + y_0 \in \operatorname{qri}(g(S) + C)$. Let $t > 0$ be fixed. Then obviously $(v(\mathrm{P}) - f(s_0) - t, -g(s_0) - y_0) \in \mathcal{E}_{v(\mathrm{P})}$. Take (r^*, y^*) an arbitrary element in $N_{\mathcal{E}_{v(\mathrm{P})}}(v(\mathrm{P}) - f(s_0) - t, -g(s_0) - y_0)$. It holds

$$r^*(u - (v(\mathrm{P}) - f(s_0) - t)) + \langle y^*, v - (-g(s_0) - y_0)\rangle \leq 0, \ \forall (u, v) \in \mathcal{E}_{v(\mathrm{P})}.$$

Choosing first in the previous inequality

$$u := v(\mathrm{P}) - f(s_0) - t/2, \quad v := -g(s_0) - y_0$$

and then

$$u := v(\mathrm{P}) - f(s_0) - (3t)/2, \quad v := -g(s_0) - y_0,$$

we obtain $+\frac{t}{2}r^* \leq 0$ and $-\frac{t}{2}r^* \leq 0$, respectively, that is $r^* = 0$. Hence $\langle y^*, v-(-g(s_0)-y_0)\rangle \leq 0, \forall (u, v) \in \mathcal{E}_{v(\mathrm{P})}$, which is nothing else than $\langle y^*, v - (-g(s_0) - y_0)\rangle \leq 0, \forall v \in -g(S) - C$. Thus $-y^* \in N_{g(S)+C}(g(s_0) + y_0)$. Since $N_{g(S)+C}(g(s_0) + y_0)$ is a linear subspace of Y^*, we have also that $y^* \in N_{g(S)+C}(g(s_0) + y_0)$ and so $\langle -y^*, v - (-g(s_0) - y_0)\rangle \leq 0, \forall v \in -g(S) - C$. Hence

$$\langle (0, -y^*), (u - (v(\mathrm{P}) - f(s_0) - t), v - (-g(s_0) - y_0))\rangle \leq 0, \forall (u, v) \in \mathcal{E}_{v(\mathrm{P})}.$$

Further, we get $-(r^*, y^*) = (0, -y^*) \in N_{\mathcal{E}_{v(\mathrm{P})}}(v(\mathrm{P}) - f(s_0) - t, -g(s_0) - y_0)$, showing that $N_{\mathcal{E}_{v(\mathrm{P})}}(v(\mathrm{P}) - f(s_0) - t, -g(s_0) - y_0)$ is a linear subspace of $\mathbb{R} \times Y^*$, that is $(v(\mathrm{P}) - f(s_0) - t, -g(s_0) - y_0) \in \operatorname{qri} \mathcal{E}_{v(\mathrm{P})}$.

(b) Assume that $(r_0, y_0) \in \operatorname{qri} \mathcal{E}_{v(\mathrm{P})}$. Take y^* an arbitrary element in the normal cone $N_{g(S)+C}(-y_0) = \{y^* \in Y^* : \langle y^*, v + y_0\rangle \leq 0, \forall v \in g(S) + C\}$. Then $(0, -y^*) \in N_{\mathcal{E}_{v(\mathrm{P})}}(r_0, y_0) = \{(r^*, y^*) \in \mathbb{R} \times Y^* : r^*(u - r_0) + \langle y^*, v - y_0\rangle \leq 0, \forall (u, v) \in \mathcal{E}_{v(\mathrm{P})}\}$. As $N_{\mathcal{E}_{v(\mathrm{P})}}(r_0, y_0)$ is a linear subspace of $\mathbb{R} \times Y^*$ we get $(0, y^*) \in N_{\mathcal{E}_{v(\mathrm{P})}}(r_0, y_0)$, that is $-\langle y^*, v + y_0\rangle \leq 0, \forall v \in g(S) + C$. This is nothing else than $-y^* \in N_{g(S)+C}(-y_0)$. This means that $N_{g(S)+C}(-y_0)$ is a linear subspace of Y^*, hence $-y_0 \in \operatorname{qri}(g(S) + C)$.

(c) This assertion is a direct consequence of the statements (a) and (b). □

Proposition 4.1 *Assume that* $O_Y \in \operatorname{qi}[(g(S)+C)-(g(S)+C)]$. *Then one has that* $N_{\operatorname{co}(\mathcal{E}_{v(\mathrm{P})}\cup\{(0,O_Y)\})}(0,O_Y)$ *is a linear subspace of* $\mathbb{R}\times Y^*$ *if and only if* $N_{\operatorname{co}(\mathcal{E}_{v(\mathrm{P})}\cup\{(0,O_Y)\})}(0,O_Y) = \{(0,O_{Y^*})\}$.

Proof. The sufficiency is trivial.

Consider that $N_{\operatorname{co}(\mathcal{E}_{v(\mathrm{P})}\cup\{(0,O_Y)\})}(0,O_Y)$ is a linear subspace of $\mathbb{R}\times Y^*$. Take $(\theta,\lambda) \in N_{\operatorname{co}(\mathcal{E}_{v(\mathrm{P})}\cup\{(0,O_Y)\})}(0,O_Y)$ arbitrary. Then $\theta u + \langle\lambda, v\rangle \leq 0$, $\forall (u,v) \in \operatorname{co}(\mathcal{E}_{v(\mathrm{P})}\cup\{0,O_Y\})$, which implies

$$-\theta(f(x)+\alpha - v(\mathrm{P})) - \langle\lambda, g(x)+y\rangle \leq 0, \forall x\in S, \forall y \in C \text{ and } \forall \alpha \geq 0. \tag{8}$$

Let $x' \in T$ be a feasible element. For $y := -g(x')$ and $x := x'$ in the above inequality we obtain $-\theta(f(x')+\alpha-v(\mathrm{P})) \leq 0, \forall\alpha\geq 0$, hence $\theta \geq 0$ (otherwise, if $\theta<0$, then when passing to the limit as $\alpha\to+\infty$ we obtain a contradiction). Since $N_{\operatorname{co}(\mathcal{E}_{v(\mathrm{P})}\cup\{(0,O_Y)\})}(0,O_Y)$ is a linear subspace of $\mathbb{R}\times Y^*$, the argument from above applies also for $(-\theta,-\lambda)$, implying $\theta\leq 0$. Finally, we get $\theta = 0$ and inequality (8) and relation $(0,-\lambda)\in N_{\operatorname{co}(\mathcal{E}_{v(\mathrm{P})}\cup\{(0,O_Y)\})}(0,O_Y)$ imply

$$\langle\lambda, g(x)+y\rangle = 0, \forall x\in S \text{ and } \forall y\in C.$$

It follows that $\langle\lambda, y\rangle = 0, \forall y \in \operatorname{cl}\operatorname{cone}[(g(S)+C)-(g(S)+C)] = Y$, that is $\lambda = O_{Y^*}$. So $(\theta,\lambda) = (0,O_{Y^*})$ and the conclusion follows. $\square$

Remark 4.1. (a) As $C - C \subseteq (g(S)+C)-(g(S)+C)$, we have the following implication

$$\operatorname{cl}(C-C) = Y \Rightarrow O_Y \in \operatorname{qi}[(g(S)+C)-(g(S)+C)].$$

(b) Since $\operatorname{cone}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\}) = \operatorname{cone}\mathcal{E}_{v(\mathrm{P})}$, we automatically get the following relation $\operatorname{cl}\operatorname{cone}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\}) = \operatorname{cl}\operatorname{cone}\mathcal{E}_{v(\mathrm{P})}$. Hence the normal cone

$N_{\operatorname{co}(\mathcal{E}_{v(\mathrm{P})}\cup\{(0,O_Y)\})}(0, O_Y)$ is a linear subspace of $\mathbb{R} \times Y^* \Leftrightarrow (0, O_Y) \in \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$

$\Leftrightarrow \operatorname{cl}\operatorname{cone}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$ is a linear subspace of $\mathbb{R} \times Y \Leftrightarrow \operatorname{cl}\operatorname{cone}\mathcal{E}_{v(\mathrm{P})}$ is a linear subspace of $\mathbb{R} \times Y$.

On the other hand, the condition $N_{\operatorname{co}(\mathcal{E}_{v(\mathrm{P})}\cup\{(0,O_Y)\})}(0, O_Y) = \{(0, O_{Y^*})\}$ is equivalent to $(0, O_Y) \in \operatorname{qi}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$, so in case $O_Y \in \operatorname{qi}[(g(S) + C) - (g(S) + C)]$, we have

$$\operatorname{cl}\operatorname{cone}\mathcal{E}_{v(\mathrm{P})} \text{ is a linear subspace of } \mathbb{R} \times Y \Leftrightarrow (0, O_Y) \in \operatorname{qi}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\}),$$

or, equivalently

$$(0, O_Y) \in \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\}) \Leftrightarrow (0, O_Y) \in \operatorname{qi}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\}).$$

(c) If the primal problem has an optimal solution (which means that $(0, O_Y) \in \mathcal{E}_{v(\mathrm{P})}$) and $O_Y \in \operatorname{qi}[(g(S) + C) - (g(S) + C)]$ we have

$$(0, O_Y) \in \operatorname{qri}\mathcal{E}_{v(\mathrm{P})} \Leftrightarrow (0, O_Y) \in \operatorname{qi}\mathcal{E}_{v(\mathrm{P})}.$$

We are able now to give the following strong duality result.

Theorem 4.1. *Suppose that* $\operatorname{cl}(C - C) = Y$ *and* $\exists \overline{x} \in S$ *such that* $g(\overline{x}) \in -\operatorname{qri} C$. *If* $(0, O_Y) \notin \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$, *then* $v(\mathrm{P}) = v(\mathrm{D_L})$ *and* $(\mathrm{D_L})$ *has an optimal solution.*

Proof. Lemma 3.2 and Lemma 4.1 ensure that $\operatorname{qri}\mathcal{E}_{v(\mathrm{P})} \neq \emptyset$, while condition $(0, O_Y) \notin$

$\operatorname{qri}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$ means actually that $\operatorname{cl}\operatorname{cone}\mathcal{E}_{v(\mathrm{P})}$ is not a linear subspace of $\mathbb{R} \times Y$. Applying Corollary 2.1, we can separate now the sets $\mathcal{E}_{v(\mathrm{P})}$ and $\{(0, O_Y)\}$. Thus there exists $(\theta, \lambda) \in \mathbb{R} \times Y^*$, $(\theta, \lambda) \neq (0, O_{Y^*})$ such that

$$\theta(f(x) + \alpha - v(\mathrm{P})) + \langle \lambda, g(x) + y\rangle \geq 0, \ \forall x \in S, \forall \alpha \geq 0, \forall y \in C. \tag{9}$$

We claim that $\lambda \in C^*$. If we suppose that $\exists y_0 \in C$ such that $\langle \lambda, y_0\rangle < 0$, then the inequality $\theta(f(x)+\alpha-v(\mathrm{P}))+\langle\lambda, g(x)\rangle+t\langle\lambda, y_0\rangle \geq 0$ is true for every $t \geq 0$ (cf. (9)) and passing to the limit as $t \to +\infty$ (for a fixed $x \in S$ and $\alpha \geq 0$) we obtain a contradiction. Similar arguments as in the proof of the Proposition 4.1 show that $\theta \geq 0$.

Let us prove that actually $\theta > 0$. Assume that $\theta = 0$. Then (9) gives us $\langle \lambda, g(x) + y\rangle \geq 0, \forall x \in S$, $\forall y \in C$. By the hypotheses, $\exists \overline{x} \in S$ such that $g(\overline{x}) \in -\operatorname{qri} C$. This together with $\lambda \in C^*$ show that $\langle \lambda, g(\overline{x})\rangle \leq 0$. Taking $y = 0_Y$ and $x = \overline{x}$ in the inequality $\langle \lambda, g(x) + y\rangle \geq 0, \forall x \in S, \forall y \in C$ we get $\langle \lambda, g(\overline{x})\rangle \geq 0$ and hence $\langle \lambda, g(\overline{x})\rangle = 0$. Also from the inequality $\langle \lambda, g(\overline{x}) + y\rangle \geq 0, \forall y \in C$ we obtain $-\lambda \in N_C(-g(\overline{x}))$. As $N_C(-g(\overline{x}))$ is a linear subspace of Y^* we get $\langle \lambda, g(\overline{x}) + y\rangle = 0, \forall y \in C$, that is $\langle \lambda, y\rangle = 0, \forall y \in C$, hence $\langle \lambda, y\rangle = 0$, $\forall y \in \operatorname{cl}(C - C) = Y$, namely $\lambda = O_{Y^*}$. Thus $(\theta, \lambda) = (0, O_{Y^*})$ and this leads to a contradiction. We must have $\theta > 0$.

Taking in (9) $\alpha = 0$ and $y = O_Y$ we obtain

$$v(\mathrm{P}) \leq f(x) + \frac{1}{\theta}\langle \lambda, g(x)\rangle, \forall x \in S.$$

With the notation $\overline{\lambda} := \frac{1}{\theta}\lambda \in C^*$ we get $v(\mathrm{P}) \le f(x) + \langle \overline{\lambda}, g(x)\rangle, \forall x \in S$. Taking the infimum with respect to $x \in S$ we have $v(\mathrm{P}) \le \inf_{x \in S}[f(x) + \langle \overline{\lambda}, g(x)\rangle]$, hence $v(\mathrm{P}) \le v(\mathrm{D_L})$. As the opposite inequality always holds, we get $v(\mathrm{P}) = v(\mathrm{D_L})$ and $\overline{\lambda}$ is an optimal solution of the dual problem $(\mathrm{D_L})$. □

In case the primal problem (P) has an optimal solution we get the following strong duality result.

Corollary 4.1. *Suppose that the primal problem has an optimal solution,* $\mathrm{cl}(C - C) = Y$ *and* $\exists \overline{x} \in S$ *such that* $g(\overline{x}) \in -\operatorname{qri} C$. *If* $(0, O_Y) \notin \operatorname{qri} \mathcal{E}_{v(\mathrm{P})}$, *then* $v(\mathrm{P}) = v(\mathrm{D_L})$ *and* $(\mathrm{D_L})$ *has an optimal solution.*

Remark 4.2. (a) In the hypotheses of Theorem 4.1 one has that $(0, O_Y) \notin \operatorname{qri} \operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$ is equivalent to $(0, O_Y) \notin \operatorname{qi} \operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$ (see Remark 4.1). Similarly, in the hypotheses of Corollary 4.1 condition $(0, O_Y) \notin \operatorname{qri} \mathcal{E}_{v(\mathrm{P})}$ is equivalent to $(0, O_Y) \notin \operatorname{qi} \mathcal{E}_{v(\mathrm{P})}$.

(b) One has that

$$(0, O_Y) \in \operatorname{qi} \operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, 0)\}) \Rightarrow O_Y \in \operatorname{qi}(g(S) + C).$$

Indeed, if $(0, O_Y) \in \operatorname{qi} \operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$, then $\operatorname{cl} \operatorname{cone} \operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\}) = \mathbb{R} \times Y$, thus $\operatorname{cl} \operatorname{cone} \mathcal{E}_{v(\mathrm{P})} = \mathbb{R} \times Y$. Since $\mathcal{E}_{v(\mathrm{P})} \subseteq \mathbb{R} \times (-g(S) - C)$, we get $\operatorname{cl} \operatorname{cone}[-g(S) - C] = Y$. The last relation is nothing else than $O_Y \in \operatorname{qi}(g(S) + C)$. Thus

$$O_Y \notin \operatorname{qi}(g(S) + C) \Rightarrow (0, O_Y) \notin \operatorname{qi} \operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\}).$$

Hence we have found a condition which guarantees the fulfillment of $(0, O_Y) \notin \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$ (which, in case $\operatorname{cl}(C - C) = Y$, is equivalent to $(0, O_Y) \notin \operatorname{qi}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$). Let us mention that one cannot substitute in the hypotheses of Theorem 4.1 $(0, O_Y) \notin \operatorname{qri}\operatorname{co}(\mathcal{E}_{v(\mathrm{P})} \cup \{(0, O_Y)\})$ by $O_Y \notin \operatorname{qi}(g(S) + C)$, since this would be in contradiction with the other assumptions (see Lemma 3.2).

(c) Coming now back to Example 3.2, it is not surprising that there strong duality does not holds, since not all the hypotheses of Corollary 4.1 are fulfilled. This is what we show in the following, namely that $(0, O_{l^2}) \in \operatorname{qi}\mathcal{E}_{v(\mathrm{P})}$. Take $(\theta, \lambda) \in N_{\mathcal{E}_{v(\mathrm{P})}}(0, O_{l^2})$. Then we have

$$\theta(-\langle c, x\rangle - \alpha) + \langle \lambda, -g(x) - y\rangle \leq 0, \forall x \in l^2, \forall \alpha \geq 0, \forall y \in l^2_+, \tag{10}$$

that is

$$\theta\left(-\sum_{n=1}^{\infty} \frac{1}{n} x_n - \alpha\right) + \sum_{n=1}^{\infty} \lambda_n \left(\frac{1}{2^n} x_n - y_n\right) \leq 0,$$

$$\forall x = (x_n)_{n\in\mathbb{N}} \in l^2, \forall \alpha \geq 0,\ \forall y = (y_n)_{n\in\mathbb{N}} \in l^2_+.$$

Setting $\alpha = 0$ and $y_n = 0$, $\forall n \in \mathbb{N}$ in the relation above we get

$$\sum_{n=1}^{\infty} \left(-\theta \frac{1}{n} + \frac{1}{2^n}\lambda_n\right) x_n \leq 0, \forall x = (x_n)_{n\in\mathbb{N}} \in l^2,$$

which implies $\lambda_n = \theta \frac{2^n}{n}$, $\forall n \in \mathbb{N}$. Since $\lambda \in l^2$, we must have $\theta = 0$ and hence $\lambda = O_{l^2}$. Thus $N_{\mathcal{E}_{v(\mathrm{P})}}(0, O_{l^2}) = \{(0, O_{l^2})\}$ and so $(0, O_{l^2}) \in \operatorname{qi}\mathcal{E}_{v(\mathrm{P})}$.

In the following example we introduce an optimization problem for which strong Lagrange duality holds. In this way we illustrate the applicability of Corollary 4.1.

Example 4.1. Let be $X = S = Y = l^2$ and $C = l^2_+$, the positive cone of l^2. For $f : l^2 \to \mathbb{R}$, $f(x) = \langle c, x\rangle$, where $c = (c_n)_{n\in\mathbb{N}}, c_n = \frac{1}{n}$, $\forall n \in \mathbb{N}$ and $g : l^2 \to l^2, g(x) = -x$, we get $T = l^2_+$ and the following optimization problem

$$\inf_{x\in T} \langle c, x\rangle.$$

Its optimal objective value $v(\mathrm{P})$ is equal to zero and $x = O_{l^2}$ is an optimal solution of (P). The conditions $\mathrm{cl}(C - C) = Y$ and $\exists \overline{x} \in S$ such that $g(\overline{x}) \in -\operatorname{qri} C$ are obviously satisfied.

We prove that $(0, O_{l^2}) \notin \operatorname{qi} \mathcal{E}_{v(\mathrm{P})}$. Indeed, by using (10) we have $(\theta, \lambda) \in N_{\mathcal{E}_{v(\mathrm{P})}}(0, O_{l^2})$ if and only if $\theta(-\langle c, x\rangle - \alpha) + \langle \lambda, x - y\rangle \leq 0, \forall x \in l^2, \forall \alpha \geq 0, \forall y \in l^2_+$, or, equivalently $\langle -\theta c + \lambda, x\rangle - \theta\alpha - \langle \lambda, y\rangle \leq 0, \forall x \in l^2, \forall \alpha \geq 0, \forall y \in l^2_+$. It is obvious that $(\theta, \lambda) := (1, c) \in N_{\mathcal{E}_{v(\mathrm{P})}}(0, O_{l^2})$, which means that $N_{\mathcal{E}_{v(\mathrm{P})}}(0, O_{l^2}) \neq \{(0, O_{l^2})\}$ or, equivalently, $(0, O_{l^2}) \notin \operatorname{qi} \mathcal{E}_{v(\mathrm{P})}$. The hypotheses of Corollary 4.1 being fulfilled, strong duality holds between (P) and ($\mathrm{D_L}$). One can easily see that $\bar{\lambda} = c$ is an optimal solution for the dual.

Remark 4.3. A correct strong duality theorem for the primal convex optimization problem with both, cone and equality constraints, and its Lagrange dual, with the regularity conditions expressed by means of the quasirelative interior, will be given in a forthcoming paper.

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