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Closedness type regularity conditions for surjectivity results involving the sum of two maximal monotone operators

Research Article

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Abstract: In this note we provide regularity conditions of closedness type which guarantee some surjectivity results concerning the sum of two maximal monotone operators by using representative functions. The first regularity condition we give guarantees the surjectivity of the monotone operator $S(\cdot + p) + T(\cdot)$, where $p \in X$ and S and T are maximal monotone operators on the reflexive Banach space X. Then, this is used to obtain sufficient conditions for the surjectivity of $S + T$ and for the situation when 0 belongs to the range of $S + T$. Several special cases are discussed, some of them delivering interesting byproducts.

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1. Introduction

The recent developments in treating monotone operators by means of convex analysis brought interesting results related to many problems involving monotone operators (cf. [4, 14]). Among them, maximal monotonicity of the sum of two maximal monotone operators [4, 6], sufficient conditions ensuring that 0 belongs to the range of a sum of maximal monotone operators and surjectivity of such a sum [4, 16]. Problems like these arise in different applications in fields

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like inverse problems, Fenchel–Rockafellar and Singer–Toland duality schemes, Clarke–Ekeland least action principle [2], variational inequalities [4, 11], Schrödinger equations and others [1]. In papers [10, 11] algorithms for determining where the sum of two maximal monotone operators takes the value 0 are given. Surjectivity issues regarding maximal monotone operators are discussed also in recent works [8, 9, 12].

In this paper we give, by using representative functions, conditions that characterize the fact that, for maximal monotone operators S and T defined on a reflexive Banach space X and $p \in X$, the monotone operator $S(\cdot + p) + T(\cdot)$ is surjective. From this we deduce characterizations of the surjectivity of $S + T$ and of the situation when 0 lies in the range of $S + T$. As main results, we introduce weak closedness type regularity conditions that guarantee the validity of the mentioned results. An example to underline the fact that these regularity conditions are indeed weaker than the interiority type ones considered in the literature, is also provided. As special cases we consider situations where T is the normal cone of a nonempty closed convex set, respectively when S and T are subdifferentials of proper convex lower semicontinuous functions. In this way we rediscover several results from the literature, like the celebrated Rockafellar's surjectivity theorem and we moreover deliver weak regularity conditions for some results known so far only under stronger hypotheses involving generalized interiors.

2. Preliminaries

First we present some notions and results from convex analysis that are necessary in order to make the paper self-contained. Let a nontrivial Hausdorff locally convex topological space be denoted by X and its dual space by X^*. The dual of X^* is said to be the *bidual* of X, being denoted by X^{**}. If X is normed, it can be identified with a subspace of X^{**}, and we denote the canonical image in X^{**} of the element $x \in X$ by x, too. By $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. Moreover, we call $c : X \times X^* \to \mathbb{R}$, $c(x, x^*) = \langle x^*, x\rangle$, the *duality product*. Denote the *indicator* function of $U \subseteq X$ by δ_U and its *support* function by σ_U.

For a function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, we denote its *domain* by $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$. We call f *proper* if $f(x) > -\infty$ for all $x \in X$ and $\operatorname{dom} f \neq \emptyset$. The *conjugate* function of f is $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup\big\{\langle x^*, x\rangle - f(x) : x \in X\big\}$. For $x \in X$ such that $f(x) \in \mathbb{R}$ we define the (*convex*) *subdifferential* of f at x by $\partial f(x) = \big\{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \text{ for all } y \in X\big\}$. When $f(x) \notin \mathbb{R}$ we take by convention $\partial f(x) = \emptyset$. The subdifferential of the indicator function of a set $U \subseteq X$ is said to be the *normal cone* of U being denoted by N_U. Between a function and its conjugate there is *Young's inequality* $f^*(x^*) + f(x) \geq \langle x^*, x\rangle$ for all $x \in X$ and all $x^* \in X^*$, fulfilled as equality by a pair $(x, x^*) \in X \times X^*$ if and only if $x^* \in \partial f(x)$. Denote also by $\operatorname{cl} f : X \to \overline{\mathbb{R}}$ the largest lower semicontinuous function everywhere less than or equal to f, i.e. the *lower semicontinuous hull* of f, and by $\operatorname{co} f : X \to \overline{\mathbb{R}}$ the largest convex function everywhere less than or equal to f, i.e. the *convex hull* of f.

When $f, g : X \to \overline{\mathbb{R}}$ are proper, we have the *infimal convolution* of f and g defined by $f\Box g : X \to \overline{\mathbb{R}}$, $f\Box g(a) = \inf\big\{f(x) + g(a - x) : x \in X\big\}$. It is said to be *exact* at $y \in X$ when the infimum at $a = y$ is attained, i.e. there exists $x \in X$ such that $f\Box g(y) = f(x) + g(y - x)$. When an infimum or a supremum is attained we write min and max instead of respectively inf and sup.

The next result can be derived from the proofs of [7, Proposition 2.2 and Theorem 3.1].

Proposition 2.1.

Consider on X^* a locally convex topology giving X as its dual space. Let proper, convex and lower semicontinuous functions $f, g : X \to \overline{\mathbb{R}}$ satisfy $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ and $p^* \in X^*$. Then $f^*\Box g^*$ is lower semicontinuous at p^* and exact at p^* if and only if

$$\inf_{x\in X}\big[f(x) + g(x) - \langle p^*, x\rangle\big] = \max_{x^*\in X^*}\big\{ - f^*(x^*) - g^*(p^* - x^*)\big\}.$$

Remark 2.2.

The continuity of either f or g at a point of $\operatorname{dom} f \cap \operatorname{dom} g$ yields the fulfillment of the equivalent statements from Proposition 2.1, even when the lower semicontinuity hypotheses imposed on f and g are removed.

Let us recall some notions and results involving monotone operators (see for instance [4, 5, 8, 14]). Further, X is a Banach space equipped with the norm $\|\cdot\|$, while the dual norm on X^* is $\|\cdot\|_*$.

A multifunction $T : X \rightrightarrows X^*$ is called a *monotone operator* provided that for any $x, y \in X$ one has $\langle y^* - x^*, y - x\rangle \geq 0$ whenever $x^* \in Tx$ and $y^* \in Ty$. The *domain* of T is $\mathrm{D}(T) = \{x \in X : Tx \neq \emptyset\}$, while its *range* is $\mathrm{R}(T) = \bigcup\{Tx : x \in X\}$. T is called *surjective* if $\mathrm{R}(T) = X^*$. A monotone operator $T : X \rightrightarrows X^*$ is called *maximal* when its *graph* $\mathrm{G}(T) = \{(x, x^*) : x \in X, x^* \in Tx\}$ is not properly included in the graph of any other monotone operator $T' : X \rightrightarrows X^*$. The subdifferential of a proper convex lower semicontinuous function on X is a typical example of a maximal monotone operator, the first to note this being Rockafellar in [13].

To a maximal monotone operator $T : X \rightrightarrows X^*$ one can attach the *Fitzpatrick function*

$$\varphi_T : X \times X^* \to \overline{\mathbb{R}}, \qquad \varphi_T(x, x^*) = \sup\left\{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in Ty\right\},$$

which is proper convex and weak×weak*-lower semicontinuous, and the so-called *Fitzpatrick family* of *representative functions*

$$\mathcal{F}_T = \left\{ f_T : X \times X^* \to \overline{\mathbb{R}} \;\middle|\; \begin{array}{c} f_T \text{ is convex and strong}\times\text{strong lower semicontinuous,} \\ c \leq f_T, \quad (x, x^*) \in \mathrm{G}(T) \Rightarrow f_T(x, x^*) = (x, x^*) \end{array} \right\}.$$

The largest element of $\mathcal{F}_T$ is $\psi_T = \mathrm{cl}_{\|\cdot\|\times\|\cdot\|_*} \mathrm{co}\left(c + \delta_{\mathrm{G}(T)}\right)$. We also have $\varphi_T(x, x^*) = \left(c + \delta_{\mathrm{G}(T)}\right)^*(x^*, x) = \psi_T^*(x^*, x)$ for all $(x, x^*) \in X \times X^*$. For $f_T \in \mathcal{F}_T$, denote by $\hat{f}_T : X \times X^* \to \overline{\mathbb{R}}$ the function defined as $\hat{f}_T(x, x^*) = f_T(x, -x^*)$, $x \in X$, $x^* \in X^*$. Note that $\hat{f}_T$ is proper, convex and lower semicontinuous, and $\hat{f}_T(x, x^*) \geq -\langle x^*, x\rangle$ and $\hat{f}_T^*(x^*, x) = f_T^*(x^*, -x)$ for all $x \in X$ and $x^* \in X^*$.

If $f : X \to \overline{\mathbb{R}}$ is a proper convex lower semicontinuous function, then the function $(x, x^*) \mapsto f(x) + f^*(x^*)$ is a representative function of the maximal monotone operator $\partial f : X \rightrightarrows X^*$ and we call it the *Fenchel representative function*. If f is also sublinear, the only representative function associated to ∂f is the Fenchel one, which coincides in this case with the Fitzpatrick function of ∂f (see [3, Theorem 5.3 and Corollary 5.9]). Other maximal monotone operators having only one representative function, the Fenchel one, are the normal cones of nonempty closed convex sets.

The following statement underlines close connections between maximal monotone operators and their representative functions.

Proposition 2.3.
Let $T : X \rightrightarrows X^*$ be a maximal monotone operator.

(i) φ_T is the smallest element of the family $\mathcal{F}_T$.

(ii) If $f_T \in \mathcal{F}_T$ one has $f_T^*(x^*, x) \geq \langle x^*, x\rangle$ for all $(x, x^*) \in X \times X^*$.

(iii) If $f_T \in \mathcal{F}_T$ and $(x, x^*) \in X \times X^*$, $(x, x^*) \in \mathrm{G}(T)$ if and only if $f_T(x, x^*) = \langle x^*, x\rangle$ and this is further equivalent to $f_T^*(x^*, x) = \langle x^*, x\rangle$.

3. Surjectivity results for the sum of two maximal monotone operators

In this main section we deal with the surjectivity results announced in the introduction. Further, let X be a reflexive Banach space, S and T be two maximal monotone operators defined on X. The first main statement of this note follows, after an observation needed in its proof.

Remark 3.1.
Let $p \in X$ and $p^* \in X^*$. Then $p^* \in \mathrm{R}\left(S(p + \cdot) + T(\cdot)\right)$ if and only if $(p, p^*) \in \mathrm{G}(S) - \mathrm{G}(-T)$, where $\mathrm{G}(-T) = \{(x, x^*) \in X \times X^* : (x, -x^*) \in \mathrm{G}(T)\}$.

Theorem 3.2.
Let $p \in X$ and $p^* \in X^*$. The following statements are equivalent:

(i) $p^* \in \mathrm{R}\big(S(p+\cdot)+T(\cdot)\big)$;

(ii) for all $f_S \in \mathcal{F}_S$ and $f_T \in \mathcal{F}_T$ one has $\operatorname{dom} f_S \cap \big(\operatorname{dom} \hat{f}_T + (p,p^*)\big) \neq \emptyset$ and the function $f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p), (\cdot,\cdot)\rangle\big)$ is lower semicontinuous at (p^*,p) and exact at (p^*,p);

(iii) there exist $f_S \in \mathcal{F}_S$ and $f_T \in \mathcal{F}_T$ with $\operatorname{dom} f_S \cap \big(\operatorname{dom} \hat{f}_T + (p,p^*)\big) \neq \emptyset$ such that the function $f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p), (\cdot,\cdot)\rangle\big)$ is lower semicontinuous at (p^*,p) and exact at (p^*,p).

Proof. Note first that the assertion (ii)⇒(iii) is immediate and one also has

$$\big(\hat{f}_T(\cdot - p, \cdot - p^*)\big)^* = \hat{f}_T^* + \langle p^*, \cdot\rangle + \langle \cdot, p\rangle. \tag{1}$$

(iii)⇒(i) Proposition 2.1 yields the equivalence of (iii) to

$$\big(f_S + \hat{f}_T(\cdot - p, \cdot - p^*)\big)^*(p^*,p) = \min_{u^*\in X^*, u \in X} \big[f_S^*(p^*-u^*, p-u) + \hat{f}_T^*(u^*,u) + \langle p^*, u\rangle + \langle u^*, p\rangle\big]. \tag{2}$$

Denoting by $(\bar{u}^*, \bar{u}) \in X^* \times X$ the point where this minimum is attained, we obtain, via Proposition 2.3,

$$\begin{aligned}\big(f_S + \hat{f}_T(\cdot - p, \cdot - p^*)\big)^*(p^*,p) &= f_S^*(p^*-\bar{u}^*, p-\bar{u}) + \hat{f}_T^*(\bar{u}^*,\bar{u}) + \langle p^*, \bar{u}\rangle + \langle \bar{u}^*, p\rangle \\ &\geq \langle p^*-\bar{u}^*, p-\bar{u}\rangle - \langle \bar{u}^*, \bar{u}\rangle + \langle p^*, \bar{u}\rangle + \langle \bar{u}^*, p\rangle = \langle p^*, p\rangle.\end{aligned} \tag{3}$$

But Proposition 2.3 yields for every $x \in X$ and $x^* \in X^*$

$$\big(f_S + \hat{f}_T(\cdot - p, \cdot - p^*)\big)(x,x^*) \geq \langle x^*, x\rangle + \langle -(x^*-p^*), x-p\rangle = \langle x^*, p\rangle + \langle p^*, x\rangle - \langle p^*, p\rangle,$$

thus $\langle p^*, p\rangle \geq \langle x^*, p\rangle + \langle p^*, x\rangle - \big(f_S + \hat{f}_T(\cdot - p, \cdot - p^*)\big)(x,x^*)$. Consequently,

$$\big(f_S + \hat{f}_T(\cdot - p, \cdot - p^*)\big)^*(p^*,p) \leq \langle p^*, p\rangle. \tag{4}$$

Together with (3) this yields

$$\big(f_S + \hat{f}_T(\cdot - p, \cdot - p^*)\big)^*(p^*,p) = \langle p^*, p\rangle,$$

and consequently the inequalities invoked to obtain (3) must be fulfilled as equalities. Therefore

$$f_S^*(p^*-\bar{u}^*, p-\bar{u}) = \langle p^*-\bar{u}^*, p-\bar{u}\rangle \qquad \text{and} \qquad \hat{f}_T^*(\bar{u}^*,\bar{u}) = \langle -\bar{u}^*, \bar{u}\rangle. \tag{5}$$

Having these, Proposition 2.3 yields then $p^*-\bar{u}^* \in S(p-\bar{u})$ and $\bar{u}^* \in T(-\bar{u})$, followed by $p^* \in S(p-\bar{u}) + T(-\bar{u})$, i.e. $p^* \in \mathrm{R}\big(S(p+\cdot)+T(\cdot)\big)$.

(i)⇒(ii) Whenever $f_S \in \mathcal{F}_S$, $f_T \in \mathcal{F}_T$, (i) yields, via Remark 3.1, $(p,p^*) \in \operatorname{dom} f_S - \operatorname{dom} \hat{f}_T$, i.e. $\operatorname{dom} f_S \cap (\operatorname{dom} \hat{f}_T + (p^*,p)) \neq \emptyset$.

For every $f_S \in \mathcal{F}_S$, $f_T \in \mathcal{F}_T$, $u \in X$ and $u^* \in X^*$ we have $f_S^*(p^*-u^*, p-u) + \hat{f}_T^*(u^*,u) + \langle (p^*,p), (u,u^*)\rangle \geq \langle p^*-u^*, p-u\rangle - \langle u^*, u\rangle + \langle p^*, u\rangle + \langle u^*, p\rangle = \langle p^*, p\rangle$. Consequently, $f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p), (\cdot,\cdot)\rangle\big)\,(p^*,p) \geq \langle p^*, p\rangle$ and since the function in the right-hand side is strong×strong continuous, its value at (p^*,p) must also be smaller than $\operatorname{cl}\big(f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p), (\cdot,\cdot)\rangle\big)\big)(p^*,p)$. But from [5, Theorem 7.6] we know, via (1), that one has $\operatorname{cl}\big(f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p), (\cdot,\cdot)\rangle\big)\big) = \big(f_S + \hat{f}_T(-(p^*,p) + (\cdot,\cdot))\big)^*$ and since (4) always holds, it follows that $\operatorname{cl}\big(f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p), (\cdot,\cdot)\rangle\big)\big)(p^*,p) \leq \langle p^*, p\rangle$. Consequently,

$$f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p), (\cdot,\cdot)\rangle\big)(p^*,p) \geq \operatorname{cl}\big(f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p), (\cdot,\cdot)\rangle\big)\big)(p^*,p) = \langle p^*, p\rangle. \tag{6}$$

Since $p^* \in \mathrm{R}\big(S(p+\cdot)+T(\cdot)\big)$, there exists $(\bar{u}^*, \bar{u}) \in X^* \times X$ fulfilling (5). Then $f_S^*(p^*-\bar{u}^*, p-\bar{u}) + \hat{f}_T^*(\bar{u}^*,\bar{u}) + \langle (p^*,p), (\bar{u},\bar{u}^*)\rangle = \langle p^*, p\rangle$, i.e.

$$f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p), (\cdot,\cdot)\rangle\big)(p^*,p) = f_S^*(p^*-\bar{u}^*, p-\bar{u}) + \hat{f}_T^*(\bar{u}^*,\bar{u}) + \langle (p^*,p), (\bar{u},\bar{u}^*)\rangle = \langle p^*, p\rangle,$$

therefore the exactness of the infimal convolution in (ii) is proven, while the lower semicontinuity follows via (6). □

From Theorem 3.2 we obtain the following surjectivity result.

Corollary 3.3.

For $p \in X$, one has $\mathrm{R}\big(S(p+\cdot)+T(\cdot)\big)=X^*$ if and only if

$$\left|\begin{array}{l}\text{for all } p^* \in X^*,\ f_S \in \mathcal{F}_S \text{ and } f_T \in \mathcal{F}_T \text{ one has } \operatorname{dom} f_S \cap \big(\operatorname{dom} \hat{f}_T + (p,p^*)\big) \neq \emptyset \text{ and}\\ \text{the function } f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p),(\cdot,\cdot)\rangle\big) \text{ is lower semicontinuous at } (p^*,p) \text{ and exact at } (p^*,p),\end{array}\right.$$

and this is further equivalent to

$$\left|\begin{array}{l}\text{for all } p^* \in X^* \text{ there exist } f_S \in \mathcal{F}_S \text{ and } f_T \in \mathcal{F}_T \text{ with } \operatorname{dom} f_S \cap \big(\operatorname{dom} \hat{f}_T + (p,p^*)\big) \neq \emptyset \text{ such that}\\ \text{the function } f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p),(\cdot,\cdot)\rangle\big) \text{ is lower semicontinuous at } (p^*,p) \text{ and exact at } (p^*,p).\end{array}\right.$$

Starting from Corollary 3.3 we are able to introduce a *sufficient condition* for the surjectivity of $S(p+\cdot)+T(\cdot)$ for a given $p \in X$.

Theorem 3.4.

Let $p \in X$. Then $\mathrm{R}\big(S(p+\cdot)+T(\cdot)\big)=X^*$ if

$$\begin{array}{l}\text{for each } p^* \in X^* \text{ there exist } f_S \in \mathcal{F}_S \text{ and } f_T \in \mathcal{F}_T \text{ with } \operatorname{dom} f_S \cap \big(\operatorname{dom} \hat{f}_T + (p,p^*)\big) \neq \emptyset \text{ such that}\\ \text{the function } f_S^* \Box \big(\hat{f}_T^* + \langle (p^*,p),(\cdot,\cdot)\rangle\big) \text{ is lower semicontinuous on } X^* \times \{p\} \text{ and exact at } (p^*,p).\end{array} \tag{RC}$$

Next we characterize the surjectivity of $S+T$ via a condition involving representative functions. The first result follows directly from Theorem 3.2, while the second one is a direct consequence of the first.

Theorem 3.5.

Let $p^* \in X^*$. The following statements are equivalent:

(i) $p^* \in \mathrm{R}(S+T)$;

(ii) *for all $f_S \in \mathcal{F}_S$ and $f_T \in \mathcal{F}_T$ one has $\operatorname{dom} f_S \cap \big(\operatorname{dom} \hat{f}_T + (0,p^*)\big) \neq \emptyset$ and the function $f_S^* \Box \big(\hat{f}_T^* + \langle p^*,\cdot\rangle\big)$ is lower semicontinuous at $(p^*,0)$ and exact at $(p^*,0)$;*

(iii) *there exist $f_S \in \mathcal{F}_S$ and $f_T \in \mathcal{F}_T$ with $\operatorname{dom} f_S \cap \big(\operatorname{dom} \hat{f}_T + (0,p^*)\big) \neq \emptyset$ such that the function $f_S^* \Box \big(\hat{f}_T^* + \langle p^*,\cdot\rangle\big)$ is lower semicontinuous at $(p^*,0)$ and exact at $(p^*,0)$.*

Corollary 3.6.

One has $\mathrm{R}(S+T)=X^*$ if and only if

$$\left|\begin{array}{l}\text{for all } p^* \in X^*,\ f_S \in \mathcal{F}_S,\ f_T \in \mathcal{F}_T \text{ one has } \operatorname{dom} f_S \cap \big(\operatorname{dom} \hat{f}_T + (0,p^*)\big) \neq \emptyset \text{ and}\\ \text{the function } f_S^* \Box \big(\hat{f}_T^* + \langle p^*,\cdot\rangle\big) \text{ is lower semicontinuous at } (p^*,0) \text{ and exact at } (p^*,0),\end{array}\right.$$

and this is further equivalent to

$$\left|\begin{array}{l}\text{for all } p^* \in X^* \text{ there exist } f_S \in \mathcal{F}_S \text{ and } f_T \in \mathcal{F}_T \text{ with } \operatorname{dom} f_S \cap \big(\operatorname{dom} \hat{f}_T + (0,p^*)\big) \neq \emptyset\\ \text{such that the function } f_S^* \Box \big(\hat{f}_T^* + \langle p^*,\cdot\rangle\big) \text{ is lower semicontinuous at } (p^*,0) \text{ and exact at } (p^*,0).\end{array}\right.$$

From Corollary 3.6 one can deduce a sufficient condition to have $S+T$ surjective.

Theorem 3.7.

One has $\mathrm{R}(S+T) = X^*$ *if*

$$\text{for all } p^* \in X^* \text{ there exist } f_S \in \mathcal{F}_S \text{ and } f_T \in \mathcal{F}_T \text{ with } \operatorname{dom} f_S \cap \big(\operatorname{dom} \hat{f}_T + (0, p^*)\big) \neq \emptyset \text{ such that the function } f_S^* \Box \big(\hat{f}_T^* + \langle p^*, \cdot\rangle\big) \text{ is lower semicontinuous on } X^* \times \{0\} \text{ and exact at } (p^*, 0). \tag{$\overline{\mathrm{RC}}$}$$

Remark 3.8.

In the literature there were given other regularity conditions guaranteeing surjectivity of $S+T$, namely, for fixed $f_S \in \mathcal{F}_S$ and $f_T \in \mathcal{F}_T$,

- (cf. [9, Corollary 2.7]) $\operatorname{dom} f_T = X \times X^*$,
- (cf. [14, Theorem 30.2]) $\operatorname{dom} f_S - \operatorname{dom} \hat{f}_T = X \times X^*$,
- (cf. [16, Corollary 4]) $\{0\} \times X^* \subseteq \operatorname{sqri}(\operatorname{dom} f_S - \operatorname{dom} \hat{f}_T)$,

where sqri denotes the *strong quasi relative interior* of a given set, respectively. It is obvious that the first one implies the second, whose fulfillment yields the third condition. This one yields

$$\big(f_S + \hat{f}_T(\cdot, \cdot - p^*)\big)^*(x^*, 0) = \min_{u^* \in X^*, u \in X} \big[f_S^*(x^* - u^*, -u) + \hat{f}_T^*(u^*, u) + \langle p^*, u\rangle\big] \qquad \text{for all} \quad x^*, p^* \in X^*,$$

which is equivalent, when $\operatorname{dom} f_S \cap (\operatorname{dom} \hat{f}_T + (0, p^*)) \neq \emptyset$ (condition fulfilled in all the three regularity conditions given above), to the fact that whenever $p^* \in X^*$ the function $f_S^* \Box \big(\hat{f}_T^* + \langle p^*, \cdot\rangle\big)$ is lower semicontinuous at $(x^*, 0)$ and exact at $(x^*, 0)$ for all $x^* \in X^*$. It is obvious that this implies ($\overline{\mathrm{RC}}$) and below we present a situation where ($\overline{\mathrm{RC}}$) holds, while the conditions for surjectivity of $S+T$ listed above do not.

Example 3.9.

Let $X = \mathbb{R}$ and consider the operators $S, T : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$Sx = \begin{cases} \{0\} & \text{if } x > 0, \\ (-\infty, 0] & \text{if } x = 0, \\ \emptyset & \text{otherwise}, \end{cases} \qquad Tx = \begin{cases} \mathbb{R} & \text{if } x = 0, \\ \emptyset & \text{otherwise}, \end{cases} \qquad x \in \mathbb{R}.$$

One notices easily that, considering the functions $f, g : \mathbb{R} \to \overline{\mathbb{R}}$, $f = \delta_{[0,+\infty)}$ and $g = \delta_{\{0\}}$, which are proper, convex and lower-semicontinuous, we have $S = \partial f$ and $T = \partial g$, thus S and T are maximal monotone. It is obvious that $\mathrm{R}(S+T) = \mathbb{R}$. The Fitzpatrick families of both S and T contain only their Fitzpatrick functions, because f and g are sublinear functions. We have

$$\varphi_S(x, x^*) = \begin{cases} 0 & \text{if } x \geq 0,\ x^* \leq 0, \\ +\infty & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \varphi_T(x, x^*) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{otherwise}. \end{cases}$$

Therefore

$$\varphi_S^*(x^*, x) = \begin{cases} 0 & \text{if } x^* \leq 0,\ x \geq 0, \\ +\infty & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \varphi_T^*(x^*, x) = \begin{cases} 0 & \text{if } x = 0, \\ +\infty & \text{otherwise}. \end{cases}$$

Then $\operatorname{dom} \varphi_S - \operatorname{dom} \hat{\varphi}_T = \mathbb{R}_+ \times \mathbb{R}$, where $\mathbb{R}_+ = [0, +\infty)$, and it is obvious that $\{0\} \times \mathbb{R}$ is not included in $\operatorname{sqri}(\operatorname{dom} \varphi_S - \operatorname{dom} \hat{\varphi}_T) = (0, +\infty) \times \mathbb{R}$. Consequently, the three conditions mentioned in Remark 3.8 fail in this situation. On the other hand, for $p^*, x, x^* \in \mathbb{R}$ one has

$$\varphi_S^* \Box \big(\hat{\varphi}_T^* + \langle p^*, \cdot\rangle\big)(x^*, x) = \begin{cases} 0 & \text{if } x \geq 0, \\ +\infty & \text{if } x < 0. \end{cases}$$

This function is lower semicontinuous on $\mathbb{R} \times \mathbb{R}_+$ and exact at all $(x^*, x) \in \mathbb{R} \times \mathbb{R}_+$. Consequently, ($\overline{\mathrm{RC}}$) is valid in this case.

Remark 3.10.

Following Remark 2.2, when one of f_S and f_T is continuous, (RC) is automatically fulfilled. It is known (see for instance [14]) that the domain of the Fitzpatrick function attached to the *duality map*

$$J : X \rightrightarrows X^*, \qquad Jx = \partial \frac{1}{2}\|x\|^2 = \left\{x^* \in X^* : \|x\|^2 = \|x^*\|_*^2 = \langle x^*, x\rangle\right\}, \qquad x \in X,$$

which is a maximal monotone operator, is the whole product space $X \times X^*$. By [15, Theorem 2.2.20] it follows that φ_J is continuous, thus by Corollary 3.3 we obtain that $S(p+\cdot) + J(\cdot)$ is surjective whenever $p \in X$. Thus we rediscover a property of maximal monotone operators. Moreover, employing Corollary 3.6 one gets that $S+J$ is surjective, rediscovering Rockafellar's classical surjectivity theorem for maximal monotone operators [14, Theorem 29.5].

The last results we derive from the main one are connected to the situation when 0 lies in the range of $S+T$.

Corollary 3.11.

One has $0 \in \mathrm{R}(S+T)$ *if and only if*

$$\left|\begin{array}{l} \text{for all } f_S \in \mathcal{F}_S \text{ and } f_T \in \mathcal{F}_T \text{ one has } \operatorname{dom} f_S \cap \operatorname{dom} \hat{f}_T \neq \emptyset \text{ and} \\ \text{the function } f_S^* \square \hat{f}_T^* \text{ is lower semicontinuous at } (0,0) \text{ and exact at } (0,0), \end{array}\right.$$

and this is further equivalent to

$$\left|\begin{array}{l} \text{there exist } f_S \in \mathcal{F}_S \text{ and } f_T \in \mathcal{F}_T \text{ with } \operatorname{dom} f_S \cap \operatorname{dom} \hat{f}_T \neq \emptyset \text{ such that} \\ \text{the function } f_S^* \square \hat{f}_T^* \text{ is lower semicontinuous at } (0,0) \text{ and exact at } (0,0). \end{array}\right.$$

From Corollary 3.11 one can deduce a sufficient condition to be sure that $0 \in \mathrm{R}(S+T)$.

Theorem 3.12.

One has $0 \in \mathrm{R}(S+T)$ *if*

$$\begin{array}{l} \text{there exist } f_S \in \mathcal{F}_S \text{ and } f_T \in \mathcal{F}_T \text{ with } \operatorname{dom} f_S \cap \operatorname{dom} \hat{f}_T \neq \emptyset \text{ such that} \\ \text{the function } f_S^* \square \hat{f}_T^* \text{ is lower semicontinuous on } X^* \times \{0\} \text{ and exact at } (0,0). \end{array} \tag{$\widetilde{\mathrm{RC}}$}$$

Remark 3.13.

Other regularity conditions guaranteeing $0 \in \mathrm{R}(S+T)$ were given in [4, Theorem 4.5], $(0,0) \in \operatorname{core}\big(\operatorname{co}(\mathrm{G}(S)) - \operatorname{co}(\mathrm{G}(-T))\big)$, where core denotes the *algebraic interior* of a given set and co its *convex hull*, and [16, Lemma 1], $(0,0) \in \operatorname{sqri}(\operatorname{dom} f_S - \operatorname{dom} \hat{f}_T)$, respectively. Following similar arguments to the ones in Remark 3.8 one can show that both yield ($\widetilde{\mathrm{RC}}$). Checking the situation from Example 3.9, we see that the condition involving sqri fails, while ($\widetilde{\mathrm{RC}}$) is fulfilled. As $\operatorname{core}\big(\operatorname{co}(\mathrm{G}(S)) - \operatorname{co}(\mathrm{G}(-T))\big) = \operatorname{int}(\mathbb{R}_+ \times (-\mathbb{R}_+) - \{0\} \times \mathbb{R}) = (0,+\infty) \times \mathbb{R}$ does not contain $(0,0)$, it is straightforward that ($\widetilde{\mathrm{RC}}$) is indeed weaker than both abovementioned conditions for $0 \in \mathrm{R}(S+T)$.

Remark 3.14.

One can notice via (1) that (2) can be rewritten when $p^* = 0$ and $p = 0$ as

$$\inf_{x \in X, x^* \in X^*} \left[f_S(x,x^*) + \hat{f}_T(x,x^*)\right] = \max_{u^* \in X^*, u \in X} \left[-f_S^*(-u^*,-u) - \hat{f}_T^*(u^*,u)\right], \tag{7}$$

i.e. there is strong duality for the convex optimization problem formulated above in the left-hand side of (7) and its Fenchel dual problem. When $(\bar{u}, \bar{u}^*) \in X \times X^*$ is an optimal solution of the dual problem, i.e. the point where the maximum in the right-hand side of (7) is attained, one obtains $\bar{u}^* \in S(\bar{u})$ and $-\bar{u}^* \in T(\bar{u})$. Employing now Proposition 2.3 we obtain $f_S(\bar{u},\bar{u}^*) = f_S^*(-\bar{u}^*,-\bar{u}) = \langle \bar{u}^*, \bar{u}\rangle$ and $\hat{f}_T(\bar{u},\bar{u}^*) = \hat{f}_T^*(\bar{u}^*,\bar{u}) = -\langle \bar{u}^*, \bar{u}\rangle$, therefore

$$f_S(\bar{u},\bar{u}^*) + \hat{f}_T(\bar{u},\bar{u}^*) = f_S^*(-\bar{u}^*,-\bar{u}) + \hat{f}_T^*(\bar{u}^*,\bar{u}) = 0.$$

Thus, the infimum in the left-hand side of (7) is attained, i.e. the primal optimization problem given there has an optimal solution, too. As strong duality for it holds, we are now in the situation called *total duality* [5, Section 17], which happens when the optimal objective values of the primal and dual coincide and both these problems have optimal solutions. Therefore we can conclude that for this kind of optimization problems when strong Fenchel duality holds the primal problem has an optimal solution, too.

Remark 3.15.
Given $p \in X$ and $p^* \in X^*$, the function $f_S^* \Box \big(\hat{f}_T^* + \langle (p^*, p), (\cdot, \cdot) \rangle\big)$ can be replaced in conditions (ii)–(iii) from Theorem 3.2 with $\big(f_S^* - \langle (p^*, p), (\cdot, \cdot) \rangle\big) \Box \hat{f}_T^*$ without altering the statement. The other conditions considered above can be correspondingly rewritten, too.

4. Applications

4.1. T is the normal cone of a closed convex set

Let $U \subseteq X$ be a nonempty closed convex set. Its normal cone N_U is a maximal monotone operator. Taking $T = N_U$, its only representative function is $f_{N_U}(x, x^*) = \delta_U(x) + \sigma_U(x^*)$, $(x, x^*) \in X \times X^*$. From our main statements we obtain in this case the following results.

Corollary 4.1.
Let $p \in X$. Then $\mathrm{R}\big(S(p + \cdot) + N_U(\cdot)\big) = X^*$ if and only if

for all $p^* \in X^*$ and $f_S \in \mathcal{F}_S$ one has $\operatorname{dom} f_S \cap \big(U \times \operatorname{dom} \sigma_{-U} + (p, p^*)\big) \neq \emptyset$ and the function

$$(y^*, y) \mapsto \inf_{x \in -U, x^* \in X^*} \Big[\big(f_S^* - \langle (p^*, p), (\cdot, \cdot) \rangle\big)(y^* - x^*, y - x) + \sigma_U(x^*)\Big] \tag{8}$$

is lower semicontinuous at (p^*, p) and the infimum within is attained when $(y^*, y) = (p^*, p)$,

and this is further equivalent to

for all $p^* \in X^*$ there exists $f_S \in \mathcal{F}_S$ with $\operatorname{dom} f_S \cap (U \times \operatorname{dom} \sigma_{-U} + (p, p^*)) \neq \emptyset$ such that the function (8) is lower semicontinuous at (p^*, p) and the infimum within is attained when $(y^*, y) = (p^*, p)$.

Corollary 4.2.
Let $p \in X$. Then $\mathrm{R}\big(S(p + \cdot) + N_U(\cdot)\big) = X^*$ if

for all $p^* \in X^*$ there exists $f_S \in \mathcal{F}_S$ with $\operatorname{dom} f_S \cap \big(U \times \operatorname{dom} \sigma_{-U} + (p, p^*)\big) \neq \emptyset$ such that the function (8) is lower semicontinuous on $X^* \times \{p\}$ and the infimum within is attained when $(y^*, y) = (p^*, p)$. (RC_U)

Corollary 4.3.
One has $0 \in \mathrm{R}(S + N_U)$ if

there exists $f_S \in \mathcal{F}_S$ with $\operatorname{dom} f_S \cap (U \times \operatorname{dom} \sigma_{-U}) \neq \emptyset$ such that the function $(y^*, y) \mapsto \inf_{x \in U}\{(f_S^*(\cdot, y + x) \Box \sigma_U)(y^*)\}$ is lower semicontinuous on $X^* \times \{0\}$ and the infimum within is attained when $(y^*, y) = (0, 0)$. (RC_0)

Remark 4.4.
A stronger than (RC_0) regularity condition for $0 \in \mathrm{R}(S + N_U)$, namely $0 \in \operatorname{core}(\operatorname{co}(\mathrm{D}(S) - U))$, was considered in [4, Corollary 5.7].

Not without importance is the question how one can equivalently characterize surjectivity of a maximal monotone operator via its representative functions. To answer this question, take $U = X$. Then $T = N_X$, i.e. $Tx = \{0\}$ for all $x \in X$, and the Fenchel representative function of N_X is $(x, x^*) \mapsto \delta_X(x) + \sigma_X(x^*) = \delta_{\{0\}}(x^*)$. Then $S + T = S$ and surjectivity of S can be characterized, via Corollary 4.1, as follows.

Corollary 4.5.

One has $\mathrm{R}(S) = X^*$ *if and only if*

$$\left|\begin{array}{l} \textit{for all } p^* \in X^* \textit{ and } f_S \in \mathcal{F}_S \textit{ the function } y^* \mapsto -\big(f_S^*(y^*, \cdot)\big)^*(p^*) \textit{ is lower} \\ \textit{semicontinuous at } p^* \textit{ and there exists } x \in X \textit{ such that } p^* \in \big(\partial f_S^*(p^*, \cdot)\big)(x), \end{array}\right.$$

and this is further equivalent to

$$\left|\begin{array}{l} \textit{for all } p^* \in X^* \textit{ there exists } f_S \in \mathcal{F}_S \textit{ such that the function } y^* \mapsto -\big(f_S^*(y^*, \cdot)\big)^*(p^*) \\ \textit{is lower semicontinuous at } p^* \textit{ and there exists } x \in X \textit{ such that } p^* \in \big(\partial f_S^*(p^*, \cdot)\big)(x). \end{array}\right.$$

Proof. Corollary 4.1 asserts the equivalence of the surjectivity of the maximal monotone operator S to the lower semicontinuity at $(p^*, 0)$ of the function $(y^*, y) \mapsto \inf_{x \in X, x^* \in X^*} \big[(f_S^* - \langle p^*, \cdot\rangle)(y^* - x^*, y + x) + \sigma_X(x^*)\big]$ concurring with the attainment of the infimum within when $(y^*, y) = (p^*, 0)$, for every $p^* \in X^*$. Taking a closer look at this function, we note that it can be simplified to $(y^*, y) \mapsto \inf_{x \in X} \big[f_S^*(y^*, y + x) - \langle p^*, y + x\rangle\big]$, which can be further reduced to $y^* \mapsto -\big(f_S^*(y^*, \cdot)\big)^*(p^*)$. For $p^* \in X^*$, the attainment of the infimum from above when $(y^*, y) = (p^*, 0)$ means actually the existence of $x \in X$ such that $f_S^*(p^*, x) - \langle p^*, x\rangle = -\big(f_S^*(p^*, \cdot)\big)^*(p^*)$, which is nothing but $p^* \in \big(\partial f_S^*(p^*, \cdot)\big)(x)$. □

Remark 4.6.

In [9, Corollary 2.2] it is said that S is surjective if $\mathrm{dom}\,(\varphi_S) = X \times X^*$. This result can be obtained as a consequence of Corollary 4.5, via Remark 2.2.

Remark 4.7.

Determining when $0 \in \mathrm{R}(S)$ is important even beyond optimization. Via Corollary 4.5 we can provide the following sufficient condition for this:

$$\left|\begin{array}{l} \text{there exists } f_S \in \mathcal{F}_S \text{ such that the function } y^* \mapsto -\big(f_S^*(y^*, \cdot)\big)^*(0) \text{ is lower} \\ \text{semicontinuous and there exists } x \in X \text{ such that } p^* \in \big(\partial f_S^*(0, \cdot)\big)(x). \end{array}\right.$$

4.2. S and T are subdifferentials

Take proper convex lower semicontinuous functions $f, g : X \to \overline{\mathbb{R}}$. Let first $T = \partial g$ and consider for it the Fenchel representative function. Then Corollary 3.3 yields the following statement.

Corollary 4.8.

Let $p \in X$. *Then* $\mathrm{R}\big(S(p + \cdot) + \partial g(\cdot)\big) = X^*$ *if and only if*

$$\left|\begin{array}{l} \textit{for all } p^* \in X^* \textit{ and } f_S \in \mathcal{F}_S \textit{ one has } \mathrm{dom}\, f_S \cap \big(\mathrm{dom}\, g \times (-\,\mathrm{dom}\, g^*) + (p, p^*)\big) \neq \emptyset \textit{ and the function} \\ f_S^* \Box \big(g(-\cdot) + g^*(\cdot) + \langle (p^*, p), (\cdot, \cdot)\rangle\big) \textit{ is lower semicontinuous at } (p^*, p) \textit{ and exact at } (p^*, p), \end{array}\right.$$

and this is further equivalent to

$$\left|\begin{array}{l} \textit{for all } p^* \in X^* \textit{ there exists } f_S \in \mathcal{F}_S \textit{ with } \mathrm{dom}\, f_S \cap \big(\mathrm{dom}\, g \times (-\,\mathrm{dom}\, g^*) + (p, p^*)\big) \neq \emptyset \textit{ such that} \\ \textit{the function } f_S^* \Box \big(g(-\cdot) + g^*(\cdot) + \langle (p^*, p), (\cdot, \cdot)\rangle\big) \textit{ is lower semicontinuous at } (p^*, p) \textit{ and exact at } (p^*, p). \end{array}\right.$$

The other statements in Section 3 can be particularized for this special case, too. However, we give here only a consequence of Theorem 3.12.

Corollary 4.9.

One has $0 \in \mathrm{R}(S+\partial g)$ *if*

$$\left|\begin{array}{l} \text{there exists } f_S \in \mathcal{F}_S \text{ with } \operatorname{dom} f_S \cap \big(\operatorname{dom} g \times (-\operatorname{dom} g^*)\big) \neq \emptyset \text{ such that the function} \\ f_S^* \Box \big(g(-\cdot)+g^*(\cdot)\big) \text{ is lower semicontinuous on } X^* \times \{0\} \text{ and exact at } (0,0). \end{array}\right.$$

Remark 4.10.

In [9, Proposition 2.9] it was proven that when g and g^* are real-valued, the monotone operator $S(p+\cdot)+\partial g(\cdot)$ is surjective whenever $p \in X$. This statement can be rediscovered as a consequence of Corollary 4.8, too. Using [15, Proposition 2.1.6] one obtains that g and g^* are continuous. Then the Fenchel representative function of ∂g is continuous and (see Remark 2.2) this yields the fulfillment of the regularity condition from Corollary 4.8. Consequently, $S(p+\cdot)+\partial g(\cdot)$ is surjective whenever $p \in X$.

Take now also $S = \partial f$, to which we associate the Fenchel representative function, too. Let the function $\hat{g} : X \to \overline{\mathbb{R}}$ be defined by $\hat{g}(x) = g(-x)$. Corollary 3.3 yields the following result.

Corollary 4.11.

Let $p \in X$. *If* $\operatorname{dom} f \cap (p+\operatorname{dom} g) \neq \emptyset$, *then* $\mathrm{R}\big(\partial f(p+\cdot)+\partial g(\cdot)\big) = X^*$ *if and only if*

$$\left|\begin{array}{l} \text{for all } p^* \in X^* \text{ one has } \operatorname{dom} f^* \cap (p^* - \operatorname{dom} g^*) \neq \emptyset, \\ \text{the function } f \Box (\hat{g}+p^*) \text{ is lower semicontinuous at } p \text{ and exact at } p \\ \text{and the function } f^* \Box (g^*+p) \text{ is lower semicontinuous at } p^* \text{ and exact at } p^*. \end{array}\right.$$

Moreover, from Corollary 4.9 one can deduce the following statement.

Corollary 4.12.

One has $0 \in \mathrm{R}(\partial f+\partial g)$ *if* $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, $\operatorname{dom} f^* \cap (-\operatorname{dom} g^*) \neq \emptyset$ *and*

$$\left|\begin{array}{l} f \Box \hat{g} \text{ is lower semicontinuous at } 0 \text{ and exact at } 0, \text{ and} \\ \text{the function } f^* \Box g^* \text{ is lower semicontinuous and exact at } 0. \end{array}\right.$$

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