Dual Representations for Convex Risk Measures via Conjugate Duality [1]

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Abstract. The aim of this paper is to give dual representations for different convex risk measures by employing their conjugate functions. To establish the formulas for the conjugates, we use on the one hand some classical results from convex analysis and on the other hand some tools from the conjugate duality theory. Some characterizations of so-called deviation measures recently given in the literature turn out to be direct consequences of our results.

Key Words. Conjugate functions, Conjugate duality, Convex risk measures, Convex deviation measures.

1. Introduction

In many practical applications, such as appear in portfolio optimization, the notion of *risk* plays an important role. It reflects the uncertainty of some processes and the biggest challenge in this context consists in quantifying it by an appropriate measure. Until now, different formulations for the so-called *risk measure* have been considered. A classical application in financial mathematics is the *portfolio optimization problem* treated by Markowitz [1], where the risk of a portfolio is measured by means of the *standard deviation* and *variance*, respectively.

In 1999 Artzner et al. [2] first gave an axiomatic definition of *coherent risk measure*. The properties stipulated for this class of measures seem to be common in many practical problems. In 2002, Rockafellar et al. [3] introduced a new class of measures, called *deviation measures*, closely related to the coherent risk measures. An important representative of this class of measures is the *variance*. It is remarkable that the coherent risk measures as well as the deviation measures fulfill some positive homogeneity and subadditivity properties. As many risk measures used in practice are not endowed with these properties, the class of coherent risk measures has been extended to the class of *convex risk measures* (see for example [4–6]) in the definition of which sublinearity is replaced by convexity. Recent papers where several theoretical results concerning convex risk measures have been given are those of Pflug ([7]) and Ruszczynski and Shapiro ([8, 9]).

In [8], some necessary and sufficient conditions for the optimal solutions of optimization problems with convex risk measures as objective functions are given, whereas in [7] the author provides some dual representations for a number of convex risk and deviation measures with practical relevance.

In this paper, we consider different convex risk and deviation measures (some of them also investigated by Pflug [7]) and calculate their conjugate functions. To this end, we use the powerful theory of *conjugate functions* from convex analysis as well as several *duality results* for convex optimization problems in separated locally convex spaces. By making use of the the Fenchel-Moreau theorem, we also give dual representations for all measures we deal with. In this way, we extend and improve the results obtained by Pflug [7].

Optimality conditions for portfolio optimization problems involving different convex risk measures have been formulated by Rockafellar and its coauthors ([10, 11]). In this context, we refer also to [12] where the necessary and sufficient optimality conditions have been derived via the duality theory. For the investigations in [12], having formulas for the conjugates of the considered measure functions is of big importance. It is worth mentioning that, in [13], in connection to a portfolio optimization problem an l_∞ function was used as risk measure, whereby the optimality conditions were derived by employing duality theory too.

The paper is organized as follows. In Section 2, we introduce some notations and preliminary results from convex analysis as well as from stochastic theory. Further, in Section 3, we introduce the notion of convex risk measure and, closely connected with it, that of convex deviation measure. Then, we furnish some examples for both classes of measures. Section 4 is devoted to the calculation of the conjugate functions of some classical convex risk and deviation measures. In Section 5 we deal with some elaborated convex risk and deviation measures and calculate their conjugates by using the general formula of the conjugate of a composite convex function. In Section 6, some dual representations for the convex risk and deviation measures considered in the previous two sections are provided and a comparison with the results obtained by Pflug in [7] is made. A conclusive section closes the paper.

2. Notations and Preliminary Results

Let $\mathcal{Z}$ be a separated locally convex space and $\mathcal{Z}^*$ its topological dual space endowed with the weak* topology. We denote by $\langle x^*, x\rangle := x^*(x)$ the value of the linear continuous functional $x^* \in \mathcal{Z}^*$ at $x \in \mathcal{Z}$.

For a set $D \subseteq \mathcal{Z}$ we denote by int(D) its *interior* and by

$$\mathrm{core(D)} = \{d \in D : \forall x \in \mathcal{Z}\ \exists \varepsilon > 0 : \forall \lambda \in [-\varepsilon, \varepsilon]\ d + \lambda x \in D\}$$

its *algebraic interior*. One always has that $\mathrm{int(D)} \subseteq \mathrm{core(D)}$. The *indicator function* $\delta_D : \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ of the set D is defined by

$$\delta_D(x) = 0, \text{ if } x \in D, \delta_D(x) = +\infty, \text{ otherwise.}$$

Given a function $f : \mathcal{Z} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, we consider the *(Fenchel-Moreau) conjugate function of f*, $f^* : \mathcal{Z}^* \to \overline{\mathbb{R}}$, defined by

$$f^*(x^*) = \sup_{x \in \mathcal{Z}} \{\langle x^*, x\rangle - f(x)\}.$$

Similarly, the *biconjugate function of f*, $f^{**} : \mathcal{Z} \to \overline{\mathbb{R}}$, is defined by

$$f^{**}(x) = \sup_{x^* \in \mathcal{Z}^*} \{\langle x^*, x\rangle - f^*(x^*)\}.$$

Further, for the function $f : \mathcal{Z} \to \overline{\mathbb{R}}$, we consider also its *epigraph*

$$\mathrm{epi}(f) = \{(x, r) : x \in \mathcal{Z}, r \in \mathbb{R} : f(x) \leq r\}$$

and its *effective domain*

$$\mathrm{dom}(f) = \{x \in \mathcal{Z} : f(x) < +\infty\}.$$

We say that f is *proper* if $\mathrm{dom}(f) \neq \emptyset$ and $f(x) > -\infty, \forall x \in \mathcal{Z}$. By the *Fenchel-Moreau theorem*, whenever $f : \mathcal{Z} \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, it holds that $f = f^{**}$.

The next result recalled here provides a sufficient condition for the formula of the conjugate of the composition of a convex function with a linear continuous mapping ([14]). In what follows, $\mathcal{U}$ is another separated locally convex space. All around this paper, we write min (max) instead of inf (sup) when the infimum (supremum) is attained.

Theorem 2.1 ([14]) Let $f : \mathcal{Z} \to \overline{\mathbb{R}}$ be a proper and convex function and $A : \mathcal{U} \to \mathcal{Z}$ a linear continuous mapping. Assume that there exists $x' \in A^{-1}(\mathrm{dom}(f))$ such that f is continuous at Ax'. Then,

$$(f \circ A)^*(u^*) = \min\{f^*(z^*) : \ A^*z^* = u^*\}, \forall u^* \in \mathcal{U}^*. \tag{1}$$

Let us notice that in the literature one can also find further sufficient conditions for (1). We refer to [14] for other *generalized interior point* conditions and to [15] for a so-called *closedness-type* condition.

In the following we turn our attention to the *Lagrange duality* in connection to the optimization problem with geometric and cone constraints

$$(P) \quad \inf_{x \in T} f(x),$$

$$T = \{x \in S : g(x) \in -C\},$$

where $\mathcal{Z}$ and $\mathcal{Y}$ are two separated locally convex spaces, the latter being partially ordered by the nonempty convex cone $C \subseteq \mathcal{Y}$, $S \subseteq \mathcal{Z}$ is a nonempty and convex set, $f : \mathcal{Z} \to \overline{\mathbb{R}}$ a proper and convex function and $g : \mathcal{Z} \to \mathcal{Y}^\bullet$ a proper and C-convex function fulfilling $\operatorname{dom} f \cap S \cap g^{-1}(-C) \neq \emptyset$. Denoting by $\leq_C$ the partial ordering induced by C on $\mathcal{Y}$, to the latter we attach an abstract maximal element with respect to $\leq_C$, denoted by ∞_C and let $\mathcal{Y}^\bullet := \mathcal{Y} \cup \{\infty_C\}$. Then for every $y \in \mathcal{Y}$ one has $y \leq_C \infty_C$, while on $\mathcal{Y}^\bullet$ the following operations are considered: $y + \infty_C = \infty_C + y = \infty_C$ and $t\infty_C = \infty_C$ for all $y \in \mathcal{Y}$ and all $t \geq 0$. Moreover, if $\lambda \in C^* = \{y^* \in \mathcal{Y}^* : \langle y^*, y \rangle \geq 0, \forall y \in C\}$, which is the *positive dual cone* of C, we let $\langle \lambda, \infty_C \rangle := +\infty$.

For the function $g : \mathcal{Z} \to \mathcal{Y}^\bullet$ we denote by $\operatorname{dom} g = \{x \in \mathcal{Z} : g(x) \in \mathcal{Y}\}$ its *domain* and by $\operatorname{epi}_C g = \{(x, y) \in \mathcal{Z} \times \mathcal{Y} : g(x) \leq_C y\}$ its *C-epigraph*. We say that g is *proper* if its domain is a nonempty set. The function g is said to be *C-convex* if $\operatorname{epi}_C g$ is a convex subset of $\mathcal{Z} \times \mathcal{Y}$, while g is said to be *C-epi closed* if $\operatorname{epi}_C g$ is a closed subset of $\mathcal{Z} \times \mathcal{Y}$ ([16]).

The *Lagrange dual* problem to (P) is

$$(\mathrm{D}) \ \sup_{\lambda \in C^*} \inf_{x \in S} \{f(x) + \langle \lambda, g(x) \rangle\}$$

and for the primal-dual pair (P) − (D) weak duality is always fulfilled. In order to formulate a strong duality result for (P) and (D) one needs to have a so-called regularity condition fulfilled. In the literature one can find, both, *generalized interior point* and *closedness-type* regularity conditions for Lagrange duality and for the relations between these two classes of conditions we refer to [17]. In this paper, we deal with two regularity conditions of the first type, which are stated in the following ([17]):

$$(\mathrm{SC}) \qquad \exists x' \in \operatorname{dom} f \cap S \text{ such that } g(x') \in -\operatorname{int}(C)$$

and

$$(\mathrm{RC}) \qquad \mathcal{Z} \text{ and } \mathcal{Y} \text{ are Fréchet spaces}, S \text{ is closed}, f \text{ is lower semicontinuous},$$
$$g \text{ is } C\text{-epi closed and } 0 \in \operatorname{core}\Big(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C\Big).$$

While (SC) is the classical *Slater constraint qualification*, for an incipient work dealing with regularity conditions involving generalizations of the interior, as happens for (RC), we refer to [18]. Denoting by $v(\mathrm{P})$ and $v(\mathrm{D})$ the optimal objective values of (P) and (D), respectively, we can state the following strong duality result ([14, 17, 18]).

Theorem 2.2 If (SC) or (RC) is fulfilled, then $v(\mathrm{P}) = v(\mathrm{D})$ and the Lagrange dual has an optimal solution.

Let us mention that in order to have strong Lagrange duality one can consider in (RC) instead of the algebraic interior more general interiority notions, like the so-called *strong quasi-relative interior*. As one will see later working with the algebraic interior is sufficient for our aims.

Consider now the *probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$, where Ω is a *basic space*, $\mathfrak{F}$ a *σ-algebra* on Ω and $\mathbb{P}$ a *probability measure* on the measurable space $(\Omega, \mathfrak{F})$. For a measurable random variable $x : \Omega \to \mathbb{R}$ the *expectation value* is defined with respect to $\mathbb{P}$ by

$$\mathbb{E}(x) = \int_{\Omega} x(\omega) d\mathbb{P}(\omega).$$

The *essential supremum* of x is $\operatorname{essup} x = \inf\{a \in \mathbb{R} : \mathbb{P}(\omega : x(\omega) > a) = 0\}$. Furthermore, for $p \in (1, +\infty)$ let L_p be the following linear space of random variables:

$$L_p := L_p(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}) = \left\{ x : \Omega \to \mathbb{R},\ x \text{ measurable}, \int_{\Omega} |x(\omega)|^p d\mathbb{P}(\omega) < +\infty \right\}.$$

The space L_p equipped with the norm $||x||_p = (\mathbb{E}(|x|^p))^{\frac{1}{p}}$ for $x \in L_p$ is a reflexive

Banach space. It is well-known that the dual space of L_p is $L_q := L_q(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R})$, where $q \in (1, +\infty)$ fulfills $\frac{1}{p} + \frac{1}{q} = 1$. The closed unit ball in L_q is denoted by $\overline{B}_q(0, 1)$.

For $x \in L_p$, $x^* \in L_q$ and $x^*x : \Omega \to \mathbb{R}$, where $(x^*x)(\omega) := x^*(\omega) \cdot x(\omega)$, one can define now

$$\langle x^*, x\rangle := \mathbb{E}(x^*x) = \int_{\Omega} x^*(\omega)x(\omega)d\mathbb{P}(\omega).$$

Equalities and inequalities between random variables are to be viewed in the sense of holding almost surely (a.s.). Thus for $x, y : \Omega \to \mathbb{R}$ when we write $x = y$ or $x \geq y$ we mean $x = y$ a.s. or $x \geq y$ a.s., respectively. For $p \in (1, +\infty)$, the cone

$$(L_p)_+ = \{x \in L_p : \ x \geq 0 \text{ a.s.}\}$$

is inducing the partial ordering denoted by "$\geq$". The dual cone of $(L_p)_+$ is $(L_q)_+$, where $q \in (1, +\infty)$ fulfills $\frac{1}{p} + \frac{1}{q} = 1$. The partial ordering induced by $(L_q)_+$ is also denoted by "$\geq$". As these orderings are given in different linear spaces, no confusion is possible.

Having a random variable $x : \Omega \to \mathbb{R}$ which takes the constant value $c \in \mathbb{R}$, i.e. $x = c$ a.s., we identify it with the real number $c \in \mathbb{R}$.

For an arbitrary random variable $x : \Omega \to \mathbb{R}$ we also define $x_-, x_+ : \Omega \to \mathbb{R}$ in the following way $x_-(\omega) := \max(-x(\omega), 0)$ for all $\omega \in \Omega$ and $x_+(\omega) := \max(x(\omega), 0)$ for all $\omega \in \Omega$, respectively. One can easily see that $x = x_+ - x_-$, $x_+ = (-x)_-$ and $x_- = (-x)_+$.

3. Risk Measures and Deviation Measures

In this section we give some formal definitions of the convex risk and deviation measures. In 2002, Föllmer and Schied [4] first introduced the convex risk measures as an extension of the well-known coherent risk measures. The latter have been introduced in [2], where for the first time an axiomatic way for defining risk measures has been given. Rockafellar and his coauthors (see [3]) introduced along the coherent risk measures the so-called deviation measures and studied the relation between these concepts. Here we deal with the broad class of convex risk measures as it was also done by Ruszczynski and Shapiro ([8]) and Pflug ([7]), respectively. We want to notice that a large number of risk functions mentioned in the literature does not have the sublinearity properties asked by the axioms of a coherent risk measure, however the properties in the definition of a convex risk measure are fulfilled. In the following definition we introduce the notion of a *convex risk measure* as it appears in [7].

Definition 3.1 The function $\rho : L_p \to \overline{\mathbb{R}}$ is called a convex risk measure if the following properties are fulfilled:

(R1) *Translation invariance:* $\rho(x + b) = \rho(x) - b, \quad \forall x \in L_p, \ \forall b \in \mathbb{R};$

(R2) *Strictness:* $\rho(x) \geq -\mathbb{E}(x), \quad \forall x \in L_p;$

(R3) *Convexity:* $\rho(\lambda x + (1-\lambda)y) \leq \lambda\rho(x) + (1-\lambda)\rho(y), \quad \forall \lambda \in [0,1], \forall x, y \in L_p.$

For certain applications it can be useful to postulate some *monotonicity* properties for the risk measure, like, for example, *the monotonicity with respect to the pointwise ordering*:

$$x \geq y \quad \Rightarrow \quad \rho(x) \geq \rho(y), \qquad \forall x, y \in L_p.$$

Closely related to the risk measure one can define the so-called *convex deviation measure.*

Definition 3.2 The function $d : L_p \to \overline{\mathbb{R}}$ is called a convex deviation measure if the following properties are fulfilled:

(D1) *Translation invariance:* $d(x+b) = d(x), \quad \forall x \in L_p, \ \forall b \in \mathbb{R};$

(D2) *Strictness:* $d(x) \geq 0, \quad \forall x \in L_p;$

(D3) *Convexity:* $d(\lambda x + (1-\lambda)y) \leq \lambda d(x) + (1-\lambda)d(y), \quad \forall \lambda \in [0,1], \forall x, y \in L_p.$

The following theorem states the connection between convex risk and convex deviation measures (see [3], [7], [11], [19]).

Theorem 3.1 The function $\rho : L_p \to \overline{\mathbb{R}}$ is a convex risk measure if and only if $d : L_p \to \overline{\mathbb{R}}, \ d(x) = \rho(x) + \mathbb{E}(x)$ for $x \in L_p$, is a convex deviation measure.

Next we give some examples of convex risk measures and corresponding deviation measures.

Example 3.1 Consider first, for $p = 2$, $\rho : L_2 \to \mathbb{R}$ defined by

$$\rho(x) = ||x - \mathbb{E}(x)||_2^2 - \mathbb{E}(x), \quad x \in L_2.$$

This is a convex risk measure and it is closely related to the classical *variance* $\sigma^2(x)$ which is its corresponding deviation measure

$$d(x) = \sigma^2(x) = ||x - \mathbb{E}(x)||_2^2, \quad x \in L_2.$$

Example 3.2 Let be again $p = 2$ and $\rho : L_2 \to \mathbb{R}$ defined by

$$\rho(x) = ||x - \mathbb{E}(x)||_2 - \mathbb{E}(x), \quad x \in L_2.$$

The related convex deviation measure is the *standard deviation* $\sigma(x)$

$$d(x) = \sigma(x) = ||x - \mathbb{E}(x)||_2, \quad x \in L_2.$$

The convex risk and deviation measures in Example 3.1 and Example 3.2 are special cases of some general classes of risk and deviation measures, respectively, that we introduce in the following.

Example 3.3 For $p \in (1, +\infty)$ and $a \geq 1$ let be the convex risk measure $\rho : L_p \to \mathbb{R}$,

$$\rho(x) = ||x - \mathbb{E}(x)||_p^a - \mathbb{E}(x), \;\; x \in L_p.$$

The corresponding convex deviation measure is $d : L_p \to \mathbb{R}$,

$$d(x) = ||x - \mathbb{E}(x)||_p^a, \;\; x \in L_p.$$

In case $p = a = 1$, d is the so-called *mean absolute deviation.*

Example 3.4 Similar to Example 3.3, for $p \in (1, +\infty)$ and $a \geq 1$ we consider the following pairs of convex risk and deviation measures, $\rho : L_p \to \mathbb{R}$ and $d : L_p \to \mathbb{R}$ defined by

$$\rho(x) = ||(x - \mathbb{E}(x))_-||_p^a - \mathbb{E}(x), \quad d(x) = ||(x - \mathbb{E}(x))_-||_p^a$$

and

$$\rho(x) = ||(x - \mathbb{E}(x))_+||_p^a - \mathbb{E}(x), \quad d(x) = ||(x - \mathbb{E}(x))_+||_p^a,$$

respectively. The deviation measures we get by taking $a = p = 1$ are the so-called *lower* and *upper semideviation*, respectively. For $p = 2$ and $a = 1$ we obtain the *standard lower* and *upper semideviation*, respectively.

4. Conjugates of Convex Deviation Measures: Case a=1

In this section we deal with formulas for the conjugate functions of some convex deviation measures, including those in Example 3.3 and Example 3.4, whenever $p \in (1, +\infty)$ and $a = 1$. Having these formulas, one can easily calculate the formulas for the conjugate functions of the corresponding risk measures. The following relation proves this, as for $x^* \in L_q$ it holds

$$\rho^*(x^*) = \sup_{x\in L_p}\{\langle x^*, x\rangle - \rho(x)\} = \sup_{x\in L_p}\{\langle x^*, x\rangle - d(x) + \mathbb{E}(x)\}$$

$$= \sup_{x\in L_p}\{\langle x^*, x\rangle - d(x) + \langle 1, x\rangle\} = \sup_{x\in L_p}\{\langle x^* + 1, x\rangle - d(x)\} = d^*(x^* + 1). \quad (2)$$

In order to derive the formulas for the conjugates of the convex deviation measures we need the following preliminary results.

Fact 4.1 Let be $f_1 : L_p \to \mathbb{R},\ f_1(x) = ||x||_p$. The conjugate function of f_1 is $f_1^* : L_q \to \overline{\mathbb{R}}$, $f_1^* = \delta_{\overline{B}_q(0,1)}$. □

Fact 4.2 Consider now $f_2 : L_p \to \mathbb{R},\ f_2(x) = ||x_-||_p$. For $x^* \in L_q$ one obtains the following formula for the conjugate function of f_2, $f_2^* : L_q \to \overline{\mathbb{R}}$,

$$-f_2^*(x^*) = \inf_{x\in L_p}\{||x_-||_p - \langle x^*, x\rangle\} = \inf_{x\in L_p}\{||\max(-x, 0)||_p - \langle x^*, x\rangle\}.$$

Having for an arbitrary $z \in L_p$ with the property $z \geq \max(-x, 0) \geq 0$ that $||z||_p \geq ||\max(-x, 0)||_p$, one gets further

$$-f_2^*(x^*) = \inf_{\substack{x\in L_p, z\in L_p,\\ z\geq \max(-x,0)}}\{||z||_p - \langle x^*, x\rangle\} = \inf_{\substack{(x,z)\in L_p\times L_p,\\ -x-z\leq 0,\\ -z\leq 0}}\{||z||_p - \langle x^*, x\rangle\}.$$

Consider the following convex optimization problem

$$(\mathrm{P_{x^*}}) \qquad \inf_{\substack{(x,z)\in L_p\times L_p,\\ -x-z\leq 0,\\ -z\leq 0}} \{||z||_p - \langle x^*, x\rangle\}.$$

To formulate it in the language of the problem (P) considered in Section 2 one has to take $\mathcal{Z} = \mathcal{Y} = S = L_p \times L_p$, $C = (L_p)_+ \times (L_p)_+$, $f : \mathcal{Z} \to \mathbb{R}$, $f(x,z) = ||z||_p - \langle x^*, x\rangle$ and $g : \mathcal{Z} \to \mathcal{Y}$, $g(x,z) = (-x-z, -z)$. Consequently, the Lagrange dual problem of $(\mathrm{P_{x^*}})$ looks like

$$(\mathrm{D_{x^*}}) \qquad \sup_{\lambda_1,\lambda_2\in(L_q)_+} \inf_{(x,z)\in L_p\times L_p} \{||z||_p - \langle x^*, x\rangle - \langle \lambda_1, z\rangle - \langle \lambda_2, x+z\rangle\}$$

or, equivalently,

$$(\mathrm{D_{x^*}}) \qquad \sup_{\lambda_1,\lambda_2\in(L_q)_+} \left\{ \inf_{x\in L_p} \{-\langle x^* + \lambda_2, x\rangle\} - (||\cdot||_p)^*(\lambda_1+\lambda_2) \right\}.$$

Since $\inf_{x\in L_p}\{-\langle x^*+\lambda_2, x\rangle\} = -\delta_{\{0\}}(x^*+\lambda_2)$ and $(||\cdot||_p)^*(\lambda_1+\lambda_2) = \delta_{\overline{B}_q(0,1)}(\lambda_1+\lambda_2)$, the optimal objective value of the Lagrange dual $(\mathrm{D_{x^*}})$ can be written as

$$v(\mathrm{D_{x^*}}) = \begin{cases} 0, & \text{if } x^* \in \left(\overline{B}_q(0,1) + (L_q)_+\right) \cap -(L_q)_+, \\ -\infty, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} 0, & \text{if } x^* \in \overline{B}_q(0,1) \cap -(L_q)_+, \\ -\infty, & \text{otherwise.} \end{cases}$$

The last equality comes in fact from the fact that the sets $\left(\overline{B}_q(0,1)+(L_q)_+\right)\cap -(L_q)_+$ and $\overline{B}_q(0,1)\cap -(L_q)_+$ coincide. As the inclusion $\overline{B}_q(0,1)\cap -(L_q)_+ \subseteq \left(\overline{B}_q(0,1) + (L_q)_+\right)\cap -(L_q)_+$ is trivial, we have to prove only the opposite one.

Let be $u^* \in \left(\overline{B}_q(0,1)+(L_q)_+\right)\cap -(L_q)_+$. Then $u^* = t^*+z^* \leq 0$, where $t^* \in \overline{B}_q(0,1)$ and $z^* \in (L_q)_+$. Since $-u^* \geq 0$ and $-z^* \leq 0$ we have $0 \leq -u^* = -t^* - z^* \leq -t^*$ and so $||u^*||_q = ||-u^*||_q \leq ||-t^*||_q = ||t^*||_q \leq 1$. Thus $u^* \in \overline{B}_q(0,1)\cap -(L_q)_+$.

In order to identify $-f_2^*(x^*)$ with the optimal objective value of $(\mathrm{D}_{\mathrm{x}^*})$ we have to prove that between $(\mathrm{P}_{\mathrm{x}^*})$ and $(\mathrm{D}_{\mathrm{x}^*})$ strong duality holds. As $(\mathrm{P}_{\mathrm{x}^*})$ is a convex optimization problem, in order to close the gap between these duals we have only to verify the fulfillment of one of the two regularity conditions stated in Section 2. Noticing that $\mathrm{int}((L_p)_+) = \emptyset$ it is clear that the classical Slater constraint qualification (SC) fails. On the other hand the functions f and g are continuous and $g(\mathrm{dom}\, f \cap S \cap \mathrm{dom}\, g) + C = L_p \times L_p$ and $(0,0)$ belongs to the algebraic interior of this set. Thus the regularity conditions (RC) is fulfilled and so by Theorem 2.2 it follows that

$$f_2^*(x^*) = -v(\mathrm{P}_{\mathrm{x}^*}) = -v(\mathrm{D}_{\mathrm{x}^*}) = \begin{cases} 0, & \text{if } ||x^*||_q \leq 1,\ x^* \leq 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3}$$

Finally we give a further formulation for the conjugate of f_2 by showing that its effective domain can be restricted to the set of those $x^* \in L_q$ which fulfill $x^* \leq 0$ and $-1 \leq \mathbb{E}(x^*) \leq 0$. (Note that $x^* \leq 0$ implies $\mathbb{E}(x^*) \leq 0$ and that $||x^*||_q \leq 1$ implies $|\mathbb{E}(x^*)| \leq 1$.) This leads to the following formula for the conjugate of f_2 (see also [7]):

$$f_2^*(x^*) = \begin{cases} 0, & \text{if } x^* \leq 0, \ \ ||x^*||_q \leq 1, \ \ -1 \leq \mathbb{E}(x^*) \leq 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4}$$

□

In the next example we deal with the conjugate function of the deviation measure $d_1 : L_p \to \mathbb{R}$, $d_1(x) = ||x - \mathbb{E}(x)||_p$, which will be derived via Theorem 2.1.

Fact 4.3 Consider $d_1 : L_p \to \mathbb{R}, \ \ d_1(x) = ||x - \mathbb{E}(x)||_p$ and $A : L_p \to L_p, \ \ Ax = x - \mathbb{E}(x)$. Here we have to interpret $\mathbb{E}(x) \in \mathbb{R}$ as a (constant) element of L_p. Denoting for $C \in \mathfrak{F}$ by $\mathbb{1}_C : \Omega \to \mathbb{R}$ the *indicator function*

$$\mathbb{1}_C(\omega) = \begin{cases} 1, & \text{if } \omega \in C, \\ 0, & \text{otherwise,} \end{cases}$$

the linear continuous mapping A can be represented as $Ax = x - \mathbb{E}(x)\mathbb{1}_\Omega$ for $x \in L_p$.

Since $d_1(x) = ||Ax||_p$ for all $x \in L_p$, in order to calculate d_1^*, we can make use of Theorem 2.1. Since $|| \cdot ||_p$ is continuous on L_p, the regularity condition stated in this result is fulfilled and so for all $x^* \in L_q$ we have

$$d_1^*(x^*) = \min\{(||\cdot||_p)^*(y^*): \quad A^*y^* = x^*\}$$

$$= \begin{cases} 0, & \text{if } \exists y^* \in L_q : A^*y^* = x^* \text{ and } ||y^*||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

From the calculation above one can see that we need the adjoint operator of A. In the following we show that A is self-adjoint, i.e. $A = A^*$. For $x \in L_p$ and $x^* \in L_q$ it holds $\langle x^*, Ax\rangle = \langle x^*, x - \mathbb{E}(x)\rangle = \langle x^*, x\rangle - \langle x^*, \mathbb{E}(x)\mathbb{1}_\Omega\rangle$. The second term can be written as follows (we apply here the *Theorem of Fubini*)

$$\langle x^*, \mathbb{E}(x)\mathbb{1}_\Omega\rangle = \int_\Omega x^*(\omega)\mathbb{E}(x)d\mathbb{P}(\omega) = \int_\Omega x^*(\omega)\left(\int_\Omega x(\tau)d\mathbb{P}(\tau)\right)d\mathbb{P}(\omega)$$

$$= \int_\Omega x(\tau)\left(\int_\Omega x^*(\omega)d\mathbb{P}(\omega)\right)d\mathbb{P}(\tau) = \int_\Omega x(\tau)\mathbb{E}(x^*)d\mathbb{P}(\tau) = \langle \mathbb{E}(x^*)\mathbb{1}_\Omega, x\rangle.$$

Consequently, $\langle x^*, Ax\rangle = \langle x^* - \mathbb{E}(x^*)\mathbb{1}_\Omega, x\rangle$ for all $x \in L_p$ and this means that $A^*x^* = x^* - \mathbb{E}(x^*)\mathbb{1}_\Omega = x^* - \mathbb{E}(x^*)$.

Thus the conjugate function of d_1 looks for $x^* \in L_q$ like

$$d_1^*(x^*) = \begin{cases} 0, & \text{if } \exists y^* \in L_q : y^* - \mathbb{E}(y^*) = x^* \text{ and } ||y^*||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

We prove now that there exists $y^* \in L_q$ such that $y^* - \mathbb{E}(y^*) = x^*$ and $||y^*||_q \leq 1$ if and only if $\mathbb{E}(x^*) = 0$ and there exists $c \in \mathbb{R}$ such that $||x^* - c||_q \leq 1$. Let be a $y^* \in L_q$

fulfilling $y^* - \mathbb{E}(y^*) = x^*$ and $||y^*||_q \leq 1$. Then $\mathbb{E}(x^*) = \mathbb{E}(y^* - \mathbb{E}(y^*)) = 0$ and for $c := -\mathbb{E}(y^*)$ one has $||x^* - c||_q \leq 1$. On the other hand, assume that $\mathbb{E}(x^*) = 0$ and that there exists $c \in \mathbb{R}$ with the property $||x^* - c||_q \leq 1$. Defining $y^* := x^* - c \in L_q$ one has $||y^*||_q \leq 1$ and $y^* - \mathbb{E}(y^*) = x^*$.

Thus the conjugate of d_1 at $x^* \in L_q$ turns out to be

$$d_1^*(x^*) = \begin{cases} 0, & \text{if } \mathbb{E}(x^*) = 0 \text{ and } \min\limits_{c\in\mathbb{R}} ||x^* - c||_q \leq 1, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

Considering the convex risk measure $\rho_1 : L_p \to \mathbb{R},\ \rho_1(x) = d_1(x) - \mathbb{E}(x) = ||x - \mathbb{E}(x)||_p - \mathbb{E}(x)$, by (2), one can easily deduce the formula for the conjugate of $\rho_1 : L_p \to \overline{\mathbb{R}}$. For $x^* \in L_q$ this looks like

$$\rho_1^*(x^*) = d_1^*(x^* + 1) = \begin{cases} 0, & \text{if } \mathbb{E}(x^*) = -1 \text{ and } \min\limits_{c\in\mathbb{R}} ||x^* - c||_q \leq 1, \\ \\ +\infty, & \text{otherwise.} \end{cases} \tag{5}$$

□

In the last application that we consider in this section we calculate the conjugate function of the convex deviation measure known also as *lower semideviation*. After that we derive the formula for the conjugate of the corresponding convex risk measure.

Fact 4.4 Let be $d_2 : L_p \to \mathbb{R},\quad d_2(x) = ||(x - \mathbb{E}(x))_-||_p$. Denoting again by $A : L_p \to L_p$ the linear continuous mapping defined by $Ax = x - \mathbb{E}(x)$ for $x \in L_p$, we have that $d_2 = f_2 \circ A$. Since f_2 is a convex and continuous function with real values, by Theorem 2.1 and (4) one has for all $x^* \in L_q$

$$d_2^*(x^*) = \min\{f_2^*(y^*) : \ A^*y^* = x^*\}$$
$$= \begin{cases} 0, & \text{if } \exists y^* \in L_q : \ A^*y^* = x^*, \ y^* \leq 0, \ ||y^*||_q \leq 1, \ -1 \leq \mathbb{E}(y^*) \leq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $A^*y^* = y^* - \mathbb{E}(y^*)$ for $y^* \in L_q$ (see Fact 4.3), for all $x^* \in L_q$ it holds

$$d_2^*(x^*) = \begin{cases} 0, & \text{if } \exists y^* \in L_q : y^* - \mathbb{E}(y^*) = x^*, y^* \leq 0, ||y^*||_q \leq 1, -1 \leq \mathbb{E}(y^*) \leq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Like in Fact 4.3 one can show that there exists $y^* \in L_q$ such that $y^* - \mathbb{E}(y^*) = x^*, \ y^* \leq 0, \ ||y^*||_q \leq 1$ and $-1 \leq \mathbb{E}(y^*) \leq 0$ if and only if $\mathbb{E}(x^*) = 0$ and there exists $c \in \mathbb{R}$ fulfilling $0 \leq c \leq 1, \ \ ||x^* - c||_q \leq 1$ and $x^* \leq c$. Thus

$$d_2^*(x^*) = \begin{cases} 0, & \text{if } \mathbb{E}(x^*) = 0 \text{ and } \exists c \in \mathbb{R} : \ 0 \leq c \leq 1, \ ||x^* - c||_q \leq 1, \ \ x^* \leq c, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us prove now that for $x^* \in L_q$ the relations

$$\mathbb{E}(x^*) = 0 \text{ and there exists } c \in \mathbb{R} \text{ such that } 0 \leq c \leq 1, \ ||x^* - c||_q \leq 1, \ \ x^* \leq c \tag{6}$$

and

$$\mathbb{E}(x^*) = 0, \ \ x^* \leq 1, \ \ ||\operatorname{essup} x^* - x^*||_q \leq 1 \tag{7}$$

are equivalent. Assuming that (6) holds, one has $x^* \leq c \leq 1$. Further we have $\operatorname{essup} x^* \leq c$ and this means that $c - x^* \geq \operatorname{essup} x^* - x^* \geq 0$, implying $1 \geq ||c - x^*||_q \geq || \operatorname{essup} x^* - x^*||_q$. Relation (7) is so proved. On the other hand, if (7) holds, one can take $c = \operatorname{essup} x^*$. That $c \leq 1$ and $x^* \leq c$ is obvious. Assuming now that $c < 0$, this would mean that $\mathbb{E}(x^*) < 0$. In conclusion, relation (6) must also hold.

This leads to the following formula for d_2^* for $x^* \in L_q$

$$d_2^*(x^*) = \begin{cases} 0, & \text{if } \mathbb{E}(x^*) = 0,\ x^* \leq 1,\ || \operatorname{essup} x^* - x^*||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{8}$$

As above the formula for the conjugate function of the corresponding convex risk measure $\rho_2 : L_p \to \overline{\mathbb{R}},\ \rho_2(x) = ||(x - \mathbb{E}(x))_-||_p - \mathbb{E}(x)$ can be also calculated. By (2) we have for all $x^* \in L_q$

$$\rho_2^*(x^*) = d_2^*(x^* + 1) = \begin{cases} 0, & \text{if } \mathbb{E}(x^*) = -1,\ x^* \leq 0,\ || \operatorname{essup} x^* - x^*||_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

$\square$

Remark 4.1 One can notice that the formulas for the conjugates of f_2 and d_2 allow us to calculate the formulas for the conjugates of the functions $x \mapsto ||x_+||_p$ and $x \mapsto ||(x - \mathbb{E}(x))_+||_p$, as these are nothing but $f_2(-x)$ and $d_2(-x)$, respectively. In general for $h : \mathcal{Z} \to \overline{\mathbb{R}}$ defined as $h(x) = f(-x)$ it holds for $x^* \in \mathcal{Z}^*$

$$h^*(x^*) = \sup_{x \in \mathcal{Z}} \{\langle x^*, x \rangle - f(-x)\} = \sup_{x \in \mathcal{Z}} \{\langle -x^*, x \rangle - f(x)\} = f^*(-x^*).$$

5. Conjugates of Convex Deviation Measures: Case a>1

In this section we extend our investigations on the conjugate functions of convex deviation measures given in Example 3.3 and Example 3.4 to the case $a > 1$ (as before, we assume that $p \in (1, +\infty)$). We use relation (2) in order to calculate the conjugate functions of the corresponding convex risk measures.

In our approach we use the very well-developed calculus existing in the theory of conjugate functions. The functions considered in this section will be viewed as compositions of a convex and increasing function with a convex function. The conjugates will be obtained by using the formula for the conjugate of a composite convex function. The theorem which provides this formula follows and is nothing else than an adaptation of Theorem 2.8.10 in [14].

Theorem 5.1 ([14]) Let $\mathcal{Z}$ be a separated locally convex space and $f : \mathcal{Z} \to \mathbb{R}$, $g : \mathbb{R} \to \overline{\mathbb{R}}$ convex functions such that g is increasing on $f(\mathcal{Z}) + [0, +\infty)$. Assume that there exists $x' \in \mathcal{Z}$ such that $f(x') \in \mathrm{dom}(g)$ and g is continuous at $f(x')$. Then

$$(g \circ f)^*(x^*) = \min_{\beta \in \mathbb{R}_+} \{g^*(\beta) + (\beta f)^*(x^*)\}, \forall x^* \in \mathcal{Z}^*. \tag{9}$$

In the following we apply Theorem 5.1 by taking f as being the convex deviation measures considered in Section 4, while $g : \mathbb{R} \to \overline{\mathbb{R}}$ is the function defined for $a > 1$ by

$$g(x) = \begin{cases} x^a, & \text{if } x \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

The set $f(L_p)+[0,+\infty)$ is nothing else than $[0,+\infty)$ and one can see that both functions f and g are convex, while g is increasing on $[0,+\infty)$. For the particular situations we treat below the regularity condition will be fulfilled, consequently, formula (9) will hold.

First of all let us furnish the formula for the conjugate function of g.

Lemma 5.1 The conjugate function of g is $g^* : \mathbb{R} \to \overline{\mathbb{R}}$,

$$g^*(\beta) = \begin{cases} (a-1)\left(\frac{\beta}{a}\right)^{\frac{a}{a-1}}, & \text{if } \beta \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Proof. By definition it holds $g^*(\beta) = \sup_{x \in \mathbb{R}} (x\beta - g(x)) = \sup_{x \geq 0} (x\beta - x^a)$. In the case $\beta \leq 0$, one gets $g^*(\beta) = 0$. In case $\beta > 0$, we consider $h : [0,+\infty) \to \mathbb{R}$, $h(x) = x\beta - x^a$. One has $h'(x) = 0 \Leftrightarrow x = \left(\frac{\beta}{a}\right)^{\frac{1}{a-1}} > 0$. Since h is concave it attains its maximum at

$x = \left(\frac{\beta}{a}\right)^{\frac{1}{a-1}} > 0$ and so

$$g^*(\beta) = \beta \left(\frac{\beta}{a}\right)^{\frac{1}{a-1}} - \left(\left(\frac{\beta}{a}\right)^{\frac{1}{a-1}}\right)^a = \beta \left(\frac{\beta}{a}\right)^{\frac{1}{a-1}} \left[1 - \frac{1}{a}\right] = (a-1)\left(\frac{\beta}{a}\right)^{\frac{a}{a-1}}.$$

In this way we get (10). □

In order to calculate the formulas for the conjugate functions of the convex deviation measures in Example 3.3 and Example 3.4 we need the following intermediate formulas.

Fact 5.1 Let be $f_3 : L_p \to \mathbb{R}$, $f_3(x) = ||x||_p^a$. For $x \in L_p$ we have $f_3(x) = (g \circ f_1)(x)$.

For a fixed $\beta \in \mathbb{R}_+$ we provide first the formula for $(\beta f_1)^*(x^*)$.

Since for $\beta > 0$ and $x^* \in L_q$ it holds (see Fact 4.1)

$$(\beta f_1)^*(x^*) = \beta f_1^* \left(\frac{1}{\beta} x^*\right) = \beta(|| \cdot ||_p)^* \left(\frac{1}{\beta} x^*\right)$$

$$= \begin{cases} 0, & \text{if } ||\frac{x^*}{\beta}||_q \leq 1, \\ +\infty, & \text{otherwise,} \end{cases} = \begin{cases} 0, & \text{if } ||x^*||_q \leq \beta, \\ +\infty, & \text{otherwise,} \end{cases}$$

and for $\beta = 0$ one has $(\beta f_1)^* = \delta_{\{0\}}$, we finally get for all $\beta \geq 0$ and all $x^* \in L_q$

$$(\beta f_1)^*(x^*) = \begin{cases} 0, & \text{if } ||x^*||_q \leq \beta, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus with Theorem 5.1 the conjugate of f_3 becomes for all $x^* \in L_q$

$$f_3^*(x^*) = \min_{\substack{\beta \geq 0, \\ ||x^*||_q \leq \beta}} g^*(\beta) = \min_{\substack{\beta \geq 0, \\ ||x^*||_q \leq \beta}} (a-1)\left(\frac{\beta}{a}\right)^{\frac{a}{a-1}} = (a-1)\left|\left|\frac{1}{a} x^*\right|\right|_q^{\frac{a}{a-1}}.$$

$\square$

Fact 5.2 Let be $f_4 : L_p \to \mathbb{R},\ f_4(x) = ||x_-||_p^a$ for $x \in L_p$. One can see that in this case $f_4 = g \circ f_2$. In order to use the relation in (9) we have to calculate first $(\beta f_2)^*$ for $\beta \geq 0$. If $\beta = 0$ one has again $(\beta f_2)^* = \delta_{\{0\}}$, while if $\beta > 0$ and $x^* \in L_q$, by (3) it yields

$$(\beta f_2)^*(x^*) = \beta f_2^*\left(\frac{1}{\beta}x^*\right) = \begin{cases} 0, & \text{if } ||x^*||_q \leq \beta,\ x^* \leq 0, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

By (9) it follows that for all $x^* \in L_q$ such that $x^* \in -(L_q)_+$ one has

$$f_4^*(x^*) = \min_{\beta \geq 0}\{g^*(\beta) + (\beta f_2)^*(x^*)\} = \min_{\substack{\beta \geq 0, \\ ||x^*||_q \leq \beta}} \left\{(a-1)\left(\frac{\beta}{a}\right)^{\frac{a}{a-1}}\right\} = (a-1)\left|\left|\frac{1}{a}x^*\right|\right|_q^{\frac{a}{a-1}}.$$

If $x^* \notin -(L_q)_+$, then $f_4^*(x^*) = +\infty$. Consequently, for all $x^* \in L_q$ we obtain

$$f_4^*(x^*) = \begin{cases} (a-1)\left|\left|\frac{1}{a}x^*\right|\right|_q^{\frac{a}{a-1}}, & \text{if } x^* \in -(L_q)_+, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

$\square$

In the next application we deal with the convex deviation measure considered in Example 3.3, $d_3 : L_p \to \mathbb{R}, d_3(x) = ||x - \mathbb{E}(x)||_p^a$.

Fact 5.3 The convex deviation measure d_3 can be written as $d_3 = g \circ d_1$. Let be

$\beta \geq 0$ and $x^* \in L_q$. If $\beta = 0$ one has $(\beta d_1)^* = \delta_{\{0\}}$, while if $\beta > 0$ it holds,

$$(\beta d_1)^*(x^*) = \beta d_1^* \left(\frac{1}{\beta} x^*\right) = \begin{cases} 0, & \text{if } \mathbb{E}(x^*) = 0 \text{ and } \min\limits_{c \in \mathbb{R}} ||x^* - c||_q \leq \beta, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

For all $x^* \in L_q$ such that $\mathbb{E}(x^*) = 0$ we have (see (9))

$$d_3^*(x^*) = \min_{\substack{\beta \geq 0, \\ \beta \geq \min\limits_{c \in \mathbb{R}} ||x^* - c||_q}} \{g^*(\beta)\} = \min_{c \in \mathbb{R}} \left\{ (a-1) \left|\left| \frac{1}{a}(x^* - c) \right|\right|_q^{\frac{a}{a-1}} \right\},$$

while $d_3^*(x^*) = +\infty$, if $\mathbb{E}(x^*) \neq 0$. In conclusion, for all $x^* \in L_q$ it holds

$$d_3^*(x^*) = \begin{cases} \min\limits_{c \in \mathbb{R}} \left\{ (a-1) \left|\left| \frac{1}{a}(x^* - c) \right|\right|_q^{\frac{a}{a-1}} \right\}, & \text{if } \mathbb{E}(x^*) = 0, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

The conjugate function of the corresponding convex risk measure $\rho_3 : L_p \to \mathbb{R}$, $\rho_3(x) = d_3(x) - \mathbb{E}(x) = ||x - \mathbb{E}(x)||_p^a - \mathbb{E}(x)$, turns out for all $x^* \in L_q$ to be (see (2))

$$\rho_3^*(x^*) = d_3^*(x^* + 1) = \begin{cases} \min\limits_{c \in \mathbb{R}} \left\{ (a-1) \left|\left| \frac{1}{a}(x^* - c) \right|\right|_q^{\frac{a}{a-1}} \right\}, & \text{if } \mathbb{E}(x^*) = -1, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

□

The last conjugate function calculated in this section is that of the convex deviation measure given in Example 3.4.

Fact 5.4 Considering $d_4 : L_p \to \mathbb{R}$, $d_4(x) = ||(x - \mathbb{E}(x))_-||_p^a$, one can see that $d_4 = g \circ d_2$. First of all we calculate the conjugate of βd_2 for $\beta \geq 0$.

Let be $\beta \geq 0$ and $x^* \in L_q$. If $\beta = 0$, then $(\beta d_2)^* = \delta_{\{0\}}$, while when $\beta > 0$ one has (see (8))

$$(\beta d_2)^*(x^*) = \beta d_2^* \left(\frac{1}{\beta} x^*\right) = \begin{cases} 0, & \text{if } \mathbb{E}(x^*) = 0, \quad x^* \leq \beta, \quad ||\operatorname{essup} x^* - x^*||_q \leq \beta, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us notice that in the case $\mathbb{E}(x^*) = 0$, if $\beta \geq ||\operatorname{essup} x^* - x^*||_q$, then $\beta \geq \mathbb{E}(\operatorname{essup} x^* - x^*) = \operatorname{essup} x^*$, which yields the following formula for the above conjugate

$$(\beta d_2)^*(x^*) = \begin{cases} 0, & \text{if } \mathbb{E}(x^*) = 0, \quad ||\operatorname{essup} x^* - x^*||_q \leq \beta, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

Consequently, by (9), for all $x^* \in L_q$ such that $\mathbb{E}(x^*) = 0$ it holds

$$d_4^*(x^*) = \inf_{\substack{\beta \geq 0, \\ \beta \geq ||\operatorname{essup} x^* - x^*||_q}} \left\{ (a-1) \left(\frac{\beta}{a}\right)^{\frac{a}{a-1}} \right\} = (a-1) \left|\left| \frac{1}{a} (\operatorname{essup} x^* - x^*) \right|\right|_q^{\frac{a}{a-1}},$$

while if $\mathbb{E}(x^*) \neq 0$, we have $d_4^*(x^*) = +\infty$. Thus for all $x^* \in L_q$ we get

$$d_4^*(x^*) = \begin{cases} (a-1) \left|\left| \frac{1}{a} (\operatorname{essup} x^* - x^*) \right|\right|_q^{\frac{a}{a-1}}, & \text{if } \mathbb{E}(x^*) = 0, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

The formula for the conjugate function of the corresponding convex risk measure $\rho_4 : L_p \to \mathbb{R}$, $\rho_4(x) = d_4(x) - \mathbb{E}(x) = ||(x - \mathbb{E}(x))_-||_p^a - \mathbb{E}(x)$, follows. For all $x^* \in L_q$ it holds (see (2))

$$\rho_4^*(x^*) = d_4^*(x^* + 1) = \begin{cases} (a-1) \left|\left| \frac{1}{a} (\operatorname{essup} x^* - x^*) \right|\right|_q^{\frac{a}{a-1}}, & \text{if } \mathbb{E}(x^*) = -1, \\ \\ +\infty, & \text{otherwise.} \end{cases}$$

$\square$

6. Dual Representation of Convex Risk Measures

In this section we give for the convex risk and deviation measures considered in the previous sections some dual representations which will follow by applying the *Fenchel-Moreau theorem*. For $p \in (1, +\infty)$ and $f : L_p \to \overline{\mathbb{R}}$ one of the treated convex risk or deviation measures, as these enjoy properness, convexity and lower semicontinuity properties, it holds

$$f(x) = f^{**}(x) = \sup_{x^* \in L_q} \{\langle x^*, x\rangle - f^*(x^*)\} = \sup_{x^* \in L_q} \{\mathbb{E}(x^* x) - f^*(x^*)\}, \forall x \in L_p. \quad (11)$$

Thus by using the formulas of the conjugates derived in the previous sections, we obtain in a very natural way the desired dual representations, which turn out to be generalizations of some recently published results by Pflug ([7]). More than that, we show the usefulness of the powerful theory of conjugate functions in this field as well.

Fact 6.1 The first convex deviation measure we investigate is $d_1 : L_p \to \mathbb{R}$, $d_1(x) = ||x - \mathbb{E}(x)||_p$. We have proven that for all $x^* \in L_q$

$$d_1^*(x^*) = \begin{cases} 0, & \text{if } \mathbb{E}(x^*) = 0 \text{ and } \min\limits_{c \in \mathbb{R}} ||x^* - c||_q \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and so, by (11),

$$d_1(x) = \sup\{\mathbb{E}(x^*x) : \quad x^* \in L_q,\ \mathbb{E}(x^*) = 0 \text{ and } \min_{c\in\mathbb{R}} ||x^* - c||_q \leq 1\}, \forall x \in L_p.$$

Analogously, by (5), we obtain

$$\rho_1(x) = \sup\left\{\mathbb{E}(x^*x) : \quad x^* \in L_q,\ \mathbb{E}(x^*) = -1 \text{ and } \min_{c\in\mathbb{R}} ||x^* - c||_q \leq 1\right\}, \forall x \in L_p.$$

□

Pflug gives in Proposition 3 in [7] in the case $p \in (1, +\infty)$ representations for the above treated convex risk and deviation measures, which are actually generalizations of the *standard deviation*. The formulas given by Pflug are not quite accurate, as he considers for instance inf instead of min. As we have seen in the previous sections, the existence of a $c \in \mathbb{R}$, such that $||x^* - c||_q \leq 1$, is indispensable. Let us also notice that Pflug uses instead of convex risk measures so-called *acceptability functionals* (we denote them like in [7] by $\mathcal{A}$). They are linked to the convex risk measures in our paper by the relation $\mathcal{A}(x) = -\rho(x)$ for $x \in L_p$.

Fact 6.2 Consider now $d_2 : L_p \to \mathbb{R}$, $d_2(x) = ||(x - \mathbb{E}(x))_-||_p$, the deviation measure investigated in Fact 4.4. For the conjugate function d_2^* it holds for $x^* \in L_q$ (see (8))

$$d_2^*(x^*) = \begin{cases} 0, & \text{if } \mathbb{E}(x^*) = 0,\ x^* \leq 1,\ ||\operatorname{essup} x^* - x^*||_q \leq 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and so one obtains the following dual representation

$$d_2(x) = \sup\{\mathbb{E}(x^* x) : \quad x^* \in L_q,\ \mathbb{E}(x^*) = 0,\ x^* \leq 1,\ ||\operatorname{essup} x^* - x^*||_q \leq 1\}, \forall x \in L_p.$$

Similary, we get

$$\rho_2(x) = \sup\{\mathbb{E}(x^* x) : \quad x^* \in L_q,\ \mathbb{E}(x^*) = -1,\ x^* \leq 0,\ ||\operatorname{essup} x^* - x^*||_q \leq 1\}, \forall x \in L_p.$$

□

The last two dual representations are actually the formulas gained in Proposition 5 in [7].

Fact 6.3 For $a > 1$ let be $d_3 : L_p \to \mathbb{R}$, $d_3(x) = ||x - \mathbb{E}(x)||_p^a$, the convex deviation measure considered in Fact 5.3. Its conjugate function $d_3^* : L_q \to \overline{\mathbb{R}}$ looks for all $x^* \in L_q$ like

$$d_3^*(x^*) = \begin{cases} \min\limits_{c \in \mathbb{R}} \left\{ (a-1) \left|\left| \frac{1}{a}(x^* - c) \right|\right|_q^{\frac{a}{a-1}} \right\}, & \text{if } \mathbb{E}(x^*) = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

By (11) we get the following dual representation for d_3

$$d_3(x) = \sup\left\{ \mathbb{E}(x^* x) - \min_{c \in \mathbb{R}} \left\{ (a-1) \left|\left| \frac{1}{a}(x^* - c) \right|\right|_q^{\frac{a}{a-1}} \right\} : \ x^* \in L_q, \mathbb{E}(x^*) = 0 \right\}, \forall x \in L_p.$$

Similarly, for the corresponding convex risk measure we obtain

$$\rho_3(x) = \sup \left\{ \mathbb{E}(x^* x) - \min_{c \in \mathbb{R}} \left\{ (a-1) \left|\left| \frac{1}{a}(x^* - c) \right|\right|_q^{\frac{a}{a-1}} \right\} : \; x^* \in L_q, \mathbb{E}(x^*) = -1 \right\}, \forall x \in L_p.$$

□

Pflug gives in Proposition 2 in [7] a similar formula just for the special case when $a = p$, where $p \in (1, +\infty)$. More than that, in the mentioned paper nothing about the attainability of the inner infimum is mentioned.

Fact 6.4 Finally, again for $a > 1$ we consider the deviation measure $d_4 : L_p \to \mathbb{R}$, $d_4(x) = ||(x - \mathbb{E}(x))_-||_p^a$. Via Fact 5.4 one has for all $x^* \in L_q$ that

$$d_4^*(x^*) = \begin{cases} (a-1)\left|\left|\frac{1}{a}(\operatorname{essup} x^* - x^*)\right|\right|_q^{\frac{a}{a-1}}, & \text{if } \mathbb{E}(x^*) = 0, \\ \\ +\infty, & \text{otherwise}, \end{cases}$$

and so d_4 can be represented as

$$d_4(x) = \sup \left\{ \mathbb{E}(x^* x) - (a-1) \left|\left| \frac{1}{a}(\operatorname{essup} x^* - x^*) \right|\right|_q^{\frac{a}{a-1}} : x^* \in L_q, \mathbb{E}(x^*) = 0 \right\}, \forall x \in L_p.$$

Further, for the corresponding convex risk measure $\rho_4 : L_p \to \mathbb{R}$, $\rho_4(x) = ||(x - \mathbb{E}(x))_-||_p^a - \mathbb{E}(x)$, we get

$$\rho_4(x) = \sup \left\{ \mathbb{E}(x^* x) - (a-1) \left|\left| \frac{1}{a}(\operatorname{essup} x^* - x^*) \right|\right|_q^{\frac{a}{a-1}} : x^* \in L_q, \; \mathbb{E}(x^*) = -1 \right\}, \forall x \in L_p.$$

The above statements generalize Proposition 4 in [7] where these formulas have been given just in the case $a = p$, where $p \in (1, +\infty)$.

7. Conclusions

In this paper, we provide dual representations for different convex risk and deviation measures by making use of their conjugate functions. For establishing the formulas for the conjugates, we employ on the one hhand some classical results from convex analysis and on the other hand some tools from the conjugate duality theory. Several dual characterizations given for deviation measures in [7] are rediscovered as consequences of our results, some of them being herewith improved.

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