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On totally Fenchel unstable functions in finite dimensional spaces

Radu Ioan Boţ · Andreas Löhne

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Abstract We give an answer to the Problem 11.6 posed by Stephen Simons in his book "From Hahn-Banach to Monotonicity": Do there exist a nonzero finite dimensional Banach space and a pair of extended real-valued, proper and convex functions which is totally Fenchel unstable? The answer is negative.

Keywords conjugate function · Fenchel duality · recession cone

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1 Introduction

Consider E a nontrivial real Banach space and E^* its topological dual space. By $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in E^*$ at $x \in E$. The *Fenchel-Moreau conjugate* of a function $f : E \to \overline{\mathbb{R}} \cup \{\pm\infty\}$ is the function $f^* : E^* \to \overline{\mathbb{R}} = \mathbb{R}$ defined by $f^*(x^*) = \sup_{x\in E}\{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in E^*$. We denote by $\mathrm{dom}(f) = \{x \in E : f(x) < +\infty\}$ its *domain*. We call f *proper* if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in E$.

Having $f, g : E \to \overline{\mathbb{R}}$ two arbitrary proper and convex functions, we say that f and g satisfy *stable Fenchel duality* if for all $x^* \in E^*$, there exists $z^* \in E^*$ such that

$$(f + g)^*(x^*) = f^*(x^* - z^*) + g^*(z^*).$$

If this property holds just for $x^* = 0$, then we obtain the classical Fenchel duality. In this case we say that f and g *satisfy Fenchel duality*. The pair f, g is called *totally Fenchel unstable* (see [4]) if f and g satisfy Fenchel duality but

$$y^*, z^* \in E^* \text{ and } (f + g)^*(y^* + z^*) = f^*(y^*) + g^*(z^*) \Longrightarrow y^* + z^* = 0.$$

The weakest *generalized interior regularity condition* that guarantees stable strong duality for f and g in case the functions are also lower semicontinuous is the celebrated Attouch-Brézis condition (cf. [1]). This condition asks that the cone generated by $\mathrm{dom}(f) - \mathrm{dom}(g)$ is a closed linear subspace.

Radu Ioan Boţ
Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany
Tel.: +49-371-53134463
Fax: +49-371-53122409
E-mail: radu.bot@mathematik.tu-chemnitz.de

Andreas Löhne
Institute of Mathematics, Martin Luther University Halle-Wittenberg, D-06099 Halle, Germany
Tel.: +49-345-5524682
Fax: +49-345-5527005
E-mail: andreas.loehne@mathematik.uni-halle.de

Obviously, stable Fenchel duality implies Fenchel duality, but the converse is not true (see the example in [2], pp. 2798-2799 and Example 11.1 in [4]). Nevertheless, each of these examples, both given in a finite dimensional setting, fails when one tries to verify total Fenchel unstability.

In the infinite dimensional setting the following example of a pair of proper and convex functions f, g, which is totally Fenchel unstable, has been proposed in Example 11.3 in [4]. Let C be a nonempty, bounded, closed and convex subset of E such that there exists an extreme point x_0 of C which is not a support point of C. Recall that if C is a convex subset of E, then $x \in C$ is a *support point* of C if there exists $x^* \in E^* \setminus \{0\}$ such that $\langle x^*, x\rangle = \sup\langle x^*, C\rangle$. We denote by $\delta_D : E \to \overline{\mathbb{R}}$ the *indicator function* of a set $D \subseteq E$ defined as

$$\delta_D(x) = \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{otherwise.} \end{cases}$$

Taking $A := x_0 - C$, $B := C - x_0$, $f := \delta_A$ and $g := \delta_B$, Simons proved in [4] that the pair f, g is totally Fenchel unstable. Let us also mention that an example of a set C and a point x_0 with the above mentioned properties was given in the space ℓ_2, following an idea due to Jonathan Borwein (see [4]).

In finite dimensional spaces a similar example cannot be given, as every extreme point of a convex set is a support point. This fact determined Stephen Simons to formulate the following open problem (Problem 11.6 in [4]).

Problem *Do there exist a nonzero finite dimensional Banach space E and $f, g : E \to \overline{\mathbb{R}}$ proper and convex functions such that the pair f, g is totally Fenchel unstable?*

We show that the answer to this question is negative. This result can be interpreted as follows:
If two proper and convex functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ satisfy Fenchel duality, then there exists at least one element $x^* \in \mathbb{R}^n \setminus \{0\}$, such that $f - \langle x^*, \cdot\rangle$ and g (or f and $g - \langle x^*, \cdot\rangle$) satisfy Fenchel duality, too.

This article contains two proofs of the result. During the reviewing process an anonymous referee proposed an alternative and much shorter proof which is presented at the end of the article in Remark 1. Nevertheless we think that the original proof is worth to mention because it approaches the problem in a more geometrical way.

2 The result

We use the following notation. For a function $f : E \to \overline{\mathbb{R}}$ we denote by $\mathrm{epi}(f) = \{(x, r) \in E \times \mathbb{R} : f(x) \le r\}$ its *epigraph* and by $\bar{f}$ its *lower semicontinuous hull* of f, namely the function of which epigraph is the closure of $\mathrm{epi}(f)$ in $E \times \mathbb{R}$, that is $\mathrm{epi}(\bar{f}) = \mathrm{cl}(\mathrm{epi}(f))$. We write $\omega(E^*, E)$ for the weak* topology on E^*. Further, when $D \subseteq \mathbb{R}^n$ is a nonempty and convex set by 0^+D we denote its *recession cone.*

The following preliminary result (see [2, Theorem 2.1]) is a direct consequence of the classical Moreau-Rockafellar theorem.

Theorem 1 *If $f, g : E \to \overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions such that* $\mathrm{dom}(f) \cap \mathrm{dom}(g) \ne \emptyset$. *Then*

$$\mathrm{epi}((f + g)^*) = \mathrm{cl}(\mathrm{epi}(f^*) + \mathrm{epi}(g^*)),$$

where the closure is taken in the product topology of $(E^*, \omega(E^*, E)) \times \mathbb{R}$.

Under the hypotheses of Theorem 1 it follows that $\mathrm{epi}(f^*)+\mathrm{epi}(g^*)$ is closed in the product topology of $(E^*, \omega(E^*, E))\times\mathbb{R}$ if and only if $\mathrm{epi}((f+g)^*) = \mathrm{epi}(f^*)+\mathrm{epi}(g^*)$. By [2, Proposition 2.2], this is equivalent to saying that f and g satisfy stable Fenchel duality.

Of course, for all $x^*, y^* \in E^*$ it holds

$$(f+g)^*(x^*) \leq f^*(x^*-y^*) + g^*(y^*). \tag{1}$$

Therefore, a pair f, g of proper and convex functions is *totally Fenchel unstable* if and only if

$$\exists y^* \in E^* : \ \ (f+g)^*(0) = f^*(-y^*) + g^*(y^*). \tag{2}$$

$$\forall x^* \in E^* \setminus \{0\}, \forall y^* \in E^* : \ \ (f+g)^*(x^*) < f^*(x^*-y^*) + g^*(y^*). \tag{3}$$

Moreover, if the pair f, g is totally Fenchel unstable one must have that $\mathrm{dom}(f)\cap \mathrm{dom}(g) \neq \emptyset$. Indeed, if this is not the case, then $f+g$ is identical $+\infty$ and thus $(f+g)^*$ is identical $-\infty$. By (2) there exists $y^* \in E^*$ such that $f^*(-y^*)+g^*(y^*) = -\infty$. But, f and g being proper we get $f^*(-y^*) > -\infty$ and $g^*(y^*) > -\infty$, a contradiction.

We give now a geometric characterization of the property that the pair f, g is totally Fenchel unstable.

Proposition 1 *Let $f, g : E \to \overline{\mathbb{R}}$ be proper functions such that* $\mathrm{dom}(f)\cap\mathrm{dom}(g) \neq \emptyset$*. Then the pair f, g is totally Fenchel unstable if and only if*

$$\mathrm{epi}((f+g)^*) \cap (\{0\}\times\mathbb{R}) = (\mathrm{epi}(f^*)+\mathrm{epi}(g^*)) \cap (\{0\}\times\mathbb{R}) \tag{4}$$

and there is no $x^* \in E^*\setminus\{0\}$ such that

$$\mathrm{epi}((f+g)^*) \cap (\{x^*\}\times\mathbb{R}) = (\mathrm{epi}(f^*)+\mathrm{epi}(g^*)) \cap (\{x^*\}\times\mathbb{R}). \tag{5}$$

Proof We want to notice first that we always have $\mathrm{epi}((f+g)^*) \supseteq \mathrm{epi}(f^*)+\mathrm{epi}(g^*)$. As $\mathrm{dom}(f)\cap\mathrm{dom}(g) \neq \emptyset$, $(f+g)^*$ never attains $-\infty$.

"$\Rightarrow$" In case $(f+g)^*(0) = +\infty$, the set $\mathrm{epi}((f+g)^*) \cap (\{0\}\times\mathbb{R})$ is empty and (4) follows automatically. In case $(f+g)^*(0) \in \mathbb{R}$, we consider an arbitrary element $r \in \mathbb{R}$ fulfilling $(f+g)^*(0) \leq r$. By (2) there exists $y^* \in E^*$ such that $f^*(-y^*)+g^*(y^*) \leq r$ and so

$$(0, r) = (-y^*, f^*(-y^*)) + (y^*, r - f^*(-y^*)) \in (\mathrm{epi}(f^*)+\mathrm{epi}(g^*)) \cap (\{0\}\times\mathbb{R}).$$

Also in this case (4) follows.

Assume now that for $x^* \in E^*\setminus\{0\}$ relation (5) is fulfilled. As (3) implies $(f+g)^*(x^*) < +\infty$, we have $(f+g)^*(x^*) \in \mathbb{R}$. In this case $(x^*, (f+g)^*(x^*)) \in \mathrm{epi}((f+g)^*)\cap(\{x^*\}\times\mathbb{R})$ and so $(x^*, (f+g)^*(x^*)) \in \mathrm{epi}(f^*)+\mathrm{epi}(g^*)$. Thus there exist $(y^*, s) \in \mathrm{epi}(f^*)$ and $(z^*, t) \in \mathrm{epi}(g^*)$ such that $y^*+z^* = x^*$ and $s+t = (f+g)^*(x^*)$. This means that $f^*(y^*)+g^*(z^*) \leq (f+g)^*(y^*+z^*)$ which contradicts (3).

"$\Leftarrow$" We prove first that Fenchel duality holds. If $(f+g)^*(0) = +\infty$ this follows automatically from (1). If $(f+g)^*(0) \in \mathbb{R}$, then $(0, (f+g)^*(0)) \in \mathrm{epi}(f^*)+\mathrm{epi}(g^*)$ and so there exist $(-z^*, s) \in \mathrm{epi}(f^*)$ and $(z^*, t) \in \mathrm{epi}(g^*)$ such that $s+t = (f+g)^*(0)$. Thus $f^*(-z^*)+g^*(z^*) \leq (f+g)^*(0)$ and the conclusion follows.

Further assume that there exist $y^*, z^* \in E^*$ such that $y^*+z^* \neq 0$ and $(f+g)^*(y^*+z^*) = f^*(y^*)+g^*(z^*)$. As (5) does not hold with equality, we get $(f+g)^*(y^*+z^*) \in \mathbb{R}$. For all $r \in \mathbb{R}$ such that $(f+g)^*(y^*+z^*) \leq r$ it holds $(y^*+z^*, r) \in (\mathrm{epi}(f^*)+\mathrm{epi}(g^*)) \cap (\{y^*+z^*\}\times\mathbb{R})$. This implies that (5) is satisfied for $x^* = y^*+z^* \neq 0$, a contradiction. □

Proposition 2 *Let $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions with* $\mathrm{int}(\mathrm{dom}(\bar{f})\cap\mathrm{dom}(\bar{g})) \neq \emptyset$*. Then the pair f, g satisfies stable Fenchel duality.*

Proof Let $x' \in \operatorname{int}(\operatorname{dom}(\bar{f}) \cap \operatorname{dom}(\bar{g})) \subseteq \operatorname{int}(\operatorname{dom}(\bar{f})) \cap \operatorname{int}(\operatorname{dom}(\bar{g}))$. We have that $\operatorname{int}(\operatorname{dom}(\bar{f})) = \operatorname{ri}(\operatorname{dom}(\bar{f})) = \operatorname{ri}(\operatorname{cl}(\operatorname{dom}(\bar{f}))) = \operatorname{ri}(\operatorname{cl}(\operatorname{dom}(f))) = \operatorname{ri}(\operatorname{dom}(f))$ and the same applies for g. This means that $x' \in \operatorname{ri}(\operatorname{dom}(f)) \cap \operatorname{ri}(\operatorname{dom}(g))$. For all $x^* \in \mathbb{R}^n$ we have $\operatorname{dom}(f) = \operatorname{dom}(f - \langle x^*, \cdot \rangle)$. By the Fenchel duality theorem [3, Theorem 31.1], there exists some $y^* \in \mathbb{R}^n$ such that

$$\begin{aligned} -(f+g)^*(x^*) &= \inf_{x \in \mathbb{R}^n} \{ f(x) - \langle x^*, x \rangle + g(x) \} \\ &= -(f - \langle x^*, \cdot \rangle)^*(-y^*) - g^*(y^*) \\ &= -f^*(x^* - y^*) - g^*(y^*). \end{aligned}$$

□

We now turn to the main result.

Theorem 2 *There are no proper convex functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that the pair f, g is totally Fenchel unstable.*

Proof We assume the contrary, namely that there exist $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$ proper convex functions such that the pair f, g is totally Fenchel unstable. By (3) it follows that $(f+g)^*(x^*) < +\infty$ for all $x^* \in \mathbb{R}^n \setminus \{0\}$. As $(f+g)^*$ is convex, we get $(f+g)^*(0) < +\infty$. As noticed above we have $\operatorname{dom}(f) \cap \operatorname{dom}(g) \neq \emptyset$, hence $(f+g)^*(0) > -\infty$.

As noticed above, $\operatorname{dom}(f) \cap \operatorname{dom}(g)$ must be nonempty. Choose some $\bar{x} \in \operatorname{dom}(f) \cap \operatorname{dom}(g) \subseteq \operatorname{dom}(\bar{f}) \cap \operatorname{dom}(\bar{g})$ and consider $L = \operatorname{aff}(\operatorname{dom}(\bar{f}) \cap \operatorname{dom}(\bar{g}) - \bar{x}) = \operatorname{lin}(\operatorname{dom}(\bar{f}) \cap \operatorname{dom}(\bar{g}) - \bar{x})$. As $\operatorname{int}(\operatorname{dom}(\bar{f}) \cap \operatorname{dom}(\bar{g})) = \emptyset$, by Proposition 2, the dimension of L is strictly less than n and this means that the orthogonal space to L, $L^\perp$ is nonzero. Of course, we have

$$\operatorname{dom}(f) \cap \operatorname{dom}(g) \subseteq \operatorname{dom}(\bar{f}) \cap \operatorname{dom}(\bar{g}) \subseteq L + \bar{x} \tag{6}$$

Theorem 1 applies to $\bar{f}$ and $\bar{g}$ and we have $f^* = \bar{f}^*$ and $g^* = \bar{g}^*$. Hence

$$\operatorname{epi}((\bar{f} + \bar{g})^*) = \operatorname{cl}(\operatorname{epi}(f^*) + \operatorname{epi}(g^*)). \tag{7}$$

It follows

$$\operatorname{epi}((f+g)^*) \supseteq \operatorname{epi}((\bar{f} + \bar{g})^*) \supseteq \operatorname{epi}(f^*) + \operatorname{epi}(g^*).$$

Since the pair f, g is totally Fenchel unstable, by Proposition 1, one has that

$$\operatorname{epi}(f+g)^* \cap (\{0\} \times \mathbb{R}) = \operatorname{epi}((\bar{f} + \bar{g})^*) \cap (\{0\} \times \mathbb{R}) = (\operatorname{epi}(f^*) + \operatorname{epi}(g^*)) \cap (\{0\} \times \mathbb{R})$$

and so $(f+g)^*(0) = (\bar{f} + \bar{g})^*(0)$. Taking an element $x^* \in L^\perp \setminus \{0\}$ we obtain

$$\begin{aligned} (f+g)^*(x^*) &= \sup_{x \in \mathbb{R}^n} \{ \langle x^*, x \rangle - f(x) - g(x) \} \\ &\overset{(6)}{=} \sup_{x \in L + \bar{x}} \{ \langle x^*, x \rangle - f(x) - g(x) \} \\ &= \langle x^*, \bar{x} \rangle + (f+g)^*(0) \\ &= \langle x^*, \bar{x} \rangle + (\bar{f} + \bar{g})^*(0) \overset{(6)}{=} (\bar{f} + \bar{g})^*(x^*). \end{aligned} \tag{8}$$

We distinguish two cases:

(a) If $\operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ is closed, we obtain from (7) and (8), $(x^*, (f+g)^*(x^*)) \in \operatorname{epi}((\bar{f} + \bar{g})^*) = \operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ and so there exist $(y^*, s) \in \operatorname{epi}(f^*)$ and $(z^*, t) \in \operatorname{epi}(g^*)$ such that $y^* + z^* = x^* \neq 0$ and $s + t = (f+g)^*(x^*)$. This means that $f^*(y^*) + g^*(z^*) \leq (f+g)^*(y^* + z^*)$. As $y^* + z^* = x^* \neq 0$ this contradicts (3).

(b) Otherwise, if $\operatorname{epi}(f^*) + \operatorname{epi}(g^*)$ is not closed, by [3, Corollary 9.1.2], there exists a direction of recession of $\operatorname{epi}(f^*)$ whose opposite direction is a direction of recession of $\operatorname{epi}(g^*)$. This can be expressed as

$$\exists\, (x^*, r) \neq 0 : \quad (x^*, r) \in 0^+ \operatorname{epi}(f^*) \quad \wedge \quad (-x^*, -r) \in 0^+ \operatorname{epi}(g^*),$$

where r can be chosen nonnegative. It follows $x^* \neq 0$, because otherwise we would have $(0, -r) \in 0^+ \operatorname{epi}(g^*)$ with $r > 0$. But g is proper and so g^* never attains $-\infty$.

Choose some y^* according to (2). Since $(f+g)^*(0), f^*(-y^*), g^*(y^*) \in \mathbb{R}$ and as $\operatorname{epi}(f^*)$ and $\operatorname{epi}(g^*)$ are nonempty convex sets, by [3, Theorem 8.1], it holds

$$\begin{aligned} \forall \lambda \geq 0: &\quad (-y^*, f^*(-y^*)) + \lambda \cdot (x^*, r) \in \operatorname{epi}(f^*) \\ \forall \mu \geq 0: &\quad (y^*, g^*(y^*)) \;\; - \mu \cdot (x^*, r) \in \operatorname{epi}(g^*) \end{aligned}$$

Adding both conditions and taking into account (2) we get

$$\forall \gamma \in \mathbb{R}: \quad (0, (f+g)^*(0)) + \gamma \cdot (x^*, r) \in \operatorname{epi}(f^*) + \operatorname{epi}(g^*). \tag{9}$$

Let $\gamma = 1$ in (9). There exist $(u^*, s) \in \operatorname{epi}(f^*)$ and $(v^*, t) \in \operatorname{epi}(g^*)$ such that $u^* + v^* = x^*$ and $s + t = (f+g)^*(0) + r$. It follows that

$$(f+g)^*(x^*) \leq f^*(u^*) + g^*(v^*) \leq s + t = (f+g)^*(0) + r. \tag{10}$$

Setting $\gamma = -1$ in (9), we obtain analogously

$$(f+g)^*(-x^*) \leq (f+g)^*(0) - r. \tag{11}$$

The conditions (10) and (11) must hold with equality. Indeed, adding both inequalities where one of them is strict, we get a contradiction to the fact that $(f+g)^*$ is convex. Hence $(f+g)^*(u^* + v^*) = f^*(u^*) + g^*(v^*)$. This contradicts (3), because of $u^* + v^* = x^* \neq 0$. □

Remark 1 We want to mention that after the paper was submitted Constantin Zălinescu provided us an alternative proof for the Problem 11.6 posed by Stephen Simons in [4].

On the other hand we find it worth mentioning the following proof for Theorem 2 proposed by one of the referees of the paper. Assuming that the pair f, g is totally Fenchel unstable, as noticed, we have $\operatorname{dom}(f) \cap \operatorname{dom}(g) \neq \emptyset$. As follows from the proof of Proposition 2, in case $\operatorname{ri}(\operatorname{dom}(f)) \cap \operatorname{ri}(\operatorname{dom}(g)) \neq \emptyset$, f and g satisfy stable Fenchel duality and the conclusion follows. Assume now that $\operatorname{ri}(\operatorname{dom}(f)) \cap \operatorname{ri}(\operatorname{dom}(g)) = \emptyset$ and consider an element z in $\operatorname{dom}(f) \cap \operatorname{dom}(g) = \operatorname{dom}(f+g)$. By [3, Theorem 31.1] there exists $z^* \in \mathbb{R}^n \setminus \{0\}$ such that

$$\langle z^*, x \rangle \leq \langle z^*, y \rangle, \ \forall x \in \operatorname{dom}(f) \ \forall y \in \operatorname{dom}(g),$$

which yields

$$\langle z^*, x \rangle \leq \langle z^*, y \rangle = \langle z^*, z \rangle, \ \forall x \in \operatorname{dom}(f) \ \forall y \in \operatorname{dom}(f) \cap \operatorname{dom}(g).$$

Thus

$$\begin{aligned} f^*(z^* - y^*) &= \sup_{x \in \operatorname{dom}(f)} \{\langle z^* - y^*, x \rangle - f(x)\} \\ &\leq \langle z^*, z \rangle + \sup_{x \in \operatorname{dom}(f)} \{\langle -y^*, x \rangle - f(x)\} = \langle z^*, z \rangle + f^*(-y^*) \end{aligned}$$

and so, using (5),

$$\begin{aligned} (f+g)^*(z^*) &= \sup_{y \in \operatorname{dom}(f+g)} \{\langle z^*, y \rangle - (f+g)(y)\} \\ &= \langle z^*, z \rangle + \sup_{y \in \operatorname{dom}(f+g)} \{-(f+g)(y)\} = \langle z^*, z \rangle + (f+g)^*(0) \\ &= \langle z^*, z \rangle + f^*(-y^*) + g^*(y^*) \geq f^*(z^* - y^*) + g^*(y^*). \end{aligned}$$

Since $(z^* - y^*) + y^* = z^* \neq 0$, this contradicts the total Fenchel instability of the pair f, g.

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