On some abstract convexity notions in real linear spaces

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Abstract. We introduce some abstract convexity notions in a real linear space and investigate which of the results from the convex analysis in topological vector spaces still work in a linear space. The differences between these abstract convexity notions and those established in spaces endowed with a topology are underlined by some examples.

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1 Introduction

Convex analysis is an important tool from the theoretical point of view, but also because of its usefulness in the optimization theory. Developing this theory in finite dimensional spaces (see [10]) or more general in locally convex spaces (see [2], [15]), soon it was realized that some of the general results remain valid in a more general setting, like metric spaces or linear spaces (so without any topology). This theory is known under the name abstract convex analysis. For an exhaustive survey of these abstract notions we refer to the books of Singer (see [13]) and Rubinov (see [12]). Many papers deal with this kind of abstract notions, see for instance [1], [4], [6], [7], [9], [11], [14].

In this paper we investigate some abstract convexity notions in the framework of real linear spaces. In a locally convex space X there is a strong connection between a lower semicontinuous function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ and its epigraph, namely f is lower semicontinuous if and only if $\text{epi}(f)$ is closed in $X \times \mathbb{R}$. But the closure of a set and the lower semicontinuity are topological notions, so in a

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real linear space the question is how to define a "lower semicontinuous" function and the "closure" of a set, in order to have a similar result between these two notions.

The aim of this paper is to verify which of the results that hold in locally convex spaces remain true in a real linear space (of course, using the abstract convexity notions).

The paper is organized as follows. In the next section we present some definitions, notations and preliminary results concerning c-convex functions that will be used later in the paper. In Section 3 we introduce the notion of a c-convex set and investigate some properties of it. Section 4 is devoted to the investigations of the connections between a c-convex function and a c-convex set.

2 Preliminaries

In the following, we consider a real linear space X and $X^{\#}$ its algebraic dual space. Let $f : X \to \overline{\mathbb{R}}$ be a given function, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

We have

- the *domain* of f: $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$,
- the *epigraph* of f: $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$,
- f is *proper* if $f(x) > -\infty \ \forall x \in X$ and $\mathrm{dom}(f) \neq \emptyset$,
- $\mathrm{co}(f) : X \to \overline{\mathbb{R}}$ is the greatest convex function majorized by f,
- $\langle x^{\#}, x\rangle := x^{\#}(x)$, where $x^{\#}(x)$ defines the value of the linear functional $x^{\#} \in X^{\#}$ at the element $x \in X$,
- $g : X \to \mathbb{R}$ is *affine* if $\exists (x^{\#}, \alpha) \in X^{\#} \times \mathbb{R}$ such that $g(x) = x^{\#}(x) + \alpha \ \forall x \in X$,
- $g \leq f \Leftrightarrow g(x) \leq f(x) \ \forall x \in X$,
- $\mathcal{A}(X, f)$ is the set of affine minorants of f on X,
- the *indicator function* of a subset A of X, defined by

$$\delta_A(x) = \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

If X is a locally convex space, it can be proved (see for instance [2]) that the following conditions are equivalent

(a) $f(x) > -\infty \ \forall x \in X$, f convex and lower semicontinuous,

(b) there exists an affine minorant of f and f is the pointwise supremum of all its affine minorants (here, an affine function is characterized by an element x^* from the topological dual X^* of X).

Regarding this result, we give the analogue notion of "lower semi - continuity" for a convex function defined on a real linear space X.

Definition 1. A function $f : X \to \overline{\mathbb{R}}$ is called **c-convex on X** if

$$f(x) = \sup\{g(x) : g \text{ is an affine minorant of } f\} \ \forall x \in X.$$

The set of all c-convex functions on X is denoted by $\Gamma(X)$. In the literature (see, for instance, [1], [6], [7], [11]) the functions defined in this way are called in different ways, existing a number of terms for this notion. As there exists an analogy between it and the notion of a lower semicontinuous (closed) convex hull of a function in locally convex spaces, we consider that the term "c-convex" is appropriate. Let us notice that a c-convex function is always convex (being the pointwise supremum of a family of affine functions).

Lemma 1. *Every affine function $g : X \to \mathbb{R}$ is c-convex on X.*

Proof. As $g \leq \sup\{h : h \text{ affine}, h \leq g\} \leq g$, one has equality and so $g \in \Gamma(X)$. □

If X is a locally convex space, the lower semicontinuous convex hull of a function $f : X \to \overline{\mathbb{R}}$, denoted by $\mathrm{cl}\big(\mathrm{co}(f)\big)$ is the function whose epigraph is the closure of $\mathrm{co}\big(\mathrm{epi}(f)\big)$ in $X \times \mathbb{R}$. It is well known that $\mathrm{cl}\big(\mathrm{co}(f)\big)$ is the greatest lower semicontinuous convex function majorized by f (see [2]). So it is natural to define an analogue notion in the case of real linear spaces, in the following way.

Definition 2. We define **the c-convex hull** of f by

$$\mathrm{cc}(f) : X \to \overline{\mathbb{R}}, \mathrm{cc}(f)(x) = \sup\{g(x) : g \in \Gamma(X),\ g \leq f\} \ \forall x \in X.$$

Other authors use for the c-convex hull the terminology of "regular hull" of f (see [4]).

An example of a space X and a function $f : X \to \overline{\mathbb{R}}$ which is c-convex but not lower semicontinuous will be given in Section 4.

The following result shows that in the definition of the c-convex hull of a function it is enough to take the supremum of the family of its affine minorants.

Lemma 2. *For $f : X \to \overline{\mathbb{R}}$ we have*

$$\mathrm{cc}(f) = \sup\{g : g \textit{ affine},\ g \leq f\}.$$

Proof. Using Lemma 1 we obtain that for all $x \in X$

$$\sup\{g(x) : g \text{ affine}, \ g \leq f\} \leq \sup\{g(x) : g \in \Gamma(X), \ g \leq f\} = \mathrm{cc}(f)(x).$$

If we suppose that there exists $x_0 \in X$ such that

$$\sup\{g(x_0) : g \text{ affine}, g \leq f\} < \mathrm{cc}(f)(x_0),$$

then there exists $r \in \mathbb{R}$ with the following property

$$\sup\{g(x_0) : g \text{ affine}, \ g \leq f\} < r < \mathrm{cc}(f)(x_0)$$

$$= \sup\{g(x_0) : g \in \Gamma(X), g \leq f\}.$$

Then

$$\forall g \text{ affine}, \ g \leq f, \ \text{ we have } g(x_0) < r \tag{1}$$

and

$$\exists g_0 \in \Gamma(X), g_0 \leq f \text{ such that } g_0(x_0) > r.$$

The function g_0 being c-convex, since $g_0(x_0) = \sup\{h(x_0) : h \text{ affine}, \ h \leq g_0\} > r$, there exists h_0 affine, $h_0 \leq g_0$ such that $h_0(x_0) > r$. But $h_0 \leq g_0 \leq f$, so h_0 is affine and $h_0 \leq f$. This implies by (1) that $h_0(x_0) < r$, contradicting $h_0(x_0) > r$.□

Proposition 1. *Let be $f : X \to \overline{\mathbb{R}}$. The following assertions are true:*

(a) $\mathrm{cc}(f) \leq \mathrm{co}(f) \leq f$,

(b) $f \in \Gamma(X) \Leftrightarrow f = \mathrm{cc}(f)$.

Proof.

(a) As $\forall x \in X$, $\mathrm{cc}(f)(x) = \sup\{g(x) : g \in \Gamma(X) \text{ and } g \leq f\} \leq f(x)$, $\mathrm{cc}(f)$ is a convex function majorized by f and the conclusion follows.

(b) If $f \in \Gamma(X)$ then $\mathrm{cc}(f)(x) = \sup\{g(x) : g \in \Gamma(X) \text{ and } g \leq f\} \geq f(x), \forall x \in X$, and by (a) we get $f = \mathrm{cc}(f)$.
If $f = \mathrm{cc}(f)$ then it is obvious that $f \in \Gamma(X)$ (see Lemma 2). □

Definition 3. Let $x^{\#} \in X^{\#}, x^{\#} \neq 0$ and $\alpha \in \mathbb{R}$. Then

(a) $H(x^{\#}, \alpha) = \{x \in X : x^{\#}(x) = \alpha\}$ is called a **hyperplane** in X,

(b) $H^{\leq}(x^{\#}, \alpha) = \{x \in X : x^{\#}(x) \leq \alpha\}$ is called a **c-half-space** in X.

Now we recall some well-known definitions (see for instance [15]). For a subset $D \subseteq X$ the *core* (or the *algebraic interior*) of D is defined by

$$\mathrm{core}(D) = \{d \in D : \forall x \in X, \exists \varepsilon > 0 \text{ such that } \forall \lambda \in [-\varepsilon, \varepsilon], d + \lambda x \in D\}.$$

The core of D relative to $\mathrm{aff}(D - D)$ is called the *intrinsic core* (or the *relative algebraic interior*) of D and is denoted by $\mathrm{icr}(D)$, that is the set

$$\{d \in D : \forall x \in \mathrm{aff}(D - D), \exists \varepsilon > 0 \text{ such that } \forall \lambda \in [-\varepsilon, \varepsilon], d + \lambda x \in D\}.$$

It is easy to see that $\mathrm{core}(D) \subseteq \mathrm{icr}(D) \subseteq D$ and $\mathrm{icr}(\{a\}) = \{a\}\ \forall a \in X$. The following separation theorem can be found in [3] (see also [5]).

Theorem 1. *Let A and B be convex subsets of X such that both $\mathrm{icr}(A)$ and $\mathrm{icr}(B)$ are nonempty. Then A and B can be separated by a hyperplane H with $A \cup B \nsubseteq H$ if and only if $\mathrm{icr}(A) \cap \mathrm{icr}(B) = \emptyset$.*

In finite dimensional spaces we have that if f is a convex function, then $f(x) = \mathrm{cl}(f)(x), \forall x \in \mathrm{ri}(\mathrm{dom}(f))$, where $\mathrm{ri}(\mathrm{dom}(f))$ is the relative interior of the domain of f (see [10]). By using Theorem 1, we show that a similar result holds also in real linear spaces, working with the intrinsic core of $\mathrm{dom}(f)$.

Theorem 2. *Let $f : X \to \overline{\mathbb{R}}$ be a convex function. Then*

$$f(x) = \mathrm{cc}(f)(x)\ \forall x \in \mathrm{icr}(\mathrm{dom}(f)).$$

Proof. If $\mathrm{icr}(\mathrm{dom}(f)) = \emptyset$ then we have nothing to prove. Consider an arbitrary element $x_0 \in \mathrm{icr}(\mathrm{dom}(f))$. We already know from Proposition 1(a) that $\mathrm{cc}(f)(x_0) \leq f(x_0)$. If we suppose that we have strict inequality, then one can find a real number r_0 such that $\mathrm{cc}(f)(x_0) < r_0 < f(x_0)$. Using Lemma 2 we obtain

$$g(x_0) < r_0, \forall g \text{ which are affine minorants of } f. \tag{2}$$

Because of $\mathrm{icr}(\mathrm{dom}(f)) \neq \emptyset$ it follows $\mathrm{icr}(\mathrm{epi}(f)) \neq \emptyset$ (see [3]). As $(x_0, r_0) \notin \mathrm{epi}(f)$, and so $(x_0, r_0) \notin \mathrm{icr}(\mathrm{epi}(f))$, we can apply Theorem 1 in order to separate the sets $\{(x_0, r_0)\}$ and $\mathrm{epi}(f)$. So $\exists (x^{\#}, \alpha) \in X^{\#} \times \mathbb{R}$, $(x^{\#}, \alpha) \neq (0, 0)$ such that

$$x^{\#}(x) + \alpha r \geq x^{\#}(x_0) + \alpha r_0, \forall (x, r) \in \mathrm{epi}(f) \tag{3}$$

and

$$x^{\#}(\overline{x}) + \alpha \overline{r} > x^{\#}(x_0) + \alpha r_0, \text{ for at least one } (\overline{x}, \overline{r}) \in \mathrm{epi}(f). \tag{4}$$

We claim that $\alpha \neq 0$. Indeed, if $\alpha = 0$ then $x^{\#} \neq 0$, $x^{\#}(x) \geq x^{\#}(x_0)\ \forall x \in \mathrm{dom}(f)$ and $x^{\#}(\overline{x}) > x^{\#}(x_0)$. As the sets $\{x_0\}$ and $\mathrm{dom}(f)$ can be separated by a hyperplane which is not containing their union, by Theorem 1 we have that

$\{x_0\} \cap \mathrm{icr}(\mathrm{dom}(f)) = \mathrm{icr}\big(\{x_0\}\big) \cap \mathrm{icr}(\mathrm{dom}(f)) = \emptyset$, which is a contradiction. Hence $\alpha \neq 0$. Moreover, α is a non-negative number (if $\alpha < 0$ then for $(x, r) := (x_0, f(x_0) + \varepsilon)$ in (3) we get $x^{\#}(x) + \alpha(f(x_0) + \varepsilon) \geq x^{\#}(x_0) + \alpha r_0 \ \forall \varepsilon > 0$, and taking the limit when $\varepsilon \to +\infty$ we obtain a contradiction). Dividing by $\alpha > 0$ in (3) we get $r \geq r_0 + (1/\alpha)x^{\#}(x_0) - (1/\alpha)x^{\#}(x) \ \forall (x, r) \in \mathrm{epi}(f)$, implying that

$$f(x) \geq r_0 + (1/\alpha)x^{\#}(x_0) - (1/\alpha)x^{\#}(x) \ \forall x \in X.$$

We define $g : X \to \mathbb{R}, g(x) = -(1/\alpha)x^{\#}(x) + r_0 + (1/\alpha)x^{\#}(x_0)$. Then g is an affine minorant of f, so by (2), $r_0 > g(x_0) = r_0$ and this is a contradiction. Hence $\mathrm{cc}(f)(x_0) = f(x_0)$. □

Remark 1. As an easy consequence of the above theorem we have

$$f(x) = \mathrm{cc}(f)(x) \ \forall x \in \mathrm{core}(\mathrm{dom}(f)),$$

if $f : X \to \overline{\mathbb{R}}$ is a convex function.

3 C-convex sets

In this section we introduce an abstract notion in a real linear space in analogy to the closed convex hull of a set in a locally convex space. Then we investigate some properties of this notion.

Definition 4. For $M \subseteq X$ we define **the c-convex hull** of M by

$$\mathrm{cc}(M) = \bigcap_{(x^{\#}, \alpha) \in (X^{\#} \setminus \{0\}) \times \mathbb{R}} \big\{ H^{\leq}(x^{\#}, \alpha) : M \subseteq H^{\leq}(x^{\#}, \alpha) \big\}.$$

We say that M is **c-convex** if and only if $M = \mathrm{cc}(M)$. As the proof of the following properties is trivial, we omit it.

(a) $\emptyset$ and X are c-convex;

(b) for every $M \subseteq X$, $M \subseteq \mathrm{co}(M) \subseteq \mathrm{cc}(M)$, where $\mathrm{co}(M)$ is the convex hull of M, that is the smallest convex set which contains M;

(c) $A \subseteq B \Rightarrow \mathrm{cc}(A) \subseteq \mathrm{cc}(B)$;

(d) For every $M \subseteq X$, $\mathrm{cc}\big(\mathrm{cc}(M)\big) = \mathrm{cc}(M)$;

(e) If $(M)_i, i \in I$, is a family of c-convex sets in X, then $\bigcap_{i \in I} M_i$ is also c-convex.

The authors of [4] use for this set introduced in Definition 4 the notion of regular hull of a set.

Lemma 3. *$H = H^{\leq}(x^{\#}, \alpha)$ is c-convex, for every $x^{\#} \in X^{\#} \setminus \{0\}$ and $\alpha \in \mathbb{R}$.*

Proof. We have the following sequence of inclusions

$$H \subseteq \operatorname{cc}(H) = \bigcap_{(y^{\#},\beta) \in (X^{\#}\setminus\{0\})\times\mathbb{R}} \left\{ G^{\leq}(y^{\#}, \beta) : H \subseteq G^{\leq}(y^{\#}, \beta) \right\} \subseteq H,$$

and the result follows. $\square$

Remark 2. If X is a locally convex space, then $\operatorname{cl}\big(\operatorname{co}(M)\big)$ is the intersection of all closed half-spaces which contain M, where $\operatorname{cl}\big(\operatorname{co}(M)\big)$ is the topological closure of $\operatorname{co}(M)$ (see for example [2]). Here, a closed half-space is characterized by an element x^* from X^*, the topological dual space of X and because $X^* \subseteq X^{\#}$, we have in general $M \subseteq \operatorname{cc}(M) \subseteq \operatorname{cl}\big(\operatorname{co}(M)\big)$. If M is convex and closed, then from the above inclusion we have that M is c-convex. If X is of finite dimension, then $X^{\#} = X^*$, so in this case $\operatorname{cc}(M) = \operatorname{cl}\big(\operatorname{co}(M)\big)$. We show by an example that if X is of infinite dimension, then the above inclusion may be strict. Consider X an infinite dimensional normed space and let $\{e_i : i \in I\}$ be a vector basis of it. We may suppose that $\mathbb{N} \subseteq I$. Obviously, $\{\big(1/\|e_i\|\big)e_i : i \in I\}$ is again a vector basis, so without lose of generality we may suppose that $\|e_i\| = 1, \forall i \in I$. Define $f_0 : \{e_i : i \in I\} \to \mathbb{R}$,

$$f_0(e_i) = \begin{cases} i, & \text{if } i \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

It is well known from the linear algebra that f_0 can be extended uniquely to a linear function on X, say $x_0^{\#}$. We claim that $x_0^{\#} \in X^{\#} \setminus X^*$. Indeed, if we suppose that $x_0^{\#}$ is continuous, then $\exists L \geq 0$ such that $|x_0^{\#}(x)| \leq L\|x\|$ $\forall x \in X$ (see Proposition 2.1.2 in [8]). But this implies, for $x = e_i, i \in \mathbb{N}$, that $i \leq L$ $\forall i \in \mathbb{N}$, which is a contradiction. Now consider the following set

$$M := \ker(x_0^{\#}) = \{x \in X : x_0^{\#}(x) = 0\}.$$

M is a subspace of X, so is convex. We have

$$M = \{x \in X : x_0^{\#}(x) \leq 0\} \cap \{x \in X : -x_0^{\#}(x) \leq 0\}$$

thus, by Lemma 3 and assertion (e), M is c-convex. Let be $x_n = e_1 - (1/n)e_n$ $\forall n \in \mathbb{N}$. It is easy to see that $x_n \in M$ $\forall n \in \mathbb{N}$. Because of $\|x_n - e_1\| = 1/n$ $\forall n \in \mathbb{N}$, we get that the limit of the sequence $\{x_n\}$ is e_1, but this element does not belong to M, so M is a c-convex set which is not topologically closed. Hence $M = \operatorname{cc}(M) \subsetneqq \operatorname{cl}(M) = \operatorname{cl}\big(\operatorname{co}(M)\big)$.

Proposition 2. *For every subsets E, F of X we have*

$$\mathrm{cc}\left(E+\mathrm{cc}(F)\right)=\mathrm{cc}(E+F),$$

where $E+F$ is the Minkowski sum of the sets E and F.

Proof. We only have to prove the inclusion

$$\mathrm{cc}\left(E+\mathrm{cc}(F)\right)\subseteq\mathrm{cc}(E+F),$$

because the reverse one is trivial. By definition,

$$\mathrm{cc}\left(E+\mathrm{cc}(F)\right)=\bigcap_{(x^{\#},\alpha)\in(X^{\#}\setminus\{0\})\times\mathbb{R}}\left\{H^{\leq}(x^{\#},\alpha):E+\mathrm{cc}(F)\subseteq H^{\leq}(x^{\#},\alpha)\right\}$$

and

$$\mathrm{cc}(E+F)=\bigcap_{(x^{\#},\alpha)\in(X^{\#}\setminus\{0\})\times\mathbb{R}}\left\{H^{\leq}(x^{\#},\alpha):E+F\subseteq H^{\leq}(x^{\#},\alpha)\right\}.$$

Let $H^{\leq}(x^{\#},\alpha)=\{x\in X:x^{\#}(x)\leq\alpha\}$ be a c-half-space with $(x^{\#},\alpha)\in(X^{\#}\setminus\{0\})\times\mathbb{R}$ such that

$$E+F\subseteq H^{\leq}(x^{\#},\alpha). \tag{5}$$

We show that

$$E+\mathrm{cc}(F)\subseteq H^{\leq}(x^{\#},\alpha). \tag{6}$$

For this, let $e\in E$ and $g\in\mathrm{cc}(F)$ be fixed. Using (5) we obtain: $e+f\in H^{\leq}(x^{\#},\alpha)$ $\forall f\in F$, so $x^{\#}(e+f)\leq\alpha$ $\forall f\in F$ or, equivalently, $x^{\#}(f)\leq\alpha-x^{\#}(e)$ $\forall f\in F$, which implies

$$F\subseteq\left\{x\in X:x^{\#}(x)\leq\alpha-x^{\#}(e)\right\}.$$

Thus F is a subset of a c-half-space, and because $g\in\mathrm{cc}(F)$, we get

$$g\in\left\{x:x^{\#}(x)\leq\alpha-x^{\#}(e)\right\}\Leftrightarrow x^{\#}(g)\leq\alpha-x^{\#}(e)$$

$$\Leftrightarrow x^{\#}(e+g)\leq\alpha\Leftrightarrow e+g\in H^{\leq}(x^{\#},\alpha).$$

Hence, the inclusion in (6) is true and this means, taking into consideration that $(x^{\#},\alpha)\in(X^{\#}\setminus\{0\})\times\mathbb{R}$ was arbitrary chosen, that $\mathrm{cc}\left(E+\mathrm{cc}(F)\right)\subseteq\mathrm{cc}(E+F)$.□

We close this section giving a result concerning the c-convexity of the cartesian product of two sets.

Proposition 3. Let X and Y be real linear spaces, $A\subseteq X$ and $B\subseteq Y$. Then

$$\mathrm{cc}(A\times B)=\mathrm{cc}(A)\times\mathrm{cc}(B).$$

Proof. A c-half-space in $X \times Y$ has the following form

$$H^{\leq}(x^{\#}, y^{\#}, \gamma) = \big\{(x,y) \in X \times Y : \langle (x^{\#}, y^{\#}), (x,y) \rangle \leq \gamma \big\}$$
$$= \big\{(x,y) \in X \times Y : x^{\#}(x) + y^{\#}(y) \leq \gamma \big\},$$

where $x^{\#} \in X^{\#}, y^{\#} \in Y^{\#}, (x^{\#}, y^{\#}) \neq (0,0)$ and $\gamma \in \mathbb{R}$.

Let $(a,b) \in \operatorname{cc}(A \times B) = \bigcap \big\{H : A \times B \subseteq H, H \text{ a c-half-space}\big\}$. Consider $H^{\leq}(x^{\#}, \alpha)$ an arbitrary c-half-space such that $A \subseteq H^{\leq}(x^{\#}, \alpha)$, with $x^{\#} \neq 0$ and $\alpha \in \mathbb{R}$. Then $A \times B \subseteq \big\{(x,y) \in X \times Y : x^{\#}(x) \leq \alpha\big\} = H^{\leq}(x^{\#}, 0, \alpha)$ and because (a,b) is in the c-convex hull of $A \times B$, we get $(a,b) \in H^{\leq}(x^{\#}, 0, \alpha)$, hence $x^{\#}(a) \leq \alpha$, which is nothing else than $a \in H^{\leq}(x^{\#}, \alpha)$. Because $H^{\leq}(x^{\#}, \alpha)$ was arbitrary chosen we obtain $a \in \operatorname{cc}(A)$. Similarly we get $b \in \operatorname{cc}(B)$, so the inclusion

$$\operatorname{cc}(A \times B) \subseteq \operatorname{cc}(A) \times \operatorname{cc}(B)$$

is true.

For the opposite inclusion, take $(a,b) \in \operatorname{cc}(A) \times \operatorname{cc}(B)$. Consider $H = H^{\leq}(x^{\#}, y^{\#}, \gamma)$ an arbitrary c-half-space in $X \times Y$ such that $A \times B \subseteq H$. If we succeed to show that $(a,b) \in H$, which is nothing else than

$$x^{\#}(a) + y^{\#}(b) \leq \gamma \tag{7}$$

then we are done. As $(x^{\#}, y^{\#}) \neq (0,0)$, we can suppose without lose of generality that $x^{\#} \neq 0$. Let $b_0 \in B$ be arbitrary. For all $a_0 \in A$ we have $(a_0, b_0) \in A \times B \subseteq H^{\leq}(x^{\#}, y^{\#}, \gamma)$, so $x^{\#}(a_0) + y^{\#}(b_0) \leq \gamma$, hence $A \subseteq \big\{x \in X : x^{\#}(x) \leq \gamma - y^{\#}(b_0)\big\}$. Since $a \in \operatorname{cc}(A)$, a must belong to the set $\big\{x \in X : x^{\#}(x) \leq \gamma - y^{\#}(b_0)\big\}$, that is $x^{\#}(a) \leq \gamma - y^{\#}(b_0)$. We treat two cases.

(1) $y^{\#} = 0$. Then $x^{\#}(a) \leq \gamma$ and (7) is fulfilled.

(2) $y^{\#} \neq 0$. Then $x^{\#}(a) + y^{\#}(b_0) \leq \gamma$. The element b_0 being arbitrary in B, we have $x^{\#}(a) + y^{\#}(b_0) \leq \gamma \ \forall b_0 \in B$, so $B \subseteq \big\{y \in Y : y^{\#}(y) \leq \gamma - x^{\#}(a)\big\}$. Using the fact that $b \in \operatorname{cc}(B)$, relation (7) follows. □

4 The connection between c-convex functions and c-convex sets

The aim of this section is to study the relations between the notions introduced in the previous sections. We start by characterizing the c-half-spaces in $X \times \mathbb{R}$.

Lemma 4. *There are three types of c-half-spaces in $X \times \mathbb{R}$, namely*

1. $\{(x,r) \in X \times \mathbb{R} : x^{\#}(x) \leq \alpha\}, x^{\#} \in X^{\#}, x^{\#} \neq 0, \alpha \in \mathbb{R}$, *called* ***vertical half-space****,*

2. $\{(x,r) \in X \times \mathbb{R} : x^{\#}(x) - r \leq \alpha\}, x^{\#} \in X^{\#}, \alpha \in \mathbb{R}$, *called* ***upper half-space***,

3. $\{(x,r) \in X \times \mathbb{R} : x^{\#}(x) - r \geq \alpha\}, x^{\#} \in X^{\#}, \alpha \in \mathbb{R}$, *called* ***lower half-space***.

Proof. The hyperplanes in $X \times \mathbb{R}$ are of the form

$$\{(x,r) \in X \times \mathbb{R} : \langle (x^{\#}, b), (x,r) \rangle = \alpha\} = \{(x,r) \in X \times \mathbb{R} : x^{\#}(x) + br = \alpha\},$$

with $x^{\#} \in X, b \in \mathbb{R}, (x^{\#}, b) \neq (0,0)$, so a c-half-space has the following form

$$H = \{(x,r) \in X \times \mathbb{R} : x^{\#}(x) + br \leq \alpha\}.$$

There are three possible cases, as follows.

(a) $b = 0$. In this case, $H = \{(x,r) \in X \times \mathbb{R} : x^{\#}(x) \leq \alpha\}, x^{\#} \neq 0$, which is a vertical half-space.

(b) $b < 0$. Dividing by $-b$ we get $H = \{(x,r) \in X \times \mathbb{R} : (-1/b)x^{\#}(x) - r \leq (-\alpha/b)\}$, which is an upper half-space.

(c) $b > 0$. Then $H = \{(x,r) \in X \times \mathbb{R} : (-1/b)x^{\#}(x) - r \geq (-\alpha/b)\}$, which is a lower half-space. □

Remark 3. Let us note that considering an arbitrary affine function $h : X \to \mathbb{R}, h(x) = x^{\#}(x) - \alpha$, for $x^{\#} \in X^{\#}$ and $\alpha \in \mathbb{R}$, the vertical half-spaces can be written as

$$\{(x,r) \in X \times \mathbb{R} : x^{\#}(x) \leq \alpha\} = \{(x,r) : h(x) \leq 0\}$$

and the upper half-spaces as

$$\{(x,r) \in X \times \mathbb{R} : x^{\#}(x) - \alpha \leq r\} = \operatorname{epi}(h),$$

respectively.

The following two results are quite natural if we take into consideration a geometric argument.

Lemma 5. *Let H be a vertical or an upper half-space in $X \times \mathbb{R}$. If for some $x \in X$ and $r \in \mathbb{R}$ we have $(x, r+\varepsilon) \in H$ $\forall \varepsilon > 0$, then $(x,r) \in H$.*

Proof. If H is a vertical half-space, the result is trivial. Now let $H = \{(x,r) \in X \times \mathbb{R} : x^{\#}(x) - \alpha \leq r\}$, with $x^{\#} \in X^{\#}$ and $\alpha \in \mathbb{R}$, be an upper half-space. By the hypothesis,

$$x^{\#}(x) - \alpha \leq r + \varepsilon \ \forall \varepsilon > 0.$$

Taking the limit when $\varepsilon \searrow 0$, we obtain $x^{\#}(x) - \alpha \leq r$, that is $(x, r) \in H$. $\square$

Lemma 6. *Let* $f : X \to \overline{\mathbb{R}}$ *be such that* $\operatorname{dom}(f) \neq \emptyset$. *Then there exists no lower half-space* H *such that* $\operatorname{epi}(f) \subseteq H$.

Proof. Assume that there exists a lower half-space $H = \{(x, r) \in X \times \mathbb{R} : x^{\#}(x) - \alpha \geq r\}$ with $x^{\#} \in X^{\#}$ and $\alpha \in \mathbb{R}$, such that $\operatorname{epi}(f) \subseteq H$. Take $y_0 \in \operatorname{dom}(f)$. Then one can find an $r_0 \in \mathbb{R}$ such that

$$r_0 > \max\{f(y_0), x^{\#}(y_0) - \alpha\} \Leftrightarrow f(y_0) < r_0 \text{ and } x^{\#}(y_0) - \alpha < r_0$$

$$\Leftrightarrow (y_0, r_0) \in \operatorname{epi}(f) \setminus H,$$

which is a contradiction. $\square$

The next proposition says that in order to obtain the c-convex hull of the epigraph of a given function having at least one affine minorant and nonempty domain, it is enough to take the intersection of the family of upper half-spaces which contain $\operatorname{epi}(f)$.

Proposition 4. *Let* $f : X \to \overline{\mathbb{R}}$ *be such that* $\{g : g \text{ affine}, g \leq f\} \neq \emptyset$ *and* $\operatorname{dom}(f) \neq \emptyset$. *Then*

$$\operatorname{cc}\big(\operatorname{epi}(f)\big) = \bigcap \big\{H : H \textit{ is an upper half-space}, \operatorname{epi}(f) \subseteq H\big\}.$$

Proof. By Lemma 6, there exist no lower half-space H such that $\operatorname{epi}(f) \subseteq H$. So

$$\operatorname{cc}\big(\operatorname{epi}(f)\big) = \bigcap \big\{H : H \text{ is a c-half-space}, \operatorname{epi}(f) \subseteq H\big\} =$$

$$\bigcap \big\{H : H \text{ is an upper half-space}, \operatorname{epi}(f) \subseteq H\big\} \bigcap$$

$$\bigcap \big\{H : H \text{ is a vertical half-space}, \operatorname{epi}(f) \subseteq H\big\}. \tag{8}$$

Let $V = \{(x, r) : h_1(x) \leq 0\}$ be a vertical half-space such that $\operatorname{epi}(f) \subseteq V$, where $h_1 : X \to \mathbb{R}$ is an affine function. We show that

$$(X \times \mathbb{R}) \setminus V \subseteq (X \times \mathbb{R}) \setminus \Bigg(\bigcap_{h \in \mathcal{A}(X,f)} \operatorname{epi}(h) \Bigg). \tag{9}$$

Let $(x_0, r_0) \notin V$, so $h_1(x_0) > 0$. By the assumptions, there exists an affine minorant $h_2 : X \to \mathbb{R}$ of f. For all $\lambda \geq 0$ and $x \in X$ we have

$$\lambda h_1(x) + h_2(x) \leq f(x). \tag{10}$$

Indeed, if $x \notin \mathrm{dom}(f)$, (10) is trivial. For $x \in \mathrm{dom}(f)$, one must have $f(x) \in \mathbb{R}$. Otherwise, if $f(x) = -\infty$, by Proposition 1(a), we have that $\mathrm{cc}(f)(x) = -\infty$ and thus, by Lemma 2, there exists no affine minorant of f. So $(x, f(x)) \in \mathrm{epi}(f) \subseteq V$, hence $h_1(x) \leq 0$ and so the inequality (10) is true. Because of $h_1(x_0) > 0$, there exists a sufficiently large λ_0 such that

$$\lambda_0 h_1(x_0) + h_2(x_0) > r_0.$$

Defining $h : X \to \mathbb{R}$ by $h(x) = \lambda_0 h_1(x) + h_2(x), \forall x \in X$, we have that h is an affine minorant of f and $(x_0, r_0) \notin \mathrm{epi}(h)$, showing that (9) is true. This implies that $\bigcap_{h \in \mathcal{A}(X,f)} \mathrm{epi}(h) \subseteq V$. V being arbitrary, we get

$$\bigcap \big\{ H : H \text{ is an upper half-space}, \mathrm{epi}(f) \subseteq H \big\} \subseteq$$

$$\bigcap \big\{ H : H \text{ is a vertical half-space}, \mathrm{epi}(f) \subseteq H \big\}$$

and by (8) the result follows. □

As we have seen in Remark 3, in the hypotheses of Proposition 4 the c-convex hull of the epigraph of f can be further written as

$$\mathrm{cc}\,\big(\mathrm{epi}(f)\big) = \bigcap_{h \in \mathcal{A}(X,f)} \big\{ \mathrm{epi}(h) : \mathrm{epi}(f) \subseteq \mathrm{epi}(h) \big\}.$$

Theorem 3. *Let $f : X \to \overline{\mathbb{R}}$ be such that $\{g : g$ affine, $g \leq f\} \neq \emptyset$. Then*

(a) $\mathrm{epi}(\mathrm{cc}(f)) = \mathrm{cc}\,\big(\mathrm{epi}(f)\big)$,

(b) $f \in \Gamma(X) \Leftrightarrow \mathrm{epi}(f) \subseteq X \times \mathbb{R}$ *is c-convex.*

Proof. (a) By Proposition 1(a) we have $\mathrm{cc}(f) \leq f$ and so

$$\mathrm{epi}(f) \subseteq \mathrm{epi}(\mathrm{cc}(f)). \tag{11}$$

We consider the following two cases.

(1) $\mathrm{dom}(f) = \emptyset$. Then $f \equiv +\infty, \mathrm{epi}(f) = \emptyset$ and thus $\mathrm{cc}\,\big(\mathrm{epi}(f)\big) = \emptyset$. Then, by Lemma 2, $\mathrm{cc}(f) = \sup\{g : g \text{ affine}, g \leq f\} = \sup\{g : g \text{ affine}\} = +\infty$, and as $\mathrm{epi}(\mathrm{cc}(f)) = \emptyset$, the equality holds.

(2) $\mathrm{dom}(f) \neq \emptyset$. By (11), we have $\mathrm{cc}\,\big(\mathrm{epi}(f)\big) \subseteq \mathrm{cc}\,\big(\mathrm{epi}(\mathrm{cc}(f))\big)$. We show that $\mathrm{epi}(\mathrm{cc}(f))$ is c-convex. If we suppose that there exists $(x_0, r_0) \in \mathrm{cc}\,\big(\mathrm{epi}(\mathrm{cc}(f))\big) \setminus \mathrm{epi}(\mathrm{cc}(f))$, then $\mathrm{cc}(f)(x_0) > r_0$, which implies by Lemma 2 that there exists an affine minorant g_0 of f such that $g_0(x_0) > r_0$. Also by Lemma 2 we have $g_0 \leq \mathrm{cc}(f)$, so $\mathrm{epi}(\mathrm{cc}(f)) \subseteq \mathrm{epi}(g_0)$. But $\mathrm{epi}(g_0)$ defines an upper half-space which contains $\mathrm{epi}(\mathrm{cc}(f))$, thus $(x_0, r_0) \in \mathrm{epi}(g_0)$, but this is a contradiction. Hence $\mathrm{epi}(\mathrm{cc}(f))$ is c-convex, so $\mathrm{cc}\,\big(\mathrm{epi}(f)\big) \subseteq \mathrm{epi}(\mathrm{cc}(f))$.

It remains to prove the reverse inclusion, namely $\operatorname{epi}(\operatorname{cc}(f)) \subseteq \operatorname{cc}\big(\operatorname{epi}(f)\big)$. Take an arbitrary $(x_1, r_1) \in \operatorname{epi}(\operatorname{cc}(f))$. Then $\operatorname{cc}(f)(x_1) \leq r_1 \Leftrightarrow h(x_1) \leq r_1$, for every affine minorant h of f, so $(x_1, r_1) \in \bigcap_{h \in \mathcal{A}(X,f)} \operatorname{epi}(h) = \operatorname{cc}\big(\operatorname{epi}(f)\big)$, where the last equality follows by Proposition 4.

(b) Using (a) and Proposition 1(b) we obtain

$$\begin{aligned} & f \in \Gamma(X) \Leftrightarrow f = \operatorname{cc}(f) \Leftrightarrow \operatorname{epi}(f) = \operatorname{epi}(\operatorname{cc}(f)) \\ & \Leftrightarrow \operatorname{epi}(f) = \operatorname{cc}\big(\operatorname{epi}(f)\big) \Leftrightarrow \operatorname{epi}(f) \text{ is c-convex.} \end{aligned}$$

$\square$

Remark 4. The direct implication in (b) is true even if $\{g : g \text{ affine}, g \leq f\} = \emptyset$. In this case, by Definition 1, $f \equiv -\infty$, $\operatorname{epi}(f) = X \times \mathbb{R}$ and thus $\operatorname{epi}(f) = \operatorname{cc}\big(\operatorname{epi}(f)\big) = X \times \mathbb{R}$.

The reverse implication does not hold in general if the function f has no affine minorants. For $f : \mathbb{R} \to \overline{\mathbb{R}}$,

$$f(x) = \begin{cases} -\infty, & \text{if } x \in (-\infty, 0], \\ +\infty, & \text{otherwise,} \end{cases}$$

we have $\operatorname{epi}(f) = (-\infty, 0] \times \mathbb{R}$ and this is a c-convex set. It is easy to see that f is not c-convex. Moreover, f is an example of a function which is lower semicontinuous and convex, but not c-convex.

In a locally convex space X, if $f : X \to \overline{\mathbb{R}}$ is convex, lower semicontinuous and $f(x) > -\infty\ \forall x \in X$, then f is c-convex. Indeed, the properties of the function f guarantee the existence of at least one affine minorant of f and $\operatorname{epi}(f)$ is a convex and closed set. This shows (see Remark 2) that $\operatorname{epi}(f)$ is a c-convex set, implying by Theorem 3(b) that f is c-convex.

Next we give another characterization of the c-convex hull of a function which has at least one affine minorant.

Corollary 1. *Let $f : X \to \overline{\mathbb{R}}$ be such that $\{g : g \text{ affine}, g \leq f\} \neq \emptyset$. Then*

$$\operatorname{cc}(f) = \inf\big\{t : (x,t) \in \operatorname{cc}\big(\operatorname{epi}(f)\big)\big\}.$$

Proof. This is an easy consequence of the above theorem, since for every function $f : X \to \overline{\mathbb{R}}$ one has $f(x) = \inf\big\{t : (x,t) \in \operatorname{epi}(f)\big\}$. $\square$

Lemma 7. If $f : X \to \overline{\mathbb{R}}$ is c-convex, then the level set

$$\{x \in X : f(x) \leq a\}$$

is c-convex $\forall a \in \mathbb{R}$.

Proof. Since f is c-convex, we have $f(x) = \sup\{g(x) : g \text{ affine}, g \leq f\}$. Let $a \in \mathbb{R}$ be arbitrary. Then $\{x \in X : f(x) \leq a\} = \bigcap_{g \in \mathcal{A}(X,f)} \{x \in X : g(x) \leq a\}$. By Lemma 3, $\{x \in X : g(x) \leq a\}$ is c-convex, for every affine function g, so the level set $\{x \in X : f(x) \leq a\}$ will be also c-convex, being the intersection of an arbitrary family of c-convex sets. □

Theorem 4. *Let A be a subset of X. Then*

$$\delta_A \in \Gamma(X), \text{ i.e. } \delta_A \text{ is c-convex, if and only if } A \text{ is c-convex.}$$

Proof. We have $\operatorname{epi}(\delta_A) = A \times [0, +\infty)$. By Proposition 3

$$\operatorname{cc}(A \times [0, +\infty)) = \operatorname{cc}(A) \times [0, +\infty).$$

Obviously, $h \equiv 0$ is an affine minorant of δ_A, hence by Theorem 3(b) we obtain

$$\delta_A \in \Gamma(X) \Leftrightarrow \operatorname{epi}(\delta_A) \text{ is c-convex} \Leftrightarrow A \times [0, +\infty) \text{ is c-convex}$$

$$\Leftrightarrow A \times [0, +\infty) = \operatorname{cc}\left(A \times [0, +\infty)\right) \Leftrightarrow A \times [0, +\infty) = \operatorname{cc}(A) \times [0, +\infty)$$

$$\Leftrightarrow A = \operatorname{cc}(A) \Leftrightarrow A \text{ is c-convex.}$$

□

Remark 5. Working in a locally convex space X, $A \subseteq X$ is closed and convex if and only if the indicator function δ_A is lower semicontinuous and convex. Using Theorem 4, we can construct a convex function defined on X which is c-convex but not lower semicontinuous. Let M be the set considered in Remark 2

$$M := \ker(x_0^{\#}) = \{x \in X : x_0^{\#}(x) = 0\}.$$

Because M is c-convex and not topologically closed, we get that δ_M is a c-convex function which is not lower semicontinuous.

Remark 6. The approach which we describe below gives a connection between the theory established in real linear spaces and the one existing in separated locally convex spaces. If we consider P the set of all seminorms defined on the real linear space X, then (X, P) becomes a separated locally convex space. The topological notions referred below are with respect to this topology, known in the literature as the "core topology". The topological dual of X is $X^{\#}$. Further, a c-convex function is closed and convex, which means, a function which is identical $-\infty$ or identical $+\infty$ or a proper lower semicontinuous convex function. Moreover, the c-convex hull of a subset M of X is nothing else than the closed convex hull of M.

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