Extending the classical vector Wolfe and Mond-Weir duality concepts via perturbations

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May 4, 2011

Abstract. Considering a general vector optimization problem, we attach to it by means of perturbation theory new vector duals. When the primal problem and the perturbation function are particularized different vector dual problems are obtained. In the special case of a constrained vector optimization problem the classical Wolfe and Mond-Weir duals to the latter, respectively, can be obtained from the general ones by using the Lagrange perturbation.

Keywords. Wolfe duality, Mond-Weir duality, conjugate functions, convex subdifferentials, vector duality

AMS subject classification (2000). 49N15, 90C25, 90C29

1 Introduction and preliminaries

The already classical duality concepts due to Wolfe and, respectively, Mond and Weir, were considered first in the scalar case, but soon they were extended for vector optimization problems, too. Thus, a flourishing literature dealing with this topic appeared, developing mainly in the differential case by means of various generalized convexity notions. This is a direction we do not embrace in this paper where we embed the Wolfe and Mond-Weir duality concepts in two classes of vector optimization problems defined via perturbations, respectively, extending thus the investigations performed in the scalar case in [1]. Even if most of the literature on vector Wolfe duality and vector Mond-Weir duality is done in finitely dimensional spaces, we work here in the very general setting of separated locally convex vector spaces. However, when the framework is particularized to the classical one from the literature we rediscover as special cases of the vector duals we introduce here the classical vector Wolfe and Mond-Weir dual problems.

Consider two separated locally convex vector spaces X and Y and their topological dual spaces X^* and Y^*, respectively, endowed with the correspond-

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ing weak* topologies, and denote by $\langle x^*, x\rangle = x^*(x)$ the value at $x \in X$ of the linear continuous functional $x^* \in X^*$. A *cone* $K \subseteq X$ is a nonempty subset of X which fulfills $\lambda K \subseteq K$ for all $\lambda \geq 0$. A cone $K \subseteq X$ is said to be *nontrivial* if $K \neq \{0\}$ and $K \neq X$ and pointed if $K \cap (-K) = \{0\}$. On Y we consider the partial ordering "$\leqq_C$" induced by the convex cone $C \subseteq Y$, defined by $z \leqq_C y \Leftrightarrow y - z \in C$ when $z, y \in Y$. We use also the notation $z \leq_C y$ to write more compact that $z \leqq_C y$ and $z \neq y$, where $z, y \in Y$. To Y we attach a greatest element with respect to "$\leqq_C$", which does not belong to Y, denoted by ∞_C and let $Y^\bullet = Y \cup \{\infty_C\}$. Then for any $y \in Y^\bullet$ one has $y \leqq_C \infty_C$ and we consider on $Y^\bullet$ the operations $y + \infty_C = \infty_C + y = \infty_C$ for all $y \in Y$ and $t \cdot \infty_C = \infty_C$ for all $t \geq 0$. The dual cone of C is $C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \ \forall y \in C\}$. By convention, $\langle v^*, \infty_C\rangle = +\infty$ for all $v^* \in C^*$. Given a subset U of X, by $\operatorname{cl}(U)$, $\operatorname{lin}(U)$, $\operatorname{aff}(U)$, $\operatorname{cone}(U)$, $\operatorname{ri}(U)$, $\dim(U)$, δ_U and σ_U we denote its *closure*, *linear hull*, *affine hull*, *conical hull*, *relative interior*, *dimension*, *indicator function* and *support function*, respectively. Moreover, if U is convex its *strong quasi relative interior* is $\operatorname{sqri}(U) = \big\{x \in U : \operatorname{cone}(U - x) \text{ is a closed linear subspace}\big\}$. In vector optimization it is often used also the *quasi interior of the dual cone* of K, $K^{*0} := \big\{x^* \in K^* : \langle x^*, x\rangle > 0 \text{ for all } x \in K\backslash\{0\}\big\}$. Note that in the literature it is a common practice to name the set from above like this and in [2, Proposition 2.1.1] and the comments following it one can find justificatory explanations. We consider also the *projection function* $\operatorname{Pr}_X : X \times Y \to X$, defined by $\operatorname{Pr}_X(x, y) = x$ for all $(x, y) \in X \times Y$.

Having a function $f : X \to \overline{\mathbb{R}}$ we use the classical notations for *domain* $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$, *epigraph* $\operatorname{epi} f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$, *lower semicontinuous hull* $\bar{f} : X \to \overline{\mathbb{R}}$ and *conjugate function* $f^* : X^* \to \overline{\mathbb{R}}$, $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$. We call f *proper* if $f(x) > -\infty$ for all $x \in X$ and $\operatorname{dom} f \neq \emptyset$. For f proper, if $f(x) \in \mathbb{R}$ the *(convex) subdifferential* of f at x is $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in X\}$, while if $f(x) = +\infty$ we take by convention $\partial f(x) = \emptyset$. Note that for $U \subseteq X$ we have for all $x \in U$ that $\partial \delta_U(x) = N_U(x)$, the latter being the *normal cone* of U at x. Between a function and its conjugate there is the *Young-Fenchel inequality* $f^*(x^*) + f(x) \geq \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in X^*$. This inequality is fulfilled as equality if and only if $x^* \in \partial f(x)$. Considering for each $\lambda \in \mathbb{R}$ the function $\lambda f : X \to \overline{\mathbb{R}}$, $(\lambda f)(x) = \lambda f(x)$ for $x \in X$, note that when $\lambda = 0$ we take $0f = \delta_{\operatorname{dom} f}$. Given a linear continuous mapping $A : X \to Y$, we have its *adjoint* $A^* : Y^* \to X^*$ given by $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle$ for any $(x, y^*) \in X \times Y^*$. For a vector function $h : X \to Y^\bullet$ one has

- h is *proper* if its *domain* $\operatorname{dom} h = \{x \in X : h(x) \in Y\}$ is nonempty,
- h is C-*convex* if $h(tx + (1-t)y) \leqq_C th(x) + (1-t)h(y) \ \forall x, y \in X \ \forall t \in [0, 1]$,
- h is C-epi-*closed* if C is closed and its C-*epigraph* $\operatorname{epi}_C h = \{(x, y) \in X \times Y : y \in h(x) + C\}$ is closed,
- h is C-*lower semicontinuous* if for every $x \in X$, each neighborhood W of zero in Y and for any $b \in Y$ satisfying $b \leqq_C h(x)$, there exists a neighborhood U of x in X such that $h(U) \subseteq b + W + Y \cup \{+\infty_C\}$.

Consider also, for $v^* \in C^*$ the function $(v^*h) : X \to \overline{\mathbb{R}}$ defined by $(v^*h)(x) = \langle v^*, h(x)\rangle$, $x \in X$. One can show that if h is C-lower semicontinuous then (v^*h) is lower semicontinuous whenever $v^* \in C^*\backslash\{0\}$. Moreover, if C is closed, then every C-lower semicontinuous vector function is also C-epi-closed, but, as [2, Example 2.2.6] shows, not all C-epi-closed vector functions are C-lower semicontinuous.

The vector optimization problems we consider in this paper consist of vector-minimizing and vector-maximizing a vector function with respect to the partial ordering induced in the image space of the vector function by a pointed convex cone. As notions of solutions for vector optimization problems we rely on the classical efficient and properly efficient solutions, the latter considered with respect to the linear scalarization. For an exhaustive review of the proper efficiency notions considered in the literature and the relations between them we refer to [2, Section 2.4].

2 General Wolfe and Mond-Weir type duals via perturbations

Let X, Y and V be separated locally convex vector spaces, with V partially ordered by the nontrivial pointed convex cone $K \subseteq V$. Let $F : X \to V^\bullet$ be a proper vector function and consider the general vector-minimization problem

$$(PVG) \qquad \operatorname*{Min}_{x\in X} F(x).$$

The solution concepts we consider for this vector optimization problem are the following ones.

Definition 1. An element $\bar{x} \in X$ is said to be an *efficient solution* to the vector optimization problem (PVG) if $\bar{x} \in \operatorname{dom} F$ and for all $x \in \operatorname{dom} F$ from $F(x) \leqq_K F(\bar{x})$ follows $F(\bar{x}) = F(x)$.

Definition 2. An element $\bar{x} \in X$ is said to be a *properly efficient solution* to the vector optimization problem (PVG) if there exists $v^* \in K^{*0}$ such that $(v^*F)(\bar{x}) \leq (v^*F)(x)$ for all $x \in X$. The set of all properly efficient solutions to (PVG) is called the *proper efficiency set of* (PVG), being denoted by $\mathcal{PE}(PVG)$. Denote also by $\operatorname{PMin}(PVG)$ the set $\cup_{x\in\mathcal{PE}(PVG)}F(x)$.

Remark 1. Every properly efficient solution to (PVG) belongs to $\operatorname{dom} F$ and it is also an efficient solution to the same vector optimization problem.

Consider now the vector perturbation function $\Phi : X \times Y \to V^\bullet$ which fulfills $\Phi(x, 0) = F(x)$ for all $x \in X$. We call Y the *perturbation space* and its elements *perturbation variables.* Then $0 \in \operatorname{Pr}_Y(\operatorname{dom}\Phi)$ and thus Φ is proper. The primal vector optimization problem introduced above can be reformulated as

$$(PVG) \qquad \operatorname*{Min}_{x \in X} \Phi(x, 0).$$

Inspired by the way conjugate dual problems are attached to a given primal problem via perturbations in the scalar case and by the investigations from [1], where we embedded the classical Wolfe and Mond-Weir duality concepts into classes of scalar dual problems obtained via perturbation theory, incorporating also ideas from different papers on Wolfe and Mond-Weir vector duality like [4,6,11,12,15,17], we attach to (PVG) the following vector dual problems with respect to properly efficient solutions

$$(DVG_W) \qquad \operatorname*{Max}_{(v^*, y^*, u, y, r) \in \mathcal{B}_W^G} h_W^G(v^*, y^*, u, y, r),$$

where

$$\mathcal{B}_W^G = \Big\{(v^*, y^*, u, y, r) \in K^{*0} \times Y^* \times X \times Y \times (K \backslash \{0\}) : (0, y^*) \in \partial(v^* \Phi)(u, y)\Big\}$$

and

$$h_W^G(v^*, y^*, u, y, r) = \Phi(u, y) - \frac{\langle y^*, y\rangle}{\langle v^*, r\rangle} r$$

and, respectively,

$$(DVG_M) \qquad \operatorname*{Max}_{(v^*, y^*, u) \in \mathcal{B}_M^G} h_M^G(v^*, y^*, u),$$

where

$$\mathcal{B}_M^G = \Big\{(v^*, y^*, u) \in K^{*0} \times Y^* \times X : (0, y^*) \in \partial(v^* \Phi)(u, 0)\Big\}$$

and

$$h_M^G(v^*, y^*, u) = \Phi(u, 0).$$

Remark 2. Fixing $r \in K\backslash\{0\}$, we can construct, starting from (DVG_W), another dual problem to (PVG), namely

$$(DVG_{W^r}) \qquad \operatorname*{Max}_{(v^*, y^*, u, y) \in \mathcal{B}_{W^r}^G} h_{W^r}^G(v^*, y^*, u, y),$$

where

$$\mathcal{B}_{W^r}^G = \Big\{(v^*, y^*, u, y) \in K^{*0} \times Y^* \times X \times Y : (0, y^*) \in \partial(v^* \Phi)(u, y), \langle v^*, r\rangle = 1\Big\}$$

and

$$h_{W^r}^G(v^*, y^*, u, y) = \Phi(u, y) - \langle y^*, y\rangle r.$$

In this way one introduces a whole family of vector duals to (PVG).

For these vector-maximization problems we consider efficient solutions, defined below for (DVG_W) and analogously for the others.

Definition 3. An element $(\bar{v}^*, \bar{y}^*, \bar{u}, \bar{y}, \bar{r}) \in \mathcal{B}_W^G$ is said to be an *efficient solution* to the vector optimization problem (DVG_W) if $(\bar{v}^*, \bar{y}^*, \bar{u}, \bar{y}, \bar{r}) \in \operatorname{dom} h_W^G$ and for all $(v^*, y^*, u, y, r) \in \mathcal{B}_W^G$ from $h_W^G(\bar{v}^*, \bar{y}^*, \bar{u}, \bar{y}, \bar{r}) \leqq_K h_W^G(v^*, y^*, u, y, r)$

follows $h_W^G(\bar{v}^*, \bar{y}^*, \bar{u}, \bar{y}, \bar{r}) = h_W^G(v^*, y^*, u, y, r)$. The set of all efficient solutions to (DVG_W) is called the *efficiency set of* (DVG_W), being denoted by $\mathcal{E}(DVG_W)$. Denote also by $\operatorname{Max}(DVG_W)$ the set $\cup_{(v^*,y^*,u,y,r)\in\mathcal{E}(DVG_W)} h_W^G(v^*, y^*, u, y, r)$, called the *maximal set* of the problem (DVG_W).

From the way the vector duals are defined above one can obtain the following results involving the images of their feasible sets via their objective functions.

Proposition 1. *It holds*

$$h_M^G(\mathcal{B}_M^G) \subseteq \bigcup_{r\in K\setminus\{0\}} h_{W^r}^G(\mathcal{B}_{W^r}^G) = h_W^G(\mathcal{B}_W^G).$$

Proof. Take $(v^*, y^*, u) \in \mathcal{B}_M^G$. Then $v^* \in K^{*0}$ and there exists $r \in K\setminus\{0\}$ such that $\langle v^*, r\rangle = 1$. Thus $(v^*, y^*, u, 0) \in \mathcal{B}_{W^r}^G$ and $h_{W^r}^G(v^*, y^*, u, 0) = h_M^G(v^*, y^*, u) = \Phi(u, 0) = F(u)$.

Let now $r \in K\setminus\{0\}$ and $(v^*, y^*, u, y) \in \mathcal{B}_{W^r}^G$. It is obvious that $(v^*, y^*, u, y, r) \in \mathcal{B}_W^G$ and $h_W^G(v^*, y^*, u, y, r) = h_{W^r}^G(v^*, y^*, u, y) = \Phi(u, y) - \langle y^*, y\rangle r$.

Finally, if $(v^*, y^*, u, y, r) \in \mathcal{B}_W^G$, then taking $s = (1/\langle v^*, r\rangle)r \in K\setminus\{0\}$, it follows $\langle v^*, s\rangle = 1$ and, consequently, $(v^*, y^*, u, y) \in \mathcal{B}_{W^s}^G$. Moreover, $h_W^G(v^*, y^*, u, y, r) = h_{W^s}^G(v^*, y^*, u, y) = \Phi(u, y) - \langle y^*, y\rangle s$. □

A situation where the inclusion from Proposition 1 is strictly fulfilled will be given later in Example 2.

Remark 3. It is a simple verification to show that if $(v^*, y^*, u, y, r), (v^*, y^*, u, y, \bar{r}) \in \mathcal{B}_W^G$ such that $\langle y^*, y\rangle \neq 0$ and $h_W^G(v^*, y^*, u, y, r) - h_W^G(v^*, y^*, u, y, \bar{r}) \in K$, then $r = \bar{r}$.

Let us prove now that for the just introduced dual problems there is weak duality.

Theorem 1. *There are no $x \in X$ and $(v^*, y^*, u, y, r) \in \mathcal{B}_W^G$ such that $F(x) \leq_K h_W^G(v^*, y^*, u, y, r)$.*

Proof. Assume to the contrary that there exist $x \in X$ and $(v^*, y^*, u, y, r) \in \mathcal{B}_W^G$ fulfilling $F(x) \leq_K h_W^G(v^*, y^*, u, y, r)$. Then $x \in \operatorname{dom} F$ and it follows

$$\left\langle v^*, \Phi(u, y) - \frac{\langle y^*, y\rangle}{\langle v^*, r\rangle} r - \Phi(x, 0)\right\rangle > 0. \tag{1}$$

On the other hand, from the feasibility of (v^*, y^*, u, y, r) to (DVG_W), it follows $(v^*\Phi)(x, 0) - (v^*\Phi)(u, y) \geq \langle y^*, 0 - y\rangle$, from which

$$\left\langle v^*, \Phi(u, y) - \frac{\langle y^*, y\rangle}{\langle v^*, r\rangle} r - \Phi(x, 0)\right\rangle \leq \langle y^*, y\rangle - \left\langle v^*, \frac{\langle y^*, y\rangle}{\langle v^*, r\rangle} r\right\rangle = 0.$$

This leads to a contradiction to the strict inequality proven above. □

By making use of Theorem 1 and Proposition 1, one can prove also the following two weak duality statements involving the other vector duals to (PVG) introduced above.

Theorem 2. *There are no $x \in X$ and $(v^*, y^*, u) \in \mathcal{B}_M^G$ such that $F(x) \leq_K h_M^G(v^*, y^*, u)$.*

Theorem 3. *Let $r \in K \backslash \{0\}$. Then there are no $x \in X$ and $(v^*, y^*, u, y) \in \mathcal{B}_{W^r}^G$ such that $F(x) \leq_K h_{W^r}^G(v^*, y^*, u, y)$.*

One of the directions in which both Wolfe and Mond-Weir duality concepts were developed is towards introducing dual problems for which strong duality holds without asking the fulfillment of a regularity condition (see [7, 14, 16]). Having the following results, (DVG_M) can be considered as such a vector dual problem to (PVG).

Proposition 2. *One always has $\mathcal{B}_M^G = \mathcal{E}(DVG_M)$ and $h_M^G(\mathcal{B}_M^G) = \text{Max}$ $(DVG_M) \subseteq \text{PMin}(PVG)$.*

Proof. If $\mathcal{B}_M^G = \emptyset$ there is nothing to prove. Assume thus that there is some $(v^*, y^*, u) \in \mathcal{B}_M^G$. Then $(v^*\Phi)^*(0, y^*) + (v^*\Phi)(u, 0) = 0$, which implies $(v^*\Phi)(u, 0) = \inf_{x \in X, y \in Y}[(v^*\Phi)(x, y) - \langle y^*, y\rangle] \leq \inf_{x \in X}(v^*\Phi)(x, 0)$. Consequently, $u \in \mathcal{PE}(PVG)$ and $\Phi(u, 0) = F(u)$ is a value taken by the objective functions of both (PVG) and (DVG_M). Assuming that $(v^*, y^*, u) \notin \mathcal{E}(DVG_M)$, a contradiction is immediately obtained by employing Theorem 2. Consequently, $\mathcal{B}_M^G = \mathcal{E}(DVG_M)$ and using that $u \in \mathcal{PE}(PVG)$ we obtain also that $h_M^G(\mathcal{B}_M^G) = \text{Max}(DVG_M) \subseteq \text{PMin}(PVG)$. □

Two immediate consequences of this assertion follow.

Corollary 1. *If $(\bar{v}^*, \bar{y}^*, \bar{u}, 0, \bar{r}) \in \mathcal{B}_W^G$, then $(\bar{v}^*, \bar{y}^*, \bar{u}, 0, \bar{r}) \in \mathcal{E}(DVG_W)$, $\bar{u} \in \mathcal{PE}(PVG)$ and $F(\bar{u}) = h_W^G(\bar{v}^*, \bar{y}^*, \bar{u}, 0, \bar{r})$.*

Proof. If $(\bar{v}^*, \bar{y}^*, \bar{u}, 0, \bar{r}) \in \mathcal{B}_W^G$, then it can be immediately verified that $F(\bar{u}) = h_W^G(\bar{v}^*, \bar{y}^*, \bar{u}, 0, \bar{r})$ and $(\bar{v}^*, \bar{y}^*, \bar{u}) \in \mathcal{B}_M^G$. By Proposition 2 it follows that $(\bar{v}^*, \bar{y}^*, \bar{u}) \in \mathcal{E}(DVG_M)$ and, consequently, $\bar{u} \in \mathcal{PE}(PVG)$. Knowing these, the efficiency of $(\bar{v}^*, \bar{y}^*, \bar{u}, 0, \bar{r})$ to (DVG_W) follows by employing Theorem 1. □

Corollary 2. *Let $\bar{r} \in K \backslash \{0\}$. If $(\bar{v}^*, \bar{y}^*, \bar{u}, 0) \in \mathcal{B}_{W^{\bar{r}}}^G$, then $(\bar{v}^*, \bar{y}^*, \bar{u}, 0, \bar{r}) \in \mathcal{E}(DVG_W)$, $(\bar{v}^*, \bar{y}^*, \bar{u}, 0) \in \mathcal{E}(DVG_{W^{\bar{r}}})$, $\bar{u} \in \mathcal{PE}(PVG)$ and $F(\bar{u}) = h_W^G(\bar{v}^*, \bar{y}^*, \bar{u}, 0, \bar{r}) = h_{W^{\bar{r}}}^G(\bar{v}^*, \bar{y}^*, \bar{u}, 0)$.*

Next we give some results involving the maximal sets of the vector duals introduced above. Combining Proposition 1 and Proposition 2, we obtain the following statement.

Proposition 3. *It holds*

$$h_M^G(\mathcal{B}_M^G) = \mathrm{Max}(DVG_M) \subseteq \mathrm{Max}(DVG_W) \subseteq \bigcup_{r \in K \setminus \{0\}} \mathrm{Max}(DVG_{W^r}).$$

Proof. From Proposition 1 and Proposition 2 it is known that $h_M^G(\mathcal{B}_M^G) = \mathrm{Max}(DVG_M) \subseteq \mathrm{PMin}(PVG) \cap h_W^G(\mathcal{B}_W^G)$. On the other hand, Theorem 1 yields that $\mathrm{PMin}(PVG) \cap h_W^G(\mathcal{B}_W^G) \subseteq \mathrm{Max}(DVG_W)$ and the first inclusion is proven.

To demonstrate the second one, let $\bar{d} \in \mathrm{Max}(DVG_W)$. This means that there exists $(\bar{v}^*, \bar{y}^*, \bar{u}, \bar{y}, \bar{r}) \in \mathcal{E}(DVG_W)$ such that $h_W^G(\bar{v}^*, \bar{y}^*, \bar{u}, \bar{y}, \bar{r}) = \bar{d}$. Taking $\bar{s} = (1/\langle \bar{v}^*, \bar{r} \rangle)\bar{r}$, we obtain that $(\bar{v}^*, \bar{y}^*, \bar{u}, \bar{y}) \in \mathcal{B}_{W^{\bar{s}}}^G$ and $h_{W^{\bar{s}}}^G(\bar{v}^*, \bar{y}^*, \bar{u}, \bar{y}) = \bar{d}$. Assuming that $(\bar{v}^*, \bar{y}^*, \bar{u}, \bar{y})$ were not efficient to $(DVG_{W^{\bar{s}}})$ would bring, via Proposition 1, a contradiction to the efficiency of $(\bar{v}^*, \bar{y}^*, \bar{u}, \bar{y}, \bar{r})$ to (DVG_W).□

Now we turn our attention to strong duality for the vector duals introduced in this paper. As usual in convex optimization, we consider regularity conditions that ensure the disappearance of the duality gap. Following [2], we introduce four types of regularity conditions, namely a classical one involving continuity

$$(RC_1^\Phi) \ \Big| \ \exists x' \in X \text{ such that } (x', 0) \in \operatorname{dom} \Phi \text{ and } \Phi(x', \cdot) \text{ is continuous at } 0,$$

a weak interiority type one

$$(RC_2^\Phi) \ \left| \ \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } \Phi \text{ is } C\text{-lower semicontinuous} \\ \text{and } 0 \in \operatorname{sqri}(\mathrm{Pr}_Y(\operatorname{dom} \Phi)), \end{array} \right.$$

another interiority type one which works in finitely dimensional spaces

$$(RC_3^\Phi) \ \Big| \ \dim(\operatorname{lin}(\mathrm{Pr}_Y(\operatorname{dom} \Phi))) < +\infty \text{ and } 0 \in \operatorname{ri}(\mathrm{Pr}_Y(\operatorname{dom} \Phi)),$$

and finally a closedness type one

$$(RC_4^\Phi) \ \left| \ \begin{array}{l} \Phi \text{ is } C\text{-lower semicontinuous and } \mathrm{Pr}_{X^* \times \mathbb{R}}(\operatorname{epi}(v^*\Phi)^*) \text{ is} \\ \text{closed in the topology } w(X^*, X) \times \mathbb{R} \text{ for all } v^* \in K^{*0}. \end{array} \right.$$

Theorem 4. *Let $\bar{r} \in K \setminus \{0\}$. Assume that Φ is a K-convex function and one of the regularity conditions (RC_i^Φ), $i \in \{1, 2, 3, 4\}$, is fulfilled. If $\bar{x} \in \mathcal{PE}(PVG)$, then there exist $\bar{v}^* \in K^{*0}$ and $\bar{y}^* \in Y^*$ such that $(\bar{v}^*, \bar{y}^*, \bar{x}, 0, \bar{r}) \in \mathcal{E}(DVG_W)$, $(\bar{v}^*, \bar{y}^*, \bar{x}, 0) \in \mathcal{E}(DVG_{W^{\bar{r}}})$, $(\bar{v}^*, \bar{y}^*, \bar{x}) \in \mathcal{E}(DVG_M)$ and $F(\bar{x}) = h_W^G(\bar{v}^*, \bar{y}^*, \bar{x}, 0, \bar{r}) = h_{W^{\bar{r}}}^G(\bar{v}^*, \bar{y}^*, \bar{x}, 0) = h_M^G(\bar{v}^*, \bar{y}^*, \bar{x})$.*

Proof. Since $\bar{x} \in \mathcal{PE}(PVG)$, there exists $\bar{v}^* \in K^{*0}$ such that $\langle \bar{v}^*, F(\bar{x}) \rangle \leq \langle \bar{v}^*, F(x) \rangle$ for all $x \in X$. As $\bar{r} \in K \setminus \{0\}$ assuming that $\langle \bar{v}^*, \bar{r} \rangle = 1$ does not imply losing the generality. From [2] it is known that each of the regularity conditions (RC_i^Φ), $i \in \{1, 2, 3, 4\}$, ensures the stability of the scalar optimization problem

$$\inf_{x \in X} (\bar{v}^*\Phi)(x, 0),$$

i.e. there exists $\bar{y}^* \in Y^*$ such that $\inf_{x \in X} (\bar{v}^*\Phi)(x, 0) = -(\bar{v}^*\Phi)^* \ (0, -\bar{y}^*)$. This relation and the inequality regarding the proper efficiency of $\bar{x}$ yield

$(\bar{v}^*\Phi)(\bar{x},0) + (\bar{v}^*\Phi)^*(0,\bar{y}^*) = 0$, which is nothing but $(0,\bar{y}^*) \in \partial(\bar{v}^*\Phi)(\bar{x},0)$. Then $(\bar{v}^*,\bar{y}^*,\bar{x}) \in \mathcal{B}_M^G$ and, moreover, $(\bar{v}^*,\bar{y}^*,\bar{x},0) \in \mathcal{B}_{W^{\bar{r}}}^G$. The conclusion follows by using Proposition 2 and Corollary 2. $\square$

Remark 4. In case $V = \mathbb{R}$ and $K = \mathbb{R}_+$, identifying $V^\bullet$ with $\mathbb{R} \cup \{+\infty\}$ and $\infty_{\mathbb{R}_+}$ with $+\infty$, and taking the function $F : X \to \overline{\mathbb{R}}$ proper we rediscover the Wolfe and Mond-Weir type scalar duality schemes from [1]. More precisely the problem (PVG) becomes then the general scalar optimization problem (PG) from the mentioned paper, while the duals (DVG_W) and (DVG_{W^r}), $r > 0$, turn out to coincide with the general scalar Wolfe type dual to (PG), denoted in [1] (DG_W), and (DVG_M) is nothing but the general scalar Mond-Weir type dual (DG_M). This sustains the way we named the vector duals introduced in this paper and the claim that we extend to vector duality the investigations from the scalar case presented in [1].

In the next sections we consider as special instances of (PVG) the two main classes of vector optimization problems, namely we work with an unconstrained and a constrained vector optimization problem, respectively. To these problems we attach vector duals that are special cases of (DVG_M), (DVG_W) and (DVG_{W^r}), $r > 0$, obtained for different choices of the perturbation vector function Φ.

3 Wolfe and Mond-Weir type vector duals for unconstrained vector optimization problems

Let X, Y and V be separated locally convex vector spaces, with V partially ordered by the nontrivial pointed convex cone $K \subseteq V$. Further, let $f : X \to V^\bullet$ and $g : Y \to V^\bullet$ be given proper vector functions and $A : X \to Y$ a linear continuous mapping such that $\operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \neq \emptyset$.

The primal unconstrained vector optimization problem we consider is

$$(PVA) \qquad \operatorname*{Min}_{x \in X} [f(x) + g(Ax)].$$

We work with properly efficient solutions in the sense of linear scalarization to (PVA), while for the vector dual we assign to it in this section we consider efficient solutions. Since (PVA) is a special case of (PVG) obtained by taking $F = f + g \circ A$, we use the approach developed in the previous section in order to deal with it via duality. More precisely, for a "good" choice of the vector perturbation function Φ we obtain vector duals to (PVA) which are special cases of (DVG_M) and (DVG_W).

In order to attach vector dual problems to (PVA), consider the vector perturbation function

$$\Phi^A : X \times Y \to V^\bullet,\ \Phi^A(x,y) = f(x) + g(Ax + y).$$

For $v^* \in K^{*0}$, $u \in X$, $y \in Y$ and $y^* \in Y^*$ one has $(0,y^*) \in \partial(v^*\Phi^A)(u,y)$ if and only if $(v^*\Phi^A)^*(0,y^*) + (v^*\Phi^A)(u,y) = \langle y^*, y\rangle$. This is further equivalent to $(v^*f)^*(-A^*y^*) + (v^*g)^*(y^*) + f(u) + g(Au+y) = \langle y^*, y\rangle$. Using the Young-Fenchel inequality, the last equality yields that $(0, y^*) \in \partial(v^*\Phi^A)(u,y)$ if and only if $y^* \in \partial(v^*g)(Au+y)$ and $-A^*y^* \in \partial(v^*f)(u)$. Now we are ready to formulate the vector duals to (PVA) that are special cases of (DVG_M) and (DVG_W), namely

$$(DVA_W) \qquad \underset{(v^*,y^*,u,y,r)\in\mathcal{B}_W^A}{\operatorname{Max}} h_W^A(v^*,y^*,u,y,r),$$

where

$$\begin{aligned}\mathcal{B}_W^A &= \Big\{(v^*,y^*,u,y,r) \in K^{*0} \times Y^* \times X \times Y \times (K\backslash\{0\}) : \\ &\qquad y^* \in (A^*)^{-1}(-\partial(v^*f)(u)) \cap \partial(v^*g)(Au+y)\Big\}\end{aligned}$$

and

$$h_W^A(v^*,y^*,u,y,r) = f(u) + g(Au+y) - \frac{\langle y^*, y\rangle}{\langle v^*, r\rangle} r$$

and

$$(DVA_M) \qquad \underset{(v^*,u)\in\mathcal{B}_M^A}{\operatorname{Max}} h_M^A(v^*,u),$$

where

$$\mathcal{B}_M^A = \Big\{(v^*,u) \in K^{*0} \times X : 0 \in (A^*)^{-1}(-\partial(v^*f)(u)) - \partial(v^*g)(Au)\Big\}$$

and

$$h_W^A(v^*,u) = f(u) + g(Au).$$

We can consider also the particularizations of the family of vector duals introduced in Remark 2. For each $r \in K\backslash\{0\}$ we have the vector dual

$$(DVA_{W^r}) \qquad \underset{(v^*,y^*,u,y)\in\mathcal{B}_{W^r}^A}{\operatorname{Max}} h_{W^r}^A(v^*,y^*,u,y),$$

where

$$\begin{aligned}\mathcal{B}_{W^r}^A &= \Big\{(v^*,y^*,u,y) \in K^{*0} \times Y^* \times X \times Y : \langle v^*, r\rangle = 1, \\ &\qquad y^* \in (A^*)^{-1}(-\partial(v^*f)(u)) \cap \partial(v^*g)(Au+y)\Big\}\end{aligned}$$

and

$$h_W^A(v^*,y^*,u,y) = f(u) + g(Au+y) - \langle y^*, y\rangle r.$$

The propositions, corollaries and theorems from Section 2, as well as Remark 3 and Remark 4 can be particularized for the framework considered in this section. We give here only the weak and strong duality statements and the connection to the scalar case.

Theorem 5. *There are no $x \in X$ and $(v^*,y^*,u,y,r) \in \mathcal{B}_W^A$ such that $f(x) + g(Ax) \leq_K h_W^A(v^*,y^*,u,y,r)$.*

Theorem 6. *There are no $x \in X$ and $(v^*, u) \in \mathcal{B}_M^A$ such that $f(x) + g(Ax) \leq_K h_M^A(v^*, u)$.*

Theorem 7. *Let $r \in K\backslash\{0\}$. Then there are no $x \in X$ and $(v^*, y^*, u, y) \in \mathcal{B}_{W^r}^A$ such that $f(x) + g(Ax) \leq_K h_{W^r}^A(v^*, y^*, u, y)$.*

For strong duality, which follows directly from Theorem 4, besides convexity assumptions which guarantee the K-convexity of the vector perturbation function we use regularity conditions, too, obtained by particularizing (RC_i^Φ), $i \in \{1, 2, 3, 4\}$, namely

$$(RC_1^A) \;\Big|\; \exists x' \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} g) \text{ such that } g \text{ is continuous at } Ax',$$

$$(RC_2^A) \;\left|\; \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } f \text{ and } g \text{ are } C\text{-lower} \\ \text{semicontinuous and } 0 \in \operatorname{sqri}(\operatorname{dom} g - A(\operatorname{dom} f)), \end{array}\right.$$

$$(RC_3^A) \;\Big|\; \dim(\operatorname{lin}(\operatorname{dom} g - A(\operatorname{dom} f))) < +\infty \text{ and } \operatorname{ri}(A(\operatorname{dom} f)) \cap \operatorname{ri}(\operatorname{dom} g) \neq \emptyset,$$

and, respectively,

$$(RC_4^A) \;\left|\; \begin{array}{l} f \text{ and } g \text{ are } C\text{-lower semicontinuous and } \operatorname{epi}(v^*f)^* + (A^* \times \mathrm{id}_{\mathbb{R}}) \\ (\operatorname{epi}(v^*g)^*) \text{ is closed in the topology } w(X^*, X) \times \mathbb{R} \text{ for all } v^* \in K^{*0}, \end{array}\right.$$

where $(A^* \times \mathrm{id}_{\mathbb{R}})(\operatorname{epi}(v^*g)^*) = \{(x^*, r) \in X^* \times \mathbb{R} : \exists y^* \in Y^* \text{ such that } A^*y^* = x^* \text{ and } (y^*, r) \in \operatorname{epi}(v^*g)^*\}$.

Theorem 8. *Let $\bar{r} \in K\backslash\{0\}$. Assume that f and g are K-convex vector functions and one of the regularity conditions (RC_i^A), $i \in \{1, 2, 3, 4\}$, is fulfilled. If $\bar{x} \in \mathcal{PE}(PVA)$, then there exist $\bar{v}^* \in K^{*0}$ and $\bar{y}^* \in Y^*$ such that $(\bar{v}^*, \bar{y}^*, \bar{x}, 0, \bar{r}) \in \mathcal{E}(DVA_W)$, $(\bar{v}^*, \bar{y}^*, \bar{x}, 0) \in \mathcal{E}(DVA_{W^{\bar{r}}})$, $(\bar{v}^*, \bar{x}) \in \mathcal{E}(DVA_M)$ and $f(\bar{x}) + g(A\bar{x}) = h_W^A(\bar{v}^*, \bar{y}^*, \bar{x}, 0, \bar{r}) = h_{W^{\bar{r}}}^A(\bar{v}^*, \bar{y}^*, \bar{x}, 0) = h_M^A(\bar{v}^*, \bar{x})$.*

Remark 5. In case $V = \mathbb{R}$ and $K = \mathbb{R}_+$, taking the functions $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$ proper we rediscover the Wolfe and Mond-Weir duality schemes for unconstrained scalar optimization problems from [1]. More precisely the problem (PVA) becomes then the unconstrained scalar optimization problem (P^A) from the mentioned paper, the duals (DVA_W) and (DVA_{W^r}), $r > 0$, turn out to coincide with the scalar Wolfe type dual to (P^A) denoted (D_W^A) and (DVA_M) is nothing but the Mond-Weir type dual (D_M^A).

4 Wolfe and Mond-Weir type vector duals for constrained vector optimization problems

Let X, Y and V be separated locally convex vector spaces, with Y partially ordered by the convex cone $C \subseteq Y$ and V partially ordered by the nontrivial pointed convex cone $K \subseteq V$. Consider the nonempty convex set $S \subseteq X$ and the proper vector functions $f : X \to V^\bullet$ and $g : X \to Y^\bullet$ fulfilling $\operatorname{dom} f \cap S \cap g^{-1}(-C) \neq \emptyset$. Let the primal vector optimization problem with geometric and cone constraints

$$(PVC) \qquad \operatorname*{Min}_{x\in\mathcal{A}} f(x),$$

where

$$\mathcal{A}=\big\{x\in S: g(x)\in -C\big\}.$$

We work with properly efficient solutions in the sense of linear scalarization to it, while for the vector dual we assign to it in this section we consider efficient solutions. Since (PVC) is a special case of (PVG) obtained by taking

$$F: X\to V^\bullet,\ F(x)=\begin{cases} f(x), & \text{if } x\in\mathcal{A},\\ \infty_K, & \text{otherwise,}\end{cases}$$

we use the approach developed in Section 2 in order to deal with it via duality. More precisely, for convenient choices of the vector perturbation function Φ we obtain vector duals to (PVC) which are special cases of (DVG_M) and (DVG_W).

Consider first the Lagrange type vector perturbation function

$$\Phi^{C_L}: X\times Y\to V^\bullet,\ \Phi^{C_L}(x,y)=\begin{cases} f(x), & \text{if } x\in S, g(x)\in y-C,\\ \infty_K, & \text{otherwise.}\end{cases}$$

For $u\in X$, $y\in Y$, $v^*\in K^{*0}$ and $y^*\in Y^*$ we have $(0,y^*)\in\partial(v^*\Phi^{C_L})(u,y)$ if and only if $(v^*\Phi^{C_L})^*(0,y^*)+(v^*\Phi^{C_L})(u,y)=\langle y^*,y\rangle$, i.e. $((v^*f)-(y^*g)+\delta_S)^*(0)+\delta_{C^*}(-y^*)+f(u)+\delta_S(u)+\delta_{-C}(g(u)-y)=\langle y^*,y\rangle$. Using that $\delta^*_{-C}=\delta_{C^*}$, this can be rewritten as $\big(((v^*f)-(y^*g)+\delta_S)^*(0)+((v^*f)-(y^*g)+\delta_S)(u)\big)+\big(\delta^*_{-C}(-y^*)+\delta_{-C}(g(u)-y)-\langle -y^*,g(u)-y\rangle\big)=0$. Having the Young-Fenchel inequality and the characterization of the subdifferential by its equality case, it follows that $(0,y^*)\in\partial(v^*\Phi^{C_L})(u,y)$ if and only if $0\in\partial((v^*f)-(y^*g)+\delta_S)(u)$, $y^*\in -C^*$ and $\delta_{-C}(g(u)-y)-\langle -y^*,g(u)-y\rangle=0$. Thus, from (DVG_W) we obtain the following vector dual to (PVC)

$$(DVC^L_{\widetilde{W}}) \qquad \operatorname*{Max}_{(v^*,y^*,u,y,r)\in\mathcal{B}^{C_L}_{\widetilde{W}}} h^{C_L}_{\widetilde{W}}(v^*,y^*,u,y,r),$$

where

$$\begin{aligned}\mathcal{B}^{C_L}_{\widetilde{W}} \;=\;& \Big\{(v^*,y^*,u,y,r)\in K^{*0}\times C^*\times S\times Y\times(K\backslash\{0\}): g(u)-y\in -C,\\ & \qquad (y^*g)(u)=\langle y^*,y\rangle, 0\in\partial((v^*f)+(y^*g)+\delta_S)(u)\Big\}\end{aligned}$$

and

$$h^{C_L}_{\widetilde{W}}(v^*,y^*,u,y,r)=f(u)+\frac{\langle y^*,y\rangle}{\langle v^*,r\rangle}r,$$

which can be equivalently rewritten as

$$(DVC^L_{W}) \qquad \operatorname*{Max}_{(v^*,y^*,u,r)\in\mathcal{B}^{C_L}_{W}} h^{C_L}_{W}(v^*,y^*,u,r),$$

where

$$\mathcal{B}^{C_L}_{W}=\Big\{(v^*,y^*,u,r)\in K^{*0}\times C^*\times S\times(K\backslash\{0\}): 0\in\partial((v^*f)+(y^*g)+\delta_S)(u)\Big\}$$

and

$$h_W^{C_L}(v^*, y^*, u, r) = f(u) + \frac{\langle y^*, g(u)\rangle}{\langle v^*, r\rangle} r,$$

further referred to as the *vector Wolfe dual of Lagrange type*, while the vector dual to (PVC) that results from (DVG_M) is

$$(DVC_M^L) \qquad \operatorname*{Max}_{(v^*,y^*,u)\in\mathcal{B}_M^{C_L}} h_M^{C_L}(v^*, y^*, u),$$

where

$$\begin{aligned}\mathcal{B}_M^{C_L} = \Big\{(v^*, y^*, u) \in K^{*0} \times C^* \times S : (y^*g)(u) \geq 0, g(u) \in -C,\\ 0 \in \partial((v^*f) + (y^*g) + \delta_S)(u)\Big\}\end{aligned}$$

and

$$h_M^{C_L}(v^*, y^*, u, r) = f(u).$$

Note that in the constraints of this dual one can replace $(y^*g)(u) \geq 0$ by $(y^*g)(u) = 0$ without altering anything since $g(u) \in -C$ and $y^* \in C^*$. Removing from $\mathcal{B}_M^{C_L}$ the constraint $g(u) \in -C$, we obtain another vector dual to (PVC), namely

$$(DVC_{MW}^L) \qquad \operatorname*{Max}_{(v^*,y^*,u)\in\mathcal{B}_{MW}^{C_L}} h_{MW}^{C_L}(v^*, y^*, u),$$

where

$$\mathcal{B}_{MW}^{C_L} = \Big\{(v^*, y^*, u) \in K^{*0} \times C^* \times S : (y^*g)(u) \geq 0, 0 \in \partial((v^*f) + (y^*g) + \delta_S)(u)\Big\}$$

and

$$h_{MW}^{C_L}(v^*, y^*, u, r) = f(u),$$

further called the *vector Mond-Weir dual of Lagrange type* to (PVC). We can consider also the particularizations of the family of vector duals introduced in Remark 2. For each $r \in K\backslash\{0\}$ we have the vector dual

$$(DVC_{W^r}^L) \qquad \operatorname*{Max}_{(v^*,y^*,u)\in\mathcal{B}_{W^r}^{C_L}} h_{W^r}^{C_L}(v^*, y^*, u),$$

where

$$\mathcal{B}_{W^r}^{C_L} = \Big\{(v^*, y^*, u) \in K^{*0} \times C^* \times X : \langle v^*, r\rangle = 1, 0 \in \partial((v^*f) + (y^*g) + \delta_S)(u)\Big\}$$

and

$$h_{W^r}^{C_L}(v^*, y^*, u) = f(u) + \langle y^*, g(u)\rangle r.$$

Remark 6. Due to the way (DVC_{MW}^L) is constructed it is clear that $h_M^{C_L}(\mathcal{B}_M^{C_L}) \subseteq h_{MW}^{C_L}(\mathcal{B}_{MW}^{C_L})$. The following example shows that there are situations when the inclusion is strict.

Example 1. Let $X=\mathbb{R}$, $Y=\mathbb{R}$, $C=\mathbb{R}_+$, $Y^\bullet=\mathbb{R}\cup\{+\infty\}$, $V=\mathbb{R}^2$, $K=\mathbb{R}^2_+$, $S=[0,+\infty)$, $f:\mathbb{R}\to\mathbb{R}^2$, $f(x)=(x,x)^T$, and $g:\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$,

$$g(x)=\begin{cases}-x, & \text{if } x>0,\\ 2, & \text{if } x=0,\\ +\infty, & \text{if } x<0.\end{cases}$$

For $\bar{v}^*=(1/2,1/2)^T$ we have $0\in\partial((\bar{v}^*f)+(0g)+\delta_S)(0)=(-\infty,1]$ and $(0g)(0)=0$, thus $(\bar{v}^*,0,0)\in\mathcal{B}^{C_L}_{MW}$, therefore $(0,0)^T\in h^{C_L}_{MW}(\mathcal{B}^{C_L}_{MW})$. On the other hand it can be shown that $\mathcal{B}^{C_L}_M=\emptyset$. Consequently, $h^{C_L}_M(\mathcal{B}^{C_L}_M)\neq h^{C_L}_{MW}(\mathcal{B}^{C_L}_{MW})$.

We give also an example where $h^{C_L}_{W^r}(\mathcal{B}^{C_L}_{W^r})\backslash h^{C_L}_M(\mathcal{B}^{C_L}_M)\neq\emptyset$, for an $r\in K\backslash\{0\}$, i.e. $h^{C_L}_M(\mathcal{B}^{C_L}_M)\neq h^{C_L}_W(\mathcal{B}^{C_L}_W)$ in general. Recall that via Proposition 1 one obtains that $h^{C_L}_M(\mathcal{B}^{C_L}_M)\subseteq\cup_{r\in K\backslash\{0\}}h^{C_L}_{W^r}(\mathcal{B}^{C_L}_{W^r})=h^{C_L}_W(\mathcal{B}^{C_L}_W)$.

Example 2. Let $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $C=\mathbb{R}^2_+$, $V=\mathbb{R}^2$, $K=\mathbb{R}^2_+$, $V^\bullet=(\mathbb{R}^2)^\bullet=\mathbb{R}^2\cup\{\infty_{\mathbb{R}^2_+}\}$, $S=\mathbb{R}_+$, $f:\mathbb{R}\to(\mathbb{R}^2)^\bullet$,

$$f(x)=\begin{cases}\begin{pmatrix}1\\1\end{pmatrix}x & \text{if } x>0,\\ \infty_{\mathbb{R}^2_+}, & \text{otherwise,}\end{cases}$$

and $g:\mathbb{R}\to\mathbb{R}^2$, $g(x)=(x-1,-x)^T$. Like in [1, Example 2], it can be shown that $h^{C_L}_M(\mathcal{B}^{C_L}_M)=\emptyset$, while for $r=(1,1)^T$, one has $\big((1/2,1/2)^T,(2,3)^T,1\big)\in\mathcal{B}^{C_L}_{W^r}$, consequently, $(-2,-2)^T\in h^{C_L}_{W^r}(\mathcal{B}^{C_L}_{W^r})$. Note that in this case $\mathcal{B}^{C_L}_{MW}=\emptyset$, too. However, the question whether $h^{C_L}_{MW}(\mathcal{B}^{C_L}_{MW})$ is in general a subset of $h^{C_L}_W(\mathcal{B}^{C_L}_W)$ is still open.

Remark 7. Assume that f is a K-convex vector function and g is a C-convex vector function. Since S is a convex set, it is a simple verification to see that the vector perturbation function Φ^{C_L} is K-convex. Denote further $\Delta_{X^3}=\{(x,x,x):x\in X\}$. When one of the following conditions (see [2])

(i) f and g are continuous at a point in $\operatorname{dom} f\cap\operatorname{dom} g\cap S$;

(ii) $\operatorname{dom} f\cap\operatorname{int}(S)\cap\operatorname{dom} g\neq\emptyset$ and f or g is continuous at a point in $\operatorname{dom} f\cap\operatorname{dom} g$;

(iii) X is a Fréchet space, S is closed, f is K-lower semicontinuous, g is C-lower semicontinuous and $0\in\operatorname{sqri}(\operatorname{dom} f\times S\times\operatorname{dom} g-\Delta_{X^3})$;

(iv) $\dim(\operatorname{lin}(\operatorname{dom} f\times S\times\operatorname{dom} g-\Delta_{X^3}))<+\infty$ and $\operatorname{ri}(\operatorname{dom} f)\cap\operatorname{ri}(S)\cap\operatorname{ri}(\operatorname{dom} g)\neq\emptyset$;

is satisfied, then, for all $v^*\in K^{*0}$ and all $y^*\in C^*$, it holds

$$\partial((v^*f)+(y^*g)+\delta_S)(x)=\partial(v^*f)(x)+\partial(y^*g)(x)+N_S(x)\ \forall x\in X.$$

Consequently, when one of these situations occurs the constraint involving the subdifferential in (DVC_W^L), $(DVC_{W^r}^L)$, (DVC_M^L) and (DVC_{MW}^L) can be modified correspondingly.

Remark 8. If $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $C = \mathbb{R}_+^m$, $V = \mathbb{R}^k$, $K = \mathbb{R}_+^k$, $f = (f_1, \ldots, f_k)^T : \mathbb{R}^n \to \mathbb{R}^k$ and $g = (g_1, \ldots, g_m)^T : \mathbb{R}^n \to \mathbb{R}^m$, and the functions f_i, $i = 1, \ldots, k$, and g_j, $j = 1, \ldots, m$, are convex, then $(DVC_{W^e}^L)$, where $e = (1, \ldots, 1)^T \in \mathbb{R}^k$, turns out to be the *nondifferentiable vector Wolfe dual* problem mentioned in the literature (see [7,9,16]), while (DVC_{MW}^L) is the *nondifferentiable vector Mond-Weir dual* problem to (PVC). In case the functions f_i, $i = 1, \ldots, k$, and g_j, $j = 1, \ldots, m$, are moreover differentiable on S which is taken to be open and the subdifferentials are replaced by gradients in the constraints, $(DVC_{W^r}^L)$ turns out to be the classical *vector Wolfe dual* problem from the literature (see [17] and, for the case $r = e$, [5,11,12,15]), while (DVC_{MW}^L) is the classical *vector Mond-Weir dual* problem to (PVC) (cf. [5,6,11,13–15]).

Like in the previous section, the results involving (PVG) and its vector duals can be particularized for the problems introduced above, however we give here only the weak and strong duality statements involving (PVC) and its vector duals of Lagrange type.

Theorem 9. *There are no $x \in \mathcal{A}$ and $(v^*, y^*, u, r) \in \mathcal{B}_W^{C_L}$ such that $f(x) \leq_K h_W^{C_L}(v^*, y^*, u, r)$.*

Theorem 10. *There are no $x \in \mathcal{A}$ and $(v^*, y^*, u) \in \mathcal{B}_M^{C_L}$ such that $f(x) \leq_K h_M^{C_L}(v^*, y^*, u)$.*

Theorem 11. *Let $r \in K \backslash \{0\}$. Then there are no $x \in \mathcal{A}$ and $(v^*, y^*, u) \in \mathcal{B}_{W^r}^{C_L}$ such that $f(x) \leq_K h_{W^r}^{C_L}(v^*, y^*, u)$.*

Analogously, one can prove also the following weak duality statement involving (PVC) and (DVC_{MW}^L).

Theorem 12. *There are no $x \in \mathcal{A}$ and $(v^*, y^*, u) \in \mathcal{B}_{MW}^{C_L}$ such that $f(x) \leq_K h_{MW}^{C_L}(v^*, y^*, u)$.*

For strong duality we particularize the regularity conditions (RC_i^Φ), $i \in \{1, 2, 3, 4\}$, obtaining (see [1,2])

$$(RC_1^{C_L}) \quad \Big| \quad \exists x' \in \operatorname{dom} f \cap S \text{ such that } g(x') \in -\operatorname{int}(C),$$

which is the classical *Slater constraint qualification* extended to the vector case,

$$(RC_2^{C_L}) \quad \left| \quad \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } S \text{ is closed, } f \text{ is } K\text{-lower} \\ \text{semicontinuous, } g \text{ is } C\text{-epi-closed and} \\ 0 \in \operatorname{sqri}\big(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C\big), \end{array} \right.$$

$$(RC_3^{C_L}) \quad \left| \quad \begin{array}{l} \dim\left(\operatorname{lin}\left(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C\right)\right) < +\infty \text{ and} \\ 0 \in \operatorname{ri}\left(g(\operatorname{dom} f \cap S \cap \operatorname{dom} g) + C\right), \end{array} \right.$$

and, respectively,

$$(RC_4^{C_L}) \quad \left| \begin{array}{l} S \text{ is closed, } f \text{ is } K\text{-lower semicontinuous, } g \text{ is } C\text{-epi-closed} \\ \text{and } \bigcup\limits_{y^* \in C^*} \operatorname{epi}((v^*f) + (y^*g) + \delta_S)^* \text{ is closed in the topology} \\ w(X^*, X) \times \mathbb{R} \text{ for all } v^* \in K^{*0}. \end{array} \right.$$

Theorem 13. *Let $\bar{r} \in K\backslash\{0\}$. Assume that f is a K-convex vector function, g is a C-convex vector function and one of the regularity conditions $(RC_i^{C_L})$, $i \in \{1,2,3,4\}$, is fulfilled. If $\bar{x} \in \mathcal{PE}(PVC)$, then there exist $\bar{v}^* \in K^{*0}$ and $\bar{y}^* \in C^*$ such that $(\bar{v}^*, \bar{y}^*, \bar{x}, \bar{r}) \in \mathcal{E}(DVC_W^L)$, $(\bar{v}^*, \bar{y}^*, \bar{x}) \in \mathcal{E}(DVC_{W^{\bar{r}}}^L) \cap \mathcal{E}(DVC_M^L) \cap \mathcal{E}(DVC_{MW}^L)$ and $f(\bar{x}) = h_W^{C_L}(\bar{v}^*, \bar{y}^*, \bar{x}, \bar{r}) = h_{W^{\bar{r}}}^{C_L}(\bar{v}^*, \bar{y}^*, \bar{x}) = h_M^{C_L}(\bar{v}^*, \bar{y}^*, \bar{x}) = h_{MW}^{C_L}(\bar{v}^*, \bar{y}^*, \bar{x})$.*

Remark 9. When $L \in \mathbb{R}^{k\times n}$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $\mathbb{R}^k$ is partially ordered by the nontrivial pointed closed convex cone $K \subseteq \mathbb{R}^k$, one can consider the *classical linear vector optimization problem* (cf. [8], see also [10])

$$(PV^l) \qquad \underset{\substack{x \in \mathbb{R}^n_+, \\ Ax = b}}{\operatorname{Min}} \, Lx.$$

Among the interesting features of this vector optimization problem are the coincidence of its efficient and properly efficient solutions and also the fact that no regularity condition is necessary in order to achieve strong duality. Of interest could be to see how do the vector duals to (PV^l) derived from (DVC_M^L), (DVC_{MW}^L) and (DVC_W^L), respectively, look like and then to compare them with the vector duals to (PV^l) treated in [3].

A second vector perturbation function that can be considered for (PVC) is the Fenchel type vector perturbation function

$$\Phi^{C_F} : X \times Y \to V^\bullet, \ \Phi^{C_F}(x,y) = \begin{cases} f(x+y), & \text{if } x \in \mathcal{A}, \\ \infty_K, & \text{otherwise.} \end{cases}$$

Using it the following vector duals obtained as special cases of (DVG_W), (DVG_M) and (DVG_{W^r}), $r \in K\backslash\{0\}$, can be attached to (PVC)

$$(DVC_W^F) \qquad \underset{(v^*, y^*, u, y, r) \in \mathcal{B}_W^{C_F}}{\operatorname{Max}} h_W^{C_F}(v^*, y^*, u, y, r),$$

where

$$\mathcal{B}_W^{C_F} = \Big\{ (v^*, y^*, u, y, r) \in K^{*0} \times C^* \times X \times Y \times (K\backslash\{0\}) : \\ y^* \in \partial(v^*f)(u+y) \cap (-N_{\mathcal{A}}(u)) \Big\}$$

and

$$h_W^{C_F}(v^*, y^*, u, y, r) = f(u+y) - \frac{\langle y^*, y\rangle}{\langle v^*, r\rangle} r,$$

further referred to as the *vector Wolfe dual of Fenchel type*, the *vector Mond-Weir dual of Fenchel type*

$$(DVC_W^F) \qquad \underset{(v^*,u)\in\mathcal{B}_M^{C_F}}{\text{Max}} h_M^{C_F}(v^*,u),$$

where

$$\mathcal{B}_M^{C_F} = \Big\{(v^*,u) \in K^{*0} \times X : 0 \in \partial(v^*f)(u) + N_{\mathcal{A}}(u)\Big\}$$

and

$$h_M^{C_F}(v^*,u) = f(u),$$

and, for $r \in K\backslash\{0\}$,

$$(DVC_{Wr}^F) \qquad \underset{(v^*,y^*,u,y)\in\mathcal{B}_{Wr}^{C_F}}{\text{Max}} h_{Wr}^{C_F}(v^*,y^*,u,y),$$

where

$$\begin{aligned}\mathcal{B}_{Wr}^{C_F} = \Big\{(v^*,y^*,u,y) \in & K^{*0} \times C^* \times X \times Y : \langle v^*, r\rangle = 1, \\ & y^* \in \partial(v^*f)(u+y) \cap (-N_{\mathcal{A}}(u))\Big\}\end{aligned}$$

and

$$h_{Wr}^{C_F}(v^*,y^*,u,y) = f(u+y) - \langle y^*, y\rangle r.$$

Rewriting (PVC) in the form of (PVA) (where g is taken to be $\delta_{\mathcal{A}}$ and A the identity operator), one can derive the vector duals of Fenchel type to (PVC) directly from the vector duals considered in Section 3. Therefore in this case we do not give again the weak and strong duality statements, since they can be obtained directly from both the general case and the unconstrained case.

The last vector perturbation function we consider in this section is the Fenchel-Lagrange type vector perturbation function $\Phi^{C_{FL}} : X \times X \times Y \to V^\bullet$,

$$\Phi^{C_{FL}}(x,z,y) = \begin{cases} f(x+z), & \text{if } x \in S, g(x) \in y - C, \\ \infty_K, & \text{otherwise.} \end{cases}$$

For $v^* \in K^{*0}$, $y^* \in Y^*$, $z^* \in X^*$, $y \in Y$ and $z \in X$, one has $(0,z^*,y^*) \in \partial\Phi^{C_{FL}}(u,z,y)$ if and only if $u \in S$, $g(u) \in y - C$ and $(v^*f)^*(z^*) + (-(y^*g) + \delta_S)^*(-z^*) + \delta_{-C^*}(y^*) + f(u+z) + \delta_{-C}(g(u)-y) + \delta_S(u) = \langle y^*, y\rangle + \langle z^*, z\rangle$, which is nothing but $u \in S$, $g(u) \in y - C$ and

$$\begin{aligned}&\big((v^*f)^*(z^*) + (v^*f)(u+z) - \langle z^*, u+z\rangle\big) + \big((-(y^*g)+\delta_S)^*(-z^*) + (-(y^*g)+\delta_S)(u) \\ &\quad -\langle -z^*, u\rangle\big) + \big(\delta_{-C}^*(-y^*) + \delta_{-C}(g(u)-y) - \langle -y^*, g(u)-y\rangle\big) = 0.\end{aligned}$$

Consequently, $(0,z^*,y^*) \in \partial\Phi^{C_{FL}}(u,z,y)$ if and only if $u \in S$, $y^* \in -C^*$, $g(u) - y \in -C$, $z^* \in \partial f(u+z) \cap (-\partial(-(y^*g) + \delta_S)(u))$ and $(y^*g)(u) = \langle y^*, y\rangle$. Consequently, the vector duals to (PVC) obtained, by making use of the vector perturbation function $\Phi^{C_{FL}}$, from the vector duals introduced in Section 2 are

$$(DVC_W^{FL}) \qquad \underset{(v^*,z^*,y^*,u,z,y,r)\in\mathcal{B}_{\widetilde{W}}^{C_{FL}}}{\text{Max}} h_{\widetilde{W}}^{C_{FL}}(v^*,z^*,y^*,u,z,y,r),$$

where

$$\begin{aligned}\mathcal{B}_{\widetilde{W}}^{C_{FL}} = \Big\{(v^*,z^*,y^*,u,z,y,r) \in K^{*0} \times X^* \times C^* \times S \times X \times Y \times (K\backslash\{0\}) : \\ g(u) - y \in -C, (y^*g)(u) = \langle y^*, y\rangle, \\ z^* \in \partial(v^*f)(u+z) \cap (-\partial((y^*g) + \delta_S))(u)\Big\}\end{aligned}$$

and

$$h_{\widetilde{W}}^{C_{FL}}(v^*, z^*, y^*, u, z, y, r) = f(u+z) - \frac{\langle y^*, y\rangle + \langle z^*, z\rangle}{\langle v^*, r\rangle} r,$$

which can be equivalently rewritten as

$$(DVC_W^{FL}) \qquad \operatorname*{Max}_{(v^*, z^*, y^*, u, z, r) \in \mathcal{B}_W^{C_{FL}}} h_W^{C_{FL}}(v^*, z^*, y^*, u, z, r),$$

where

$$\begin{aligned}\mathcal{B}_W^{C_{FL}} = \Big\{&(v^*, z^*, y^*, u, z, r) \in K^{*0} \times X^* \times C^* \times S \times X \times (K \backslash \{0\}) : \\ &z^* \in \partial(v^* f)(u+z) \cap (-\partial((y^* g) + \delta_S))(u)\Big\}\end{aligned}$$

and

$$h_W^{C_{FL}}(v^*, z^*, y^*, u, z, r) = f(u+z) + \frac{(y^* g)(u) - \langle z^*, z\rangle}{\langle v^*, r\rangle} r,$$

which is the *vector Wolfe dual of Fenchel-Lagrange type*,

$$(DVC_M^{FL}) \qquad \operatorname*{Max}_{(v^*, y^*, u) \in \mathcal{B}_M^{C_{FL}}} h_M^{C_{FL}}(v^*, y^*, u),$$

where

$$\begin{aligned}\mathcal{B}_M^{C_{FL}} = \Big\{&(v^*, y^*, u) \in K^{*0} \times C^* \times S : (y^* g)(u) \geq 0, g(u) \in -C, \\ &0 \in \partial(v^* f)(u) + \partial((y^* g) + \delta_S)(u)\Big\}\end{aligned}$$

and

$$h_M^{C_{FL}}(v^*, y^*, u) = f(u)$$

and, for each $r \in K \backslash \{0\}$,

$$(DVC_{W^r}^{FL}) \qquad \operatorname*{Max}_{(v^*, z^*, y^*, u, z) \in \mathcal{B}_{W^r}^{C_{FL}}} h_{W^r}^{C_{FL}}(v^*, z^*, y^*, u, z),$$

where

$$\begin{aligned}\mathcal{B}_{W^r}^{C_{FL}} = \Big\{&(v^*, z^*, y^*, u, z) \in K^{*0} \times X^* \times C^* \times S \times X : \langle v^*, r\rangle = 1, \\ &z^* \in \partial(v^* f)(u+z) \cap (-\partial((y^* g) + \delta_S))(u)\Big\}\end{aligned}$$

and

$$h_{W^r}^{C_{FL}}(v^*, z^*, y^*, u, z, r) = f(u+z) + \big((y^* g)(u) - \langle z^*, z\rangle\big) r.$$

Note that in the constraints of (DVC_M^{FL}) one can replace $(y^* g)(u) \geq 0$ by $(y^* g)(u) = 0$ without altering anything. Removing the constraint $g(u) \in -C$ from $\mathcal{B}_M^{C_{FL}}$, one obtains from (DVC_M^{FL}) the *vector Mond-Weir dual of Fenchel-Lagrange type* to (PVC)

$$(DVC_{MW}^{FL}) \qquad \operatorname*{Max}_{(v^*, y^*, u) \in \mathcal{B}_{MW}^{C_{FL}}} h_{MW}^{C_{FL}}(v^*, y^*, u),$$

where

$$\mathcal{B}_{MW}^{C_{FL}} = \Big\{(v^*, y^*, u) \in K^{*0} \times C^* \times S : (y^* g)(u) \geq 0, 0 \in \partial(v^* f)(u) + \partial((y^* g) + \delta_S)(u)\Big\}$$

and

$$h_{MW}^{C_{FL}}(v^*, y^*, u) = f(u).$$

Remark 10. Note that $h_M^{C_{FL}}(\mathcal{B}_M^{C_{FL}}) \subseteq h_{MW}^{C_{FL}}(\mathcal{B}_{MW}^{C_{FL}})$. The inclusion is actually strict, since in the situation presented in Example 1 we have $h_M^{C_{FL}}(\mathcal{B}_M^{C_{FL}}) = \emptyset$ and $h_{MW}^{C_{FL}}(\mathcal{B}_{MW}^{C_{FL}}) \neq \emptyset$.

Remark 11. For sufficient conditions to "split" the subdifferentials from the Fenchel-Lagrange type vector duals, analogous to the ones delivered in Remark 7 for the Lagrange type vector duals, we refer to [2].

The results involving (PVG) and its vector duals can be particularized for the Fenchel-Lagrange type vector duals, but here we give only the weak and strong duality statements involving (PVC) and these vector duals.

Theorem 14. *There are no $x \in \mathcal{A}$ and $(v^*, z^*, y^*, u, z, r) \in \mathcal{B}_W^{C_{FL}}$ such that $f(x) \leq_K h_W^{C_{FL}}(v^*, z^*, y^*, u, z, r)$.*

Theorem 15. *There are no $x \in \mathcal{A}$ and $(v^*, y^*, u) \in \mathcal{B}_M^{C_{FL}}$ such that $f(x) \leq_K h_M^{C_{FL}}(v^*, y^*, u)$.*

Theorem 16. *Let $r \in K\setminus\{0\}$. Then there are no $x \in \mathcal{A}$ and $(v^*, z^*, y^*, u, z) \in \mathcal{B}_{W^r}^{C_{FL}}$ such that $f(x) \leq_K h_{W^r}^{C_{FL}}(v^*, z^*, y^*, u, z)$.*

Analogously, one can prove also the following weak duality statement involving (PVC) and (DVC_{MW}^{FL}).

Theorem 17. *There are no $x \in \mathcal{A}$ and $(v^*, y^*, u) \in \mathcal{B}_{MW}^{C_{FL}}$ such that $f(x) \leq_K h_{MW}^{C_{FL}}(v^*, y^*, u)$.*

For strong duality, besides some convexity hypotheses which ensure the K-convexity of the vector perturbation function $\Phi^{C_{FL}}$ we also particularize the regularity conditions (RC_i^{Φ}), $i \in \{1, 2, 3, 4\}$, obtaining (see [1,2])

$(RC_1^{C_{FL}})$ | $\exists x' \in \operatorname{dom} f \cap S$ such that f is continuous at x' and $g(x') \in -\operatorname{int}(C)$,

$(RC_2^{C_{FL}})$ | X and Y are Fréchet spaces, S is closed, f is K-lower semicontinuous, g is C-epi-closed and $0 \in \operatorname{sqri}\big(\operatorname{dom} f \times C - \operatorname{epi}_{(-C)}(-g) \cap (S \times Y)\big)$,

$(RC_3^{C_{FL}})$ | $\dim\Big(\operatorname{lin}\Big(\operatorname{dom} f \times C - \operatorname{epi}_{(-C)}(-g) \cap (S \times Z)\Big)\Big) < +\infty$ and $0 \in \operatorname{ri}\Big(\operatorname{dom} f \times C - \operatorname{epi}_{(-C)}(-g) \cap (S \times Z)\Big)$.

and, respectively,

$(RC_4^{C_{FL}})$ | S is closed, f is K-lower semicontinuous, g is C-epi-closed and $\operatorname{epi}(v^*f)^* + \bigcup\limits_{z^* \in C^*} \operatorname{epi}((z^*g)+\delta_S)^*$ is closed in the topology $w(X^*,X)\times\mathbb{R}$ for every $v^* \in K^{*0}$.

Theorem 18. *Let $\bar{r} \in K\backslash\{0\}$. Assume that f is a K-convex vector function, g is a C-convex vector function and one of the regularity conditions $(RC_i^{C_{FL}})$, $i \in \{1,2,3,4\}$, is fulfilled. If $\bar{x} \in \mathcal{PE}(PVC)$, then there exist $\bar{v}^* \in K^{*0}$, $\bar{z}^* \in X^*$, $\bar{y}^* \in C^*$ and $\bar{z} \in X$ such that $(\bar{v}^*, \bar{z}^*, \bar{y}^*, \bar{x}, \bar{z}, \bar{r}) \in \mathcal{E}(DVC_W^{FL})$, $(\bar{v}^*, \bar{z}^*, \bar{y}^*, \bar{x}, \bar{z}) \in \mathcal{E}(DVC_{W^{\bar{r}}}^{FL})$, $(\bar{v}^*, \bar{y}^*, \bar{x}) \in \mathcal{E}(DVC_M^{FL}) \cap \mathcal{E}(DVC_{MW}^{FL})$ and $f(\bar{x}) = h_W^{C_{FL}}(\bar{v}^*, \bar{z}^*, \bar{y}^*, \bar{x}, \bar{z}, \bar{r}) = h_{W^{\bar{r}}}^{C_{FL}}(\bar{v}^*, \bar{z}^*, \bar{y}^*, \bar{x}, \bar{z},) = h_M^{C_{FL}}(\bar{v}^*, \bar{y}^*, \bar{x}) = h_{MW}^{C_{FL}}(\bar{v}^*, \bar{y}^*, \bar{x})$.*

Remark 12. In case $V=\mathbb{R}$ and $K=\mathbb{R}_+$, taking the functions $f: X \to \overline{\mathbb{R}}$ and $g: X \to Y^\bullet$ proper we rediscover the Wolfe and Mond-Weir duality schemes for constrained scalar optimization problems from [1]. More precisely the problem (PVC) becomes then the constrained scalar optimization problem (P^C) from the mentioned paper and the vector duals considered in this section turn out to be to the duals introduced to (P^C) in [1].

Besides the inclusion relations that can be obtained as particularizations of Proposition 1 and the ones given in Remark 6 and Remark 10, there are other connections between the images of the feasible sets of the vector duals to (PVC) introduced in this section through their objective functions. In the following we prove some of them. First we deal with the vector duals obtained from (DVG_M).

Theorem 19. *It holds*

(a) $h_M^{C_{FL}}(\mathcal{B}_M^{C_{FL}}) \subseteq h_M^{C_F}(\mathcal{B}_M^{C_F})$;

(b) $h_M^{C_{FL}}(\mathcal{B}_M^{C_{FL}}) \subseteq h_M^{C_L}(\mathcal{B}_M^{C_L})$;

(c) $h_{MW}^{C_{FL}}(\mathcal{B}_{MW}^{C_{FL}}) \subseteq h_{MW}^{C_L}(\mathcal{B}_{MW}^{C_L})$.

Proof. (a) Let $(v^*, y^*, u) \in \mathcal{B}_M^{C_{FL}}$. This means that $(v^*, y^*, u) \in K^{*0} \times C^* \times S$, $(y^*g)(u) \geq 0$, $g(u) \in -C$ and $0 \in \partial(v^*f)(u) + \partial((y^*g)+\delta_S)(u)$. Then $u \in \mathcal{A}$ and $\partial((y^*g)+\delta_S)(u) \subseteq N_{\mathcal{A}}(u)$, which yields $0 \in \partial(v^*f)(u) + N_{\mathcal{A}}(u)$. Consequently, $(v^*, u) \in \mathcal{B}_M^{C_F}$. As $h_M^{C_{FL}}(v^*, y^*, u) = f(u) = h_M^{C_F}(v^*, u)$, we are done.

(b) Let $(v^*, y^*, u) \in \mathcal{B}_M^{C_{FL}}$. Then $u \in S$, $v^* \in K^{*0}$, $y^* \in C^*$, $(y^*g)(u) \geq 0$ and $g(u) \in -C$ and, because $0 \in \partial(v^*f)(u) + \partial((y^*g)+\delta_S)(u) \subseteq \partial\big((v^*f)(u) + (y^*g)+\delta_S\big)(u)$, it follows that $(v^*, y^*, u) \in \mathcal{B}_M^{C_L}$. As $h_M^{C_{FL}}(v^*, y^*, u) = f(u) = h_M^{C_L}(v^*, y^*, u)$, the conclusion follows.

(c) The proof is analogous to the one in (b). □

The question if similar inclusions are valid for the vector Wolfe type duals is very natural, but has a negative answer, even if the primal problem is convex.

The following examples (adapted from the scalar case treated in [1]) demonstrate this. Fixing $r \in K\backslash\{0\}$, they can be used also to show the similar facts for the vector duals obtained as special cases of (DVG_{W^r}).

Example 3. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $V = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $V^\bullet = \mathbb{R}^2 \cup \{\infty_{\mathbb{R}^2_+}\}$,

$$S = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, \begin{array}{ll} 3 \leq x_2 \leq 4, & \text{if } x_1 = 0, \\ 1 \leq x_2 \leq 4, & \text{if } x_1 \in (0, 2] \end{array} \right\},$$

$$f : \mathbb{R}^2 \to (\mathbb{R}^2)^\bullet,\ f(x_1, x_2) = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2, & \text{if } x_1 \leq 0, \\ \infty_{\mathbb{R}^2_+}, & \text{otherwise,} \end{cases}$$

and $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x_1, x_2) = 0$. Then for $\bar{v}^* = (1/2, 1/2)^T$ and any $y^* \in \mathbb{R}_+$ we get $(0, 0) \in \partial((\bar{v}^* f) + (y^* g) + \delta_S)(0, 3)$, thus $(3, 3)^T \in h_W^{C_L}(\mathcal{B}_W^{C_L})$. Trying to find $(v^*, z^*, y^*, u, z, r) \in \mathcal{B}_W^{C_{FL}}$ such that $(3, 3)^T \in h_W^{C_{FL}}(\mathcal{B}_W^{C_{FL}})$ leads to a contradiction, consequently, $h_W^{C_L}(\mathcal{B}_W^{C_L}) \not\subseteq h_W^{C_{FL}}(\mathcal{B}_W^{C_{FL}})$.

Example 4. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $V = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $V^\bullet = \mathbb{R}^2 \cup \{\infty_{\mathbb{R}^2_+}\}$, $S = \mathbb{R}$,

$$f : \mathbb{R} \to (\mathbb{R}^2)^\bullet,\ f(x) = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x, & \text{if } x > 0, \\ \infty_{\mathbb{R}^2_+}, & \text{otherwise,} \end{cases}$$

and

$$g : \mathbb{R} \to \mathbb{R},\ g(x) = \begin{cases} -x, & \text{if } x \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for all $v^* = (v_1^*, v_2^*)^T \in \operatorname{int}(\mathbb{R}^2_+)$ and $y^* \geq 0$ one has

$$\partial((v^* f) + (z^* g) + \delta_S)(u) = \partial(v^* f)(u) = \begin{cases} \{v_1^* + v_2^*\}, & \text{if } u > 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Consequently, $\mathcal{B}_W^{C_L} = \emptyset$. On the other hand it can be shown that $\big((1/2, 1/2)^T, 1, 1, 0, 1, (1, 1)^T\big) \in \mathcal{B}_W^{C_{FL}}$, thus $(0, 0)^T \in h_W^{C_{FL}}(\mathcal{B}_W^{C_{FL}})$. Therefore, $h_W^{C_{FL}}(\mathcal{B}_W^{C_{FL}}) \not\subseteq h_W^{C_L}(\mathcal{B}_W^{C_L})$.

Example 5. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $V = \mathbb{R}^2$, $K = \mathbb{R}^2_+$,

$$S = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, \begin{array}{ll} 3 \leq x_2 \leq 4, & \text{if } x_1 = 0, \\ 1 \leq x_2 \leq 4, & \text{if } x_1 \in (0, 2] \end{array} \right\},$$

$f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (x_2, x_2)^T$ and $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x_1, x_2) = x_1$. Since, for $\bar{v}^* = (1/2, 1/2)^T$ it holds $(0, 1)^T \in \partial(\bar{v}^* f)(0, 3) \cap (-N_{\mathcal{A}}(0, 3))$, it follows that $(3, 3)^T \in h_W^{C_F}(\mathcal{B}_W^{C_F})$. On the other hand, assuming that $(3, 3)^T \in h_W^{C_{FL}}(\mathcal{B}_W^{C_{FL}})$ leads to a contradiction, consequently, $h_W^{C_F}(\mathcal{B}_W^{C_F}) \not\subseteq h_W^{C_{FL}}(\mathcal{B}_W^{C_{FL}})$.

Example 6. Consider again the situation from Example 1. We have $\mathcal{A} = (0, +\infty)$, $N_{\mathcal{A}}(u) = \{0\}$ for all $u \in \mathcal{A}$, $\partial(v^* f)(u) = \{v_1^* + v_2^*\}$ for all $v^* = (v_1^*, v_2^*)^T \in \operatorname{int}(\mathbb{R}_+^2)$ and $u \in \mathbb{R}$, thus $\partial(v^* f)(u + z) \cap (-N_{\mathcal{A}}(u)) = \emptyset$ for all $u \in S$ and all $z \in \mathbb{R}$. Consequently, $\mathcal{B}_W^{C_F} = \emptyset$. On the other hand, it can be shown that $\big((1/2, 1/2)^T, 0, 1, 0, 1, (1,1)^T\big) \in h_W^{C_{FL}}(\mathcal{B}_W^{C_{FL}})$, thus $(0,0)^T \in h_W^{C_{FL}}(\mathcal{B}_W^{C_{FL}})$. Therefore, $h_W^{C_{FL}}(\mathcal{B}_W^{C_{FL}}) \nsubseteq h_W^{C_F}(\mathcal{B}_W^{C_F})$.

Remark 13. One can wonder why we considered in this paper only the classical vector duality scheme involving properly efficient solutions to the primal vector optimization problems and efficient solutions to their duals. In the literature it was unsuccessfully claimed (see [2, Remark 6.2.6]) that vector Wolfe or Mond-Weir strong vector duality statements could be given from which properly efficient solutions to the vector duals were obtained. Since these allegations were not properly proven, we avoided this vector duality scheme from our investigations. Another possible vector duality scheme involves efficient solutions to both primal and dual vector optimization problems. Hints regarding its possible development within our framework can be found in the next section.

5 Conclusions and further challenges

Following our investigations from [1], we propose two new duality schemes for general vector optimization problems, based on the classical Wolfe and Mond-Weir duality approaches. Then, particularizing the primal vector optimization problem to be first unconstrained, then constrained, and carefully choosing the vector perturbation functions we obtain new vector duals, among which are rediscovered also the classical nondifferentiable vector Wolfe and Mond-Weir dual problems. Weak and strong duality statements are given for the primal-dual pairs of vector problems. Moreover, different inclusion relations involving the images of the feasible sets of some vector duals through their objective functions and, respectively, the maximal sets of some vector duals are derived. We provide also some examples showing that in some cases no inclusion relations exist.

Investigations similar to the ones performed in this paper can be made with respect to weakly efficient solutions, too. We did not include them here since everything works analogously, the only changes consisting in reformulating the duals by taking the variable v^* to belong to $K^* \setminus \{0\}$ and, for the Wolfe type vector duals, $r \in \operatorname{int}(K)$, and in the fact that instead of efficient and properly efficient solutions we deal then only with weakly efficient solutions.

Another interesting direction of research can be developed starting from the observation that for a fixed $v^* \in K^{*0}$ one can show that an element $\bar{x} \in X$ is efficient to (PVG) if and only if it is an optimal solution of the scalar optimization problem

$$(EP) \qquad \inf_{\substack{F(\bar{x}) - F(x) \in K, \\ x \in X}} (v^* F)(x).$$

Having different scalar duals assigned to this scalar optimization problem, one can use them to formulate vector optimization dual problems with respect to efficient solutions to (PVG). More precisely, the strong duality statements regarding (PVG) and these new vector duals would ask the existence of an efficient solution to (PVG), besides convexity hypotheses and regularity conditions, in order to obtain efficient solutions to the vector duals.

Starting with the investigations from this paper, different other interesting problems can be posed. A first one is how can be formulated a dual problem to (PVG) which becomes (DVC_{MW}^L) in a particular case. Then, one can try to give weak and strong duality statements for the primal-dual pairs of optimization problems considered in this paper when the functions involved are differentiable on an open set S and the subdifferentials are replaced by gradients in the duals, by using generalized convexity notions like quasiconvexity, pseudoconvexity, even invexity. It were interesting to find out how can be obtained via strong duality an efficient solution to (DVG_W) for which the variable y needs not be equal to 0. Do there exist inclusions involving the maximal sets of the vector duals from Section 4 or it can be proven by counterexamples that such inclusions do not hold in general? Nevertheless, converse duality for (PVG) and its duals or for the particular cases studied in this paper, could be investigated, too.

Acknowledgements. Research partially supported by DFG (German Research Foundation), project WA 922/1-3. The authors are grateful to an anonymous referee for some interesting questions concerning further developments of the investigations presented in this paper.

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