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Lower semicontinuous type regularity conditions for subdifferential calculus

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We give a lower semicontinuous type regularity condition and a closedness type one which turn out to be necessary and sufficient for the fulfillment of two different formulae involving the ε-subdifferential of a perturbation function, respectively. These regularity conditions prove to be sufficient also for having formulae for the classical subdifferential of a perturbation function. Some recently published results concerning ε-subdifferentials are rediscovered as special cases.

Keywords: Conjugate functions, convex subdifferentials, regularity conditions, stable strong duality, composed convex functions

AMS Subject Classification: 90C46, 49N15, 90C25

1. Introduction

There are different types of subdifferentials considered in the literature, introduced mostly in order to extend the classical notion of gradient to nondifferentiable functions. In this paper we work with the classical *subdifferential*, also known as the *convex subdifferential* or the *Fenchel subdifferential* and with its generalization the ε-subdifferential.

The main results which we exhibit here concern the way how the ε-subdifferential of $\Phi(\cdot, 0)$ can be written more explicitly, where Φ is a proper convex lower semicontinuous perturbation function of two variables taking extended real values. More precisely, we give two different formulae for $\partial_\varepsilon \Phi(\cdot, 0)$ and we equivalently characterize their fulfillment by means of so-called *regularity conditions*. One of these conditions, the weakest one, requires the lower semicontinuity of an infimal value function. The other one consists in asking a set to be closed and it belongs to the recently introduced class of *closedness type regularity conditions*. These regularity conditions turn out to be sufficient for having two different formulae concerning the subdifferential of $\Phi(\cdot, 0)$ fulfilled, too.

Nevertheless, a discussion on the differences that exist between the two regularity conditions is provided. The closedness type regularity condition is actually equivalent to the so-called *stable strong duality* regarding the optimization problem

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$$(P) \qquad \inf_{x\in X} \Phi(x,0),$$

where $\Phi : X \times Y \to \overline{\mathbb{R}}$ and X and Y are separated locally convex vector spaces, and its conjugate dual

$$(D) \qquad \sup_{y^*\in Y^*} -\Phi^*(0,y^*),$$

while the lower semicontinuous type one holds if and only if there is only stable zero duality gap for the same primal-dual pair of optimization problems.

Recall that perturbation functions are used in convex optimization for stating basic results in general frameworks and to assign conjugate duals to optimization problems. By particularizing the optimization problem (P) and by appropriately choosing the perturbation function Φ one can obtain or rediscover different statements in duality. When Φ is taken to be a perturbation function for a convex optimization problem with the objective function consisting in the sum of a convex function with a composition of convex functions or for a problem with a convex objective function and both geometric and cone-inequality constraints, the main statements of the paper deliver interesting particular results, some not known yet, some recently given in works like [2, 4–9, 11–13, 15].

2. Preliminaries

2.1. *Notions*

Consider two separated locally convex vector spaces X and Y and their topological dual spaces X^* and Y^*, endowed with the corresponding weak* topologies, and denote by $\langle x^*, x\rangle = x^*(x)$ the value of the continuous linear functional $x^* \in X^*$ at $x \in X$. The nonempty convex cone $C \subseteq Y$ induces on Y the partial ordering "$\leq_C$", defined by $z \leq_C y \Leftrightarrow y - z \in C$, when $z, y \in Y$. To Y we attach a greatest element with respect to "$\leq_C$" denoted by ∞_C which does not belong to Y. Thus we extend the space Y to $Y^\bullet = Y \cup \{\infty_C\}$. Then for every $y \in Y^\bullet$ one has $y \leq_C \infty_C$ and we consider on $Y^\bullet$ the operations $y + \infty_C = \infty_C + y = \infty_C$ for $y \in Y$ and $t \cdot \infty_C = \infty_C$ for $t \geq 0$. A function $g : Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is said to be *C-increasing* if for $y, z \in Y$ such that $z \leq_C y$ one has $g(z) \leq g(y)$. Such a function can be extended by convention to $Y^\bullet$ with the additional value $g(\infty_C) = +\infty$. The *dual cone* of the cone C is $C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \ \forall y \in C\}$. By convention we take $\langle \lambda, \infty_C\rangle = +\infty$ for all $\lambda \in C^*$. Given a subset U of X, by $\mathrm{cl}(U)$, δ_U and σ_U we denote its *closure*, its *indicator function* and its *support function*, respectively. We use also the *projection function* $\mathrm{Pr}_X : X \times Y \to X$, defined by $\mathrm{Pr}_X(x, y) = x$ for all $(x, y) \in X \times Y$.

Having a function $f : X \to \overline{\mathbb{R}}$ we use the classical notations for its *domain* $\mathrm{dom}(f) = \{x \in X : f(x) < +\infty\}$, its *epigraph* $\mathrm{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ and its *conjugate function* $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$. We call f *proper* if $f(x) > -\infty$ for all $x \in X$ and $\mathrm{dom}(f) \neq \emptyset$. For f proper and $\varepsilon \geq 0$, if $f(x) \in \mathbb{R}$ the *ε-subdifferential* of f at x is $\partial_\varepsilon f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle - \varepsilon \ \forall y \in X\}$, while if $f(x) = +\infty$ we take by convention $\partial_\varepsilon f(x) = \emptyset$. The ε-normal cone of a set $U \subseteq X$ at $x \in X$ is $N_U^\varepsilon(x) = \partial_\varepsilon \delta_U(x)$. If f is proper denote by $\partial f(x) = \partial_0 f(x)$ the *subdifferential* of f at x. Regarding a function and its conjugate we have *Young's inequality* $f^*(x^*) + f(x) \geq \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in X^*$. It can be proven that for $x \in X$, $x^* \in X^*$ and $\varepsilon \geq 0$ one has $f(x) + f^*(x^*) \leq \langle x^*, x\rangle + \varepsilon$ if and only if $x^* \in \partial_\varepsilon f(x)$. If $0 \leq \varepsilon \leq \eta$ it holds $\partial_\varepsilon f(x) \subseteq \partial_\eta f(x)$ and $\cap_{\mu>\varepsilon}\partial_\mu f(x) = \partial_\varepsilon f(x)$ for all $x \in X$.

Considering for each $\lambda \in \mathbb{R}$ the function $(\lambda f) : X \to \overline{\mathbb{R}}$, $(\lambda f)(x) = \lambda f(x)$ for $x \in X$, note that when $\lambda = 0$ we have $(0f) = \delta_{\text{dom}(f)}$. Given the proper functions $f_i : X \to \overline{\mathbb{R}}$, $i = 1, \ldots, n$, their *infimal convolution* is the function $f_1 \Box \ldots \Box f_n : X \to \overline{\mathbb{R}}$, $(f_1 \Box \ldots \Box f_n)(a) = \inf \left\{ \sum_{i=1}^n f_i(x_i) : x_i \in X, i = 1, \ldots, n, \sum_{i=1}^n x_i = a \right\}$. The *lower semicontinuous hull* of $f : X \to \overline{\mathbb{R}}$ is the function $\text{cl}(f) : X \to \overline{\mathbb{R}}$ which has as epigraph $\text{cl}(\text{epi}(f))$. Given a linear continuous mapping $A : X \to Y$, we have its *adjoint* $A^* : Y^* \to X^*$ defined by $\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle$ for $(x, y^*) \in X \times Y^*$. For a proper function $f : X \to \overline{\mathbb{R}}$ we define also the *infimal function of f through A* as $Af : Y \to \overline{\mathbb{R}}$, $Af(y) = \inf \left\{ f(x) : x \in X, Ax = y \right\}$ for $y \in Y$.

Some of the notions given for functions taking values in the extended real space $\overline{\mathbb{R}}$ can be generalized to vector functions as follows. A vector function $h : X \to Y^\bullet$ is said to be *proper* if its *domain* $\text{dom}(h) = \{x \in X : h(x) \in Y\}$ is nonempty and it is called *C-convex* if

$$h(tx + (1-t)y) \leq_C th(x) + (1-t)h(y) \ \forall x, y \in X \ \forall t \in [0, 1].$$

Lower semicontinuity can be extended to vector functions in different ways. Here we consider two notions of generalized lower semicontinuity. We say that h is *C-epi-closed* if its *C-epigraph* $\text{epi}_C(h) = \{(x, y) \in X \times Y : y \in h(x) + C\}$ is closed, and h is called *star C-lower semicontinuous* if the function $(\lambda h) : X \to \overline{\mathbb{R}}$ defined by $(\lambda h)(x) = \langle \lambda, h(x) \rangle$ is lower semicontinuous for all $\lambda \in C^*$.

Remark 1 Every star C-lower semicontinuous function is also C-epi-closed. The reverse statement does not hold in general (see [1]).

Given a function of two variables $\Phi : X \times Y \to \overline{\mathbb{R}}$, its *infimal value function* is $h_\Phi : Y \to \overline{\mathbb{R}}$, defined by $h_\Phi(y) = \inf_{x \in X} \Phi(x, y)$. When Φ is convex, h_Φ is convex, too.

Having a primal-dual pair of optimization problems we call *zero duality gap* the situation when the optimal objective values of the primal and dual problems coincide. If there is zero duality gap and the dual problem has an optimal solution we talk about *strong duality.* When strong duality (zero duality gap) remains valid for every linear perturbation of the objective function of the primal problem we say that there is *stable strong duality* (*stable zero duality gap*).

2.2. *Results*

We give now some results needed later in our investigations. Let us mention that for an attained infimum (supremum) instead of inf (sup) we write min (max). Let $\Phi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex lower semicontinuous function fulfilling $0 \in \text{Pr}_Y(\text{dom}(\Phi))$. We begin with a result given in [14] (see also [2]).

LEMMA 2.1 *For all $x^* \in X^*$ it holds*

$$(\Phi(\cdot, 0))^*(x^*) = \text{cl}\left(\inf_{y^* \in Y^*} \Phi^*(\cdot, y^*) \right)(x^*). \tag{1}$$

By making use of epigraphs, this statement turns into the following one (cf. [2]).

LEMMA 2.2 *It holds*

$$\text{epi}((\Phi(\cdot, 0))^*) = \text{cl}\left(\text{epi}\left(\inf_{y^* \in Y^*} \Phi^*(\cdot, y^*) \right) \right) = \text{cl}\left(\bigcup_{y^* \in Y^*} \text{epi}(\Phi^*(\cdot, y^*)) \right).$$

A consequence of Lemma 2.2 follows (see [2, 7]).

LEMMA 2.3 *The set* $\Pr_{X^*\times\mathbb{R}}(\mathrm{epi}(\Phi^*))$ *is closed if and only if*

$$(\Phi(\cdot,0))^*(x^*) = \min_{y^*\in Y^*} \Phi^*(x^*,y^*) \ \forall x^* \in X^*.$$

We recall also the following assertion from [15].

LEMMA 2.4 *For all* $\varepsilon \geq 0$ *and all* $x \in X$ *it holds*

$$\partial_\varepsilon \Phi(\cdot,0)(x) = \bigcap_{\eta>0} \mathrm{cl}(\Pr_{X^*}(\partial_{\varepsilon+\eta}\Phi(x,0))). \tag{2}$$

Remark 2 Formula (2) is given in [15, Theorem 2.6.3] for $x \in \mathrm{dom}\,\Phi(\cdot,0)$. With the conventions we made, it can be extended to the whole space X.

For various choices of Φ one can derive from the lemmata given above different interesting results. For instance, in [2] there are *Moreau-Rockafellar type statements* for different convexity preserving combinations of functions such as the sum of a convex function with the composition of convex functions, and *stable strong duality* statements for optimization problems involving such functions. Considering some perturbations used there, one can obtain new formulae for the ε-subdifferential of the sum of a convex function with the composition of convex functions, as seen in Proposition 2.5 below.

The first particular instance of (P) we consider is an unconstrained composed convex optimization problem. Let $f : X \to \overline{\mathbb{R}}$ be a proper convex lower semicontinuous function, $g : Y \to \overline{\mathbb{R}}$ a proper convex lower semicontinuous C-increasing function, for which we suppose by convention that $g(\infty_C) = +\infty$, and $h : X \to Y^\bullet$ a proper C-convex star C-lower semicontinuous function. Assume the feasibility condition $\big(h(\mathrm{dom}(f)) + C\big) \cap \mathrm{dom}(g) \neq \emptyset$ fulfilled. For the optimization problem

$$(P^C) \qquad \inf_{x\in X}\{f(x) + g(h(x))\},$$

we consider the perturbation functions

$$\Phi_1 : X \times Y \to \overline{\mathbb{R}},\ \Phi_1(x,y) = f(x) + g(h(x) - y),$$

and

$$\Phi_2 : X \times Y \times X \to \overline{\mathbb{R}},\ \Phi_2(x,y,z) = f(x+z) + g(h(x) - y).$$

In [2] we have shown that both these perturbation functions are proper, convex and lower semicontinuous and thus Lemma 2.4 can be employed to deliver the following statement.

PROPOSITION 2.5 *For all* $\varepsilon \geq 0$ *and all* $x \in X$ *it holds*

$$\partial_\varepsilon(f + g\circ h)(x) = \bigcap_{\eta>0} \mathrm{cl}\left(\bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0,\\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta}} \bigcup_{y^*\in C^*\cap\partial_{\varepsilon_2}g(h(x))} \partial_{\varepsilon_1}(f + (y^*h))(x) \right)$$

$$= \bigcap_{\eta>0} \mathrm{cl}\left(\bigcup_{\substack{\varepsilon_1,\varepsilon_2,\varepsilon_3\geq 0,\\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon+\eta}} \bigcup_{y^*\in C^*\cap\partial_{\varepsilon_2}g(h(x))} \Big(\partial_{\varepsilon_1}f(x) + \partial_{\varepsilon_3}(y^*h)(x)\Big) \right). \tag{3}$$

Remark 3 Note that for $\varepsilon = 0$ relation (3) can be simplified to

$$\partial(f + g \circ h)(x) = \bigcap_{\eta>0} \operatorname{cl}\Big(\bigcup_{y^* \in C^* \cap \partial_\eta g(h(x))} \partial_\eta (f + (y^* h))(x) \Big)$$

$$= \bigcap_{\eta>0} \operatorname{cl}\Big(\bigcup_{y^* \in C^* \cap \partial_\eta g(h(x))} \big(\partial_\eta f(x) + \partial_\eta (y^* h)(x)\big) \Big).$$

The second particular instance of (P) taken into consideration is a constrained composed convex optimization problem. Let $U \subseteq X$ be a nonempty convex closed set and consider the proper C-convex C-epi-closed function $w : X \to Y^\bullet$ such that the feasibility condition $\operatorname{dom}(f) \cap U \cap w^{-1}(-C) \neq \emptyset$ is satisfied. To the constrained optimization problem

$$(P^P) \qquad \inf_{\substack{x \in U, \\ w(x) \in -C}} f(x),$$

we assign the perturbation functions

$$\Phi_3 : X \times Y \to \overline{\mathbb{R}},\ \Phi_3(x, y) = f(x) + \delta_{\{(x,y) \in U \times Y : w(x) - y \in -C\}}(x, y),$$

$$\Phi_4 : X \times Y \times X \to \overline{\mathbb{R}},\ \Phi_4(x, y, z) = f(x + z) + \delta_{\{(x,y) \in U \times Y : w(x) - y \in -C\}}(x, y),$$

and $\Phi_5 : X \times Y \times X \times X \to \overline{\mathbb{R}}$,

$$\Phi_5(x, y, z, t) = f(x + z) + \delta_{\{(x,y) \in X \times Y : w(x) - y \in -C\}}(x, y) + \delta_U(x + t).$$

All these perturbation functions are proper, convex and lower semicontinuous (see [2]). Consequently, from Lemma 2.4 we obtain the following statements.

Proposition 2.6 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_\varepsilon (f + \delta_{w^{-1}(-C) \cap U})(x) = \bigcap_{\eta>0} \operatorname{cl}\left(\bigcup_{\substack{y^* \in C^*, \\ w(x) \in -C}} \partial_{\varepsilon + \eta + (y^* w)(x)} (f + (y^* w) + \delta_U)(x) \right)$$

$$= \bigcap_{\eta>0} \operatorname{cl}\left(\bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0, y^* \in C^*, w(x) \in -C, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \eta + (y^* w)(x)}} \big(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} ((y^* w) + \delta_U)(x)\big) \right)$$

$$= \bigcap_{\eta>0} \operatorname{cl}\left(\bigcup_{\substack{\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0, y^* \in C^*, w(x) \in -C, \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon + \eta + (y^* w)(x)}} \big(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} (y^* w)(x) + N_U^{\varepsilon_3}(x)\big) \right).$$

Finally, we particularize (P) to be an unconstrained convex optimization problem. If $\varphi : X \times Y \to \overline{\mathbb{R}}$ is proper convex lower semicontinuous and $A : X \to Y$ is linear continuous, we can obtain from Lemma 2.4 a formula for $\partial_\varepsilon \varphi(\cdot, A\cdot)(x)$, when $\varepsilon \geq 0$ and $x \in X$, by attaching the perturbation function

$$\Phi_6 : X \times Y \to \overline{\mathbb{R}},\ \Phi_6(x, y) = \varphi(x, Ax + y),$$

to the unconstrained optimization problem

$$(P^{\varphi}) \qquad \inf_{x\in X} \varphi(x, Ax).$$

PROPOSITION 2.7 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_{\varepsilon}\varphi(\cdot, A\cdot)(x) = \bigcap_{\eta>0} \operatorname{cl}\Big(\mathrm{Pr}_{X^*}\Big\{(x^* + A^*y^*, y^*) : (x^*, y^*) \in \partial_{\varepsilon+\eta}\varphi(x, Ax)\Big\}\Big).$$

The results in Proposition 2.5, Proposition 2.6 and Proposition 2.7 can be further particularized, rediscovering formulae from [11, 12, 15].

A natural question is under which circumstances can the "closures" that appear in all these formulae be removed. We give some answers in the next section by providing equivalent characterizations of lower semicontinuous type for these formulae stated without involving lower semicontinuous hulls.

3. Equivalent lower semicontinuous type characterizations of formulae for ε-subdifferentials

Unless otherwise specified, the functions and sets considered in this section are taken as defined in Subsection 2.2.

THEOREM 3.1 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_{\varepsilon}\Phi(\cdot, 0)(x) = \bigcap_{\eta>0} \mathrm{Pr}_{X^*}(\partial_{\varepsilon+\eta}\Phi(x, 0))$$

if and only if the regularity condition

$$(LSC) \qquad \inf_{y^*\in Y^*} \Phi^*(\cdot, y^*) \text{ is lower semicontinuous,}$$

is fulfilled.

Proof Lemma 2.2 yields that the condition (LSC) is equivalent to $\operatorname{epi}((\Phi(\cdot, 0))^*) = \operatorname{epi}\big(\inf_{y^*\in Y^*} \Phi^*(\cdot, y^*)\big)$. Note that in general it holds $\operatorname{epi}((\Phi(\cdot, 0))^*) \supseteq \operatorname{epi}\big(\inf_{y^*\in Y^*} \Phi^*(\cdot, y^*)\big)$, so (LSC) is equivalent to $\operatorname{epi}((\Phi(\cdot, 0))^*) \subseteq \operatorname{epi}\big(\inf_{y^*\in Y^*} \Phi^*(\cdot, y^*)\big)$, too.

"$\Rightarrow$" Take an arbitrary pair $(x^*, r) \in \operatorname{epi}((\Phi(\cdot, 0))^*)$. This means actually that $(\Phi(\cdot, 0))^*(x^*) \leq r$. Let $x \in \operatorname{dom}(\Phi(\cdot, 0))$ and $\varepsilon = r + \Phi(x, 0) - \langle x^*, x\rangle \geq 0$. Then $(\Phi(\cdot, 0))^*(x^*) + \Phi(x, 0) \leq \langle x^*, x\rangle + \varepsilon$, i.e. $x^* \in \partial_{\varepsilon}\Phi(\cdot, 0)(x)$. Using the hypothesis, whenever $\eta > 0$ there exists $y^*_{\eta} \in Y^*$ for which $(x^*, y^*_{\eta}) \in \partial_{\varepsilon+\eta}\Phi(x, 0)$. Fixing an $\eta > 0$, we get $\Phi(x, 0) + \Phi^*(x^*, y^*_{\eta}) \leq \langle x^*, x\rangle + \varepsilon + \eta$. Consequently, $\Phi(x, 0) + \inf_{y^*\in Y^*} \Phi^*(x^*, y^*) \leq \langle x^*, x\rangle + \varepsilon + \eta$ for all $\eta > 0$. Letting η tend toward 0, it follows $\Phi(x, 0) + \inf_{y^*\in Y^*} \Phi^*(x^*, y^*) \leq \langle x^*, x\rangle + \varepsilon$ and, replacing ε by $r + \Phi(x, 0) - \langle x^*, x\rangle$, we get $\inf_{y^*\in Y^*} \Phi^*(x^*, y^*) \leq r$, i.e. $(x^*, r) \in \operatorname{epi}\big(\inf_{y^*\in Y^*} \Phi^*(\cdot, y^*)\big)$.

"$\Leftarrow$" Let $\varepsilon \geq 0$. From (2) one can easily deduce that $\cap_{\eta>0} \mathrm{Pr}_{X^*}(\partial_{\varepsilon+\eta}\Phi(x, 0))$ is a subset of $\partial_{\varepsilon}\Phi(\cdot, 0)(x)$ for all $x \in X$. Note that this inclusion holds in the most general framework. It remains to show the opposite one.

Let $x \in X$. If $\Phi(x, 0) = +\infty$, then $\partial_{\varepsilon}\Phi(\cdot, 0)(x) = \partial_{\varepsilon+\eta}\Phi(x, 0) = \emptyset$ for all $\eta > 0$. Assume thus further that $\Phi(x, 0) \in \mathbb{R}$. For $x^* \in \partial_{\varepsilon}\Phi(\cdot, 0)(x)$, one has $\Phi(x, 0) + (\Phi(\cdot, 0))^*(x^*) \leq \langle x^*, x\rangle + \varepsilon$. By (1), the condition (LSC) is equivalent to $(\Phi(\cdot, 0))^*(x^*) = \inf_{y^*\in Y^*} \Phi^*(\cdot, y^*)(x^*)$ for all $x^* \in X^*$. Using this in the previous inequality, one gets $\Phi(x, 0) + \inf_{y^*\in Y^*} \Phi^*(x^*, y^*) \leq \langle x^*, x\rangle + \varepsilon$. Since for each

$\eta > 0$ there is a $y^*_\eta \in Y^*$ for which $\Phi^*(x^*, y^*_\eta) \leq \inf_{y^* \in Y^*} \Phi^*(\cdot, y^*)(x^*) + \eta$, fixing an $\eta > 0$ we obtain $\Phi(x, 0) + \Phi^*(x^*, y^*_\eta) \leq \langle x^*, x\rangle + \varepsilon + \eta$, i.e. $(x^*, y^*_\eta) \in \partial_{\varepsilon+\eta}\Phi(x, 0)$. Consequently, $x^* \in \Pr_{X^*}(\partial_{\varepsilon+\eta}\Phi(x, 0))$ whenever $\eta > 0$. Therefore $x^* \in \cap_{\eta>0} \Pr_{X^*}(\partial_{\varepsilon+\eta}\Phi(x, 0))$. ■

Taking as Φ the six perturbation functions considered in the preliminaries, respectively, we obtain equivalent characterizations of formulae for ε-subdifferentials of different combinations of convex functions where lower semicontinuous hulls play no role, as follows.

Proposition 3.2 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_\varepsilon(f + g \circ h)(x) = \bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0,\\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta}} \bigcup_{y^*\in C^*\cap\partial_{\varepsilon_2} g(h(x))} \partial_{\varepsilon_1}(f + (y^*h))(x)$$

if and only if the regularity condition

$$(LSC_1^C) \qquad \inf_{y^*\in C^*}\{g^*(y^*) + (f + (y^*h))^*(\cdot)\} \text{ is lower semicontinuous,}$$

is fulfilled.

Proof The hypotheses imposed on f, g and h ensure the properness, convexity and lower semicontinuity of Φ_1. Note that $\Phi_1(x, 0) = f(x) + g(h(x))$ for all $x \in X$. Let be $\varepsilon \geq 0$. Then $\partial_\varepsilon\Phi_1(\cdot, 0)(x) = \partial_\varepsilon(f + g \circ h)(x)$ for all $x \in X$ and all $\varepsilon \geq 0$.

Let us see, for arbitrarily chosen $x \in X$, $\varepsilon \geq 0$ and $\eta > 0$, how can $\Pr_{X^*}(\partial_{\varepsilon+\eta}\Phi_1(x, 0))$ be written in a simpler way. By definition, $x^* \in X^*$ belongs to this set if and only if there exists $y^* \in Y^*$ such that $\Phi_1(x, 0) + \Phi_1^*(x^*, y^*) \leq \langle x^*, x\rangle + \varepsilon + \eta$, i.e. $f(x) + g(h(x)) + g^*(-y^*) + (f + (-y^*h))^*(x^*) \leq \langle x^*, x\rangle + \varepsilon + \eta$ and $y^* \in -C^*$. This can be rewritten as $(f(x) + (-y^*h)(x) + (f + (-y^*h))^*(x^*) - \langle x^*, x\rangle) + (g(h(x)) + g^*(-y^*) - (-y^*h)(x)) \leq \varepsilon + \eta$ and $y^* \in -C^*$, which means that there are some $\varepsilon_1, \varepsilon_2 \geq 0$ fulfilling $\varepsilon_1 + \varepsilon_2 = \varepsilon + \eta$ such that $x^* \in \partial_{\varepsilon_1}(f + (-y^*h))(x)$ and $-y^* \in \partial_{\varepsilon_2} g(h(x)) \cap (-C^*)$. Thus, $x^* \in \Pr_{X^*}(\partial_{\varepsilon+\eta}\Phi_1(x, 0))$ if and only if there are some $\varepsilon_1, \varepsilon_2 \geq 0$ fulfilling $\varepsilon_1 + \varepsilon_2 = \varepsilon + \eta$ such that $x^* \in \cup_{y^*\in C^*\cap\partial_{\varepsilon_2} g(h(x))} \partial_{\varepsilon_1}(f + (y^*h))(x)$. Then

$$\Pr_{X^*}(\partial_{\varepsilon+\eta}\Phi_1(x, 0)) = \bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0,\\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta}} \bigcup_{y^*\in C^*\cap\partial_{\varepsilon_2} g(h(x))} \partial_{\varepsilon_1}(f + (y^*h))(x). \tag{4}$$

As (4) holds whenever $\eta > 0$, considering in both sides the intersection regarding all $\eta > 0$ and noting that (LSC) turns out to become in this case exactly (LSC_1^C), the desired equivalence follows by Theorem 3.1. ■

Analogously can be proven the following statements, too, with Φ_i, $i \in \{2, \ldots, 6\}$, as perturbation functions, respectively.

Proposition 3.3 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_\varepsilon(f + g \circ h)(x) = \bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_1,\varepsilon_2,\varepsilon_3\geq 0,\\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon+\eta}} \bigcup_{y^*\in C^*\cap\partial_{\varepsilon_2} g(h(x))} \Big(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_3}(y^*h)(x)\Big)$$

if and only if the regularity condition

$$(LSC_2^C) \quad \inf_{z^*\in X^*, y^*\in C^*}\{f^*(z^*) + g^*(y^*) + (y^*h)^*(\cdot - z^*)\} \text{ is lower semicontinuous,}$$

is fulfilled.

Proposition 3.4 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_\varepsilon(f + \delta_{w^{-1}(-C)\cap U})(x) = \bigcap_{\eta>0} \bigcup_{\substack{y^*\in C^*,\\ w(x)\in -C}} \partial_{\varepsilon+\eta+(y^*w)(x)}(f + (y^*w) + \delta_U)(x)$$

if and only if the regularity condition

$$(LSC_1^P) \qquad \inf_{y^*\in C^*} (f + (y^*w) + \delta_U)^* \text{ is lower semicontinuous,}$$

is fulfilled.

Proposition 3.5 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_\varepsilon(f + \delta_{w^{-1}(-C)\cap U})(x) = \bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0, y^*\in C^*, w(x)\in -C,\\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta+(y^*w)(x)}} \big(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}((y^*w) + \delta_U)(x)\big)$$

if and only if the regularity condition

$$(LSC_2^P) \qquad \inf_{y^*\in C^*} (f^*\Box((y^*w) + \delta_U)^*) \text{ is lower semicontinuous,}$$

is fulfilled.

Proposition 3.6 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_\varepsilon(f + \delta_{w^{-1}(-C)\cap U})(x) = \bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_1,\varepsilon_2,\varepsilon_3\geq 0, y^*\in C^*, w(x)\in -C,\\ \varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon+\eta+(y^*w)(x)}} \big(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2}(y^*w)(x) + N_U^{\varepsilon_3}(x)\big)$$

if and only if the regularity condition

$$(LSC_3^P) \qquad \inf_{y^*\in C^*} (f^*\Box(y^*w)^*\Box\sigma_U) \text{ is lower semicontinuous,}$$

is fulfilled.

Proposition 3.7 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_\varepsilon\varphi(\cdot, A\cdot)(x) = \bigcap_{\eta>0} \mathrm{Pr}_{X^*}\Big\{(x^* + A^*y^*, y^*) : (x^*, y^*) \in \partial_{\varepsilon+\eta}\varphi(x, Ax)\Big\}$$

if and only if the regularity condition

$$(LSC_1^\varphi) \qquad \inf_{y^*\in Y^*} \varphi^*(\cdot - A^*y^*, y^*) \text{ is lower semicontinuous,}$$

is fulfilled.

For the following statement, which can be obtained as a consequence from each of Proposition 3.2, Proposition 3.3 and Proposition 3.7, let $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ be proper convex lower semicontinuous functions and $A : X \to Y$ a linear continuous mapping fulfilling the feasibility condition $\mathrm{dom}(f) \cap A^{-1}(\mathrm{dom}(g)) \neq \emptyset$.

Corollary 3.8 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_\varepsilon(f + g \circ A)(x) = \bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0,\\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta}} \big(\partial_{\varepsilon_1} f(x) + A^*\partial_{\varepsilon_2} g(Ax)\big)$$

if and only if the regularity condition

$$(LSC^A) \qquad \inf_{y^* \in Y^*} \{g^*(y^*) + f^*(\cdot - A^* y^*)\} \text{ is lower semicontinuous,}$$

is fulfilled.

This assertion can be further particularized to the following statement, where $f, g : X \to \overline{\mathbb{R}}$ are proper convex lower semicontinuous functions with their domains fulfilling $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$.

Corollary 3.9 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_\varepsilon (f+g)(x) = \bigcap_{\eta > 0} \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \eta}} \big(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x)\big)$$

if and only if the regularity condition

$$(LSC^S) \qquad f^* \square g^* \text{ is lower semicontinuous,}$$

is fulfilled.

4. Other formulae for ε-subdifferentials and ε-optimality conditions

Unless otherwise specified, the functions and sets considered in this section are taken as defined in Subsection 2.2 and Section 3, respectively.

From Lemma 2.1 we know that the condition (LSC) is equivalent to $(\Phi(\cdot, 0))^*(x^*) = \inf_{y^* \in Y^*} \Phi^*(\cdot, y^*)(x^*)$ for all $x^* \in X^*$, while the situation when the infimum in the right-hand side of this equality is for all $x^* \in X^*$ attained can be equivalently characterized, via Lemma 2.3, by the validity of the following closedness type regularity condition

$$(RC) \qquad \mathrm{Pr}_{X^* \times \mathbb{R}}(\mathrm{epi}(\Phi^*)) \text{ is closed.}$$

As we have seen in Theorem 3.1, (LSC) is equivalent to a formula for $\partial_\varepsilon \Phi(\cdot, 0)(x)$, with $x \in X$ and $\varepsilon \geq 0$. To the natural question if is it possible to give another formula, this time equivalent to (RC), for this ε-subdifferential, we answer with the following statement.

Theorem 4.1 *For all $\varepsilon \geq 0$ and all $x \in X$ it holds*

$$\partial_\varepsilon \Phi(\cdot, 0)(x) = \mathrm{Pr}_{X^*}(\partial_\varepsilon \Phi(x, 0)) \tag{5}$$

if and only if (RC) is fulfilled.

Proof The condition (RC) is equivalent to $\mathrm{epi}((\Phi(\cdot, 0))^*) = \cup_{y^* \in Y^*} \mathrm{epi}(\Phi^*(\cdot, y^*))$. Since in general it holds $\mathrm{epi}((\Phi(\cdot, 0))^*) \supseteq \cup_{y^* \in Y^*} \mathrm{epi}(\Phi^*(\cdot, y^*))$, we can notice that (RC) is actually equivalent to $\mathrm{epi}((\Phi(\cdot, 0))^*) \subseteq \cup_{y^* \in Y^*} \mathrm{epi}(\Phi^*(\cdot, y^*))$.

"$\Rightarrow$" Take an arbitrary pair $(x^*, r) \in \mathrm{epi}((\Phi(\cdot, 0))^*)$. This means actually that $(\Phi(\cdot, 0))^*(x^*) \leq r$. Let $x \in \mathrm{dom}(\Phi(\cdot, 0))$ and $\varepsilon = r + \Phi(x, 0) - \langle x^*, x \rangle \geq 0$. Then $(\Phi(\cdot, 0))^*(x^*) + \Phi(x, 0) \leq \langle x^*, x \rangle + \varepsilon$, i.e. $x^* \in \partial_\varepsilon \Phi(\cdot, 0)(x) = \mathrm{Pr}_{X^*}(\partial_\varepsilon \Phi(x, 0))$. Thus there exists $y^*_\varepsilon \in Y^*$ for which $(x^*, y^*_\varepsilon) \in \partial_\varepsilon \Phi(x, 0)$, i.e. $\Phi(x, 0) + \Phi^*(x^*, y^*_\varepsilon) \leq \langle x^*, x \rangle + \varepsilon = \langle x^*, x \rangle + r + \Phi(x, 0) - \langle x^*, x \rangle$. Consequently, $\Phi^*(x^*, y^*_\varepsilon) \leq r$, i.e. $(x^*, r) \in \mathrm{epi}(\Phi^*(\cdot, y^*_\varepsilon)) \subseteq \cup_{y^* \in Y^*} \mathrm{epi}(\Phi^*(\cdot, y^*))$.

"$\Leftarrow$" Let $\varepsilon \geq 0$ and $x \in X$. Choose an arbitrary element $x^* \in \mathrm{Pr}_{X^*}(\partial_\varepsilon \Phi(x,0))$. Then there exists $y^*_{x^*} \in Y^*$ for which $(x^*, y^*_{x^*}) \in \partial_\varepsilon \Phi(x,0)$, i.e. $\Phi(x,0) + \Phi^*(x^*, y^*_{x^*}) \leq \langle x^*, x\rangle + \varepsilon$. Since we get from (1) that $(\Phi(\cdot,0))^*(x^*) \leq \Phi^*(x^*, y^*_{x^*})$, it follows $\Phi(x,0) + (\Phi(\cdot,0))^*(x^*) \leq \langle x^*, x\rangle + \varepsilon$, i.e. $x^* \in \partial_\varepsilon \Phi(\cdot,0)(x)$. Thus $\mathrm{Pr}_{X^*}(\partial_\varepsilon \Phi(x,0)) \subseteq \partial_\varepsilon \Phi(\cdot,0)(x)$ and note that this inclusion holds in the most general setting.

Now let us check the opposite inclusion. If $\Phi(x,0) = +\infty$, then $\partial_\varepsilon \Phi(\cdot,0)(x) = \partial_\varepsilon \Phi(x,0) = \emptyset$. Assume thus further that $\Phi(x,0) \in \mathbb{R}$. For $x^* \in \partial_\varepsilon \Phi(\cdot,0)(x)$, one has $\Phi(x,0) + (\Phi(\cdot,0))^*(x^*) \leq \langle x^*, x\rangle + \varepsilon$. By Lemma 2.3, the condition (RC) is equivalent to $(\Phi(\cdot,0))^*(x^*) = \min_{y^* \in Y^*} \Phi^*(\cdot, y^*)(x^*)$ for all $x^* \in X^*$, thus for every $x^* \in X^*$ there exists $y^*_{x^*} \in Y^*$ for which $(\Phi(\cdot,0))^*(x^*) = \Phi^*(x^*, y^*_{x^*})$. Using this in the previous inequality, one gets $\Phi(x,0) + \Phi^*(x^*, y^*_{x^*}) \leq \langle x^*, x\rangle + \varepsilon$, i.e. $(x^*, y^*_{x^*}) \in \partial_\varepsilon \Phi(x,0)$. Consequently, $x^* \in \mathrm{Pr}_{X^*}(\partial_\varepsilon \Phi(x,0))$. ■

Remark 4 As noted in the beginning of the section, the difference between the conditions (RC) and (LSC) can be clearer observed by comparing the way these regularity conditions can be equivalently written as formulae for the conjugate of $\Phi(\cdot,0)$. The formula equivalent to (LSC) consists of an infimum, while the one which is equivalent to (RC) means that the same infimum is also attained. Thus it is clear that (RC) is stronger than (LSC). When Φ is a perturbation function for a convex optimization problem, (RC) is equivalent to the so-called *stable strong duality*, while (LSC) turns out to mean that for the mentioned primal problem and its conjugate dual obtained by using Φ there is *stable zero duality gap*. An example which underlines the difference between these two situations can be found in [5], for instance. The difference between these two conditions can be seen also when we equivalently characterize them as formulae for the ε-subdifferential of $\Phi(\cdot,0)$, since (RC) can be equivalently written also as $\partial_\varepsilon \Phi(\cdot,0)(x) = \cap_{\eta \geq 0} \mathrm{Pr}_{X^*}(\partial_{\varepsilon+\eta} \Phi(x,0))$ for all $x \in X$. Comparing this to the formula given in Theorem 3.1, we see that the difference consists in the set where η takes values from.

Remark 5 Taking as Φ the six perturbation functions considered in the preliminaries one can obtain equivalent characterizations of the other formulae considered in Section 3 from which the variable η disappears. In this way we rediscover statements from [4, 6, 13]. For other choices of the perturbation function one can rediscover statements from [8, 9], too. We leave these to the interested reader as exercises.

There are different byproducts of the main results in this paper, Theorem 3.1 and Theorem 4.1, some of them listed in the following.

THEOREM 4.2 *If the regularity condition (LSC) is fulfilled, then for all $x \in X$ it holds*

$$\partial \Phi(\cdot,0)(x) = \bigcap_{\eta > 0} \mathrm{Pr}_{X^*}(\partial_\eta \Phi(x,0)).$$

THEOREM 4.3 *If the regularity condition (RC) is fulfilled, then for all $x \in X$ it holds*

$$\partial \Phi(\cdot,0)(x) = \mathrm{Pr}_{X^*}(\partial \Phi(x,0)).$$

Remark 6 Taking Φ to be the perturbation functions Φ_i, $i \in \{1, \ldots, 6\}$, respectively, new formulae for subdifferentials of convexity preserving combinations of functions can be obtained from Theorem 4.2 and Theorem 4.3, some of them rediscovering statements from [2, 5]. The closedness type regularity conditions obtained in this way guarantee different subdifferential formulae and turn out to be weaker than the interiority type regularity conditions considered in the literature for the same purposes (see for instance [15]). Let us also mention that in [10], without lower semicontinuity assumptions on the functions involved, stable strong duality statements are shown to yield special cases of (5).

In [15, Theorem 2.6.2(ii)] it is given a formula for $\partial_\varepsilon h_\Phi^*$, when Φ is proper and convex. Taking Φ to be moreover lower semicontinuous and imposing the feasibility condition $0 \in \mathrm{Pr}_{X^*}(\mathrm{dom}(\Phi^*))$, noting that $h_\Phi^* = \Phi^*(0,\cdot)$ we can obtain from Theorem 3.1 and Theorem 4.1 the following formulae for $\partial_\varepsilon h_\Phi^*$.

Proposition 4.4 *For all $\varepsilon \geq 0$ and all $y^* \in Y^*$ it holds $\partial_\varepsilon h_\Phi^*(y^*) = \bigcap_{\eta>0} \mathrm{Pr}_Y(\partial_{\varepsilon+\eta}\Phi^*(0,\cdot)(y^*))$ if and only if h_Φ is lower semicontinuous.*

Proposition 4.5 *For all $\varepsilon \geq 0$ and all $y^* \in Y^*$ it holds $\partial_\varepsilon h_\Phi^*(y^*) = \mathrm{Pr}_Y(\partial_\varepsilon \Phi^*(0,\cdot)(y^*))$ if and only if $\mathrm{Pr}_{Y\times\mathbb{R}}(\mathrm{epi}(\Phi))$ is closed.*

Concerning the convex optimization problem

$$(P) \qquad \inf_{x\in X} \Phi(x,0),$$

for a fixed $\varepsilon \geq 0$, an element $\bar{x} \in X$ is said to be an *ε-solution* to (P) if $0 \in \partial_\varepsilon \Phi(\cdot,0)(\bar{x})$. From Theorem 3.1 and Theorem 4.1 we deduce the following ε-optimality conditions for (P).

Theorem 4.6

(i) If $\varepsilon \geq 0$, assuming that the regularity condition (LSC) is fulfilled and that $\bar{x} \in X$ is an ε-solution to (P), then for each $\eta > 0$ there exists $\bar{y}_\eta^* \in Y^*$ such that $(0, \bar{y}_\eta^*) \in \partial_{\varepsilon+\eta}\Phi(\bar{x},0)$.

(ii) If $\varepsilon \geq 0$, $\bar{x} \in X$ and for each $\eta > 0$ there exists $\bar{y}_\eta^* \in Y^*$ such that $(0,\bar{y}_\eta^*) \in \partial_{\varepsilon+\eta}\Phi(\bar{x},0)$, then $\bar{x}$ is an ε-solution to (P).

Theorem 4.7

(i) If $\varepsilon \geq 0$, assuming that the regularity condition (RC) is fulfilled and that $\bar{x} \in X$ is an ε-solution to (P), then there exists $\bar{y}^* \in Y^*$ such that $(0,\bar{y}^*) \in \partial_\varepsilon\Phi(\bar{x},0)$.

(ii) If $\bar{x} \in X$ and $\bar{y}^* \in Y^*$ such that $(0,\bar{y}^*) \in \partial_\varepsilon\Phi(\bar{x},0)$ for $\varepsilon \geq 0$, then $\bar{x}$ is an ε-solution to (P).

Remark 7 In Theorem 4.7 we generalize [4, Theorem 4] and its special cases.

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References

[1] R.I. Boţ, S.-M. Grad, and G. Wanka, *On strong and total Lagrange duality for convex optimization problems*, J. Math. Anal. Appl. 337 (2008), pp. 1315–1325.

[2] R.I. Boţ, S.-M. Grad, and G. Wanka, *Generalized Moreau-Rockafellar results for composed convex functions*, Optimization 58 (2009), to appear.

[3] R.I. Boţ, S.-M. Grad, and G. Wanka, *New regularity conditions for Lagrange and Fenchel-Lagrange duality in infinite dimensional spaces*, Math. Inequal. Appl. 12 (2009), pp. 171–189.

[4] R.I. Boţ, I.B. Hodrea, and G. Wanka, *ε-optimality conditions for composed convex optimization problems*, J. Approx. Theory 153 (2008), pp. 108–121.

[5] R.I. Boţ and G. Wanka, *A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces*, Nonlinear Anal. 64 (2006), pp. 2787–2804.

[6] R.S. Burachik and V. Jeyakumar, *A new geometric condition for Fenchel's duality in infinite dimensional spaces*, Math. Program. 104 (2005), pp. 229–233.
[7] R.S. Burachik, V. Jeyakumar, and Z.-Y. Wu, *Necessary and sufficient conditions for stable conjugate duality*, Nonlinear Anal. 64 (2006), pp. 1998–2006.
[8] N. Dinh, T.T.A. Nghia, and G. Vallet, *A closedness condition and its applications to DC programs with convex constraints*, Optimization, to appear, DOI: 10.1080/02331930801951348.
[9] N. Dinh, G. Vallet, and T.T.A. Nghia, *Farkas-type results and duality for DC programs with convex constraints*, J. Convex Anal. 15 (2008), pp. 235–262.
[10] J.-B. Hiriart-Urruty, *ε-subdifferential calculus*, in *Convex analysis and optimization (London, 1980)*, Research Notes in Mathematics Vol. 57, Pitman, Boston, 1982, pp. 43–92.
[11] J.-B. Hiriart-Urruty, M. Moussaoui, A. Seeger, and M. Volle, *Subdifferential calculus without qualification conditions, using approximate subdifferentials: a survey*, Nonlinear Anal. 24 (1995), pp. 1727–1754.
[12] J.-B. Hiriart-Urruty and R.R. Phelps, *Subdifferential calculus using ε-subdifferentials*, J. Funct. Anal. 118 (1993), pp. 154–166.
[13] V. Jeyakumar, A.M. Rubinov, and Z.-Y. Wu, *Generalized Fenchel's conjugation formulas and duality for abstract convex functions*, J. Optim. Theory Appl. 132 (2007), pp. 441–458.
[14] T. Precupanu, *Closedness conditions for the optimality of a family of nonconvex optimization problems*, Math. Operationsforsch. Statist. Ser. Optim. 15 (1984), pp. 339–346.
[15] C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific, River Edge, 2002.