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Generalized Moreau-Rockafellar results for composed convex functions

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We give two generalized Moreau-Rockafellar-type results for the sum of a convex function with a composition of convex functions in separated locally convex spaces. Then we equivalently characterize the stable strong duality for composed convex optimization problems through two new regularity conditions, which also guarantee two formulae of the subdifferential of the mentioned sum of functions. We also treat some special cases, rediscovering older results in the literature. A discussion on the topological assumptions for the vector function used in the composition closes the paper.

Keywords: conjugate functions, Moreau-Rockafellar results, regularity conditions, stable strong duality, composed convex functions

AMS Subject Classification: 90C25; 49N15; 65K10

1. Introduction

With this paper we cover some missing results concerning composed convex functions in convex analysis. Our scope with this article is twofold. First we give two generalized Moreau-Rockafellar-type results for composed functions of type $f+g\circ h$. Depending on how much we perturb this function, we obtain two formulae for its conjugate. For a simpler perturbation function, with a single perturbation variable in the argument of g, f and h remain coupled in the final formula. On the other hand, when using a more complicated perturbation function, with two perturbation variables, i.e. perturbing the arguments of both f and g, f and h are separated in the formula of $(f+g\circ h)^*$. Note that these formulae are always valid, not requiring the fulfillment of any sufficient condition. By using these formulae we prove also some characterizations of the epigraph of $(f+g\circ h)^*$. The other aim of the paper is to equivalently characterize two known formulae for the conjugate of $f+g\circ h$ through closedness-type regularity conditions involving epigraphs, which are sufficient to yield two formulae for $\partial(f+g\circ h)$, too. The formulae for $(f+g\circ h)^*$ can be seen also as stable strong duality statements. When particularizing the functions involved we rediscover older Moreau-Rockafellar-type results known in the literature, including the classical one, respectively stable strong duality statements for the Fenchel, Lagrange and Fenchel-Lagrange duals. Moreover, we give formulae for the conjugate of the supremum of infinitely many functions and the epigraph of this conjugate.

We work in separated locally convex spaces and the functions f and g are taken proper, convex and lower semicontinuous, with g also C-increasing, while h is

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considered to be proper, C-convex and star C-lower semicontinuous. We gave in [1] similar characterizations for $(f + g \circ h)^*$, but there we have considered for h a more relaxed topological property, namely C-epi-closedness. We observe that the regularity conditions that equivalently characterize the mentioned formulae in [1] are stronger than the ones obtained here, thus we end this paper with a discussion on the topological hypothesis for h, where we formulate also an open problem for the reader.

2. Preliminaries

Consider two separated locally convex vector spaces X and Y and their topological dual spaces X^* and Y^*, endowed with the corresponding weak* topologies, and denote by $\langle x^*, x\rangle = x^*(x)$ the value at $x \in X$ of the linear continuous functional $x^* \in X^*$. Take Y to be partially ordered by the nonempty closed convex cone C, i.e. on Y there is the partial order "$\leq_C$", defined by $z \leq_C y \Leftrightarrow y - z \in C$, $z, y \in Y$. To Y we attach a greatest element with respect to "$\leq_C$" which does not belong to Y, denoted by ∞_Y and let $Y^\bullet = Y \cup \{\infty_Y\}$. Then for any $y \in Y^\bullet$ one has $y \leq_C \infty_Y$ and we consider on $Y^\bullet$ the following operations: $y + \infty_Y = \infty_Y + y = \infty_Y$ and $t \cdot \infty_Y = \infty_Y$ for all $y \in Y$ and all $t \geq 0$. A function $g : Y^\bullet \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is said to be C-increasing if $g(\infty_Y) = +\infty$ and for $y, z \in Y^\bullet$ such that $z \leq_C y$ one has $g(z) \leq g(y)$. The dual cone of C is $C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0\ \forall y \in C\}$. By convention, let $\langle y^*, \infty_Y\rangle = +\infty$ whenever $y^* \in C^*$.

Given a subset U of X, by $|U|$, $\mathrm{cl}(U)$, $\mathrm{co}(U)$, δ_U and σ_U we denote its *cardinality*, its *closure*, its *convex hull*, its *indicator function* and *support function*, respectively. We use also the *projection function* $\mathrm{Pr}_X : X \times Y \to X$, defined by $\mathrm{Pr}_X(x, y) = x$ $\forall (x, y) \in X \times Y$ and the *identity function* on X, $\mathrm{id}_X : X \to X$ with $\mathrm{id}_X(x) = x$ $\forall x \in X$.

For a function $f : X \to \overline{\mathbb{R}}$ we use the classical notations for *domain* $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$, *epigraph* $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$, *conjugate function regarding the set* $U \subseteq X$ $f^*_U : X^* \to \overline{\mathbb{R}}$, $f^*_U(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in U\}$ and *subdifferential* at x, where $f(x) \in \mathbb{R}$, $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle\ \forall y \in X\}$. Between a function and its conjugate regarding some set $U \subseteq X$ there is *Young's inequality* $f^*_U(x^*) + f(x) \geq \langle x^*, x\rangle$ $\forall x \in U$ $\forall x^* \in X^*$. When $U = X$ the conjugate regarding the set U is actually the classical (Fenchel-Moreau) conjugate function of f denoted by f^*. We call f *proper* if $f(x) > -\infty$ $\forall x \in X$ and $\mathrm{dom}(f) \neq \emptyset$. Considering for each $\lambda \in \mathbb{R}$ the function $(\lambda f) : X \to \overline{\mathbb{R}}$, $(\lambda f)(x) = \lambda f(x)$ $\forall x \in X$, note that when $\lambda = 0$ we take $(0f) = \delta_{\mathrm{dom}(f)}$. Given two proper functions $f, g : X \to \overline{\mathbb{R}}$, we have the *infimal convolution* of f and g defined by $f \Box g : X \to \overline{\mathbb{R}}$, $\big(f \Box g\big)(a) = \inf\{f(x) + g(a - x) : x \in X\}$. The *lower semicontinuous hull* of f is $\mathrm{cl}(f) : X \to \overline{\mathbb{R}}$, the function which has as epigraph $\mathrm{cl}(\mathrm{epi}(f))$, and the *lower semicontinuous convex hull* of f is $\mathrm{cl}(\mathrm{co}(f)) : X \to \overline{\mathbb{R}}$, the function which has as epigraph $\mathrm{cl}(\mathrm{co}(\mathrm{epi}(f)))$. The conjugate function of the conjugate of a function $f : X \to \overline{\mathbb{R}}$ is said to be the *biconjugate* of f and it is denoted by $f^{**} : X \to \overline{\mathbb{R}}$, $f^{**}(x) = \sup\{\langle x^*, x\rangle - f^*(x^*) : x^* \in X^*\}$.

Lemma 2.1: *(Fenchel-Moreau) Let* $f : X \to \overline{\mathbb{R}}$ *be a convex function such that* $\mathrm{cl}(f)$ *is proper. Then* $f^{**} = \mathrm{cl}(f)$.

There are notions given for functions with extended real values that can be generalized also for functions having their ranges in infinite dimensional spaces.

For a function $h : X \to Y^\bullet$ one has

- the *domain*: $\mathrm{dom}(h) = \{x \in X : h(x) \in Y\}$,

· h is *proper*: $\text{dom}(h) \neq \emptyset$,
· h is *C-convex*: $h(tx + (1-t)y) \leq_C th(x) + (1-t)h(y)$ $\forall x, y \in X$ $\forall t \in [0,1]$,
· the *C-epigraph* $\text{epi}_C(h) = \{(x,y) \in X \times Y : y \in h(x) + C\}$,
· h is C-epi-*closed* if $\text{epi}_C(h)$ is closed (cf. [8]),
· h is *star C-lower semicontinuous*: $(\lambda h) : X \to \overline{\mathbb{R}}$, $(\lambda h)(x) = \langle \lambda, h(x) \rangle$, $x \in X$, is lower semicontinuous $\forall \lambda \in C^*$ (cf. [7]).

Remark 1 : There are other extensions of lower semicontinuity for functions taking values in infinite dimensional spaces used in convex optimization. We mention here just the C-lower semicontinuity, introduced in [9] and refined in [6]. When a function is C-lower semicontinuous it is automatically star C-lower semicontinuous, too, and every star C-lower semicontinuous function is also C-epi-closed. The reverse statements do not hold in general (see [3, 9]).

Given a linear continuous mapping $A : X \to Y$, we have its *adjoint* $A^* : Y^* \to X^*$ given by $\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle$ for any $(x, y^*) \in X \times Y^*$. For the proper function $f : X \to \overline{\mathbb{R}}$ we define also the *infimal function of f through A* as $Af : Y \to \overline{\mathbb{R}}$, $Af(y) = \inf\{f(x) : x \in X, Ax = y\}$, $y \in Y$.

Given a function $\Phi : X \times Y \to \overline{\mathbb{R}}$, the *infimal value function* of its conjugate is $\eta_\Phi : X^* \to \overline{\mathbb{R}}$, $\eta_\Phi(x^*) = \inf_{y^* \in Y^*} \Phi^*(x^*, y^*)$. Since Φ^* is convex, η_Φ is convex, too. We give now a result which can be obtained from [10] and plays an important role in proving the main statements in this paper.

Theorem 2.2 : *Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex lower semicontinuous function with $0 \in \text{Pr}_Y(\text{dom}(\Phi))$. For each $x^* \in X^*$ one has*

$$(\Phi(\cdot,0))^*(x^*) = \sup_{x \in X}\{\langle x^*, x \rangle - \Phi(x,0)\} = \text{cl}\Big(\inf_{y^* \in Y^*} \Phi^*(\cdot, y^*)\Big)(x^*). \tag{1}$$

Proof: First we determine the conjugate of η_Φ. For all $x \in X$ there is

$$\begin{aligned}\eta_\Phi^*(x) &= \sup_{x^* \in X^*}\Big\{\langle x^*, x \rangle - \inf_{y^* \in Y^*} \Phi^*(x^*, y^*)\Big\} \\ &= \sup_{\substack{x^* \in X^*,\\ y^* \in Y^*}}\{\langle x^*, x \rangle - \Phi^*(x^*, y^*)\} = \Phi^{**}(x,0).\end{aligned}$$

As Φ is proper, convex and lower semicontinuous, we get further $\eta_\Phi^*(x) = \Phi(x,0)$ $\forall x \in X$. Let us prove now that $\text{cl}(\eta_\Phi)$ is proper. Assuming that it takes everywhere the value $+\infty$ we obtain that its conjugate, which coincides with η_Φ^*, is everywhere $-\infty$. This contradicts the properness of Φ. The other possibility of $\text{cl}(\eta_\Phi)$ to be improper is to take somewhere the value $-\infty$. Because $0 \in \text{Pr}_Y(\text{dom}(\Phi))$, there is some $x_0 \in X$ such that $\Phi(x_0, 0) < +\infty$. By Young's inequality one has $\eta_\Phi(x^*) = \inf_{y^* \in Y^*} \Phi^*(x^*, y^*) \geq \langle x^*, x_0 \rangle - \Phi(x_0, 0)$ $\forall x^* \in X^*$. As $\langle \cdot, x_0 \rangle - \Phi(x_0, 0)$ is a continuous function we get $\text{cl}(\eta_\Phi)(x^*) \geq \langle x^*, x_0 \rangle - \Phi(x_0, 0) > -\infty$ $\forall x^* \in X^*$. Consequently, $\text{cl}(\eta_\Phi)$ is everywhere greater than $-\infty$, therefore it is proper.

The first equality in (1) arises from the definition of the conjugate function. To obtain the second one we apply Lemma 2.1 for η_Φ. From the calculations above one gets $\eta_\Phi^{**} = (\Phi(\cdot, 0))^*$ and we are done. □

A consequence of this statement follows, by giving similar characterizations for the epigraphs of the functions involved in (1).

Theorem 2.3 : *Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex lower semicontinuous* *function with* $0 \in \Pr_Y(\mathrm{dom}(\Phi))$. *Then*

$$\mathrm{epi}((\Phi(\cdot,0))^*) = \mathrm{cl}\left(\mathrm{epi}\left(\inf_{y^*\in Y^*} \Phi^*(\cdot,y^*)\right)\right) = \mathrm{cl}\left(\bigcup_{y^*\in Y^*} \mathrm{epi}(\Phi^*(\cdot,y^*))\right).$$

Proof: Whenever $(x^*,r) \in \cup_{y^*\in Y^*} \mathrm{epi}(\Phi^*(\cdot,y^*))$ it is clear that $\inf_{y^*\in Y^*} \Phi^*(x^*, y^*) \le r$, thus $(x^*,r) \in \mathrm{epi}\,(\inf_{y^*\in Y^*} \Phi^*(\cdot,y^*))$.

If $(x^*,r) \in \mathrm{epi}\,(\inf_{y^*\in Y^*} \Phi^*(\cdot,y^*))$, then for each $\varepsilon > 0$ there is an $y^* \in Y^*$ such that $\Phi^*(x^*,y^*) \le r+\varepsilon$. Thus $(x^*,r+\varepsilon) \in \cup_{y^*\in Y^*} \mathrm{epi}(\Phi^*(\cdot,y^*))$ $\forall \varepsilon > 0$, which yields $(x^*,r) \in \mathrm{cl}\,(\cup_{y^*\in Y^*} \mathrm{epi}(\Phi^*(\cdot,y^*)))$. Then we get

$$\bigcup_{y^*\in Y^*} \mathrm{epi}(\Phi^*(\cdot,y^*)) \subseteq \mathrm{epi}\left(\inf_{y^*\in Y^*} \Phi^*(\cdot,y^*)\right) \subseteq \mathrm{cl}\left(\bigcup_{y^*\in Y^*} \mathrm{epi}(\Phi^*(\cdot,y^*))\right),$$

which implies that the closures of these two sets coincide. Since the previous theorem yields $\mathrm{epi}((\Phi(\cdot,0))^*) = \mathrm{cl}\,(\mathrm{epi}\,(\inf_{y^*\in Y^*} \Phi^*(\cdot,y^*)))$, we are done. □

From these two theorems we can get the following statement, given also in [5].

Lemma 2.4: *Let* $\Phi : X \times Y \to \overline{\mathbb{R}}$ *be a proper convex lower semicontinuous function with* $0 \in \Pr_Y(\mathrm{dom}(\Phi))$. *Then* $\Pr_{X^*\times\mathbb{R}}(\mathrm{epi}(\Phi^*))$ *is closed if and only if*

$$\sup_{x\in X}\{\langle x^*,x\rangle - \Phi(x,0)\} = \min_{y^*\in Y^*} \Phi^*(x^*,y^*) \;\forall x^* \in X^*.$$

Let us mention that for an attained infimum (supremum) instead of inf (sup) we write min (max).

3. Moreau-Rockafellar results for composed functions

The main results in this paper are given for the following framework. Consider the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$, the proper convex lower semicontinuous C-increasing function $g : Y^\bullet \to \overline{\mathbb{R}}$ with $g(\infty_Y) = +\infty$ and the proper C-convex star C-lower semicontinuous function $h : X \to Y^\bullet$. We impose moreover the feasibility condition $(h(\mathrm{dom}(f)) + C) \cap \mathrm{dom}(g) \neq \emptyset$.

Remark 1: Since $g : Y \to \overline{\mathbb{R}}$ is C-increasing, $g^*(y^*) = +\infty$ $\forall y^* \notin C^*$.

We formulate and prove two generalized Moreau-Rockafellar-type formulae involving composed functions, namely the conjugate function of $f + g \circ h$. To this end, we attach to $f + g \circ h$ a so-called *perturbation function* $\Phi : X \times Z \to \overline{\mathbb{R}}$, where Z is a separated locally convex space, which is a function that fulfills $\Phi(x,0) = (f + g \circ h)(x)$ $\forall x \in X$. We use the results introduced earlier for two different perturbation functions attached to $f + g \circ h$.

Take first the perturbation function

$$\Phi_1 : X \times Y \to \overline{\mathbb{R}},\; \Phi_1(x,y) = f(x) + g(h(x) - y).$$

It is proper and convex and its conjugate function turns out, via Remark **1** from this section, to be

$$\Phi_1^* : X^* \times Y^* \to \overline{\mathbb{R}},\; \Phi_1^*(x^*,y^*) = g^*(-y^*) + (f + (-y^*h))^*(x^*) \;\forall (x^*,y^*) \in X^* \times -C^*,$$

and $\Phi_1^*(x^*,y^*) = +\infty$ otherwise. Then the biconjugate of Φ_1 is $\Phi_1^{**} : X \times Y \to \overline{\mathbb{R}}$, which at each pair $(x,y) \in X \times Y$ takes the value (by Lemma 2.1 and Remark **1** from this section)

$$\begin{aligned}
\Phi_1^{**}(x,y) &= \sup_{(x^*,y^*)\in X^*\times -C^*} \{\langle x^*,x\rangle + \langle y^*,y\rangle - g^*(-y^*) - (f+(-y^*h))^*(x^*)\} \\
&= \sup_{y^*\in -C^*} \Big\{\langle y^*,y\rangle - g^*(-y^*) + \sup_{x^*\in X^*}\{\langle x^*,x\rangle - (f+(-y^*h))^*(x^*)\}\Big\} \\
&= \sup_{y^*\in -C^*} \{\langle y^*,y\rangle - g^*(-y^*) + (f+(-y^*h))^{**}(x)\} \\
&= \sup_{y^*\in C^*} \{\langle y^*,-y\rangle - g^*(y^*) + (f+(y^*h))(x)\} \\
&= f(x) + \sup_{y^*\in C^*} \{\langle y^*,-y\rangle - g^*(y^*) + (y^*h)(x)\} \\
&= f(x) + \sup_{y^*\in C^*} \{\langle y^*,h(x)-y\rangle - g^*(y^*)\} \\
&= f(x) + g^{**}(h(x)-y) = f(x) + g(h(x)-y) = \Phi_1(x,y). \qquad (2)
\end{aligned}$$

Here the assumption h star C-lower semicontinuous is essential for obtaining the fourth equality, because this hypothesis ensures the lower semicontinuity of $f+(y^*h)$ for all $y^*\in C^*$. This function is thus proper, convex and lower semicontinuous, therefore it coincides with its biconjugate. Note that (2) yields that the function Φ_1 is lower semicontinuous, too. As it is also proper and convex, using Theorem 2.2 we obtain the first Moreau-Rockafellar-type formula for $(f+g\circ h)^*$.

Theorem 3.1 : *One has*

$$(f+g\circ h)^* = \mathrm{cl}\Big(\inf_{y^*\in C^*}\{g^*(y^*) + (f+(y^*h))^*(\cdot)\}\Big). \qquad (3)$$

Corollary 3.2: *It holds*

$$\begin{aligned}
\mathrm{epi}((f+g\circ h)^*) &= \mathrm{cl}\Big(\mathrm{epi}\Big(\inf_{y^*\in C^*}\{g^*(y^*) + (f+(y^*h))^*(\cdot)\}\Big)\Big) \\
&= \mathrm{cl}\Big(\bigcup_{y^*\in \mathrm{dom}(g^*)}\big((0,g^*(y^*)) + \mathrm{epi}((f+(y^*h))^*)\big)\Big).
\end{aligned}$$

Proof: The first equality follows directly from Theorem 3.1. For the second one, we use Theorem 2.3. We have $(x^*,r)\in \cup_{y^*\in Y^*}\mathrm{epi}(\Phi_1^*(\cdot,y^*))$ if and only if there is some $y^*\in Y^*$ such that $\Phi_1^*(x^*,y^*)\le r$, which is nothing but the existence of a $y^*\in -C^*$ for which $g^*(-y^*)+(f+(-y^*h))^*(x^*)\le r$. Using Remark **1** from this section, this turns out to be equivalent to the existence of a $y^*\in\mathrm{dom}(g^*)$ fulfilling $g^*(y^*)+(f+(y^*h))^*(x^*)\le r$, inequality meaning actually that $(x^*,r-g^*(y^*))\in \mathrm{epi}((f+(y^*h))^*)$. This can be rewritten as $(x^*,r)\in(0,g^*(y^*))+\mathrm{epi}((f+(y^*h))^*)$. By Theorem 2.3 we have then

$$\mathrm{epi}((f+g\circ h)^*) = \mathrm{epi}((\Phi_1(\cdot,0))^*) = \mathrm{cl}\Big(\bigcup_{y^*\in\mathrm{dom}(g^*)}\big((0,g^*(y^*)) + \mathrm{epi}((f+(y^*h))^*)\big)\Big).$$

□

Further we apply Lemma 2.4 for Φ_1 and we obtain an equivalent characterization through epigraphs of a formula of the conjugate of the function $f+g\circ h$, which acts as a regularity condition for the formula of the subdifferential of the mentioned function.

Theorem 3.3: *For each $x^* \in X^*$ we have*

$$(f + g \circ h)^*(x^*) = \min_{y^* \in C^*} \left\{ g^*(y^*) + (f + (y^*h))^*(x^*) \right\} \tag{4}$$

if and only if the regularity condition

$$(RC_1) \qquad \bigcup_{y^* \in \operatorname{dom}(g^*)} \big((0, g^*(y^*)) + \operatorname{epi}((f + (y^*h))^*)\big) \textit{ is closed,}$$

is fulfilled.

Remark 2: The result in Theorem 3.3 can be seen also as the equivalent characterization by the closedness-type regularity condition involving epigraphs (RC_1) of the *stable strong duality* statement for the primal composed convex optimization problem

$$(P) \qquad \inf_{x \in X} \{f(x) + g \circ h(x)\},$$

and its conjugate dual problem

$$(D) \qquad \sup_{y^* \in C^*} \{-g^*(y^*) - (f + (y^*h))^*(0)\}.$$

Corollary 3.4: *If (RC_1) holds, for all $x \in \operatorname{dom}(f) \cap h^{-1}(\operatorname{dom}(g))$ one has*

$$\partial(f + g \circ h)(x) = \bigcup_{\lambda \in \partial g(h(x))} \partial\big(f + (\lambda h)\big)(x). \tag{5}$$

Remark 3: As we have proven, the closedness-type regularity condition we use in order to have (5), (RC_1), is equivalent to (4). On the other hand, according to Proposition 4.11 in [6], the interiority-type conditions considered so far for (5) in the literature imply (4), without being equivalent to it, as Example 3.5 in [1] shows.

Consider now another perturbation function, namely

$$\Phi_2 : X \times Y \times X \to \overline{\mathbb{R}},\ \Phi_2(x, y, z) = f(x + z) + g(h(x) - y).$$

It is proper and convex, too, and its conjugate function is $\Phi_2^* : X^* \times Y^* \times X^* \to \overline{\mathbb{R}}$,

$$\Phi_2^*(x^*, y^*, z^*) = g^*(-y^*) + f^*(z^*) + (-y^*h)^*(x^* - z^*)\ \forall (x^*, y^*, z^*) \in X^* \times -C^* \times X^*,$$

and $\Phi_2^*(x^*, y^*, z^*) = +\infty$ otherwise. By similar calculations to the ones used to determine Φ_1^{**}, one can show, using again that h is star C-lower semicontinuous, that Φ_2 coincides with its biconjugate, thus it is lower semicontinuous, too.

We give now another extended Moreau-Rockafellar-type formula for $(f + g \circ h)^*$, provable by using Theorem 2.2, and other characterizations of the epigraph of this conjugate function which follow from Theorem 2.3.

Theorem 3.5: *One has*

$$\begin{aligned} (f + g \circ h)^* &= \operatorname{cl}\left(\inf_{\substack{z^* \in X^*,\\ y^* \in C^*}} \{f^*(z^*) + g^*(y^*) + (y^*h)^*(\cdot - z^*)\} \right) \\ &= \operatorname{cl}\Big(\inf_{y^* \in C^*} \{g^*(y^*) + f^* \square (y^*h)^*(\cdot)\} \Big). \end{aligned} \tag{6}$$

Corollary 3.6: *There holds*

$$\begin{aligned}\operatorname{epi}((f+g\circ h)^*) &= \operatorname{cl}\left(\operatorname{epi}\left(\inf_{\substack{z^*\in X^*,\\ y^*\in C^*}}\{f^*(z^*)+g^*(y^*)+(y^*h)^*(\cdot - z^*)\}\right)\right)\\ &= \operatorname{cl}\Big(\bigcup_{\substack{z^*\in X^*,\\ y^*\in \operatorname{dom}(g^*)}}\big((z^*,f^*(z^*))+(0,g^*(y^*))+\operatorname{epi}((y^*h)^*)\big)\Big)\\ &= \operatorname{cl}\Big(\operatorname{epi}(f^*)+\bigcup_{y^*\in \operatorname{dom}(g^*)}\big((0,g^*(y^*))+\operatorname{epi}((y^*h)^*)\big)\Big).\end{aligned}$$

Proof: The first two equalities follow analogously to the ones in Corollary 3.2. To prove the last one, note that, whenever $y^*\in \operatorname{dom}(g^*)$, for each $z^*\in X^*$ there is $(z^*,f^*(z^*))\in \operatorname{epi}(f^*)$, thus $\cup_{z^*\in X^*}\big((z^*,f^*(z^*))+(0,g^*(y^*))+\operatorname{epi}((y^*h)^*)\big)\subseteq \operatorname{epi}(f^*)+(0,g^*(y^*))+\operatorname{epi}((y^*h)^*)$. To prove the opposite inclusion, let $(x^*,r)\in \operatorname{epi}(f^*)+(0,g^*(y^*))+\operatorname{epi}((y^*h)^*)$. Then there is some $z^*\in X^*$ such that $f^*(z^*)+g^*(y^*)+(y^*h)^*(x^*-z^*)\le r$. Consequently, $(x^*-z^*,r-f^*(z^*)-g^*(y^*))\in \operatorname{epi}((y^*h)^*)$, which yields $(x^*,r)\in\cup_{z^*\in X^*}\big((z^*,f^*(z^*))+(0,g^*(y^*))+\operatorname{epi}((y^*h)^*)\big)$. Thus for all $y^*\in C^*$ one has

$$\bigcup_{z^*\in X^*}\big((z^*,f^*(z^*))+(0,g^*(y^*))+\operatorname{epi}((y^*h)^*)\big)=\operatorname{epi}(f^*)+(0,g^*(y^*))+\operatorname{epi}((y^*h)^*),$$

and the third desired equality follows at once. □

Remark 4: Regarding the terms in the right-hand sides of (3) and (6), it can be easily proven that

$$\inf_{y^*\in C^*}\{g^*(y^*)+(f+(y^*h))^*(x^*)\}\le \inf_{\substack{z^*\in X^*,\\ y^*\in C^*}}\{f^*(z^*)+g^*(y^*)+(y^*h)^*(x^*-z^*)\}$$

and

$$\bigcup_{y^*\in \operatorname{dom}(g^*)}\big((0,g^*(y^*))+\operatorname{epi}((f+(y^*h))^*)\big)\supseteq \bigcup_{\substack{z^*\in X^*,\\ y^*\in \operatorname{dom}(g^*)}}\big((z^*,f^*(z^*))+(0,g^*(y^*))+\operatorname{epi}((y^*h)^*)\big).$$

Though, as shown above, the closures of these functions and sets, respectively, coincide.

One can introduce another regularity condition which yields (RC_1) without being always implied by it, whose fulfillment ensures a formula for $\partial(f+g\circ h)$ where the functions f and h appear separated, as indicated in the following.

Theorem 3.7: *For each $x^*\in X^*$ we have*

$$(f+g\circ h)^*(x^*)=\min_{\substack{y^*\in C^*,\\ z^*\in X^*}}\big\{g^*(y^*)+f^*(z^*)+(y^*h)^*(x^*-z^*)\big\} \tag{7}$$

if and only if the regularity condition

$$(RC_2) \qquad \operatorname{epi}(f^*)+\bigcup_{y^*\in \operatorname{dom}(g^*)}\big((0,g^*(y^*))+\operatorname{epi}((y^*h)^*)\big) \text{ is closed,}$$

is fulfilled.

Remark 5: The result proven above can be seen also as the equivalent characterization by a regularity condition of the stable strong duality statement for the primal composed convex optimization problem (P) and another conjugate dual problem attached to it, which can be seen as a refinement of (D),

$$(\bar{D}) \qquad \sup_{\substack{y^* \in C^*,\\ z^* \in X^*}} \{-g^*(y^*) - f^*(z^*) - (y^*h)^*(-z^*)\}.$$

Corollary 3.8: *If (RC_2) holds, for all $x \in \mathrm{dom}(f) \cap h^{-1}(\mathrm{dom}(g))$ one has*

$$\partial(f + g \circ h)(x) = \partial f(x) + \bigcup_{\lambda \in \partial g(h(x))} \partial(\lambda h)(x). \tag{8}$$

Remark 6: The formula (8) was also given in other papers, like [6], but under more restrictive regularity conditions of interiority-type which, together with the assumption that h is continuous at a point of $\mathrm{dom}(f)$, imply (7), without being equivalent to it. The regularity condition we give in order to obtain (8), (RC_2), is equivalent to (7) and it yields (8) without requiring additional continuity hypotheses on h.

4. Special cases

In this section we treat some special choices of the functions f, g and h, rediscovering older results from the literature, among which also the classical Moreau-Rockafellar formula.

4.1. *Compositions with linear functions*

Consider the proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$, the proper convex lower semicontinuous function $g : Y \to \overline{\mathbb{R}}$ and the linear continuous operator $A : X \to Y$ fulfilling $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$. Taking h to be A and $C = \{0\}$ (therefore $C^* = Y^*$) we see that we are in a special case of the general framework of this paper. For each $y^* \in Y^*$ and any $x^* \in X^*$ one has $(f + (y^*A))^*(x^*) = f^*(x^* - A^*y^*)$. Note also that

$$(y^*A)^*(x^*) = \begin{cases} 0, & \text{if } A^*y^* = x^*,\\ +\infty, & \text{otherwise.} \end{cases}$$

Therefore in this case (3) and (6) collapse into the following result.

Theorem 4.1: *For each $x^* \in X^*$ there is*

$$(f + g \circ A)^*(x^*) = \mathrm{cl}\Big(\inf_{y^* \in Y^*} \{f^*(\cdot - A^*y^*) + g^*(y^*)\}\Big)(x^*) = \mathrm{cl}(f^* \Box A^*g^*)(x^*).$$

Proof: The first equality follows directly from Theorem 3.1 (or Theorem 3.5), by applying the formulae given above. The second equality is a consequence of Theorem 3.5, which yields

$$(f + g \circ A)^*(x^*) = \mathrm{cl}\left(\inf_{\substack{z^* \in X^*,\\ y^* \in C^*}} \{f^*(z^*) + g^*(y^*) + (y^*A)^*(\cdot - z^*)\}\right)(x^*).$$

For each $x^* \in X^*$ we have

$$\begin{aligned}\inf_{\substack{z^*\in X^*,\\ y^*\in C^*}} \{f^*(z^*) + g^*(y^*) + (y^*A)^*(x^* - z^*)\} &= \inf_{\substack{z^*\in X^*, y^*\in C^*,\\ A^*y^*=x^*-z^*}} \{f^*(z^*) + g^*(y^*)\} \\ &= \inf_{z^*\in X^*} \left\{ f^*(z^*) + \inf_{\substack{y^*\in C^*,\\ A^*y^*=x^*-z^*}} g^*(y^*) \right\} \\ &= \inf_{z^*\in X^*} \{f^*(z^*) + A^*g^*(x^* - z^*)\} \\ &= (f^*\Box A^*g^*)(x^*),\end{aligned}$$

which leads to the desired conclusion. □

In the following consequence of this statement we denote by $(A^* \times \mathrm{id}_{\mathbb{R}})(\mathrm{epi}(g^*))$ the image of the set $\mathrm{epi}(g^*)$ through the function $(A^* \times \mathrm{id}_{\mathbb{R}}) : Y^* \times \mathbb{R} \to X^* \times \mathbb{R}$, defined by $(A^* \times \mathrm{id}_{\mathbb{R}})(y^*, r) = (A^*y^*, r)$.

Corollary 4.2: *It holds*

$$\begin{aligned}\mathrm{epi}((f + g \circ A)^*) &= \mathrm{cl}\left(\mathrm{epi}\left(\inf_{y^*\in Y^*} (g^*(y^*) + f^*(\cdot - A^*y^*)) \right) \right) \\ &= \mathrm{cl}\left(\mathrm{epi}(f^*) + (A^* \times \mathrm{id}_{\mathbb{R}})(\mathrm{epi}(g^*)) \right) \\ &= \mathrm{cl}\left(\mathrm{epi}(f^*\Box A^*g^*) \right).\end{aligned}$$

Proof: The first equality follows directly from Corollary 3.2. Note that $\mathrm{epi}(y^*A)^* = \{(A^*y^*)\} \times \mathbb{R}_+ \ \forall y^* \in Y^*$. Thus $\cup_{y^*\in\mathrm{dom}(g^*)}((0, g^*(y^*)) + \mathrm{epi}((y^*A)^*)) = (A^* \times \mathrm{id}_{\mathbb{R}})(\mathrm{epi}(g^*))$ and the second equality follows by Corollary 3.6. For the third one we use the first equality in Corollary 3.6 and the calculations in the proof of Theorem 4.1. □

Specializing further A to be the identity operator we have to take $X = Y$ and we rediscover the following results (cf. [4, 11, 12]), the first of them being known as the classical *Moreau-Rockafellar formula.*

Corollary 4.3: *If f and g are proper convex lower semicontinuous functions defined on X with values in $\overline{\mathbb{R}}$ such that* $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$, *then*

(a) $(f + g)^* = \mathrm{cl}(f^*\Box g^*)$,
(b) $\mathrm{epi}((f + g)^*) = \mathrm{cl}(\mathrm{epi}(f^*\Box g^*)) = \mathrm{cl}(\mathrm{epi}(f^*) + \mathrm{epi}(g^*))$.

Like in the general case, we turn now our attention to stable strong duality. In the framework considered in the beginning of the subsection, the primal problem (P) turns into

$$(P^A) \qquad \inf_{x\in X} \{f(x) + g(A(x))\},$$

while its mentioned duals turn both into

$$(D^A) \qquad \sup_{y^*\in Y^*} \{-f^*(A^*y^*) - g^*(-y^*)\},$$

which is the classical Fenchel dual problem to (P^A). Because of the continuity of A it follows that in this case the regularity conditions (RC_1) and (RC_2) collapse both into

$$(RC^A) \qquad \operatorname{epi}(f^*) + (A^* \times \operatorname{id}_{\mathbb{R}})(\operatorname{epi}(g^*)) \text{ is closed.}$$

This condition is valid if and only if there is stable strong duality for (P^A) and (D^A). Moreover, this regularity condition coincides with the one given in Theorem 3.1 in [4] and then rediscovered in Theorem 5.4 in [1] to equivalently characterize $(f + g \circ A)^*$.

4.2. *The case $g = \delta_{-C}$*

Working in the hypotheses of Section 3 we consider now the function g to be δ_{-C}, which is proper, convex, lower semicontinuous and C-increasing, while the feasibility condition becomes $h(\operatorname{dom}(f)) \cap (-C) \neq \emptyset$. The conjugate of g is $g^* = \sigma_{-C} = \delta_{C^*}$, thus $\operatorname{dom}(g^*) = C^*$. Let $U \subseteq X$ be nonempty, convex and closed and take $h : X \to Y^\bullet$ to be defined as follows

$$h(x) = \begin{cases} w(x), \text{ if } x \in U, \\ \infty_Y, \ \text{ otherwise,} \end{cases}$$

where $w : X \to Y^\bullet$ is a proper, C-convex and C-epi-closed function.

The perturbation functions considered earlier become $\Phi_1^C(x,y) = f(x) + \delta_{\{(x,y)\in U\times Y: w(x)-y\in -C\}}(x,y)\ \forall (x,y) \in X \times Y$ and, respectively, $\Phi_2^C(x,y,z) = f(x+z) + \delta_{\{(x,y)\in U\times Y: w(x)-y\in -C\}}(x,y)\ \forall (x,y,z) \in X \times Y \times X$. By construction Φ_1^C and Φ_2^C are obviously proper and convex. Because w is C-epi-closed and U is closed, one has that h is C-epi-closed, too. Then it is straightforward that Φ_1^C and Φ_2^C are lower semicontinuous functions. Note that in this case it is not necessary to consider the function h to be star C-lower semicontinuous in order to be able to apply the theory developed earlier for the case when the perturbation functions are lower semicontinuous. From Theorem 2.2 one can derive the new Moreau-Rockafellar-type formulae listed below. Note that if h were star C-lower semicontinuous they could be obtained from Theorem 3.1 and Theorem 3.5.

Theorem 4.4: *For each $x^* \in X^*$ there is*

$$\sup_{\substack{x\in U,\\ w(x)\in -C}} \{\langle x^*, x\rangle - f(x)\} = (f + \delta_{-C}(w))_U^*(x^*) = \operatorname{cl}\Big(\inf_{y^*\in C^*} (f + (y^*w))_U^*\Big)(x^*)$$

$$= \operatorname{cl}\Big(\inf_{y^*\in C^*} f^* \square (y^*w)_U^*\Big)(x^*).$$

Corollary 4.5: *It holds*

$$\begin{aligned} \operatorname{epi}((f + \delta_{-C}(w))_U^*) &= \operatorname{cl}\Big(\operatorname{epi}\Big(\inf_{y^*\in C^*} (f + (y^*w))_U^*\Big)\Big) \\ &= \operatorname{cl}\Big(\bigcup_{y^*\in C^*} \operatorname{epi}((f + (y^*w))_U^*)\Big) \\ &= \operatorname{cl}\Big(\bigcup_{y^*\in C^*} \operatorname{epi}(f^* \square (y^*w)_U^*)\Big) \\ &= \operatorname{cl}\Big(\operatorname{epi}(f^*) + \bigcup_{y^*\in C^*} \operatorname{epi}((y^*w)_U^*)\Big). \end{aligned}$$

From these results one can obtain characterizations via epigraphs of the stable strong duality statements for the problem

$$(P^C) \qquad \inf_{\substack{x\in U,\\ w(x)\in -C}} f(x),$$

and its duals

$$(D^C) \qquad \sup_{y^*\in C^*} \{-(f+(y^*w))_U^*(0)\},$$

and

$$(\bar{D}^C) \qquad \sup_{\substack{y^*\in C^*,\\ z^*\in X^*}} \{-f^*(z^*) - (y^*w)_U^*(-z^*)\}.$$

Notice that (D^C) is actually the Lagrange dual problem to (P^C), while $(\bar{D}^C)$ is its Fenchel-Lagrange dual. The regularity conditions used in this paper turn out to be in this case

$$(RC_1^C) \qquad \bigcup_{y^*\in C^*} \operatorname{epi}((f+(y^*w))_U^*) \text{ is closed},$$

and, respectively,

$$(RC_2^C) \qquad \operatorname{epi}(f^*) + \bigcup_{y^*\in C^*} \operatorname{epi}((y^*w)_U^*) \text{ is closed},$$

which are actually the conditions used in [2] to equivalently characterize the stable strong duality for (P^C) and (D^C), respectively for (P^C) and $(\bar{D}^C)$. Like in this special case, also in [2] the function w has been supposed to be C-epi-closed.

5. The conjugate of the supremum of infinitely many functions

In this section we discuss the Moreau-Rockafellar representation of the conjugate of the supremum of infinitely many functions and give formulae of the epigraph of this conjugate function.

Let T be a possibly infinite index set and let $\mathbb{R}^T$ be the space of all functions $y : T \to \mathbb{R}$, endowed with the product topology and with the operations being the usual pointwise ones. For simplicity, denote $y_t = y(t)\ \forall y \in \mathbb{R}^T\ \forall t \in T$. The dual space of $\mathbb{R}^T$ is $(\mathbb{R}^T)^*$, the *space of generalized finite sequences* $\lambda = (\lambda_t)_{t\in T}$ such that $\lambda_t \in \mathbb{R}\ \forall t \in T$, and with only finitely many λ_t different from zero. The positive cone in $\mathbb{R}^T$ is $\mathbb{R}^T_+ = \{y \in \mathbb{R}^T : y_t = y(t) \geq 0\ \forall t \in T\}$, and its dual is the positive cone in $(\mathbb{R}^T)^*$, namely $(\mathbb{R}^T_+)^* = \{y^* = (y^*_t)_{t\in T} \in (\mathbb{R}^T)^* : y^*_t \geq 0\ \forall t \in T\}$. Take $g : \mathbb{R}^T \to \overline{\mathbb{R}}$, $g(y) = \sup_{t\in T} y_t$, which is a proper, convex, lower semicontinuous and $\mathbb{R}^T_+$-increasing function and the proper convex lower semicontinuous functions $h_t : X \to \overline{\mathbb{R}}$, $t \in T$, such that $\operatorname{dom}\big(\sup_{t\in T} h_t\big) \neq \emptyset$. Consider the function

$$h : X \to (\mathbb{R}^T)^\bullet,\ h(x) = \begin{cases} (h_t(x))_{t\in T}, & \text{if } x \in \bigcap_{t\in T} \operatorname{dom}(h_t),\\ \infty_{\mathbb{R}^T}, & \text{otherwise.}\end{cases}$$

One can easily see that h is proper, $\mathbb{R}^T_+$-convex and $\mathbb{R}^T_+$-epi-closed.

Note that for all $x \in X$ there is

$$\sup_{t\in T} h_t(x) = \sup_{\substack{S\subseteq T,\\ |S|<+\infty}} \sup_{\substack{y_s^*>0 \forall s\in S,\\ \sum_{s\in S} y_s^*=1}} \sum_{s\in S} y_s^* h_s(x).$$

For all the finite subsets S of T and for any $y_s^* > 0\ \forall s \in S$ fulfilling $\sum_{s\in S} y_s^* = 1$, the function $x \mapsto \sum_{s\in S} y_s^* h_s(x)$ is proper, convex and lower semicontinuous. By Lemma 2.1 it is equal to its biconjugate, thus for all $x \in X$ there is

$$\sup_{t\in T} h_t(x) = \sup_{\substack{S\subseteq T,\\ |S|<+\infty}} \sup_{\substack{y_s^*>0 \forall s\in S,\\ \sum_{s\in S} y_s^*=1}} \left(\sum_{s\in S} y_s^* h_s\right)^{**}(x) = \left(\inf_{\substack{S\subseteq T, |S|<+\infty,\\ y_s^*>0\forall s\in S,\\ \sum_{s\in S} y_s^*=1}} \left(\sum_{s\in S} y_s^* h_s\right)^*\right)^*(x).$$

Proposition 5.1: *The function*

$$\eta : X^* \to \overline{\mathbb{R}},\ \eta(x^*) = \inf_{\substack{S\subseteq T, |S|<+\infty,\\ y_s^*>0\forall s\in S,\\ \sum_{s\in S} y_s^*=1}} \left(\sum_{s\in S} y_s^* h_s\right)^*(x^*)$$

is proper and convex.

Proof: Whenever $S \subseteq T$ with S finite and $y_s^* > 0\ \forall s \in S$ with $\sum_{s\in S} y_s^* = 1$, the function $\left(\sum_{s\in S} y_s^* h_s\right)^*$ is proper, thus η cannot be identical $+\infty$. Assuming the existence of some $x^* \in X^*$ where $\eta(x^*) = -\infty$, it follows that $\sup_{t\in T} h_t$ is identical $+\infty$, which contradicts the feasibility hypothesis $\operatorname{dom}\left(\sup_{t\in T} h_t\right) \neq \emptyset$. Therefore η is proper.

In order to prove its convexity, let $\lambda \in [0, 1]$ and $x_1^*, x_2^* \in X^*$. What we have to show is

$$\eta(\lambda x_1^* + (1-\lambda)x_2^*) \leq \lambda\eta(x_1^*) + (1-\lambda)\eta(x_2^*). \tag{9}$$

If $\lambda \in \{0, 1\}$ or $\eta(x_1^*) = +\infty$ or $\eta(x_2^*) = +\infty$, (9) is valid. Let further $\lambda \in (0, 1)$ and $\eta(x_1^*), \eta(x_2^*) \in \mathbb{R}$. Then there are some $\alpha, \beta \in \mathbb{R}$ such that $\eta(x_1^*) < \alpha$ and $\eta(x_2^*) < \beta$. Thus there exist the finite subsets S_1 and S_2 of T and $y_s^* > 0\ \forall s \in S_1 \cup S_2$ such that $\sum_{s\in S_1} y_s^* = 1$, $\sum_{s\in S_2} y_s^* = 1$, $\left(\sum_{s\in S_1} y_s^* h_s\right)^*(x_1^*) < \alpha$ and $\left(\sum_{s\in S_2} y_s^* h_s\right)^*(x_2^*) < \beta$. Thus

$$\eta(\lambda x_1^* + (1-\lambda)x_2^*) \leq \left(\lambda \sum_{s\in S_1} y_s^* h_s + (1-\lambda) \sum_{s\in S_2} y_s^* h_s\right)^*(\lambda x_1^* + (1-\lambda)x_2^*)$$

$$\leq \left(\sum_{s\in S_1} \lambda y_s^* h_s\right)^*(\lambda x_1^*) + \left(\sum_{s\in S_2} (1-\lambda) y_s^* h_s\right)^*((1-\lambda)x_2^*) < \lambda\alpha + (1-\lambda)\beta.$$

If α converges towards $\eta(x_1^*)$ and β towards $\eta(x_2^*)$, (9) turns out to hold in this case, too. As α, x_1^* and x_2^* were arbitrarily chosen, it follows that η is convex. □

The function $\operatorname{cl}(\eta)$ is convex and lower semicontinuous, and its properness can be proven similarly to the one of η. Applying again Lemma 2.1 one gets

$$\left(\sup_{t\in T} h_t\right)^* = \text{cl}\left(\inf_{\substack{S\subseteq T, |S|<+\infty,\\ y_s^*>0 \forall s\in S,\\ \sum_{s\in S} y_s^*=1}} \left(\sum_{s\in S} y_s^* h_s\right)^*\right). \tag{10}$$

Since

$$\bigcup_{\substack{S\subseteq T, |S|<+\infty,\\ y_s^*>0 \forall s\in S,\\ \sum_{s\in S} y_s^*=1}} \text{epi}\left(\left(\sum_{s\in S} y_s^* h_s\right)^*\right) \subseteq \text{epi}\left(\inf_{\substack{S\subseteq T, |S|<+\infty,\\ y_s^*>0 \forall s\in S,\\ \sum_{s\in S} y_s^*=1}} \left(\sum_{s\in S} y_s^* h_s\right)^*\right)$$

$$\subseteq \text{cl}\left(\bigcup_{\substack{S\subseteq T, |S|<+\infty,\\ y_s^*>0 \forall s\in S,\\ \sum_{s\in S} y_s^*=1}} \text{epi}\left(\left(\sum_{s\in S} y_s^* h_s\right)^*\right)\right),$$

it follows

$$\text{epi}\left(\left(\sup_{t\in T} h_t\right)^*\right) = \text{cl}\left(\text{epi}\left(\inf_{\substack{S\subseteq T, |S|<+\infty,\\ y_s^*>0 \forall s\in S,\\ \sum_{s\in S} y_s^*=1}} \left(\sum_{s\in S} y_s^* h_s\right)^*\right)\right)$$

$$= \text{cl}\left(\bigcup_{\substack{S\subseteq T, |S|<+\infty,\\ y_s^*>0 \forall s\in S,\\ \sum_{s\in S} y_s^*=1}} \text{epi}\left(\left(\sum_{s\in S} y_s^* h_s\right)^*\right)\right).$$

On the other hand, for all $x^* \in X^*$ there is

$$\left(\sup_{t\in T} h_t\right)^*(x^*) \le \inf_{y^*\in\mathcal{P}} (y^*h)^*(x^*) \le \inf_{\substack{S\subseteq T, |S|<+\infty\\ y_s^*>0\ \forall s\in S\\ \sum_{s\in S} y_s^*=1}} \left(\sum_{s\in S} y_s^* h_s\right)^*(x^*), \tag{11}$$

where $\mathcal{P} = \{y^* \in (\mathbb{R}_+^T)^* : \sum_{t\in T} y_t^* = 1\}$.

As the first inequality in (11) is obvious, we prove only the second one. Let be $x^* \in X^*$. If the infimum in the right-hand side of the desired inequality is equal to $+\infty$ there is nothing to prove. Otherwise, take an arbitrary $r \in \mathbb{R}$ strictly greater than this infimum. Thus there is a finite subset S of T and some $y_s^* > 0$ for all $s \in S$ with $\sum_{s\in S} y_s^* = 1$ for which $(\sum_{s\in S} y_s^* h_s)^*(x^*) < r$. Considering a $\bar{y}^* \in \mathcal{P}$ which satisfies $\bar{y}_s^* = y_s^*$ when $s \in S$ and $\bar{y}_s^* = 0$ otherwise, it follows that $\inf_{y^*\in\mathcal{P}}(y^*h)^*(x^*) \le (\bar{y}^*h)^*(x^*) \le (\sum_{s\in S} y_s^* h_s)^*(x^*) < r$. Since r was arbitrarily chosen, (11) follows.

By taking (10) and (11) into consideration it yields

$$\left(\sup_{t\in T} h_t\right)^*(x^*) = \operatorname{cl}\left(\inf_{y^*\in\mathcal{P}} (y^*h)^*\right)(x^*).$$

On the other hand, we have

$$\underset{y^*\in\mathcal{P}}{\cup} \operatorname{epi}(y^*h)^* \subseteq \operatorname{epi}\left(\inf_{y^*\in\mathcal{P}} (y^*h)^*\right) \subseteq \operatorname{cl}\left(\underset{y^*\in\mathcal{P}}{\cup} \operatorname{epi}(y^*h)^*\right).$$

The relations above lead to

$$\operatorname{epi}\left(\left(\sup_{t\in T} h_t\right)^*\right) = \operatorname{cl}\left(\underset{y^*\in\mathcal{P}}{\cup} \operatorname{epi}((y^*h)^*)\right) = \operatorname{cl}\left(\operatorname{epi}\left(\inf_{y^*\in\mathcal{P}} (y^*h)^*\right)\right). \tag{12}$$

For every finite subset S of T and all $y_s^* > 0\ \forall s \in S$ with $\sum_{s\in S} y_s^* = 1$, there is

$$\operatorname{epi}\left(\sum_{s\in S} y_s^* h_s\right)^* = \operatorname{cl}\left(\sum_{s\in S} \operatorname{epi}\left((y_s^* h_s)^*\right)\right) = \operatorname{cl}\left(\sum_{s\in S} y_s^* \operatorname{epi}\left(h_s^*\right)\right)$$
$$\subseteq \operatorname{cl}\left(\operatorname{co}\left(\underset{t\in T}{\cup} \operatorname{epi}(h_t^*)\right)\right),$$

whence

$$\operatorname{epi}\left(\left(\sup_{t\in T} h_t\right)^*\right) \subseteq \operatorname{cl}\left(\operatorname{co}\left(\underset{t\in T}{\cup} \operatorname{epi}(h_t^*)\right)\right). \tag{13}$$

Taking an arbitrary element $(x^*, r) \in \operatorname{co}\left(\cup_{t\in T} \operatorname{epi}(h_t^*)\right)$, there is a finite subset S of T and some $y_s^* > 0$, $s \in S$ with $\sum_{s\in S} y_s^* = 1$ for which $(x^*, r) \in \sum_{s\in S} y_s^* \operatorname{epi}(h_s^*) \subseteq \operatorname{epi}\left(\left(\sum_{s\in S} y_s^* h_s\right)^*\right)$. This yields $\left(\sum_{s\in S} y_s^* h_s\right)^*(x^*) \leq r$. For any $\bar{y}^* \in (\mathbb{R}_+^T)^*$ fulfilling $\sum_{t\in T} \bar{y}_t^* = 1$ which satisfies $\bar{y}_s^* = y_s^*$ when $s \in S$ and $\bar{y}_s^* = 0$ otherwise, there is

$$(\bar{y}^*h)^*(x^*) = \left(\sum_{s\in S} y_s^* h_s + \delta_{\underset{s\in T\setminus S}{\cap} \operatorname{dom}(h_s)}\right)^*(x^*) \leq \left(\sum_{s\in S} y_s^* h_s\right)^*(x^*) \leq r.$$

This means that $(x^*, r) \in \operatorname{epi}((\bar{y}^*h)^*)$, which yields

$$\operatorname{co}\left(\underset{t\in T}{\cup} \operatorname{epi}(h_t^*)\right) \subseteq \bigcup_{y^*\in\mathcal{P}} \operatorname{epi}((y^*h)^*). \tag{14}$$

Consequently,

$$\underset{t\in T}{\cup} \operatorname{epi}(h_t^*) \subseteq \operatorname{epi}\left(\inf_{t\in T} h_t^*\right) \subseteq \operatorname{cl}\left(\operatorname{epi}\left(\inf_{t\in T} h_t^*\right)\right) \subseteq \operatorname{cl}\left(\operatorname{co}\left(\operatorname{epi}\left(\inf_{t\in T} h_t^*\right)\right)\right),$$

and

$$\operatorname{cl}\left(\operatorname{co}\left(\operatorname{epi}\left(\inf_{t\in T} h_t^*\right)\right)\right) = \operatorname{cl}\left(\operatorname{co}\left(\underset{t\in T}{\cup} \operatorname{epi}(h_t^*)\right)\right) = \operatorname{epi}\left(\operatorname{cl}\left(\operatorname{co}\left(\inf_{t\in T} h_t^*\right)\right)\right).$$

Therefore we rediscover the known formulae

$$\left(\sup_{t\in T} h_t\right)^* = \mathrm{cl}\left(\mathrm{co}\left(\inf_{t\in T} h_t^*\right)\right)$$

and, via (12), (13) and (14),

$$\mathrm{epi}\left(\left(\sup_{t\in T} h_t\right)^*\right) = \mathrm{cl}\left(\mathrm{co}\left(\bigcup_{t\in T} \mathrm{epi}(h_t^*)\right)\right).$$

One can also show that for all $x^* \in X^*$ there is

$$\inf_{y^*\in\mathcal{P}} (y^*h)^*(x^*) \leq \inf_{\substack{S\subseteq T, |S|<+\infty, \\ y_s^*>0 \forall s\in S, \\ \sum_{s\in S} y_s^*=1}} \left(\sum_{s\in S} y_s^* h_s\right)^* (x^*).$$

Let $x^* \in X^*$. If the infimum in the right-hand side of the inequality above is equal to $+\infty$ there is nothing to prove. Otherwise, there is some $r \in \mathbb{R}$ greater than this infimum. Thus there is a finite subset S of T and some $y_s^* > 0$, $s \in S$ with $\sum_{s\in S} y_s^* = 1$ for which $(\sum_{s\in S} y_s^* h_s)^*(x^*) < r$. Considering a $\bar{y}^* \in \mathcal{P}$ which satisfies $\bar{y}_s^* = y_s^*$ when $s \in S$ and $\bar{y}_s^* = 0$ otherwise, it follows that the infimum in the left-hand side is less than or equal to $(\sum_{s\in S} y_s^* h_s)^*(x^*)$. Since r was arbitrarily chosen, the inequality follows.

Remark 1 : From (12) and (14) one obtains that

$$\left(\sup_{t\in T} h_t\right)^* (x^*) = \min_{y^*\in\mathcal{P}} (y^*h)^*(x^*) = \min_{\substack{S\subseteq T, |S|<+\infty, \\ y_s^*>0 \forall s\in S, \\ \sum_{s\in S} y_s^*=1}} \left(\sum_{s\in S} y_s^* h_s\right)^* (x^*) \ \forall x^* \in X^*$$

holds if

$$\mathrm{epi}\left(\left(\sup_{t\in T} h_t\right)^*\right) = \mathrm{co}\left(\bigcup_{t\in T} \mathrm{epi}(h_t^*)\right),$$

the latter meaning actually that $\mathrm{co}\left(\cup_{t\in T} \mathrm{epi}(h_t^*)\right)$ is closed.

6. Discussion and an open problem

We equivalently characterized in this paper the formulae (4) and, respectively, (7) with regularity conditions involving epigraphs. The discussion will be further carried on only for the formula (4), because for (7) the things work similarly.

Before proceeding, we need some prerequisites. A set $U \subseteq X$ is said to be *closed regarding the closed subspace* $Z \subseteq X$ if $U \cap Z = \mathrm{cl}(U) \cap Z$. Note that we always have

$$U \cap Z \subseteq \mathrm{cl}(U \cap Z) \subseteq \mathrm{cl}(U) \cap Z. \tag{15}$$

In the preliminary section we have introduced two different extensions of lower semicontinuity to vector functions, namely C-epi-closedness and star C-lower semicontinuity. It is known that the star C-lower semicontinuous functions are always C-epi-closed and next we give an example which shows that the opposite assertion does not always hold.

Example 6.1 Consider the function

$$g : \mathbb{R} \to (\mathbb{R}^2)^\bullet = \mathbb{R}^2 \cup \{\infty_{\mathbb{R}^2}\},\ g(x) = \begin{cases} (\frac{1}{x}, x), & \text{if } x > 0, \\ \infty_{\mathbb{R}^2}, & \text{otherwise.} \end{cases}$$

One can verify that g is $\mathbb{R}^2_+$-convex and $\mathbb{R}^2_+$-epi-closed, but not star $\mathbb{R}^2_+$-lower semicontinuous. For instance, for $\lambda = (0,1)^T \in (\mathbb{R}^2_+)^* = \mathbb{R}^2_+$ one has

$$((0,1)^T g)(x) = \begin{cases} x, & \text{if } x > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

which is not lower semicontinuous.

In this paper we have shown that (4) is valid if and only if

(RC_1) $$\bigcup_{y^* \in \operatorname{dom}(g^*)} \big((0, g^*(y^*)) + \operatorname{epi}((f + (y^*h))^*)\big) \text{ is closed.}$$

The formula (4) was equivalently characterized also in [1], where the assumption of star C-lower semicontinuity considered here for h was relaxed to C-epi-closedness, with another regularity condition, namely

(RC_1') $\{0\} \times \operatorname{epi}(g^*) + \bigcup_{\lambda \in C^*} \big\{(a, -\lambda, r) : (a, r) \in \operatorname{epi}\big((f + (\lambda h))^*\big)\big\}$ is closed regarding the closed subspace $X^* \times \{0\} \times \mathbb{R}$.

Denoting by $\mathcal{M}$ the set asked to be closed regarding the closed subspace $X^* \times \{0\} \times \mathbb{R}$ in (RC_1'), it can be proven that (RC_1) is equivalent to saying that $\mathcal{M} \cap (X^* \times \{0\} \times \mathbb{R})$ is closed. Using (15), it follows that (RC_1') implies (RC_1).

We noticed that by strengthening the initial topological assumptions on the function h, namely by considering it star C-lower semicontinuous instead of C-epi-closed we obtain that (4) is equivalent to a condition that is weaker than the one which equivalently characterizes (4) in the framework of [1]. Thus, one "loses" something by restricting the hypotheses, but there is a "gain" in the regularity condition which equivalently characterizes (4). In each of these contexts, the formula in discussion is the weakest regularity condition known to us that ensures the subdifferential formula (5). Thus it is up to the user to decide what is more important in each specific situation: weaker hypotheses or weaker regularity conditions.

A similar discussion can be made also for (7), equivalently characterized in this paper, where h is taken star C-lower semicontinuous, by (RC_2) and in [1] where h is considered C-epi-closed by the condition

(RC_2') $\{0\} \times \operatorname{epi}(g^*) + \big\{(p, 0, r) : (p, r) \in \operatorname{epi}(f^*)\big\} + \bigcup_{\lambda \in C^*} \big\{(p, -\lambda, r) : (p, r) \in \operatorname{epi}\big(\lambda h\big)^*\big\}$ is closed regarding the closed subspace $X^* \times \{0\} \times \mathbb{R}$.

We conclude this discussion by challenging the reader to provide an example.

Conjecture. Let the separated locally convex spaces X and Y, the latter partially ordered by a closed convex cone C, a proper convex lower semicontinuous function $f : X \to \overline{\mathbb{R}}$, a proper convex lower semicontinuous C-increasing function

$g : Y \to \overline{\mathbb{R}}$ and a proper C-convex C-epi-closed function $h : X \to Y^\bullet$ which is not star C-lower semicontinuous, such that $\big(h(\mathrm{dom}(f)) + C\big) \cap \mathrm{dom}(g) \neq \emptyset$. We conjecture that it is possible to choose X, Y, f, g and h such that (RC_1) is fulfilled, but (RC'_1) fails.

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