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Enlargements of positive sets

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ABSTRACT

In this paper we introduce the notion of enlargement of a positive set in SSD spaces. To a maximally positive set A we associate a family of enlargements $\mathbb{E}(A)$ and characterize the smallest and biggest element in this family with respect to the inclusion relation. We also emphasize the existence of a bijection between the subfamily of closed enlargements of $\mathbb{E}(A)$ and the family of so-called representative functions of A. We show that the extremal elements of the latter family are two functions recently introduced and studied by Stephen Simons. In this way we extend to SSD spaces some former results given for monotone and maximally monotone sets in Banach spaces.

1. Introduction

The notion of positive set with respect to a quadratic form defined on a so-called *symmetrically self-dual Banach space* (*Banach SSD space*) has been introduced by Stephen Simons in [21] (see also [22]) as an extension of a monotone set in a Banach space. A number of known results coming from the theory of monotone operators have been successfully generalized to this framework. In his investigations Simons has mainly used some techniques based on the extension of the notion of Fitzpatrick function from the theory of monotone sets to a similar concept for positive sets. These investigations have been continued by the same author in [23], where notions and results recently introduced in the theory of monotone operators in general Banach spaces have known an appropriate generalization to positive sets in Banach SSD spaces.

In analogy to the enlargement of a monotone operator we introduce and study in this article the notion of enlargement of a positive set in SSD spaces. In this way we extend to SSD spaces some results given in the literature for monotone and maximally monotone sets.

In Section 2 we recall the definition of an SSD space B along with some examples given in [23] and give a calculus rule for the quadratic form $q : B \to \mathbb{R}$ considered on it.

In Section 3 we introduce the notion of enlargement of a q-positive set (positive with respect to q) $A \subseteq B$ as being the multifunction $E : \mathbb{R}_+ \rightrightarrows B$ fulfilling $A \subseteq E(\varepsilon)$ for all $\varepsilon \geqslant 0$. In connection with this notion we introduce the so-called *transportation formula* and provide a characterization of enlargements which fulfill it. We also associate to A a family of enlargements $\mathbb{E}(A)$ for which we provide, in case A is q-maximally positive, the smallest and the biggest element with respect to the partial ordering inclusion relation of the graphs. In this way we extend to SSD spaces some former results given in [11,12,25].

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In Section 4 we assume that B is a Banach SSD space and deal with $\mathbb{E}_c(A)$, a subfamily of $\mathbb{E}(A)$, containing those enlargements of $\mathbb{E}(A)$ having a closed graph. For $\mathbb{E}_c(A)$ we point out, in case A is maximally q-positive, the smallest and the biggest element with respect to the partial ordering inclusion relation of the graphs, too, as well as a bijection between this subfamily and the set of so-called *representative functions of A*

$$\mathcal{H}(A) = \{h : B \to \overline{\mathbb{R}}:\ h \text{ convex, lower semicontinuous},\ h \geqslant q \text{ on } B,\ h = q \text{ on } A\}.$$

These results generalize to Banach SSD spaces the ones given in [12,25] for enlargements of maximally monotone operators. We also show that the smallest and the biggest element of $\mathcal{H}(A)$ are nothing else than the functions Φ_A and ${}^*\Theta_A$ considered in [23] and provide some characterizations of these functions beyond the ones given in the mentioned paper. We close the paper by giving a characterization of the additive enlargements in $\mathbb{E}_c(A)$, in case A is a maximally q-positive set, which turns out to be helpful when showing the existence of enlargements having this property.

2. Preliminary notions and results

Consider X a real separated locally convex space and X^* its topological dual space. The notation $\omega(X^*, X)$ stands for the weak* topology induced by X on X^*, while by $\langle x, x^*\rangle$ we denote the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. For a subset C of X we denote by $\overline{C}$ and $\operatorname{co}(C)$ its *closure* and *convex hull*, respectively. We also consider the *indicator function* of the set C, denoted by δ_C, which is zero for $x \in C$ and $+\infty$ otherwise.

For a function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we denote by $\operatorname{dom}(f) = \{x \in X:\ f(x) < +\infty\}$ its *domain* and by $\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R}:\ f(x) \leqslant r\}$ its *epigraph*. We call f *proper* if $\operatorname{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. By $\operatorname{cl}(f)$ we denote the *lower semicontinuous hull* of f, namely the function of which epigraph is the closure of $\operatorname{epi}(f)$ in $X \times \mathbb{R}$, that is $\operatorname{epi}(\operatorname{cl}(f)) = \operatorname{cl}(\operatorname{epi}(f))$. The function $\operatorname{co}(f)$ is the largest convex function majorized by f.

Having $f : X \to \overline{\mathbb{R}}$ a proper function, for $x \in \operatorname{dom}(f)$ we define the *ε-subdifferential* of f at x, where $\varepsilon \geqslant 0$, by

$$\partial_\varepsilon f(x) = \big\{x^* \in X^*:\ f(y) - f(x) \geqslant \langle y - x, x^*\rangle - \varepsilon \text{ for all } y \in X\big\}.$$

For $x \notin \operatorname{dom}(f)$ we take $\partial_\varepsilon f(x) := \emptyset$. The set $\partial f(x) := \partial_0 f(x)$ is then the classical *subdifferential* of f at x.

The *Fenchel–Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) = \sup_{x\in X}\{\langle x, x^*\rangle - f(x)\}$ for all $x^* \in X^*$. Next we mention some important properties of conjugate functions. We have the so-called *Young–Fenchel inequality* $f^*(x^*) + f(x) \geqslant \langle x, x^*\rangle$ for all $(x, x^*) \in X \times X^*$. If f is proper, then f is convex and lower semicontinuous if and only if $f^{**} = f$ (see [13,28]). As a consequence we have that in case f is convex and $\operatorname{cl}(f)$ is proper, then $f^{**} = \operatorname{cl}(f)$ (cf. [28, Theorem 2.3.4]).

One can give the following characterizations for the subdifferential and ε-subdifferential of a proper function f by means of conjugate functions (see [13,28]):

$$x^* \in \partial f(x) \quad \Leftrightarrow \quad f(x) + f^*(x^*) = \langle x, x^*\rangle$$

and, respectively,

$$x^* \in \partial_\varepsilon f(x) \quad \Leftrightarrow \quad f(x) + f^*(x^*) \leqslant \langle x, x^*\rangle + \varepsilon.$$

Definition 1. (Cf. [23, Definition 1.2].) (i) We say that $(B, \lfloor\cdot,\cdot\rfloor)$ is a **symmetrically self-dual space** (**SSD space**) if B is a nonzero real vector space and $\lfloor\cdot,\cdot\rfloor : B \times B \to \mathbb{R}$ is a symmetric bilinear form. We consider the quadratic form $q : B \to \mathbb{R}$, $q(b) = \frac{1}{2}\lfloor b, b\rfloor$ for all $b \in B$.

(ii) A subset A of B is said to be **q-positive** if $A \neq \emptyset$ and $q(b - c) \geqslant 0$ for all $b, c \in A$. We say that A is **maximally q-positive** if A is q-positive and maximal (with respect to the inclusion) in the family of q-positive subsets of B.

Remark 1. (i) In every SSD space B the following calculus rule is fulfilled:

$$q(\alpha b + \gamma c) = \alpha^2 q(b) + \gamma^2 q(c) + \alpha\gamma\lfloor b, c\rfloor \quad \text{for all } \alpha, \gamma \in \mathbb{R} \text{ and } b, c \in B. \tag{1}$$

(ii) Let B be an SSD space and $A \subseteq B$ be a q-positive set. Then A is maximally q-positive if and only if for all $b \in B$ the implication below holds

$$q(b - c) \geqslant 0 \quad \text{for all } c \in A \ \Rightarrow\ b \in A.$$

Example 1. (Cf. [23].) (a) If B is a Hilbert space with inner product $(b, c) \mapsto \langle b, c\rangle$ then B is an SSD space with $\lfloor b, c\rfloor = \langle b, c\rangle$ and $q(b) = \frac{1}{2}\|b\|^2$ and every subset of B is q-positive.

(b) If B is a Hilbert space with inner product $(b, c) \mapsto \langle b, c\rangle$ then B is an SSD space with $\lfloor b, c\rfloor = -\langle b, c\rangle$ and $q(b) = -\frac{1}{2}\|b\|^2$ and the q-positive sets are the singletons.

(c) $\mathbb{R}^3$ is an SSD space with $\lfloor (b_1, b_2, b_3), (c_1, c_2, c_3)\rfloor = b_1c_2 + b_2c_1 + b_3c_3$ and $q(b_1, b_2, b_3) = b_1b_2 + \frac{1}{2}b_3^2$. See [23] for a discussion regarding the q-positive sets in this setting.

(d) Consider X a nonzero Banach space and $B := X \times X^*$. For all $b = (x, x^*)$ and $c = (y, y^*) \in B$ we set $\lfloor b, c\rfloor := \langle x, y^*\rangle + \langle y, x^*\rangle$. Then B is an SSD space, $q(b) = \langle x, x^*\rangle$ and $q(b - c) = \langle x - y, x^* - y^*\rangle$. Hence for $A \subseteq B$ we have that A is q-positive exactly when A is a nonempty monotone subset of $X \times X^*$ in the usual sense and A is maximally q-positive exactly when A is a maximally monotone subset of $X \times X^*$ in the usual sense. The classical example of a maximal monotone operator is the subdifferential of a proper, convex and lower semicontinuous function (see [19]). However, there exist maximal monotone operators which are not subdifferentials (see [22]).

Let us consider in the following an arbitrary SSD space B and a function $f : B \to \overline{\mathbb{R}}$. We write $f^{@}$ for the conjugate of f with respect to the pairing $\lfloor \cdot,\cdot \rfloor$, that is $f^{@}(c) = \sup_{b\in B}\{\lfloor b, c\rfloor - f(b)\}$. We write $\mathcal{P}(f) := \{b \in B\colon\ f(b) = q(b)\}$. If f is proper, convex, $f \geqslant q$ on B and $\mathcal{P}(f) \neq \emptyset$, then $\mathcal{P}(f)$ is a q-positive subset of B (see [23, Lemma 1.9]). Conditions under which $\mathcal{P}(f)$ is maximally q-positive are given in [23, Theorem 2.9].

3. Enlargements of positive sets in SSD spaces

In this section we introduce the concept of enlargement of a positive set and study some of its algebraic properties.

Definition 2. Let B be an SSD space. Given A a q-positive subset of B, we say that the multifunction $E : \mathbb{R}_+ \rightrightarrows B$ is an **enlargement** of A if

$$A \subseteq E(\varepsilon) \quad \text{for all } \varepsilon \geqslant 0.$$

Example 2. Let B be an SSD space and A a q-positive subset of B. The multifunction $E^A : \mathbb{R}_+ \rightrightarrows B$,

$$E^A(\varepsilon) := \bigl\{b \in B\colon\ q(b - c) \geqslant -\varepsilon \text{ for all } c \in A\bigr\}$$

is an enlargement of A. Let us notice that A is maximally q-positive if and only if $A = E^A(0)$. Moreover, in the framework of Example 1(d), for the graph of E^A we have $G(E^A) = \{(\varepsilon, x, x^*)\colon\ \langle x - y, x^* - y^*\rangle \geqslant -\varepsilon \text{ for all } (y, y^*) \in A\}$, hence in this case $G(E^A) = G(B^A)$, where $B^A : \mathbb{R}_+ \times X \rightrightarrows X^*$, $B^A(\varepsilon, x) = \{x^*\colon\ \langle x - y, x^* - y^*\rangle \geqslant -\varepsilon,\ \forall (y, y^*) \in A\}$, was defined in [9] and studied in [7–9,11,12,25].

The following definition extends to SSD spaces a notion given in [10,11] (see also [12, Definition 2.3]).

Definition 3. We say that the multifunction $E : \mathbb{R}_+ \rightrightarrows B$ satisfies the **transportation formula** if for every $\varepsilon_1, \varepsilon_2 \geqslant 0$, $b^1 \in E(\varepsilon_1)$, $b^2 \in E(\varepsilon_2)$ and every $\alpha_1, \alpha_2 \geqslant 0$, $\alpha_1 + \alpha_2 = 1$ we have $\varepsilon := \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_1\alpha_2 q(b^1 - b^2) \geqslant 0$ and $\alpha_1 b^1 + \alpha_2 b^2 \in E(\varepsilon)$.

Proposition 3. *Let B be an SSD space and $A \subseteq B$ be a maximally q-positive set. Then E^A satisfies the transportation formula.*

Proof. Take $\varepsilon_1, \varepsilon_2 \geqslant 0$, $b^1 \in E^A(\varepsilon_1)$, $b^2 \in E^A(\varepsilon_2)$ and $\alpha_1, \alpha_2 \geqslant 0$, $\alpha_1 + \alpha_2 = 1$. We have to show that

$$q\bigl(\alpha_1 b^1 + \alpha_2 b^2 - c\bigr) \geqslant -\alpha_1\varepsilon_1 - \alpha_2\varepsilon_2 - \alpha_1\alpha_2 q\bigl(b^1 - b^2\bigr) \quad \text{for all } c \in A, \tag{2}$$

and

$$\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_1\alpha_2 q\bigl(b^1 - b^2\bigr) \geqslant 0. \tag{3}$$

Let c be an arbitrary element in A. By using the inequalities $q(b^1 - c) \geqslant -\varepsilon_1$ and $q(b^2 - c) \geqslant -\varepsilon_2$ and the calculus rule (1) we obtain

$$\begin{aligned} q\bigl(\alpha_1 b^1 + \alpha_2 b^2 - c\bigr) &= q\bigl(\alpha_1\bigl(b^1 - c\bigr) + \alpha_2\bigl(b^2 - c\bigr)\bigr) = \alpha_1^2 q\bigl(b^1 - c\bigr) + \alpha_2^2 q\bigl(b^2 - c\bigr) + \alpha_1\alpha_2\bigl\lfloor b^1 - c, b^2 - c\bigr\rfloor \\ &= \alpha_1^2 q\bigl(b^1 - c\bigr) + \alpha_2^2 q\bigl(b^2 - c\bigr) + \alpha_1\alpha_2\bigl(q\bigl(b^1 - c\bigr) + q\bigl(b^2 - c\bigr) - q\bigl(b^1 - b^2\bigr)\bigr) \\ &= \alpha_1 q\bigl(b^1 - c\bigr) + \alpha_2 q\bigl(b^2 - c\bigr) - \alpha_1\alpha_2 q\bigl(b^1 - b^2\bigr) \geqslant -\alpha_1\varepsilon_1 - \alpha_2\varepsilon_2 - \alpha_1\alpha_2 q\bigl(b^1 - b^2\bigr). \end{aligned}$$

Let us suppose that $\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_1\alpha_2 q(b^1 - b^2) < 0$. From (2) we obtain

$$q\bigl(\alpha_1 b^1 + \alpha_2 b^2 - c\bigr) > 0 \quad \text{for all } c \in A. \tag{4}$$

Since A is maximally q-positive we get $\alpha_1 b^1 + \alpha_2 b^2 \in A$ (cf. Remark 1(ii)). By choosing $c := \alpha_1 b^1 + \alpha_2 b^2 \in A$ in (4) we get $0 > 0$, which is a contradiction. Hence (3) is also fulfilled and the proof is complete. □

The following result establishes a connection between the transportation formula and convexity (see also [25, Lemma 3.2]).

Proposition 4. *Let B be an SSD space, $E : \mathbb{R}_+ \rightrightarrows B$ a multifunction and define the function $\Psi : \mathbb{R} \times B \to \mathbb{R} \times B$, $\Psi(\epsilon, b) = (\epsilon + q(b), b)$ for all $(\epsilon, b) \in \mathbb{R} \times B$. The following statements are equivalent*:

(i) *E satisfies the transportation formula*;
(ii) *E satisfies the* ***generalized transportation formula*** *(or the* ***n-point transportation formula****), that is for all* $n \geqslant 1$, $\varepsilon_i \geqslant 0$, $b^i \in E(\varepsilon_i)$ *and* $\alpha_i \geqslant 0$, $i = 1, \dots, n$, *with* $\sum_{i=1}^n \alpha_i = 1$ *we have* $\varepsilon := \sum_{i=1}^n \alpha_i \varepsilon_i + \sum_{i=1}^n \alpha_i q(b^i - \sum_{j=1}^n \alpha_j b^j) \geqslant 0$ *and* $\sum_{i=1}^n \alpha_i b^i \in E(\varepsilon)$;
(iii) $\Psi(G(E))$ *is convex.*

Proof. We notice first that Ψ is a bijective function with inverse $\Psi^{-1} : \mathbb{R} \times B \to \mathbb{R} \times B$, $\Psi^{-1}(\epsilon, b) = (\epsilon - q(b), b)$.

(ii) ⇒ (i) Take $\varepsilon_1, \varepsilon_2 \geqslant 0$, $b^1 \in E(\varepsilon_1)$, $b^2 \in E(\varepsilon_2)$ and $\alpha_1, \alpha_2 \geqslant 0$, $\alpha_1 + \alpha_2 = 1$. Then $\varepsilon := \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_1 q(b^1 - (\alpha_1 b^1 + \alpha_2 b^2)) + \alpha_2 q(b^2 - (\alpha_1 b^1 + \alpha_2 b^2)) \geqslant 0$ and $\alpha_1 b^1 + \alpha_2 b^2 \in E(\varepsilon)$. Since

$$\varepsilon = \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_1\alpha_2^2 q\big(b^1 - b^2\big) + \alpha_2\alpha_1^2 q\big(b^1 - b^2\big) = \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_1\alpha_2 q\big(b^1 - b^2\big),$$

this implies that E satisfies the transportation formula.

(i) ⇒ (iii) Let be $(\mu_1, b^1), (\mu_2, b^2) \in \Psi(G(E))$ and $\alpha_1, \alpha_2 \geqslant 0$ with $\alpha_1 + \alpha_2 = 1$. Then there exist $\varepsilon_1, \varepsilon_2 \geqslant 0$ such that $\mu_1 = \varepsilon_1 + q(b^1)$, $b^1 \in E(\varepsilon_1)$ and $\mu_2 = \varepsilon_2 + q(b^2)$, $b^2 \in E(\varepsilon_2)$. By (i) we have that $\varepsilon := \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_1\alpha_2 q(b^1 - b^2) \geqslant 0$ and $\alpha_1 b^1 + \alpha_2 b^2 \in E(\varepsilon)$. Using (1) we further get that

$$\begin{aligned}\varepsilon + q\big(\alpha_1 b^1 + \alpha_2 b^2\big) &= \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_1\alpha_2\big(q\big(b^1\big) + q\big(b^2\big) - \lfloor b^1, b^2 \rfloor\big) + \alpha_1^2 q\big(b^1\big) + \alpha_2^2 q\big(b^2\big) + \alpha_1\alpha_2\lfloor b^1, b^2 \rfloor \\ &= \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_1 q\big(b^1\big) + \alpha_2 q\big(b^2\big) = \alpha_1\mu_1 + \alpha_2\mu_2.\end{aligned}$$

Thus

$$\alpha_1\big(\mu_1, b^1\big) + \alpha_2\big(\mu_2, b^2\big) = \big(\varepsilon + q\big(\alpha_1 b^1 + \alpha_2 b^2\big), \alpha_1 b^1 + \alpha_2 b^2\big) = \Psi\big(\varepsilon, \alpha_1 b^1 + \alpha_2 b^2\big) \in \Psi\big(G(E)\big)$$

and this provides the convexity of $\Psi(G(E))$.

(iii) ⇒ (ii) Let be $n \geqslant 1$, $\varepsilon_i \geqslant 0$, $b^i \in E(\varepsilon_i)$ and $\alpha_i \geqslant 0$, $i = 1, \dots, n$, with $\sum_{i=1}^n \alpha_i = 1$. This means that $(\varepsilon_i + q(b^i), b^i) \in \Psi(G(E))$ for $i = 1, \dots, n$. Denote $b := \sum_{i=1}^n \alpha_i b^i$ and $\varepsilon := \sum_{i=1}^n \alpha_i(\varepsilon_i + q(b^i)) - q(b)$. By using the convexity of $\Psi(G(E))$ one has $(\varepsilon + q(b), b) = \sum_{i=1}^n \alpha_i(\varepsilon_i + q(b^i), b^i) \in \Psi(G(E))$, which implies that $\Psi^{-1}(\varepsilon + q(b), b) = (\varepsilon, b) \in G(E)$. From here it follows that $\varepsilon = \sum_{i=1}^n \alpha_i\varepsilon_i + \sum_{i=1}^n \alpha_i q(b^i) - q(b) \geqslant 0$ and $b = \sum_{i=1}^n \alpha_i b^i \in E(\varepsilon)$. To conclude the proof we have only to show that $\sum_{i=1}^n \alpha_i q(b^i - b) = \sum_{i=1}^n \alpha_i q(b^i) - q(b)$. Indeed

$$\begin{aligned}\sum_{i=1}^n \alpha_i q\big(b^i - b\big) &= \sum_{i=1}^n \alpha_i q\big(b^i\big) + \sum_{i=1}^n \alpha_i q(b) - \sum_{i=1}^n \alpha_i \lfloor b^i, b \rfloor = \sum_{i=1}^n \alpha_i q\big(b^i\big) + q(b) - \lfloor b, b \rfloor \\ &= \sum_{i=1}^n \alpha_i q\big(b^i\big) + q(b) - 2q(b) = \sum_{i=1}^n \alpha_i q\big(b^i\big) - q(b)\end{aligned}$$

and, consequently, the generalized transportation formula holds. □

As in [25, Definition 3.3] (see also [12, Definition 2.5]) one can introduce a family of enlargements associated to a positive set.

Definition 4. Let B be an SSD space and $A \subseteq B$ be a q-positive set. We define $\mathbb{E}(A)$ as being the family of multifunctions $E : \mathbb{R}_+ \rightrightarrows B$ satisfying the following properties:

(r1) E is an enlargement of A, that is

$$A \subseteq E(\varepsilon) \quad \text{for all } \varepsilon \geqslant 0;$$

(r2) E is nondecreasing:

$$0 \leqslant \varepsilon_1 \leqslant \varepsilon_2 \quad \Rightarrow \quad E(\varepsilon_1) \subseteq E(\varepsilon_2);$$

(r3) E satisfies the transportation formula.

If A is maximally q-positive then E^A satisfies the properties (r1)–(r3) (cf. Example 2 and Proposition 3, while (r2) is obviously satisfied), hence in this case the family $\mathbb{E}(A)$ is nonempty. Define the multifunction $E_A : \mathbb{R}_+ \rightrightarrows B$, $E_A(\varepsilon) := \bigcap_{E \in \mathbb{E}(A)} E(\varepsilon)$ for all $\varepsilon \geqslant 0$.

Proposition 5. *Let B be an SSD space and $A \subseteq B$ be a maximally q-positive set. Then*

(i) $E_A, E^A \in \mathbb{E}(A)$;
(ii) E_A *and* E^A *are, respectively, the smallest and the biggest elements in* $\mathbb{E}(A)$ *with respect to the partial ordering inclusion relation of the graphs, that is* $G(E_A) \subseteq G(E) \subseteq G(E^A)$ *for all* $E \in \mathbb{E}(A)$.

Proof. (i) That $E^A \in \mathbb{E}(A)$ was pointed out above. The statement $E_A \in \mathbb{E}(A)$ follows immediately, if we take into consideration the definition of E_A.

(ii) E_A is obviously the smallest element in $\mathbb{E}(A)$. We prove in the following that E^A is the biggest element in $\mathbb{E}(A)$. Suppose that E^A is not the biggest element in $\mathbb{E}(A)$, namely that there exist $E \in \mathbb{E}(A)$ and $(\varepsilon, b) \in G(E) \setminus G(E^A)$. Since $(\varepsilon, b) \notin G(E^A)$, there exists $c \in A$ such that $q(b-c) < -\varepsilon$. Let $\lambda \in (0,1)$ be fixed. As E satisfies (r1), we have $c \in A \subseteq E(0)$, that is $(0,c) \in G(E)$. As $(\varepsilon, b), (0,c) \in G(E)$, $\lambda \in (0,1)$ and E satisfies the transportation formula, we obtain $\lambda\varepsilon + \lambda(1-\lambda)q(b-c) \geqslant 0$, hence $\varepsilon + (1-\lambda)q(b-c) \geqslant 0$. Since this inequality must hold for arbitrary $\lambda \in (0,1)$ we get $\varepsilon + q(b-c) \geqslant 0$, which is a contradiction. □

Lemma 6. *Let B be an SSD space, $A \subseteq B$ a maximally q-positive set and $E \in \mathbb{E}(A)$. Then*

$$E(0) = \bigcap_{\varepsilon>0} E(\varepsilon) = A.$$

Proof. By using the properties (r1), (r2), Proposition 5, the definition of E^A and Example 2 we get

$$A \subseteq E(0) \subseteq \bigcap_{\varepsilon>0} E(\varepsilon) \subseteq \bigcap_{\varepsilon>0} E^A(\varepsilon) = E^A(0) = A$$

and the conclusion follows. □

4. Enlargements of positive sets in Banach SSD spaces

We start by recalling the definition of a Banach SSD space, concept introduced by Stephen Simons, and further we study some topological properties of enlargements of positive sets in this framework. By following the suggestion of Heinz Bauschke we propose calling this class of spaces *Simons spaces* and hope that Stephen Simons agrees to this proposal.

Definition 5. We say that B is a **Banach SSD space** (**Simons space**) if B is an SSD space and $\|\cdot\|$ is a norm on B with respect to which B is a Banach space with norm-dual B^*,

$$\frac{1}{2}\|\cdot\|^2 + q \geqslant 0 \quad \text{on } B \tag{5}$$

and there exists $\iota : B \to B^*$ linear and continuous such that

$$\big\langle b, \iota(c)\big\rangle = \lfloor b, c\rfloor \quad \text{for all } b, c \in B. \tag{6}$$

Remark 2. (i) From (6) we obtain $|\lfloor b, c\rfloor| \leqslant \|\iota\|\|b\|\|c\|$. Hence for $(b,c), (\bar{b}, \bar{c}) \in B \times B$ it holds $|\lfloor b, c\rfloor - \lfloor \bar{b}, \bar{c}\rfloor| = |\lfloor b-\bar{b}, c-\bar{c}\rfloor + \lfloor \bar{b}, c-\bar{c}\rfloor + \lfloor b-\bar{b}, \bar{c}\rfloor| \leqslant \|\iota\|(\|b-\bar{b}\|\|c-\bar{c}\| + \|\bar{b}\|\|c-\bar{c}\| + \|b-\bar{b}\|\|\bar{c}\|)$. The function $(b,c) \mapsto \lfloor b, c\rfloor$ is, consequently, continuous and from here one gets immediately the continuity of q, $\lfloor \cdot, c\rfloor$ and $\lfloor b, \cdot\rfloor$ for all $b, c \in B$.

(ii) For a function $f : B \to \overline{\mathbb{R}}$ we have $f^@(c) = \sup_{b\in B}\{\langle b, \iota(c)\rangle - f(b)\} = f^*(\iota(c))$, that is $f^@ = f^* \circ \iota$ on B.

Example 7. (a) The SSD spaces considered in Example 1(a)–(c) are Banach SSD spaces (Simons spaces) (see [23, Remark 2.2]).

(b) Consider again the framework of Example 1(d), that is X is a nonzero Banach space and $B := X \times X^*$. The canonical embedding of X into X^{**} is defined by $\widehat{\ } : X \to X^{**}$, $\langle x^*, \widehat{x}\rangle := \langle x, x^*\rangle$ for all $x \in X$ and $x^* \in X^*$. The dual of B (with respect to the norm topology) is $X^* \times X^{**}$ under the pairing

$$\langle b, c^*\rangle = \langle x, y^*\rangle + \langle x^*, y^{**}\rangle \quad \text{for all } b = (x, x^*) \in B,\ c^* = (y^*, y^{**}) \in B^*.$$

Then $X \times X^*$ is a Banach SSD space (Simons space), where $\iota : X \times X^* \to X^* \times X^{**}$, $\iota(x, x^*) := (x^*, \widehat{x})$ for all $(x, x^*) \in X \times X^*$.

For $E : \mathbb{R}_+ \rightrightarrows B$ we define $\overline{E} : \mathbb{R}_+ \rightrightarrows B$ by $\overline{E}(\varepsilon) := \{b \in B:\ (\varepsilon, b) \in \overline{G(E)}\}$. E is said to be closed if $E = \overline{E}$. One can see that E is closed if and only if $G(E)$ is closed. For $A \subseteq B$, consider also the subfamily $\mathbb{E}_c(A) = \{E \in \mathbb{E}(A):\ E \text{ is closed}\}$.

Proposition 8. *Let B be a Banach SSD space (Simons space) and $A \subseteq B$ be a maximally q-positive set. Then*

(i) *If $E \in \mathbb{E}(A)$ then $\overline{E} \in \mathbb{E}_c(A)$.*
(ii) *If $E \in \mathbb{E}_c(A)$ then $E(\varepsilon)$ is closed, for all $\varepsilon \geqslant 0$.*
(iii) *$\overline{E_A}$ and E^A are, respectively, the smallest and the biggest elements in $\mathbb{E}_c(A)$, with respect to the partial ordering inclusion relation of the graphs, that is $G(\overline{E_A}) \subseteq G(E) \subseteq G(E^A)$ for all $E \in \mathbb{E}_c(A)$.*

Proof. (i) Let be $E \in \mathbb{E}(A)$. One can note that the continuity of the function q implies that if E satisfies the transportation formula, then $\overline{E}$ satisfies this formula, too. Further, if E is nondecreasing, then $\overline{E}$ is also nondecreasing. Hence the first assertion follows.

(ii) The second statement of the proposition is a consequence of the fact that E is closed if and only if $G(E)$ is closed.

(iii) Employing once more the continuity of the function q we get that $G(E^A)$ is closed. Combining Propositions 5 and 8(i) we obtain $\overline{E_A}, E^A \in \mathbb{E}_c(A)$. The proof of the minimality, respectively, maximality of these elements presents no difficulty. □

In the following we establish a one-to-one correspondence between $\mathbb{E}_c(A)$ and a family of convex functions with certain properties. This is done be extending the techniques used in [12, Section 3] to Banach SSD spaces (Simons spaces).

Consider B a Banach SSD space (Simons space). To $A \subseteq B \times \mathbb{R}$ we associate the so-called *lower envelope of A* (cf. [1]), defined as $\mathrm{le}_A : B \to \overline{\mathbb{R}}$, $\mathrm{le}_A(b) = \inf\{r \in \mathbb{R}:\ (b,r) \in A\}$. Obviously, $A \subseteq \mathrm{epi}(\mathrm{le}_A)$. If, additionally, A is closed and has an *epigraphical structure*, that is $(b,r_1) \in A \Rightarrow (b,r_2) \in A$ for all $r_2 \in [r_1,+\infty)$, then $A = \mathrm{epi}(\mathrm{le}_A)$.

Let us consider now $E : \mathbb{R}_+ \rightrightarrows B$ and define $\lambda_E : B \to \overline{\mathbb{R}}$, $\lambda_E(b) = \inf\{\varepsilon \geqslant 0:\ b \in E(\varepsilon)\}$. It is easy to observe that $\lambda_E(b) = \inf\{r \in \mathbb{R}:\ (b,r) \in G(E^{-1})\}$, where $E^{-1} : B \rightrightarrows \mathbb{R}_+$ is the *inverse of the multifunction E*. One has $G(E^{-1}) = \{(b,\varepsilon):\ (\varepsilon,b) \in G(E)\}$. Hence λ_E is the lower envelope of $G(E^{-1})$. We have $G(E^{-1}) \subseteq \mathrm{epi}(\lambda_E)$. If E is closed and nondecreasing, then $G(E^{-1})$ is closed and has an epigraphical structure, so in this case $G(E^{-1}) = \mathrm{epi}(\lambda_E)$. As in [12, Proposition 3.1] we obtain the following result.

Proposition 9. *Let B be a Banach SSD space* (*Simons space*) *and $E : \mathbb{R}_+ \rightrightarrows B$ a multifunction which is closed and nondecreasing. Then*:

(i) $G(E^{-1}) = \mathrm{epi}(\lambda_E)$;
(ii) λ_E *is lower semicontinuous*;
(iii) $\lambda_E \geqslant 0$;
(iv) $E(\varepsilon) = \{b \in B:\ \lambda_E(b) \leqslant \varepsilon\}$ *for all* $\varepsilon \geqslant 0$.

Moreover, λ_E is the only function from B to $\overline{\mathbb{R}}$ satisfying (iii) *and* (iv).

Given $E : \mathbb{R}_+ \rightrightarrows B$, we define the function $\Lambda_E : B \to \overline{\mathbb{R}}$, $\Lambda_E := \lambda_E + q$. Let us notice that Λ_E is the lower envelope of $\Psi(G(E^{-1}))$ (the function Ψ was defined in Proposition 4) and $\mathrm{epi}(\Lambda_E) = \Psi(\mathrm{epi}(\lambda_E))$. From these observations, Propositions 9(i) and 4 we obtain the following result.

Corollary 10. *Let B be Banach SSD space* (*Simons space*) *and $E : \mathbb{R}_+ \rightrightarrows B$ a closed and nondecreasing enlargement of the maximally q-positive set $A \subseteq B$. Then $E \in \mathbb{E}(A)$ if and only if Λ_E is convex.*

Proposition 11. *Let B be a Banach SSD space* (*Simons space*)*, $A \subseteq B$ a maximally q-positive set and $E \in \mathbb{E}_c(A)$. Then Λ_E is convex, lower semicontinuous, $\Lambda_E \geqslant q$ on B and $A \subseteq \mathcal{P}(\Lambda_E)$.*

Proof. The first three assertions follow from Corollary 10 and Proposition 9(ii) and (iii). Take an arbitrary $b \in A$. Since E is an enlargement of A we get $b \in E(0)$, hence $\lambda_E(b) = 0$ and the conclusion follows. □

To every maximally q-positive set we introduce the following family of convex functions.

Definition 6. Let B be a Banach SSD space (Simons space) and $A \subseteq B$ be a maximally q-positive set. We define $\mathcal{H}(A)$ as the family of convex lower semicontinuous functions $h : B \to \overline{\mathbb{R}}$ such that

$$h \geqslant q \quad \text{on } B \quad \text{and} \quad A \subseteq \mathcal{P}(h).$$

Remark 3. Combining Propositions 11 and 9(i) we obtain that the map $E \mapsto \Lambda_E$ is one-to-one from $\mathbb{E}_c(A)$ to $\mathcal{H}(A)$.

For $h \in \mathcal{H}(A)$ we define the multifunction $A_h : \mathbb{R}_+ \rightrightarrows B$,

$$A_h(\varepsilon) := \big\{b \in B:\ h(b) \leqslant \varepsilon + q(b)\big\} \quad \text{for all } \varepsilon \geqslant 0.$$

Proposition 12. *Let B be a Banach SSD space* (*Simons space*) *and $A \subseteq B$ be a maximally q-positive set. If $h \in \mathcal{H}(A)$, then $A_h \in \mathbb{E}_c(A)$ and $\Lambda_{A_h} = h$.*

Proof. Take an arbitrary $h \in \mathcal{H}(A)$. The properties of the function h imply that A_h is a closed enlargement of A. Obviously A_h is nondecreasing. Trivially, $A_h(\varepsilon) = \{b \in B:\ l(b) \leqslant \varepsilon\}$, where $l : B \to \overline{\mathbb{R}}$, $l := h - q$. By Proposition 9 we get $\lambda_{A_h} = l$, implying $\Lambda_{A_h} = h$. The convexity of h and Corollary 10 guarantee that $A_h \in \mathbb{E}_c(A)$. □

As a consequence of the above results we obtain a bijection between the family of closed enlargements (which satisfy conditions (r1)–(r3) from Definition 4) associated to a maximally q-positive set and the family of convex functions introduced in Definition 6.

Theorem 13. *Let B be a Banach SSD space (Simons space) and $A \subseteq B$ be a maximally q-positive set. The map*

$$\mathbb{E}_c(A) \to \mathcal{H}(A),$$
$$E \mapsto \Lambda_E$$

is a bijection, with inverse given by

$$\mathcal{H}(A) \to \mathbb{E}_c(A),$$
$$h \mapsto A_h.$$

Moreover, $A_{\Lambda_E} = E$ for all $E \in \mathbb{E}_c(A)$ and $\Lambda_{A_h} = h$ for all $h \in \mathcal{H}(A)$.

The following corollary shows that there exists a closed connection between an element $h \in \mathcal{H}(A)$ and the maximally q-positive set A.

Corollary 14. *Let B be a Banach SSD space (Simons space) and $A \subseteq B$ be a maximally q-positive set. Take $h \in \mathcal{H}(A)$. Then $A = \mathcal{P}(h)$.*

Proof. We have $A \subseteq \mathcal{P}(h)$ by the definition of $\mathcal{H}(A)$. Take an arbitrary $b \in \mathcal{P}(h)$. Define $E := A_h$. Then $b \in E(0)$. Applying Theorem 13 we get $E \in \mathbb{E}_c(A)$. Further, by Lemma 6 we have $E(0) = A$, hence $b \in A$ and the proof is complete. □

Remark 4. In what follows, we call an arbitrary element h of $\mathcal{H}(A)$ a **representative functions** of A. The word "representative" is justified by Corollary 14. Since for A a q-positive set, we have $A \neq \emptyset$ (see Definition 1(ii)), every representative function of A is proper.

Corollary 15. *Let B be a Banach SSD space (Simons space) and $A \subseteq B$ be a maximally q-positive set. Take $E \in \mathbb{E}_c(A)$ and $b^1 \in E(\varepsilon_1)$, $b^2 \in E(\varepsilon_2)$, where $\varepsilon_1, \varepsilon_2 \geqslant 0$ are arbitrary. Then*

$$q\big(b^1 - b^2\big) \geqslant -(\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2})^2.$$

Proof. By Theorem 13, there exists a representative function $h \in \mathcal{H}(A)$ such that $E = A_h$. By using the definition of A_h and applying [23, Lemma 1.6] we obtain

$$-q\big(b^1 - b^2\big) \leqslant \Big[\sqrt{(h-q)\big(b^1\big)} + \sqrt{(h-q)\big(b^2\big)}\Big]^2 \leqslant (\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2})^2$$

and the proof is complete. □

Remark 5. The above lower bound is established in [11, Corollary 3.12], where B is taken as in Example 7(b), for the enlargement B^A (see Example 2). Here we generalize this result to Banach SSD spaces (Simons spaces) and to an arbitrary $E \in \mathbb{E}_c(A)$.

In what follows we investigate the properties of the functions $\Lambda_{E^A}, \Lambda_{\overline{E_A}}$ and rediscover in this way the functions introduced and studied by S. Simons in [21–23] (see Proposition 18(iii) below).

Corollary 16. *Let B be a Banach SSD space (Simons space) and $A \subseteq B$ be a maximally q-positive set.*

(i) *The functions $\Lambda_{E^A}, \Lambda_{\overline{E_A}} \in \mathcal{H}(A)$ and are, respectively, the minimum and the maximum of this family, that is*

$$\Lambda_{E^A} \leqslant h \leqslant \Lambda_{\overline{E_A}} \quad \text{for all } h \in \mathcal{H}(A). \tag{7}$$

(ii) *Conversely, if $h : B \to \overline{\mathbb{R}}$ is a convex lower semicontinuous function such that*

$$\Lambda_{E^A} \leqslant h \leqslant \Lambda_{\overline{E_A}}, \tag{8}$$

then $h \in \mathcal{H}(A)$.

(iii) *It holds $\mathcal{H}(A) = \{h : B \to \overline{\mathbb{R}}:\ h$ convex, lower semicontinuous and $\Lambda_{E^A} \leqslant h \leqslant \Lambda_{\overline{E_A}}\}$.*

Proof. (i) This follows immediately from Theorem 13 and Proposition 8.

(ii) If $h : B \to \overline{\mathbb{R}}$ is a convex lower semicontinuous function satisfying (8), then (since $\Lambda_{E^A} \in \mathcal{H}(A)$)

$$h \geqslant \Lambda_{E^A} \geqslant q \quad \text{on } B. \tag{9}$$

Further, for $b \in A$ we obtain (employing that $\Lambda_{\overline{E_A}} \in \mathcal{H}(A)$) that $h(b) \leqslant \Lambda_{\overline{E_A}}(b) = q(b)$. In view of (9) it follows that $b \in \mathcal{P}(h)$, hence $h \in \mathcal{H}(A)$.

(iii) This characterization of $\mathcal{H}(A)$ is a direct consequence of (i) and (ii). □

Definition 7. (Cf. [23].) Let B be a Banach SSD space (Simons space) and $A \subseteq B$ be a q-positive set. We define the function $\Theta_A : B^* \to \overline{\mathbb{R}}$,

$$\Theta_A(b^*) := \sup_{a \in A}\bigl[\langle a, b^*\rangle - q(a)\bigr] \quad \text{for all } b^* \in B^*.$$

We define the function $\Phi_A : B \to \overline{\mathbb{R}}$,

$$\Phi_A := \Theta_A \circ \iota$$

and also the function ${}^*\Theta_A : B \to \overline{\mathbb{R}}$,

$${}^*\Theta_A(c) := \sup_{b^* \in B^*}\bigl[\langle c, b^*\rangle - \Theta_A(b^*)\bigr] \quad \text{for all } c \in B.$$

We denote by $a \vee b$ the maximum value between $a, b \in \overline{\mathbb{R}}$. The following properties of the functions defined above appear in [23, Lemma 2.13 and Theorem 2.16]. The property (vii) is a direct consequence of (i)–(vi).

Lemma 17. *Let B be a Banach SSD space (Simons space) and $A \subseteq B$ a q-positive set. Then:*

(i) *For all $b \in B$, $\Phi_A(b) = \sup_{a \in A}[\lfloor a, b\rfloor - q(a)] = q(b) - \inf_{c \in A} q(b - c)$.*
(ii) *Φ_A is proper, convex, lower semicontinuous and $A \subseteq \mathcal{P}(\Phi_A)$.*
(iii) *$({}^*\Theta_A)^* = \Theta_A$ and $({}^*\Theta_A)^@ = \Phi_A$.*
(iv) *${}^*\Theta_A$ is proper, convex, lower semicontinuous, ${}^*\Theta_A \geqslant \Phi_A^@ \geqslant \Phi_A \vee q$ on B and*

$${}^*\Theta_A = \Phi_A^@ = q \quad \text{on } A.$$

(v) *${}^*\Theta_A = \sup\{h : B \to \overline{\mathbb{R}}:\ h$ proper, convex, lower semicontinuous, $h \leqslant q$ on $A\}$.*

If, additionally, A is maximally q-positive, then:

(vi) *${}^*\Theta_A \geqslant \Phi_A^@ \geqslant \Phi_A \geqslant q$ on B and $A = \mathcal{P}({}^*\Theta_A) = \mathcal{P}(\Phi_A^@) = \mathcal{P}(\Phi_A)$.*
(vii) *${}^*\Theta_A, \Phi_A^@, \Phi_A \in \mathcal{H}(A)$.*

Next we give other characterizations of the function ${}^*\Theta_A$ and establish the connection between $\Lambda_{E^A}, \Lambda_{\overline{E_A}}$ and $\Phi_A, {}^*\Theta_A$, respectively.

Proposition 18. *Let B be a Banach SSD space (Simons space) and $A \subseteq B$ a q-positive set. Then:*

(i) *${}^*\Theta_A = \sup\{h : B \to \overline{\mathbb{R}}:\ h$ proper, convex, lower semicontinuous, $h \geqslant q$ on B and $A \subseteq \mathcal{P}(h)\}$.*
(ii) *${}^*\Theta_A = \operatorname{cl}\operatorname{co}(q + \delta_A)$.*

If, additionally, A is maximally q-positive, then:

(iii) *$\Lambda_{E^A} = \Phi_A$ and $\Lambda_{\overline{E_A}} = {}^*\Theta_A$.*
(iv) *If $h : B \to \overline{\mathbb{R}}$ is a function such that $h \in \mathcal{H}(A)$, then $h^@ \in \mathcal{H}(A)$.*

Proof. (i) We have $\{h : B \to \overline{\mathbb{R}}:\ h$ proper, convex, lower semicontinuous, $h \geqslant q$ on B and $A \subseteq \mathcal{P}(h)\} \subseteq \{h : B \to \overline{\mathbb{R}}:\ h$ proper, convex, lower semicontinuous, $h \leqslant q$ on $A\}$ hence from Lemma 17(v) we get

$$\sup\bigl\{h : B \to \overline{\mathbb{R}}:\ h \text{ proper, convex, lower semicontinuous, } h \geqslant q \text{ on } B \text{ and } A \subseteq \mathcal{P}(h)\bigr\} \leqslant {}^*\Theta_A.$$

On the other hand, by Lemma 17(iv), ${}^*\Theta_A$ is proper, convex and lower semicontinuous and it fulfills ${}^*\Theta_A \geqslant q$ on B and $A \subseteq \mathcal{P}({}^*\Theta_A)$. Thus the equality follows.

(ii) Since ${}^*\Theta_A \leqslant q$ on A we have ${}^*\Theta_A \leqslant q + \delta_A$ on B, hence

$${}^*\Theta_A \leqslant \operatorname{cl}\operatorname{co}(q + \delta_A) \leqslant q + \delta_A. \tag{10}$$

The above inequality shows that $\operatorname{cl}\operatorname{co}(q+\delta_A)$ is a proper, convex, lower semicontinuous function such that $\operatorname{cl}\operatorname{co}(q+\delta_A)\leqslant q$ on A. Applying Lemma 17(v) we obtain ${}^*\Theta_A \geqslant \operatorname{cl}\operatorname{co}(q+\delta_A)$, which combined with (10) delivers the desired result.

(iii) From Lemma 17(i) and the definition of E^A we obtain

$$\begin{aligned} b\in E^A(\varepsilon) \quad &\Leftrightarrow \quad q(b-c)\geqslant -\varepsilon \quad \text{for all } c\in A \quad \Leftrightarrow \quad \inf_{c\in A} q(b-c)\geqslant -\varepsilon \\ &\Leftrightarrow \quad q(b)-\Phi_A(b)\geqslant -\varepsilon \quad \Leftrightarrow \quad \Phi_A(b)\leqslant \varepsilon + q(b). \end{aligned}$$

This is nothing else than $E^A = A_{\Phi_A}$. Theorem 13 implies that $\Lambda_{E^A} = \Lambda_{A_{\Phi_A}} = \Phi_A$.

The equality $\Lambda_{\overline{E_A}} = {}^*\Theta_A$ follows from (i) and Corollary 16.

(iv) From (iii), Corollary 16 and [23, Theorem 2.15(b)] we get $h^@ \geqslant q$ on B and $\mathcal{P}(h)=\mathcal{P}(h^@)=A$. The function $h^@$ is proper, convex and the lower semicontinuity follows from the definition of $h^@$ and Remark 2, hence $h^@\in\mathcal{H}(A)$. $\square$

Remark 6. Proposition 18(iv) is a generalization of [12, Theorem 5.3] to Banach SSD spaces (Simons spaces).

Remark 7. In general, the functions ${}^*\Theta_A$ and $\Phi_A^@$ are not identical. A striking example in this sense was given by C. Zălinescu (see [23, Remark 2.14]) for B a Banach space and $\lfloor\cdot,\cdot\rfloor = 0$ on $B\times B$. An alternative example, originally due to M.D. Voisei and C. Zălinescu, is given below (see Example 19).

Before we present this example, we need the following remark.

Remark 8. Consider again the particular setting of Examples 1(d) and 7(b), namely when $B = X\times X^*$, where X is a nonzero Banach space. Let A be a nonempty monotone subset of $X\times X^*$. In this case $q(x,x^*)=\langle x,x^*\rangle$ for all $(x,x^*)\in X\times X^*$ and the function $\Theta_A : X^*\times X^{**}\to\overline{\mathbb{R}}$ is defined by

$$\Theta_A(x^*,x^{**}) = \sup_{(s,s^*)\in A}\bigl[\langle s,x^*\rangle + \langle s^*,x^{**}\rangle - \langle s,s^*\rangle\bigr] \quad \text{for all } (x^*,x^{**})\in X^*\times X^{**}.$$

The function $\Phi_A : X\times X^*\to\overline{\mathbb{R}}$ has the following form:

$$\Phi_A(x,x^*) = \sup_{(s,s^*)\in A}\bigl[\langle s,x^*\rangle + \langle x,s^*\rangle - \langle s,s^*\rangle\bigr] \quad \text{for all } (x,x^*)\in X\times X^*,$$

that is $\Phi_A = (q+\delta_A)^@$. Φ_A is the Fitzpatrick function of A. Introduced by S. Fitzpatrick in [14] in 1988 and rediscovered after some years in [12,16], it proved to be very important in the theory of maximal monotone operators, revealing important connections between convex analysis and monotone operators (see [2–6,12,15,17,18,20,22,24,26] and the references therein). Applying the Fenchel–Moreau Theorem we obtain

$$\Phi_A^@ = \operatorname{cl}_{s\times w^*}\operatorname{co}(q+\delta_A) \tag{11}$$

(the closure is taken with respect to the strong-weak* topology on $X\times X^*$). The function ${}^*\Theta_A : X\times X^*\to\overline{\mathbb{R}}$ is defined by

$${}^*\Theta_A(y,y^*) = \sup_{(x^*,x^{**})\in X^*\times X^{**}}\bigl[\langle y,x^*\rangle + \langle y^*,x^{**}\rangle - \Theta_A(x^*,x^{**})\bigr] \quad \text{for all } (y,y^*)\in X\times X^*.$$

Example 19. As in [27, page 5], consider E a nonreflexive Banach space, $X := E^*$ and $A := \{0\}\times\widehat{E}$, which is a monotone subset of $X\times X^*$. Let us notice that for A we have $q+\delta_A=\delta_A$. By applying Proposition 18(ii) we obtain: ${}^*\Theta_A = \operatorname{cl}\operatorname{co}\delta_A = \delta_A$ (the closure is taken with respect to the strong topology of $X\times X^*$). Further, by using (11) and the Goldstine Theorem we get $\Phi_A^@ = \operatorname{cl}_{s\times w^*}\operatorname{co}\delta_A = \delta_{\{0\}\times E^{**}} \neq {}^*\Theta_A$ (the closure is considered with respect to the strong-weak* topology of $X\times X^*$).

In the last part of the paper we deal with another subfamily of $\mathbb{E}(A)$, namely the one of closed and *additive* enlargements. In this way we extend the results from [12,25] to Banach SSD spaces (Simons spaces).

Definition 8. Let B be a Banach SSD space (Simons space). We say that the multifunction $E:\mathbb{R}_+ \rightrightarrows B$ is additive if for all $\varepsilon_1,\varepsilon_2\geqslant 0$ and $b^1\in E(\varepsilon_1)$, $b^2\in E(\varepsilon_2)$ one has

$$q\bigl(b^1-b^2\bigr)\geqslant -(\varepsilon_1+\varepsilon_2).$$

In case $A\subseteq B$ is a maximally q-positive set we denote by $\mathbb{E}_{ca}(A)=\{E\in\mathbb{E}_c(A):\ E \text{ is additive}\}$.

We have the following characterization of the set $\mathbb{E}_{ca}(A)$.

Theorem 20. *Let B be a Banach SSD space (Simons space), $A\subseteq B$ a maximally q-positive set and $E\in\mathbb{E}_c(A)$. Then:*

$$E\in\mathbb{E}_{ca}(A) \quad \Leftrightarrow \quad \Lambda_E^@ \leqslant \Lambda_E.$$

Proof. Assume first that $E \in \mathbb{E}_{ca}(A)$ and take b^1, b^2 two arbitrary elements in B. By Proposition 9(iii) follows that $\lambda_E(b^1) \geqslant 0$ and $\lambda_E(b^2) \geqslant 0$. We claim that

$$q(b^1 - b^2) \geqslant -(\lambda_E(b^1) + \lambda_E(b^2)).$$

In case $\lambda_E(b^1) = +\infty$ or $\lambda_E(b^2) = +\infty$ (or both), this fact is obvious. If $\lambda_E(b^1)$ and $\lambda_E(b^2)$ are finite, the inequality above follows by using that (cf. Proposition 9(i)) $(b_1, \lambda_E(b^1))$, $(b_2, \lambda_E(b^2)) \in \operatorname{epi}(\lambda_E) = G(E^{-1})$ and that E is additive. Consequently, for all $b^1, b^2 \in B$,

$$\lambda_E(b^1) + q(b^1) \geqslant \lfloor b^1, b^2 \rfloor - (\lambda_E(b^2) + q(b^2)) \quad \Leftrightarrow \quad \Lambda_E(b^1) \geqslant \lfloor b^1, b^2 \rfloor - \Lambda_E(b^2).$$

This means that for all $b^1 \in B$, $\Lambda_E^{@}(b^1) \leqslant \Lambda_E(b^1)$.

Assume now that $\Lambda_E^{@} \leqslant \Lambda_E$ and take arbitrary $\varepsilon_1, \varepsilon_2 \geqslant 0$ and $b^1 \in E(\varepsilon_1)$, $b^2 \in E(\varepsilon_2)$. This means that $\lambda_E(b^1) \leqslant \varepsilon_1$ and $\lambda_E(b^2) \leqslant \varepsilon_2$. Since $\Lambda_E(b^1) \geqslant \Lambda_E^{@}(b^1)$, one has

$$\lambda_E(b^1) + q(b^1) \geqslant \lfloor b^1, b^2 \rfloor - (\lambda_E(b^2) + q(b^2))$$

and from here

$$q(b^1 - b^2) \geqslant -(\lambda_E(b^1) + \lambda_E(b^2)) \geqslant -(\varepsilon_1 + \varepsilon_2).$$

This concludes the proof. □

One can use the theorem above for providing an element in the set $\mathbb{E}_{ca}(A)$ in case A is a maximally q-positive set.

Proposition 21. *Let B be a Banach SSD space (Simons space) and $A \subseteq B$ a maximally q-positive set. Then $\overline{E_A} \in \mathbb{E}_{ca}(A)$.*

Proof. By Proposition 18(iii) and Lemma 17(iii)–(iv) we have $(\Lambda_{\overline{E_A}})^{@} = ({}^*\Theta_A)^{@} = \Phi_A \leqslant {}^*\Theta_A = \Lambda_{\overline{E_A}}$. Theorem 20 guarantees that $\overline{E_A} \in \mathbb{E}_{ca}(A)$. □

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