A New Fenchel Dual Problem in Vector Optimization

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Abstract

We introduce a new Fenchel dual for vector optimization problems inspired by the form of the Fenchel dual attached to the scalarized primal multiobjective problem. For the vector primal-dual pair we prove weak and strong duality. Furthermore, we recall two other Fenchel-type dual problems introduced in the past in the literature, in the vector case, and make a comparison among all three duals. Moreover, we show that their sets of maximal elements are equal.

Key Words. conjugate functions, Fenchel duality, vector optimization, weak and strong duality

AMS subject classification. 49N15, 90C29, 90C46

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1 Introduction

Multiobjective optimization problems have generated a great deal of interest during the last years, not only from a theoretical point of view, but also from a practical one, due to their applicability in different fields, like economics and engineering. In general, when dealing with scalar optimization problems, the duality theory proves to be an important tool for giving some dual characterizations for the optimal solutions of the primal problem. Similar characterizations can also be given for multiobjective optimization problems, namely for problems having a vector function as objective function.

An overview on the literature dedicated to this field shows that the general interest was centered on multiobjective problems with geometric and inequality constraints. The duality theories developed for these problems are extensions of the *classical Lagrange duality approach*. We recall here the concepts developed by Mond and Weir in [14], [15] (the formulation of which being based on the optimality conditions provided by strong Lagrange duality). Tanino, Nakayama and Sawaragi investigated in [12] the duality for vector optimization problems in finite dimensional spaces using the *perturbation approach*, the duals obtained in this case being also Lagrange-type duals. They extended to the vector case the conjugate theory from scalar optimization (see for example [11]). In Jahn's paper [8] the Lagrange dual appears explicitly in the formulation of the feasible set of the multiobjective dual.

Another approach which we mention here is due to Boţ and Wanka, who constructed a vector dual (cf. [4]) using the so-called *Fenchel-Lagrange dual* of a scalar convex optimization problem. This is a combination of the *classical Lagrange and Fenchel* duals and was treated in papers like [1], [2] and [3].

With respect to *vector-type Fenchel duality concepts* the bibliography is not very rich. We mention in this direction the works of Breckner and Kolumbán [5] and [6] as well as the ones due to Gerstewitz and Göpfert (cf. [7]) and Malivert (cf. [10]).

The primal problem treated in this paper has as objective function *the sum* of a vector function, with another one, which is the composition of a vector function, with a linear operator. For it we propose a *Fenchel-type dual* which extends the well-known scalar Fenchel dual from [11]. We prove *weak and strong duality* and compare the new dual to two other from the literature.

The paper is organized as follows. In Section 2 we introduce some preliminary notions and results and formulate the primal vector optimization problem we deal with. In Section 3 we attach to the vector primal problem a scalarized optimization problem and consider its scalar Fenchel dual problem. For the scalar primal-dual pair we derive via the strong duality theorem necessary and sufficient *optimality conditions*. Inspired by the formulation of the scalarized dual we define in Section 4 the new *vector dual problem* and prove for the vector primal-dual pair weak and strong duality. In Section 5 we present two other Fenchel-type dual problems, one inspired by Breckner and Kolumban's paper [6], while the other one is constructed in a similar manner to Jahn's multiobjective problem considered in [8] by making a slight change in the definition of the feasible set of the first one. For these two primal-dual pairs we also provide *weak and strong duality theorems*. The *image sets* of the three duals are closely connected, as it is proved in Section 6, where the existence of some relations of inclusion between these sets is proved. Moreover, we illustrate by some examples that in general these inclusions are strict. Finally, we show that even though this happens, the *sets of maximal elements* of the image sets coincide.

2 Preliminary notions and results

In this section we present some notions and preliminary results used throughout the paper. We also introduce the primal vector optimization problem and consider two notions of solutions appropriate for vector optimization.

All the vectors considered are column vectors. For $x = (x_1, ..., x_n)^T$ and $y = (y_1, ..., y_n)^T$ in $\mathbb{R}^n$ by $x^T y = \sum_{i=1}^{n} x_i y_i$ we denote the usual *inner product*. Having a function $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ its *effective domain* is denoted by $\operatorname{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The function f is said to be *proper* if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\operatorname{dom}(f) \neq \emptyset$. Its *epigraph* is the set $\operatorname{epi}(f) = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$ and its *conjugate function* is defined by $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, $f^*(p) = \sup\left\{p^T x - f(x) : x \in \mathbb{R}^n\right\}$.

The function f is *polyhedral* if $\operatorname{epi}(f)$ is a *polyhedral set*, which means that it can be written as the intersection of a finite family of closed half-spaces.

For the operations in the extended real-line $\overline{\mathbb{R}}$, along the usual *addition* and *multiplication*, we consider the following conventions which are typical in the convex analysis

$$(+\infty) + (-\infty) = +\infty, 0(+\infty) = +\infty, 0(-\infty) = 0.$$

Having a nonempty subset C of $\mathbb{R}^n$, $\text{int}(C)$ denotes its *interior*, while $\text{ri}(C)$ denotes its *relative interior*. For a linear operator $A : \mathbb{R}^n \to \mathbb{R}^k$ its *adjoint* $A^* : \mathbb{R}^k \to \mathbb{R}^n$ is the linear operator defined by $(A^*y)^T x = y^T (Ax)$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k$.

Before coming to the formulation of the primal vector optimization problem let us make some considerations regarding the set $\overline{\mathbb{R}}^m$, $m \geq 1$, which is defined as being $\underbrace{\overline{\mathbb{R}} \times ... \times \overline{\mathbb{R}}}_{m}$. For $d', d'' \in \overline{\mathbb{R}}^m$ we have that $d' = d''$ if and only if $d'_i = d''_i$ for all $i \in \{1, ..., m\}$. For elements in $\overline{\mathbb{R}}^m$ operations like *addition* and *multiplication with an element in* $\overline{\mathbb{R}}$ are to be understand *componentwise*. Even if in our investigations only the addition occurs, one can notice that, due to the conventions made above, both operations are well-defined. Extending the concept of *partial order induced by the nonnegative orthant* $\mathbb{R}^m_+$ *on* $\mathbb{R}^m$ we say that for two elements $x, y \in \overline{\mathbb{R}}^m$ one has

$$x \geqq y \text{ if and only if } x_i \geq y_i \text{ for all } i \in \{1, ..., m\}.$$

The primal problem, for which a new Fenchel duality concept is here considered, is the following vector optimization problem

$$(P^A) \qquad v\text{-} \min_{x \in \mathbb{R}^n} (f(x) + (g \circ A)(x)).$$

Throughout this paper we assume that f and g are two vector functions

$$f = (f_1, f_2, ...f_m)^T \quad \text{and} \quad g = (g_1, g_2, ..., g_m)^T,$$

such that $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g_i : \mathbb{R}^k \to \overline{\mathbb{R}}$ are *proper* and *convex* for $i \in \{1, ..., m\}$ and that $A : \mathbb{R}^n \to \mathbb{R}^k$ is a linear operator. Furthermore, we suppose that for

$I, J \subseteq \{1, ..., m\}$ the functions $f_i, i \in I$, and $g_j, j \in J$, are in fact *polyhedral* and that the following *regularity condition* is fulfilled

$$(RC^A) \quad \exists\ x' \in \bigcap_{i \in I} \mathrm{dom}(f_i) \quad \cap \bigcap_{l \in \{1,...,m\} \setminus I} \mathrm{ri}(\mathrm{dom}(f_l)) \ \text{ such that}$$
$$Ax' \in \bigcap_{j \in J} \mathrm{dom}\,(g_j) \quad \cap \bigcap_{t \in \{1,...,m\} \setminus J} \mathrm{ri}\,(\mathrm{dom}\,(g_t))\,.$$

For the vector optimization problem (P^A) different notions of solutions have been introduced and studied in the literature. We use in this paper the *Pareto-efficient* and the *properly efficient solutions*, which are defined below. One can notice that similar investigations can be done for the *weakly efficient solutions*, too.

Definition 1 *An element $\overline{x} \in \mathbb{R}^n$ is said to be Pareto-efficient to the problem (P^A) if from*

$$f(\overline{x}) + (g \circ A)(\overline{x}) \geqq f(x) + (g \circ A)(x) \text{ for } x \in \mathbb{R}^n$$

follows that

$$f(\overline{x}) + (g \circ A)(\overline{x}) = f(x) + (g \circ A)(x).$$

Definition 2 *An element $\overline{x} \in \mathbb{R}^n$ is said to be properly efficient to the problem (P^A) if there exists $\lambda = (\lambda_1, ..., \lambda_m)^T$ in $\mathrm{int}(\mathbb{R}^m_+)$ such that*

$$\sum_{i=1}^m \lambda_i \Big(f_i(\overline{x}) + (g_i \circ A)(\overline{x}) \Big) \leq \sum_{i=1}^m \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big).$$

Remark 1. If the *feasibility condition* $\bigcap_{i=1}^m \mathrm{dom}(f_i) \cap A^{-1}(\bigcap_{i=1}^m \mathrm{dom}(g_i)) \neq \emptyset$ is fulfilled (which is the case when (RC^A) holds), then for every properly efficient element $\bar{x}$ of (P^A) one has that $f(\overline{x}) + (g \circ A)(\overline{x}) \in \mathbb{R}^m$. One should also notice that every properly efficient element is efficient, but the reverse claim does not hold in general.

3 Duality for the scalarized problem

In order to be able to formulate a vector dual problem to (P^A), we develop first a duality theory for the following scalar optimization problem (motivated by the definition of a properly efficient solution), with $\lambda \in \operatorname{int}(\mathbb{R}^m_+)$ arbitrarily chosen,

$$(P^A_\lambda) \quad \inf_{x\in\mathbb{R}^n} \sum_{i=1}^m \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big).$$

The *classical Fenchel dual problem* to (P^A_λ) is (cf. [11, Corollary 31.2.1])

$$\sup_{q\in\mathbb{R}^k} \left(- \left(\sum_{i=1}^m \lambda_i f_i \right)^* (-A^*q) - \left(\sum_{i=1}^m \lambda_i g_i \right)^* (q) \right).$$

With regard to our purposes this dual problem has the drawback that the functions involved don't appear separately. Therefore we consider as dual problem to (P^A_λ) a refinement of it, namely

$$\left(D^A_\lambda\right) \quad \sup_{\substack{p_i\in\mathbb{R}^n, q_i\in\mathbb{R}^k, i=1,\dots m,\\ \sum\limits_{i=1}^m \lambda_i(p_i + A^*q_i)=0}} \sum_{i=1}^m \lambda_i \Big(- f_i^*(p_i) - g_i^*(q_i) \Big).$$

One should notice that it is possible to obtain (D^A_λ) as *conjugate dual* to (P^A_λ) by employing the *perturbation approach* for an appropriate *perturbation function* (cf. [1, 11, 13]). In the following we prove that $\left(D^A_\lambda\right)$ is indeed a *dual problem* of $\left(P^A_\lambda\right)$, namely, that *weak duality* always holds, while, under the convexity assumptions and assuming the fulfillment of the regularity condition (RC^A), *strong duality* holds.

For the scalar problems (P^A_λ) and (D^A_λ) we denote by $v\left(P^A_\lambda\right)$ and $v\left(D^A_\lambda\right)$ their *optimal objective values*, respectively.

Theorem 1 *(scalar weak duality) It holds* $v\left(P^A_\lambda\right) \geq v\left(D^A_\lambda\right)$.

Proof. Let us consider $x \in \mathbb{R}^n$, $p = (p_1, ..., p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$ and $q = (q_1, ..., q_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k$ such that $\sum\limits_{i=1}^{m} \lambda_i (p_i + A^* q_i) = 0$. From Fenchel-Young's inequality we get $f_i(x) + f_i^*(p_i) - p_i^T x \geq 0$ and $(g_i \circ A)(x) + g_i^*(q_i) - (A^* q_i)^T x \geq 0$, for all $i \in \{1, ..., m\}$. Then

$$\sum_{i=1}^{m} \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big) \geq \sum_{i=1}^{m} \lambda_i \Big(- f_i^*(p_i) - g_i^*(q_i) \Big) +$$

$$+ \sum_{i=1}^{m} \lambda_i \Big(p_i + A^* q_i \Big)^T x = \sum_{i=1}^{m} \lambda_i \Big(- f_i^*(p_i) - g_i^*(q_i) \Big).$$

As x and (p, q) are arbitrary feasible elements to (P_λ^A) and (D_λ^A), respectively, the conclusion follows. ■

Remark 2. One can easily notice that in the proof of the theorem above neither the convexity assumptions for the functions involved nor the regularity condition (RC^A) has been used. Nevertheless, for having strong duality one needs these assumptions to be fulfilled.

Theorem 2 *(scalar strong duality) Let $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g_i : \mathbb{R}^k \to \overline{\mathbb{R}}$ be proper and convex functions, $i \in \{1, ..., m\}$, and $I, J \subseteq \{1, ..., m\}$ be the sets of indices for which the functions $f_i, i \in I$, and $g_j, j \in J$, are polyhedral. If the regularity condition (RC^A) is fulfilled, then $v\left(P_\lambda^A\right) = v\left(D_\lambda^A\right)$ and $\left(D_\lambda^A\right)$ has an optimal solution.*

Proof. First we notice that

$$-v(P_\lambda^A) = \sup_{x \in \mathbb{R}^n} \left[- \sum_{i=1}^{m} \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big) \right] = \left[\sum_{i=1}^{m} \lambda_i \Big(f_i + (g_i \circ A) \Big) \right]^* (0)$$

$$= \left[\sum_{i \in I} \lambda_i f_i + \sum_{j \in J} \lambda_j (g_j \circ A) + \sum_{l \in \{1,..,m\} \setminus \ I} \lambda_l f_l + \sum_{t \in \{1,...,m\} \setminus J} \lambda_t (g_t \circ A) \right]^* (0).$$

The functions $\lambda_l f_l$, $l \in \{1,...,m\}\backslash I$, and $\lambda_t(g_t \circ A)$, $t \in \{1,...,m\}\backslash J$, are proper and convex, while the functions $\lambda_i f_i$, $i \in I$, and $\lambda_j(g_j \circ A)$, $j \in J$, are proper and polyhedral. From (RC^A) we have that there exists $x' \in \mathbb{R}^n$ such that

$$x' \in \bigcap_{i\in I} \operatorname{dom}(\lambda_i f_i) \cap \bigcap_{j\in J} \operatorname{dom}\left(\lambda_j(g_j \circ A)\right) \cap \bigcap_{l\in\{1,...,m\}\backslash I} \operatorname{ri}\left(\operatorname{dom}(\lambda_l f_l)\right)$$

and

$$Ax' \in \bigcap_{t\in\{1,...,m\}\backslash J} \operatorname{ri}\left(\operatorname{dom}(g_t)\right).$$

For $t \in \{1,...,m\}\backslash J$, as $Ax' \in \operatorname{ri}\left(\operatorname{dom}(g_t)\right)$, by [11, Theorem 6.7],

$$x' \in A^{-1}\left(\operatorname{ri}(\operatorname{dom}(g_t))\right) = \operatorname{ri}\left(A^{-1}(\operatorname{dom}(g_t))\right) = \operatorname{ri}(\operatorname{dom}(g_t \circ A)).$$

Therefore

$$\begin{aligned} x' \in & \bigcap_{i\in I} \operatorname{dom}(\lambda_i f_i) \cap \bigcap_{j\in J} \operatorname{dom}(\lambda_j(g_j \circ A)) \cap \\ & \bigcap_{l\in\{1,...,m\}\backslash I} \operatorname{ri}(\operatorname{dom}(\lambda_l f_l)) \cap \bigcap_{t\in\{1,...,m\}\backslash J} \operatorname{ri}(\operatorname{dom}(\lambda_t(g_t \circ A))). \end{aligned}$$

Consequently, by [11, Theorem 20.1], there exist $\overline{p}_i \in \mathbb{R}^n, \overline{v}_i \in \mathbb{R}^n, i \in \{1,...,m\}$, such that $\sum\limits_{i=1}^m (\bar{p}_i + \bar{v}_i) = 0$ and

$$\begin{aligned} -v(P_\lambda^A) &= \inf_{\substack{p_i\in\mathbb{R}^n, v_i\in\mathbb{R}^n,\\ i\in\{1,...,m\}\\ \sum_{i=1}^m (p_i+v_i)=0}} \left\{ \sum_{i\in I}(\lambda_i f_i)^*(p_i) + \sum_{j\in J}\Big(\lambda_j(g_j\circ A)\Big)^*(v_j) \right. \\ &\quad \left. + \sum_{l\in\{1,...,m\}\backslash I}(\lambda_l f_l)^*(p_l) + \sum_{t\in\{1,...,m\}\backslash J}\Big(\lambda_t(g_t\circ A)\Big)^*(v_t) \right\} \\ &= \sum_{i\in I}(\lambda_i f_i)^*(\overline{p}_i) + \sum_{j\in J}\Big(\lambda_j(g_j\circ A)\Big)^*(\overline{v}_j) \\ &\quad + \sum_{l\in\{1,...,m\}\backslash I}(\lambda_l f_l)^*(\overline{p}_l) + \sum_{t\in\{1,...,m\}\backslash J}\Big(\lambda_t(g_t\circ A)\Big)^*(\overline{v}_t). \end{aligned}$$

Applying now statement [11, Theorem 16.3] for the proper and convex functions $\lambda_t g_t, t \in \{1, ..., m\} \backslash J$, and taking into consideration the remark after [11, Corollary 31.2.1] for the proper and polyhedral functions $\lambda_j g_j, j \in J$, there exist $\overline{q}_i \in \mathbb{R}^k, i \in \{1, ..., m\}$, such that $A^* \overline{q}_i = \overline{v}_i$ and $\big(\lambda_i (g_i \circ A)\big)^* (\overline{v}_i) = \big(\lambda_i g_i\big)^* (\overline{q}_i)$. Then

$$-v(P_\lambda^A) \quad = \quad \sum_{l=1}^m (\lambda_i f_i)^* (\overline{p}_i) + \sum_{i=1}^m (\lambda_i g_i)^* (\overline{q}_i) \text{ and } \sum_{i=1}^m (\overline{p}_i + A^* \overline{q}_i) = 0.$$

As

$$(\lambda_i f_i)^* (\overline{p}_i) = \lambda_i f_i^* \left(\frac{1}{\lambda_i} \overline{p}_i \right) \quad \text{and} \quad (\lambda_i g_i)^* (\overline{q}_i) = \lambda_i q_i^* \left(\frac{1}{\lambda_i} \overline{q}_i \right) \quad \forall i \in \{1, ..., m\},$$

by denoting $\overline{p}_i := \frac{1}{\lambda_i} \overline{p}_i$ and $\overline{q}_i := \frac{1}{\lambda_i} \overline{q}_i, i \in \{1, ..., m\}$, one has

$$v(P_\lambda^A) = - \sum_{l=1}^m \lambda_i f_i^* (\overline{p}_i) - \sum_{i=1}^m \lambda_i g_i^* (\overline{q}_i), \text{ where } \sum_{i=1}^m \lambda_i \left(\overline{p}_i + A^* \overline{q}_i \right) = 0.$$

By Theorem 1 we have that $v(P_\lambda^A) = v(D_\lambda^A)$ and, consequently, $(\overline{p}, \overline{q})$ with $\overline{p} = (\overline{p}_1, ..., \overline{p}_m)$ and $\overline{q} = (\overline{q}_1, ..., \overline{q}_m)$ is an optimal solution of the dual. ■

The next theorem states the necessary and sufficient *optimality conditions* one can derive from the theorem above for the primal-dual pair (P_λ^A)-(D_λ^A). They will play a decisive role when proving the vector strong duality result in the next section.

Theorem 3 *a) Assume that the hypotheses of Theorem 2 are fulfilled. If $\overline{x} \in \mathbb{R}^n$ is an optimal solution of $\big(P_\lambda^A\big)$, then there exists $(\overline{p}, \overline{q})$, $\overline{p} = (\overline{p}_1, ..., \overline{p}_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$, $\overline{q} = (\overline{q}_1, ..., \overline{q}_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k$, an optimal solution of $\big(D_\lambda^A\big)$,* *such that*

$$\begin{cases} (i) & f_i(\overline{x}) + f_i^*(\overline{p}_i) = \overline{p}_i^T \overline{x}, \quad \forall i \in \{1, ..., m\}; \\ (ii) & (g_i \circ A)(\overline{x}) + g_i^*(\overline{q}_i) = (A^* q_i)^T \overline{x}, \quad \forall i \in \{1, ..., m\}; \\ (iii) & \sum\limits_{i=1}^{m} \lambda_i (\overline{p}_i + A^* \overline{q}_i) = 0. \end{cases}$$

b) If $\overline{x} \in \mathbb{R}^n$ and $(\overline{p}, \overline{q})$ with $\overline{p} = (\overline{p}_1, ..., \overline{p}_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$ and $\overline{q} = (\overline{q}_1, ..., \overline{q}_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k$ are such that (i), (ii) and (iii) are fulfilled, then they are optimal solutions to $\left(P_\lambda^A\right)$ and $\left(D_\lambda^A\right)$, respectively, and $v(P_\lambda^A) = v(D_\lambda^A)$.

Proof. a) Since $\overline{x}$ is an optimal solution of $\left(P_\lambda^A\right)$,

$$v(P_\lambda^A) = \sum_{i=1}^{m} \lambda_i \Big(f_i(\overline{x}) + (g_i \circ A)(\overline{x}) \Big).$$

Further, by Theorem 2, we obtain the existence of an optimal solution $(\overline{p}, \overline{q})$ to $\left(D_\lambda^A\right)$, $\overline{p} = (\overline{p}_1, ..., \overline{p}_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n$ and $\overline{q} = (\overline{q}_1, ..., \overline{q}_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k$, fulfilling $\sum\limits_{i=1}^{m} \lambda_i (\overline{p}_i + A^* \overline{q}_i) = 0$ and

$$\sum_{i=1}^{m} \lambda_i \Big(f_i(\overline{x}) + (g_i \circ A)(\overline{x}) \Big) = \sum_{i=1}^{m} \lambda_i \Big(- f_i^*(\overline{p}_i) - {g_i}^*(\overline{q}_i) \Big).$$

Thus

$$\sum_{i=1}^{m} \lambda_i \Big(f_i(\overline{x}) + (g_i \circ A)(\overline{x}) + f_i^*(\overline{p}_i) + {g_i}^*(\overline{q}_i) \Big) = 0 \iff$$

$$\begin{aligned} 0 \quad = \quad & \sum_{i=1}^{m} \lambda_i \Big(f_i(\overline{x}) + f_i^*(\overline{p}_i) - \overline{p}_i^T \overline{x} \Big) + \\ & \sum_{i=1}^{m} \lambda_i \Big((g_i \circ A)(\overline{x}) + {g_i}^*(\overline{q}_i) - (A^* \overline{q}_i)^T \overline{x} \Big). \end{aligned}$$

But, for each $i \in \{1, ..., m\}$, $f_i(\overline{x}) + f_i^*(\overline{p}_i) - \overline{p}_i^T \overline{x} \geq 0$ and $(g_i \circ A)(\overline{x}) + {g_i}^*(\overline{q}_i) - (A^* \overline{q}_i)^T \overline{x} \geq 0$ due to Fenchel-Young's inequality. Thus we have obtained that a sum of terms, each greater than or equal to zero is zero. Consequently, each of them must be zero. Hence the relations (i), (ii) and (iii) hold.

b) All the calculations and transformations done within part a) may be carried out backwards, starting from the conditions (i), (ii) and (iii). ■

4 The new vector dual problem

By using the results obtained in the previous section, we are now able to formulate a *Fenchel-type multiobjective dual* to (P^A). The dual (D^A) will be a vector maximum problem, therefore efficient solutions in this sense are considered for it. For the primal-dual pair $(P^A) - (D^A)$ weak and strong duality results will be proved.

Let us define (D^A) as being

$$\left(D^A\right) \quad v - \max_{(p,q,\lambda,t)\in\mathcal{B}} h\left(p,q,\lambda,t\right),$$

where

$$\mathcal{B} = \left\{ \begin{array}{ll} (p,q,\lambda,t): & p = (p_1,...,p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, \\ & q = (q_1,...,q_m) \in \mathbb{R}^k \times ... \times \mathbb{R}^k, \\ & \lambda = (\lambda_1,...,\lambda_m)^T \in \operatorname{int}(\mathbb{R}^m_+), \\ & t = (t_1,...,t_m)^T \in \mathbb{R}^m, \\ & \sum\limits_{i=1}^m \lambda_i \left(p_i + A^* q_i\right) = 0, \sum\limits_{i=1}^m \lambda_i t_i = 0 \end{array} \right\}$$

and h is defined by

$$h\left(p,q,\lambda,t\right) = \begin{pmatrix} h_1\left(p,q,\lambda,t\right) \\ ... \\ h_m\left(p,q,\lambda,t\right) \end{pmatrix},$$

with

$$h_i\left(p,q,\lambda,t\right) = -f_i^*\left(p_i\right) - g_i^*\left(q_i\right) + t_i \text{ for all } i \in \{1,...,m\}.$$

Definition 3 *An element* $\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right) \in \mathcal{B}$ *is said to be Pareto-efficient to the* $\left(D^A\right)$ *if from*

$$h\left(p,q,\lambda,t\right) \geqq h\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right) \ \ \textit{for} \ \ (p,q,\lambda,t) \in \mathcal{B}$$

follows that $h\left(p,q,\lambda,t\right) = h\left(\overline{p},\overline{q},\overline{\lambda},\overline{t}\right)$.

Between $\left(P^A\right)$ and $\left(D^A\right)$ the following weak duality assertion holds.

Theorem 4 *(vector weak duality) There exist no* $x \in \mathbb{R}^n$ *and no* $(p,q,\lambda,t) \in \mathcal{B}$ *such that*

$$h\left(p,q,\lambda,t\right) \geqq f\left(x\right) + \left(g \circ A\right)\left(x\right) \ \ \textit{and} \ h\left(p,q,\lambda,t\right) \neq f\left(x\right) + (g \circ A)(x).$$

Proof. We proceed by contradiction, assuming that there exist $x \in \mathbb{R}^n$ and $(p,q,\lambda,t) \in \mathcal{B}$ such that

$$h\left(p,q,\lambda,t\right) \geqq f(x) + \left(g \circ A\right)\left(x\right)$$

and $h\left(p,q,\lambda,t\right) \neq f\left(x\right) + \left(g \circ A\right)\left(x\right)$. This means that

$$h_i\left(p,q,\lambda,t\right) \geq f_i(x) + \left(g_i \circ A\right)\left(x\right) \qquad \forall i \in \{1,...,m\}$$

and that there exists at least one $j \in \{1,...,m\}$ such that

$$h_j\left(p,q,\lambda,t\right) > f_j\left(x\right) + \left(g_j \circ A\right)\left(x\right).$$

Therefore

$$\sum_{i=1}^{m} \lambda_i h_i(p,q,\lambda,t) > \sum_{i=1}^{m} \lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big).$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{m}\lambda_i h_i(p,q,\lambda,t) &= \sum_{i=1}^{m}\lambda_i\Big(-f_i^*(p_i)-g_i^*(q_i)+t_i\Big)\\ &= \sum_{i=1}^{m}\lambda_i\Big(-f_i^*(p_i)-g_i^*(q_i)\Big)+\sum_{i=1}^{m}\lambda_i t_i\\ &= \sum_{i=1}^{m}\lambda_i\Big(-f_i^*(p_i)-g_i^*(q_i)\Big).\end{aligned}$$

Applying again Fenchel-Young's inequality, one has that for each $i\in\{1,...,m\}$ $f_i^*(p_i)\geq p_i^T x - f_i(x)$ and $g_i^*(q_i)\geq (A^*q_i)^T x-(g_i\circ A)(x)$. Further we obtain

$$\begin{aligned}\sum_{i=1}^{m}\lambda_i h_i(p,q,\lambda,t) &\leq \sum_{i=1}^{m}\lambda_i\Big(f_i(x)-p_i^T x+(g_i\circ A)(x)-(A^*q_i)^T x\Big)\\ &= \sum_{i=1}^{m}\lambda_i\Big(f_i(x)+(g_i\circ A)(x)\Big),\end{aligned}$$

which is a contradiction to the strict inequality from above. This concludes the proof. ■

Remark 3. As in the scalar case, for proving the weak duality theorem neither convexity assumptions for the functions involved nor the regularity condition (RC^A) has been used.

Theorem 5 *(vector strong duality) Let* $f_i:\mathbb{R}^n\to\overline{\mathbb{R}}$ *and* $g_i:\mathbb{R}^k\to\overline{\mathbb{R}}$ *be proper and convex functions,* $i\in\{1,...,m\}$*, and* $I,J\subseteq\{1,...,m\}$ *be the sets of indices for which the functions* $f_i, i\in I$*, and* $g_j, j\in J$*, are polyhedral. If the regularity condition* (RC^A) *is fulfilled and* $\overline{x}$ *is a properly efficient solution to* (P^A)*, then there exists an efficient solution* $(\overline{p},\overline{q},\overline{\lambda},\overline{t})$ *to* (D^A) *such that* $f(\overline{x})+(g\circ A)(\overline{x})=h(\overline{p},\overline{q},\overline{\lambda},\overline{t})\in\mathbb{R}^m$.

Proof. Let $\overline{x}$ be a properly efficient solution to (P^A). Then, according to Definition 2, there exists $\overline{\lambda}\in\text{int}(\mathbb{R}^m_+)$ such that $\overline{x}$ is an optimal solution to the scalar optimization problem

$$(P^A_{\bar\lambda}) \quad \inf_{x\in\mathbb{R}^n} \sum_{i=1}^m \bar\lambda_i \Big(f_i(x) + (g_i \circ A)(x) \Big).$$

As we are working under the assumption that $\left(RC^A\right)$ holds, Theorem 3 ensures the existence of an optimal solution to $\left(D^A_{\bar\lambda}\right)$, $(\overline{p}, \overline{q})$ such that the optimality conditions (i), (ii) and (iii) are satisfied. Moreover, as mentioned in Remark 1, $f(\overline{x}) + (g \circ A)(\overline{x}) \in \mathbb{R}^m$. Let us define

$$\bar t_i := \left(\overline{p}_i + A^*\overline{q}_i\right)^T \overline{x} \in \mathbb{R} \quad \text{for all} \quad i \in \{1, ..., m\}.$$

Since

$$\sum_{i=1}^m \overline{\lambda}_i \left(\overline{p}_i + A^*\overline{q}_i\right) = 0 \text{ and } \sum_{i=1}^m \overline{\lambda}_i \bar t_i = \sum_{i=1}^m \lambda_i \left(\overline{p}_i + A^*\overline{q}_i\right)^T \overline{x} = 0,$$

there is $\left(\overline{p}, \overline{q}, \overline{\lambda}, \bar t\right) \in \mathcal{B}$, which means that this element is feasible to $\left(D^A_{\bar\lambda}\right)$.

Moreover, by the optimality conditions (i), (ii) and (iii) from Theorem 3, for each $i \in \{1, ..., m\}$ one has

$$\begin{aligned} h_i\left(\overline{p}, \overline{q}, \overline{\lambda}, \bar t\right) &= -f_i^*\left(\overline{p}_i\right) - g_i^*\left(\overline{q}_i\right) + \bar t_i \\ &= f_i\left(\overline{x}\right) - \overline{p}_i^T \overline{x} + \left(g_i \circ A\right)\left(\overline{x}\right) - \left(A^*\overline{q}_i\right)^T \overline{x} + \bar t_i \\ &= f_i\left(\overline{x}\right) + \left(g_i \circ A\right)\left(\overline{x}\right). \end{aligned}$$

In order to finish the proof it remains to show that $\left(\overline{p}, \overline{q}, \overline{\lambda}, \bar t\right)$ is Pareto-efficient to $\left(D^A\right)$. If this were not the case, then there would exist $(p, q, \lambda, t) \in \mathcal{B}$ such that $h(p, q, \lambda, t) \geqq h\left(\overline{p}, \overline{q}, \overline{\lambda}, \bar t\right)$ and $h(p, q, \lambda, t) \neq h\left(\overline{p}, \overline{q}, \overline{\lambda}, \bar t\right) = f(\overline{x}) + (g \circ A)(\overline{x})$. As this contradicts the weak duality theorem (Theorem 4), it leads to the desired conclusion. ■

Remark 4. In the particular case when $n = 1$ (we denote f_1 and g_1 by f and g, respectively) our dual proves to be exactly the *Fenchel dual problem* (cf. [11]) to the primal scalar optimization problem

$$\inf_{x \in \mathbb{R}^n} (f(x) + g(Ax)).$$

In this case $\lambda_1 > 0$, $t_1 = 0$ and denoting $p := p_1$ and $q := q_1$, the dual becomes

$$\sup_{\substack{p \in \mathbb{R}^n,\, q \in \mathbb{R}^k \\ p + A^* q = 0}} \Big\{ - f^*(p) - g^*(q) \Big\},$$

which is nothing else than

$$\sup_{q \in \mathbb{R}^k} \Big\{ - f^*(-A^* q) - g^*(q) \Big\}.$$

This means that the vector duality concept developed in this section is a natural extension of the classical Fenchel duality.

5 Other two Fenchel-type vector dual problems

In this section we consider two other Fenchel-type dual problems for the primal vector optimization problem (P^A), in the particular case $A : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be the *identical operator*. The two duals have in common the fact that the Fenchel dual of the scalarized primal problem is involved in the formulation of the feasible set. In the following section we study the relationship between these two duals and the problem (D^A). One can notice that these investigations can be easily extended to the general case when A is a linear operator that is not necessarily the identical one. We opted for this special setting due to the simpleness of the calculations.

Consider the primal problem

$$(P) \qquad v\text{-}\min_{x\in\mathbb{R}^n} (f(x)+g(x)),$$

where f and g are as in Section 2 supposing that $n=k$ and $A:\mathbb{R}^n\to\mathbb{R}^n$ is the identical operator. Therefore the regularity condition (RC^A) becomes

$$(RC) \qquad \bigcap_{i\in I}\text{dom}(f_i) \quad \cap \quad \bigcap_{j\in J}\text{dom}(g_j) \quad \cap$$

$$\bigcap_{l\in\{1,\ldots,m\}\setminus I} \text{ri}\,(\text{dom}\,(f_l)) \quad \cap \quad \bigcap_{t\in\{1,\ldots,m\}\setminus J} \text{ri}\,(\text{dom}\,(g_t)) \neq \emptyset.$$

The first vector dual problem we consider here is denoted by (D_1) and is a particular case of the one introduced by Breckner and Kolumbán in [6]. It has the following formulation

$$(D_1) \qquad v-\max_{(\lambda,p,d)\in\mathcal{B}_1} h^1(\lambda,p,d)$$

with the objective function $h^1(\lambda,p,d)=d$ and the feasible set

$$\mathcal{B}_1=\left\{\begin{array}{c}(\lambda,p,d)\in\text{int}(\mathbb{R}^m_+)\times\mathbb{R}^n\times\mathbb{R}^m: \\ \\ \lambda^T d=-\left(\sum\limits_{i=1}^m \lambda_i f_i\right)^*(p)-\left(\sum\limits_{i=1}^m \lambda_i g_i\right)^*(-p)\end{array}\right\}.$$

The weak and strong duality theorems for the vector primal-dual pair $(P)-(D_1)$ are consequences of [6, Proposition 2.1] and [6, Theorem 3.1], respectively.

Theorem 6 *(vector weak duality for (D_1)) There exist no $x\in\mathbb{R}^n$ and no $(\lambda,p,d)\in\mathcal{B}_1$ such that $d\geqq f(x)+g(x)$ and $d\neq f(x)+g(x)$.*

Theorem 7 *(vector strong duality for (D_1)) Let $f_i,g_i:\mathbb{R}^n\to\overline{\mathbb{R}}$ be proper and convex functions, $i\in\{1,\ldots,m\}$, and $I,J\subseteq\{1,\ldots,m\}$ be the sets of indices for* *which the functions $f_i, i \in I$, and $g_j, j \in J$, are polyhedral. If the regularity condition (RC) is fulfilled and $\overline{x} \in \mathbb{R}^n$ is a properly efficient solution to (P), then there exists an efficient solution $\left(\overline{\lambda}, \overline{p}, \overline{d}\right)$ to (D_1) such that $f(\overline{x}) + g(\overline{x}) = \overline{d} \in \mathbb{R}^m$.*

It is worth mentioning that the idea of considering in the formulation of the feasible set of the vector dual problem the dual optimization problem of the scalarized primal problem can be also found by Jahn in [8]. In this paper the author provides for the multiobjective problem with geometric and cone constraints a vector dual by employing in the formulation of the feasible set of the latter the *Lagrange dual* of the scalarized primal problem. Different to Breckner and Kolumbán, for Jahn's dual the *equality* is replaced by an *inequality*. Following the same scheme we introduce a further vector dual problem to (P) as being

$$(D_2) \qquad v - \max_{(\lambda,p,d)\in\mathcal{B}_2} h^2(\lambda, p, d)$$

with the objective function $h^2(\lambda, p, d) = d$ and the feasible set

$$\mathcal{B}_2 = \left\{ \begin{array}{c} (\lambda, p, d) \in \operatorname{int}(\mathbb{R}^m_+) \times \mathbb{R}^n \times \mathbb{R}^m : \\ \\ \lambda^T d \leq -\left(\sum\limits_{i=1}^m \lambda_i f_i\right)^*(p) - \left(\sum\limits_{i=1}^m \lambda_i g_i\right)^*(-p) \end{array} \right\}.$$

The weak and strong duality theorems for the primal-dual pair $(P) - (D_2)$ follows.

Theorem 8 *(vector weak duality for (D_2)) There exist no $x \in \mathbb{R}^n$ and no $(\lambda, p, d) \in \mathcal{B}_2$ such that $d \geqq f(x) + g(x)$ and $d \neq f(x) + g(x)$.*

Theorem 9 *(vector strong duality for (D_2)) Let $f_i, g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper and convex functions, $i \in \{1, ..., m\}$, and $I, J \subseteq \{1, ..., m\}$ be the sets of indices for which the functions $f_i, i \in I$, and $g_j, j \in J$, are polyhedral. If the regularity condition (RC) is fulfilled and $\overline{x} \in \mathbb{R}^n$ is a properly efficient solution to (P), then there exists an efficient solution $\left(\overline{\lambda}, \overline{p}, \overline{d}\right)$ to (D_2) such that $f\left(\overline{x}\right) + g\left(\overline{x}\right) = \overline{d} \in \mathbb{R}^m$.*

We omit giving the proofs of the last two theorems, as Theorem 8 is an easy consequence of Theorem 6, while Theorem 9 follows from Theorem 7 and Theorem 13 (the proof of the latter will be given in the next section).

6 A comparison of the image sets of the vector duals

The aim of this section is to provide some relations between the image sets of the vector duals (D), (D_1) and (D_2) of the problem (P), where the first one is (take in the formulation of (D^A) $n = k$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ the identical operator)

$$(D) \quad v - \max_{(p,q,\lambda,t) \in \mathcal{B}} h\left(p, q, \lambda, t\right),$$

where

$$\mathcal{B} = \left\{ \begin{array}{ll} (p, q, \lambda, t) : & p = (p_1, ..., p_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, \\ & q = (q_1, ..., q_m) \in \mathbb{R}^n \times ... \times \mathbb{R}^n, \\ & \lambda = (\lambda_1, ..., \lambda_m)^T \in \operatorname{int}(\mathbb{R}^m_+), \\ & t = (t_1, ..., t_m)^T \in \mathbb{R}^m, \\ & \sum\limits_{i=1}^m \lambda_i \left(p_i + q_i\right) = 0, \sum\limits_{i=1}^m \lambda_i t_i = 0 \end{array} \right\}$$

and h is defined by

$$h(p,q,\lambda,t) = \begin{pmatrix} h_1(p,q,\lambda,t) \\ \dots \\ h_m(p,q,\lambda,t) \end{pmatrix},$$

with

$$h_i(p,q,\lambda,t) = -f_i^*(p_i) - g_i^*(q_i) + t_i \text{ for all } i \in \{1,...,m\}.$$

Throughout this section we assume that $f_i, g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ are proper and convex for $i \in \{1,...,m\}$ and that $I, J \subseteq \{1,...,m\}$ are the sets of indices for which the functions $f_i, i \in I$, and $g_j, j \in J$, are polyhedral. Furthermore, we assume that the regularity condition (RC) is satisfied.

Proposition 10 *The following relations among the image sets of the three duals hold:*

$$h^1(\mathcal{B}_1) \subseteq h(\mathcal{B}) \cap \mathbb{R}^m \subseteq h^2(\mathcal{B}_2).$$

Proof. We start with the first relation of inclusion. Let $d \in h^1(\mathcal{B}_1)$. Then there exists $\lambda \in \text{int}(\mathbb{R}^m_+)$ and $p \in \mathbb{R}^n$ such that $(\lambda, p, d) \in \mathcal{B}_1$. Furthermore,

$$\sum_{i=1}^m \lambda_i d_i = -\left(\sum_{i=1}^m \lambda_i f_i\right)^*(p) - \left(\sum_{i=1}^m \lambda_i g_i\right)^*(-p).$$

Since (RC) is fulfilled, we apply [11, Theorem 20.1], obtaining in this way the existence of $p_i \in \mathbb{R}^n, q_i \in \mathbb{R}^n$, $i \in \{1,...,m\}$, such that $\sum\limits_{i=1}^m \lambda_i p_i = p$, $\sum\limits_{i=1}^m \lambda_i q_i = -p$ and

$$\lambda^T d = -\sum_{i=1}^m \lambda_i f_i^*(p_i) - \sum_{i=1}^m \lambda_i g_i^*(q_i).$$

For $t_i := d_i + f_i^*(p_i) + g_i^*(q_i)$ for all $i \in \{1,...,m\}$ we have that $\sum\limits_{i=1}^m \lambda_i(p_i+q_i) = 0$ and $\sum\limits_{i=1}^m \lambda_i t_i = 0$. Thus $d = h(p,q,\lambda,t) \in h(\mathcal{B}) \cap \mathbb{R}^m$ and so $h^1(\mathcal{B}_1) \subseteq h(\mathcal{B}) \cap \mathbb{R}^m$.

We come now to the second relation of inclusion. Let $(p, q, \lambda, t) \in \mathcal{B}$ be such that $h(p, q, \lambda, t) \in h(\mathcal{B}) \cap \mathbb{R}^m$. For $\overline{p} := \sum\limits_{i=1}^{m} \lambda_i p_i$ and $d := h(p, q, \lambda, t)$ we have

$$\lambda^T d = \lambda^T h(p, q, \lambda, t) = \sum_{i=1}^{m} \lambda_i \left(-f_i^*(p_i) - g_i^*(q_i)\right)$$
$$\leq - \left(\sum_{i=1}^{m} \lambda_i f_i\right)^* (\overline{p}) - \left(\sum_{i=1}^{m} \lambda_i g_i\right)^* (-\overline{p}).$$

Hence $(\lambda, \overline{p}, d) \in \mathcal{B}_2$ and $h(p, q, \lambda, t) = d \in h^2(\mathcal{B}_2)$. Thus $h(\mathcal{B}) \cap \mathbb{R}^m \subseteq h^2(\mathcal{B}_2)$.

■

In the following we give two examples which prove that the inclusions among the image sets in Proposition 10 are in general strict, i.e.

$$h^1(\mathcal{B}_1) \underset{\neq}{\subset} h(\mathcal{B}) \cap \mathbb{R}^m \underset{\neq}{\subset} h^2(\mathcal{B}_2).$$

Example 11 Consider the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(x) = (x - 1, -x - 1)^T \quad \text{and} \quad g(x) = (x, -x)^T \text{ for all } x \in \mathbb{R}.$$

We prove that $h(\mathcal{B}) \cap \mathbb{R}^m \underset{\neq}{\subset} h^2(\mathcal{B}_2)$.

For $\lambda = (1, 1)^T$, $p = 0$ and $d = (-2, -2)^T$, there is $(\lambda, p, d) \in \mathcal{B}_2$ and $d \in h^2(\mathcal{B}_2)$, since

$$\lambda^T d = -2 - 2 = -4 < -2 = -(f_1 + f_2)^*(p) - (g_1 + g_2)^*(-p).$$

We show now that $d \notin h(\mathcal{B})$. Let us suppose by contradiction that there exists $(p', q', \lambda', t') \in \mathcal{B}$ such that $h(p', q', \lambda', t') = d$. This means that

$$h_i(p', q', \lambda', t') = -f_i^*\left(p_i'\right) - g_i^*\left(q_i'\right) + t_i' = -2 \text{ for } i \in \{1, 2\}.$$

and one must necessarily have that $p_1' = 1, p_2' = -1, q_1' = 1$ and $q_2' = -1$. Moreover, $\sum\limits_{i=1}^{2} \lambda_i' (p_i' + q_i') = 0$, which means that $\lambda_1' - \lambda_2' = 0$. We obtain $-f_i^*\left(p_i'\right) - g_i^*\left(q_i'\right) + t_i' = -1 + t_i' = -2$ for $i \in \{1, 2\}$, meaning that $t_1' = t_2' = -1$.

Since we have supposed that $(p', q', \lambda', t') \in \mathcal{B}$, the equality $\sum_{i=1}^{2} \lambda'_i t'_i = -\lambda'_1 - \lambda'_2 = -2\lambda'_1$ must hold. But this is a contradiction to the fact that $\lambda' \in \mathrm{int}(\mathbb{R}^2_+)$.

Consequently, for $d = (-2,-2)^T \in h^2(\mathcal{B}_2)$, there exists no $(p', q', \lambda', t') \in \mathcal{B}$ such that $h(p', q', \lambda', t') = d$, which shows that $h(\mathcal{B}) \cap \mathbb{R}^m \underset{\neq}{\subset} h^2(\mathcal{B}_2)$.

Example 12 Consider now the functions $f, g : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(x) = \left(2x^2 - 1, x^2\right)^T \quad \text{and} \quad g(x) = (-2x, -x+1)^T \text{ for all } x \in \mathbb{R}.$$

We prove that $h^1(\mathcal{B}_1) \underset{\neq}{\subset} h(\mathcal{B}) \cap \mathbb{R}^m$.

For $p = (3,0)\,, q = (-2,-1)\,,\ \lambda = (1,1)^T$ and $t = \left(\frac{3}{8}, -\frac{3}{8}\right)^T$ we have both relations $\sum_{i=1}^{2} \lambda_i (p_i + q_i) = 0$ and $\sum_{i=1}^{2} \lambda_i t_i = 0$ fulfilled. Thus $(p, q, \lambda, t) \in \mathcal{B}$ and $h(p, q, \lambda, t) = \left(-\frac{14}{8}, \frac{5}{8}\right)^T \in h(\mathcal{B}) \cap \mathbb{R}^m$.

Suppose now that there exists $(\lambda', p', d') \in \mathcal{B}_1$ such that $d' = h(p, q, \lambda, t) = \left(-\frac{14}{8}, \frac{5}{8}\right)^T$. Then

$$\lambda'^T d = \inf_{x \in \mathbb{R}} \left\{-p'x + x^2\left(2\lambda'_1 + \lambda'_2\right) - \lambda'_1\right\} + \inf_{x \in \mathbb{R}} \left\{x\left(p' - 2\lambda'_1 - \lambda'_2\right) + \lambda'_2\right\}.$$

This means that

$$-\frac{14}{8}\lambda'_1 + \frac{5}{8}\lambda'_2 = -\frac{2\lambda'_1 + \lambda'_2}{4} - \lambda'_1 + \lambda'_2$$

which is equivalent to $2\lambda'_1 + \lambda'_2 = 0$, obviously a contradiction to $\lambda' \in \mathrm{int}(\mathbb{R}^2_+)$. Therefore there exists no $(\lambda', p', d') \in \mathcal{B}_1$ such that $d' = h(p, q, \lambda, t)$. Hence $h^1(\mathcal{B}_1) \underset{\neq}{\subset} h(\mathcal{B}) \cap \mathbb{R}^m$.

In what follows, we study the relations among the sets of maximal elements of the image sets treated in this section. They are defined as

$$v - \max h(\mathcal{B}) = \left\{ \begin{array}{c} d \in \mathbb{R}^m : \exists (p, q, \lambda, t) \in \mathcal{B} \text{ efficient to } (D) \\ \text{such that } \ d = h(p, q, \lambda, t) \end{array} \right\}$$

for the problem (D), while $v-\max h^1(\mathcal{B}_1)$ and $v-\max h^2(\mathcal{B}_2)$, respectively, are defined analogously.

Theorem 13 *It holds* $v-\max h^1(\mathcal{B}_1)=v-\max h^2(\mathcal{B}_2)$.

Proof. "$v-\max h^1(\mathcal{B}_1)\subseteq v-\max h^2(\mathcal{B}_2)$ " Let $\left(\overline{\lambda},\overline{p},\overline{d}\right)\in\mathcal{B}_1$ be such that $\overline{d}\in v-\max h^1(\mathcal{B}_1)$. We suppose that $\overline{d}\notin v-\max h^2(\mathcal{B}_2)$. Since $\overline{d}\in h^2(\mathcal{B}_2)$, there exists $(\lambda,p,d)\in\mathcal{B}_2$ such that $d\geqq\overline{d}$ and $d\neq\overline{d}$. Further, having that $(\lambda,p,d)\in\mathcal{B}_1$ this would contradict $\overline{d}\in v-\max h^1(\mathcal{B}_1)$. So $(\lambda,p,d)\notin\mathcal{B}_1$, which means

$$\lambda^T d<-\left(\sum_{i=1}^m\lambda_i f_i\right)^*(p)-\left(\sum_{i=1}^m\lambda_i g_i\right)^*(-p).$$

Thus there exists $\widetilde{d}\in d+\mathbb{R}^m_+\backslash\{0\}$ fulfilling

$$\lambda^T\widetilde{d}=-\left(\sum_{i=1}^m\lambda_i f_i\right)^*(p)-\left(\sum_{i=1}^m\lambda_i g_i\right)^*(-p).$$

It follows that $\left(\lambda,p,\widetilde{d}\right)\in\mathcal{B}_1$. But in this case $\widetilde{d}\geqq\overline{d}$ and $\widetilde{d}\neq\overline{d}$, which is a contradiction to the maximality of $\overline{d}$ in $h^1(\mathcal{B}_1)$. Therefore we must have

$$v-\max h^1(\mathcal{B}_1)\subseteq v-\max h^2(\mathcal{B}_2).$$

"$v-\max h^2(\mathcal{B}_2)\subseteq v-\max h^1(\mathcal{B}_1)$". Let $\left(\overline{\lambda},\overline{p},\overline{d}\right)\in\mathcal{B}_2$ be such that $\overline{d}\in v-\max h^2(\mathcal{B}_2)$. We start by proving that $\left(\overline{\lambda},\overline{p},\overline{d}\right)\in\mathcal{B}_1$. Assuming the contrary, one has $\overline{\lambda}^T\overline{d}<-\left(\sum\limits_{i=1}^m\overline{\lambda}_i f_i\right)^*(\overline{p})-\left(\sum\limits_{i=1}^m\overline{\lambda}_i g_i\right)^*(-\overline{p})$. Therefore there exists $\widetilde{d}\in\overline{d}+\mathbb{R}^m_+\backslash\{0\}$, such that

$$\overline{\lambda}^T\overline{d}<\overline{\lambda}^T\widetilde{d}=-\left(\sum_{i=1}^m\overline{\lambda}_i f_i\right)^*(\overline{p})-\left(\sum_{i=1}^m\overline{\lambda}_i g_i\right)^*(-\overline{p}).$$

As $\left(\overline{\lambda},\overline{p},\widetilde{d}\right)\in\mathcal{B}_2$ and $\widetilde{d}\in\overline{d}+\mathbb{R}^m_+\backslash\{0\}$ we have obtained a contradiction to the maximality of $\overline{d}$ in $h^2(\mathcal{B}_2)$ for (D_2). This means that $\overline{d}\in h_1(\mathcal{B}_1)$.

Let us suppose now that $\overline{d} \notin v-\max h^1(\mathcal{B}_1)$. Then there exists $(\lambda, p, d) \in \mathcal{B}_1$ such that $d \geqq \overline{d}$ and $d \neq \overline{d}$. Since $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $d \in h^2(\mathcal{B}_2)$ this provides a contradiction to the maximality of $\overline{d}$ in $h^2(\mathcal{B}_2)$. Thus $v - \max h^2(\mathcal{B}_2) \subseteq v - \max h^1(\mathcal{B}_1)$. ∎

Remark 5. Let us emphasize the fact that in the proof of Theorem 13 neither the convexity assumptions on the functions involved nor the regularity condition (RC) has been used. In conclusion the sets of maximal elements of the problems (D_1) and (D_2) are always identical.

By Proposition 10 we have that the set $h(\mathcal{B}) \cap \mathbb{R}^m$ is placed in between the sets $h^1(\mathcal{B}_1)$ and $h^2(\mathcal{B}_2)$. Since, as proved above, the sets of the maximal elements of the latter coincide, one can easily conclude that

$$v - \max h^1(\mathcal{B}_1) = v - \max h(\mathcal{B}) = v - \max h^2(\mathcal{B}_2).$$

This happens even though in general $h^1(\mathcal{B}_1) \underset{\neq}{\subset} h(\mathcal{B}) \cap \mathbb{R}^m \underset{\neq}{\subset} h^2(\mathcal{B}_2)$.

Acknowledgements. The authors are indebted to an anonymous reviewer for valuable suggestions and comments which improved the quality of the paper.

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