# On linear vector optimization duality in infinite-dimensional spaces*

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**Dedicated to Professor Franco Giannessi on the occasion of his 75th birthday**

**Abstract.** In this paper we extend to infinite-dimensional spaces a vector duality concept recently considered in the literature in connection to the classical vector minimization linear optimization problem in a finite-dimensional framework. Weak, strong and converse duality for the vector dual problem introduced with this respect are proven and we also investigate its connections to some classical vector duals considered in the same framework in the literature.

**Keywords.** linear vector duality, cones, vector optimization

**AMS mathematics subject classification.** 90C05, 90C25, 90C29.

## 1 Introduction and preliminaries

Different vector dual problems have been attached in the literature to the classical linear vector optimization problem in finite-dimensional spaces and in [2, Section 5.5] we investigated the connections between them. Worth mentioning is that for some of these classical vector duals strong duality statements were available only when a constant vector that appears in the constraints of the primal problem was taken different to the zero vector. Then, inspired by a vector dual considered in [3] for the case when the image space of the objective function of the primal problem is partially ordered by the corresponding nonnegative orthant, we introduced in [1] a new vector dual to the classical linear vector optimization problem, for the situation when an arbitrary pointed convex cone partially ordered the mentioned image space, overcoming the drawbacks of the mentioned duals. We have provided duality assertions for this vector

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dual regardless of the choices of the elements involved and we compared the image set of this new dual and its set of maximal elements with the others considered in the literature. The scope of this paper is to extend this vector dual to infinite-dimensional spaces.

Consider the separated locally convex vector spaces $X$, $Z$ and $V$ and their topological dual spaces $X^*$, $Z^*$ and, respectively, $V^*$, endowed with the corresponding weak$^*$ topologies. Denote by $\langle x^*, x\rangle = x^*(x)$ the value at $x \in X$ of the linear continuous functional $x^* \in X^*$. A nonempty set $S \subseteq X$ is called *cone* if $\lambda S \subseteq S$ for all $\lambda \geq 0$. The cone $S \subseteq X$ is said to be *nontrivial* if $S \neq \{0\}$ and $S \neq X$, and *pointed* if $S \cap (-S) = \{0\}$. On $Z$ we consider the partial ordering "$\leqq_C$" induced by the convex cone $C \subseteq Z$, defined by $z \leqq_C y \Leftrightarrow y - z \in C$ when $z, y \in Z$. We use also the notation $z \leq_C y$ to write more compactly that $z \leqq_C y$ and $z \neq y$, where $z, y \in Z$. The dual cone of $C$ is $C^* = \{y^* \in Z^* : \langle y^*, y\rangle \geq 0 \ \forall y \in C\}$. Let also $V$ be partially ordered by the nontrivial pointed convex cone $K \subseteq V$.

Given a subset $U$ of $X$, by $\mathrm{cl}(U)$, $\mathrm{lin}(U)$, $\mathrm{cone}(U)$ and $\mathrm{ri}(U)$ we denote its *closure*, *linear hull*, *conical hull* and *relative interior* (that is the interior relative to the closure of its affine hull), respectively. Moreover, if $U$ is convex its *strong quasi relative interior* is $\mathrm{sqri}(U) = \big\{x \in U : \mathrm{cone}(U - x)$ is a closed linear subspace$\big\}$. The *quasi interior of the dual cone* of $K$ is $K^{*0} := \big\{v^* \in K^* : \langle v^*, v\rangle > 0$ for all $v \in K\backslash\{0\}\big\}$. If $K$ is closed, then $K^{*0} = \{x^* \in K^* : \mathrm{cl}(\mathrm{cone}(K^* - x^*)) = X^*\}$.

With $\mathcal{L}(X, V)$ we denote the set of the linear continuous mappings $L : X \to V$. Given a linear continuous mapping $L \in \mathcal{L}(X, V)$, we have its *adjoint* $L^* : V^* \to X^*$ given by $\langle L^*v^*, x\rangle = \langle v^*, Lx\rangle$ for any $(x, v^*) \in X \times V^*$.

The vector optimization problems we consider in this paper consist of vector-minimizing or vector-maximizing a vector function with respect to the partial ordering induced in its image space by a pointed convex cone. As notions of solutions for vector optimization problems we rely on the classical efficient and properly efficient solutions, the latter considered with respect to the linear scalarization.

Let $M \subseteq V$ be a nonempty set. An element $\overline{v} \in M$ is said to be a *minimal element of $M$ (regarding the partial ordering induced by $K$)* if there exits no $v \in M$ satisfying $v \leq_K \overline{v}$. The set of all minimal elements of $M$ is denoted by $\mathrm{Min}(M, K)$. Even if in the literature there are several concepts of proper minimality for a given set, we deal here only with the properly minimal elements of a set in the sense of linear scalarization. An element $\overline{v} \in M$ is said to be a *properly minimal element of $M$ (in the sense of linear scalarization)* if there exists a $\lambda \in K^{*0}$ such that $\langle \lambda, \overline{v}\rangle \leq \langle \lambda, v\rangle$ for all $v \in M$. The set of all properly minimal elements of $M$ (in the sense of linear scalarization) is denoted by $\mathrm{PMin}(M, K)$. It can be shown that every properly minimal element of $M$ is also minimal, but the reverse assertion fails in general. Corresponding maximality notions are defined by using the definitions from above. The elements of the set $\mathrm{Max}(M, K) := \mathrm{Min}(M, -K)$ are called *maximal elements of $M$*.

## 2 Vector duals and relations between them

The primal *linear vector optimization problem* we consider is

$$(P) \qquad \underset{x\in\mathcal{A}}{\operatorname{Min}}\, Lx, \\ \mathcal{A} = \{x \in S : Ax - b \in C\}$$

where $L \in \mathcal{L}(X,V)$, $A \in \mathcal{L}(X,Z)$, $b \in Z$, and $S \subseteq X$ and $C \subseteq Z$ are convex cones. In case $X = \mathbb{R}^n$, $Z = \mathbb{R}^m$, $V = \mathbb{R}^k$, $S = \mathbb{R}^n_+$ and $C = \{0\}$, where the linear continuous mappings $L$ and $A$ can be identified with the matrices $L \in \mathbb{R}^{k\times n}$ and, respectively, $A \in \mathbb{R}^{m\times n}$, $(P)$ becomes the so-called *classical linear vector optimization problem*

$$(CP) \qquad \underset{x\in\mathcal{A}}{\operatorname{Min}}\, Lx. \\ \mathcal{A} = \{x \in \mathbb{R}^n_+ : Ax = b\}$$

An element $\overline{x} \in \mathcal{A}$ is said to be a *properly efficient solution* to $(P)$ if $L\overline{x} \in \operatorname{PMin}(L(\mathcal{A}), K)$, i.e. there exists $\lambda \in K^{*0}$ such that $\langle \lambda, L\overline{x}\rangle \leq \langle \lambda, Lx\rangle$ for all $x \in \mathcal{A}$. An element $\overline{x} \in \mathcal{A}$ is said to be an *efficient solution* to $(P)$ if $L\overline{x} \in \operatorname{Min}(L(\mathcal{A}), K)$, i.e. there exists no $x \in \mathcal{A}$ such that $Lx \leq_K L\overline{x}$. A properly efficient solution $\overline{x}$ to $(P)$ is also efficient to $(P)$.

*Remark 1.* In general not all the efficient solutions to $(P)$ are also properly efficient to it. However, we have shown in [1, Theorem 1] that when $X = \mathbb{R}^n$, $Z = \mathbb{R}^m$, $V = \mathbb{R}^k$, $S = \mathbb{R}^n_+$ and $C = \{0\}$ the efficient and properly efficient solutions to $(P)$ coincide. Note that this result remains valid when $S$ and $C$ are arbitrary polyhedral cones.

*Remark 2.* In the literature there were proposed several concepts of properly efficient solutions to a vector optimization problem. For an exhaustive review of the proper efficiency notions considered in the literature and the relations between them we refer to [2, Section 2.4]. In [2, Proposition 2.4.16] we have shown that the properly efficient solutions (in the sense of linear scalarization) are properly efficient solutions in the senses of Geoffrion, Hurwicz, Borwein, Benson, Henig and Lampe and generalized Borwein, respectively, too. If $V$ is normed and $K$ is closed and has a compact base, then according to [4] all these types of properly efficient solutions coincide. This is the case for instance when $V = \mathbb{R}^k$ and $K = \mathbb{R}^k_+$.

Although different issues on linear vector duality were already investigated by Gale, Kuhn and Tucker back in the fifties, the first relevant contributions to the study of duality for $(P)$ were brought by Isermann for the case where $X = \mathbb{R}^n$, $Z = \mathbb{R}^m$, $V = \mathbb{R}^k$, $S = \mathbb{R}^n_+$, $C = \{0\}$ and $K = \mathbb{R}^k_+$, followed by Jahn, who considered the problem $(P)$ in the general case treated in this paper, bringing into attention two vector dual problems to it, namely (see [6, 7]) the so-called *vector abstract dual* to $(P)$

$$(D^J) \qquad \underset{(\lambda,U)\in\mathcal{B}^J}{\operatorname{Max}}\, h^J(\lambda, U),$$

where

$$\mathcal{B}^J = \left\{(\lambda, U) \in K^{*0} \times \mathcal{L}(Z, V) : U^*\lambda \in C^* \text{ and } (L - U \circ A)^*\lambda \in S^*\right\}$$

and

$$h^J(\lambda, U) = Ub,$$

and, respectively, the *vector Lagrange-type dual*

$$(D^L) \quad \underset{(\lambda,\eta,v)\in\mathcal{B}^L}{\mathrm{Max}} h^L(\lambda, \eta, v),$$

where

$$\mathcal{B}^L = \left\{(\lambda, \eta, v) \in K^{*0} \times C^* \times V : \langle\lambda, v\rangle \leq \langle\eta, b\rangle \text{ and } L^*\lambda - A^*\eta \in S^*\right\}$$

and

$$h^L(\lambda, \eta, v) = v.$$

When $b \neq 0$ the maximal sets of the images of the feasible sets through the corresponding objective functions of these vector duals coincide, but the disadvantage of $(D^J)$ in relation to $(D^L)$ can be noticed in case $b = 0$ when no strong duality statement can be obtained for the first one, unlike the other. More recently, in the finite-dimensional case considered by Isermann but with an arbitrary nontrivial pointed convex cone $K \subseteq \mathbb{R}^k$ instead of $\mathbb{R}^k_+$, a vector dual to $(CP)$ was proposed in [5] for which the duality assertions were shown via complicated set-valued optimization techniques. Nevertheless, in the very recent paper [1] we have introduced a direct generalization of a vector dual introduced for $K = \mathbb{R}^k_+$ in [3] to the framework of [5], providing moreover a complete analysis of all the mentioned vector duals to $(CP)$ in that setting.

The latter vector dual to $(P)$ can be extended to the framework of this paper as

$$(D) \quad \underset{(\lambda,U,v)\in\mathcal{B}}{\mathrm{Max}} h(\lambda, U, v),$$

where

$$\mathcal{B} = \left\{(\lambda, U, v) \in K^{*0} \times \mathcal{L}(Z, V) \times V : \langle\lambda, v\rangle = 0, U^*\lambda \in C^*, (L - U \circ A)^*\lambda \in S^*\right\}$$

and

$$h(\lambda, U, v) = Ub + v.$$

Let us see now what inclusions involving the images of the feasible sets through their objective functions of the vector duals to $(P)$ considered in this paper can be established, extending the scheme in [1, Section 4].

**Proposition 1.** *It holds $h^J(\mathcal{B}^J) \subseteq h(\mathcal{B})$.*

**Proof.** Let $d \in h^J(\mathcal{B}^J)$. Thus, there exists $(\lambda, U) \in \mathcal{B}^J$ such that $d = Ub$. It is easy to notice that $(\lambda, U, 0) \in \mathcal{B}$. Thus, $h(\lambda, U, 0) = Ub = d$, i.e. $d \in h(\mathcal{B})$.□

**Proposition 2.** *It holds $h(\mathcal{B}) \subseteq h^L(\mathcal{B}^L)$.*

**Proof.** Let $d \in h(\mathcal{B})$. Thus, there exist $(\lambda, U, v) \in \mathcal{B}$ such that $d = h(\lambda, U, v) = Ub + v$. Let $\eta := U^*\lambda$. Then $\langle \lambda, d \rangle = \langle \lambda, Ub + v \rangle = \langle U^*\lambda, b \rangle + \langle \lambda, v \rangle = \langle \eta, b \rangle$, while $L^*\lambda - A^*\eta = L^*\lambda - A^*U^*\lambda = (L - U \circ A)^*\lambda \in S^*$. Consequently, $(\lambda, \eta, d) \in \mathcal{B}^L$ and, moreover, $d \in h^L(\mathcal{B}^L)$. $\square$

Now let us investigate the sets of maximal elements of these sets with respect to $K$.

**Theorem 1.** *It holds*

$$\mathrm{Max}(h^J(\mathcal{B}^J), K) \subseteq \mathrm{Max}(h(\mathcal{B}), K) = \mathrm{Max}(h^L(\mathcal{B}^L), K)$$

*and the inclusion becomes equality when $b \neq 0$.*

**Proof.** Assume the existence of a $d \in \mathrm{Max}(h(\mathcal{B}), K) \setminus \mathrm{Max}(h^L(\mathcal{B}^L), K)$. Then there exist a $\overline{d} \in h^L(\mathcal{B}^L)$, such that $d \leq_K \overline{d}$, and $(\lambda, \eta, \overline{d}) \in \mathcal{B}^L$ such that $\overline{d} = h^L(\lambda, \eta, \overline{d})$ and $\langle \lambda, \overline{d} \rangle = \langle \eta, b \rangle$. There exists also $k \in K \setminus \{0\}$ such that $\langle \lambda, k \rangle = 1$. Let $U \in \mathcal{L}(Z, V)$ be defined by $Uz := \langle \eta, z \rangle k$, for $z \in Z$. Then $U^*\lambda = \eta \in C^*$. Moreover, $(L - U \circ A)^*\lambda = L^*\lambda - A^*\eta \in S^*$. Taking $v := \overline{d} - Ub$, one gets $\langle \lambda, v \rangle = 0$. Consequently, $(\lambda, U, v) \in \mathcal{B}$ and thus $\overline{d} \in h(\mathcal{B})$. But since $d \in \mathrm{Max}(h(\mathcal{B}), K)$ and $d \leq_K \overline{d}$ a contradiction is attained, therefore $\mathrm{Max}(h(\mathcal{B}), K) \subseteq \mathrm{Max}(h^L(\mathcal{B}^L), K)$.

Take now $d \in \mathrm{Max}(h^L(\mathcal{B}^L), K)$. Then there exists $(\lambda, \eta, d) \in \mathcal{B}^L$ such that $\langle \lambda, d \rangle \leq \langle \eta, b \rangle$. From the maximality of $d$ in $h^L(\mathcal{B}^L)$ it follows that one actually has $\langle \lambda, d \rangle = \langle \eta, b \rangle$. Defining $U$ and $v$ like above, one can directly verify that $(\lambda, U, v) \in \mathcal{B}$ and $d \in h(\mathcal{B})$. By Proposition 2 it follows that $d \in \mathrm{Max}(h(\mathcal{B}), K)$, whence $\mathrm{Max}(h^L(\mathcal{B}^L), K) \subseteq \mathrm{Max}(h(\mathcal{B}), K)$, too.

Therefore $\mathrm{Max}(h(\mathcal{B}), K) = \mathrm{Max}(h^L(\mathcal{B}^L), K)$ and what remained yet unproven follows via [2, Theorem 4.5.2]. $\square$

## 3 Weak, strong and converse vector duality

We give in the following weak, strong and converse duality results for the primal-dual pair of vector optimization problems $(P) - (D)$. The first one holds in the most general framework.

**Theorem 2.** (weak duality for $(D)$) *There exist no $x \in \mathcal{A}$ and $(\lambda, U, v) \in \mathcal{B}$ such that $Lx \leq_K Ub + v$.*

**Proof.** Assume the existence of $x \in \mathcal{A}$ and $(\lambda, U, v) \in \mathcal{B}$ such that $Lx \leq_K Ub + v$. Then $0 < \langle \lambda, Ub + v - Lx \rangle = \langle \lambda, Ub - Lx \rangle = \langle \lambda, Ub - U \circ Ax + U \circ Ax - Lx \rangle = \langle U^*\lambda, b - Ax \rangle - \langle (L - U \circ A)^*\lambda, x \rangle \leq 0$. As this cannot happen, the assumption we made is false. $\square$

In order to prove strong and converse duality for $(D)$ we consider the following regularity condition

$$(RC) \quad \Big| \quad \exists x' \in S \text{ such that } Ax' - b \in \operatorname{int}(C).$$

**Theorem 3.** (strong duality for $(D)$) *If $\overline{x}$ is a properly efficient solution to $(P)$ and $(RC)$ is fulfilled, there exists $(\overline{\lambda}, \overline{U}, \overline{v}) \in \mathcal{B}$, an efficient solution to $(D)$, such that $L\overline{x} = \overline{U}b + \overline{v}$.*

**Proof.** Since $\overline{x}$ is properly efficient to $(P)$, there exists $\overline{\lambda} \in K^{*0}$ such that $\langle \overline{\lambda}, L\overline{x} \rangle \leq \langle \overline{\lambda}, Lx \rangle$ for all $x \in \mathcal{A}$. The fulfillment of $(RC)$ yields that for the scalar optimization problem

$$\inf_{x \in \mathcal{A}} \langle \overline{\lambda}, Lx \rangle$$

and its Lagrange dual

$$\sup_{\eta \in C^*} \inf_{x \in S} \big\{ \langle \overline{\lambda}, Lx \rangle + \langle \eta, b - Ax \rangle \big\},$$

which can be equivalently written as

$$\sup_{\substack{\eta \in C^*, \\ L^*\overline{\lambda} - A^*\eta \in S^*}} \langle \eta, b \rangle,$$

there is strong duality, i.e. their optimal objective values coincide and the dual has an optimal solution, say $\overline{\eta} \in C^*$. Consequently, as $\overline{x}$ solves the primal problem, one gets $\langle \overline{\lambda}, L\overline{x} \rangle = \langle \overline{\eta}, b \rangle$, where $L^*\overline{\lambda} - A^*\overline{\eta} \in S^*$.

As $\overline{\lambda} \in K^{*0}$, there exists $k \in K \backslash \{0\}$ such that $\langle \overline{\lambda}, k \rangle = 1$. Let $\overline{U} \in \mathcal{L}(Z, V)$ be defined by $\overline{U}z := \langle \overline{\eta}, z \rangle k$ for $z \in Z$, and $\overline{v} := L\overline{x} - \overline{U}b \in V$. Then $\langle \overline{\lambda}, \overline{v} \rangle = \langle \overline{\lambda}, L\overline{x} - \overline{U}b \rangle = \langle \overline{\lambda}, L\overline{x} \rangle - \langle \overline{\eta}, b \rangle = 0$, $\overline{U}^*\overline{\lambda} = \overline{\eta} \in C^*$ and $(L - \overline{U} \circ A)^*\overline{\lambda} = L^*\overline{\lambda} - A^*\overline{\eta} \in S^*$. Consequently, $(\overline{\lambda}, \overline{U}, \overline{v}) \in \mathcal{B}$ and $\overline{U}b + \overline{v} = \overline{U}b + L\overline{x} - \overline{U}b = L\overline{x}$. Assuming that $(\overline{\lambda}, \overline{U}, \overline{v})$ is not efficient to $(D)$, i.e. the existence of another feasible solution $(\lambda, U, v) \in \mathcal{B}$ satisfying $\overline{U}b + \overline{v} \leq_K Ub + v$, it follows $L\overline{x} \leq_K Ub + v$, which contradicts Theorem 2. Consequently, $(\overline{\lambda}, \overline{U}, \overline{v})$ is an efficient solution to $(D)$ for which $L\overline{x} = \overline{U}b + \overline{v}$. $\square$

*Remark 3.* In case $\operatorname{int}(C) = \emptyset$ and $X$ and $Z$ are Fréchet spaces, $S$ is closed and $C$ is closed one can assume instead of $(RC)$ that $b \in \operatorname{sqri}(A(S) - C)$. If the linear subspace $\operatorname{lin}(A(S) - C)$ has a finite dimension, the regularity condition can be replaced by $b \in \operatorname{ri}(A(S) - C)$. When $X$ and $Z$ are finite-dimensional, the result in Theorem 3 remains valid under the hypothesis $b \in A(\operatorname{ri}(S)) - \operatorname{ri}(C)$ and in this condition one can replace the relative interiors of the cones which are actually nonnegative orthants with the cones themselves.

*Remark 4.* If $\overline{x} \in \mathcal{A}$ and $(\overline{\lambda}, \overline{U}, \overline{v}) \in \mathcal{B}$ fulfill $L\overline{x} = \overline{U}b + \overline{v}$, then the *complementarity conditions* $\langle (L - \overline{U} \circ A)^*\overline{\lambda}, \overline{x} \rangle = 0$ and $\langle \overline{U}^*\overline{\lambda}, A\overline{x} - b \rangle = 0$ are fulfilled.

Like in the finite-dimensional case, a converse duality statement for $(D)$ can be provided, too.

**Theorem 4.** (converse duality) *If $(\overline{\lambda}, \overline{U}, \overline{v}) \in \mathcal{B}$ is an efficient solution to $(D)$, $(RC)$ is fulfilled and $L(\mathcal{A}) + K$ is closed, there exists $\overline{x} \in \mathcal{A}$, a properly efficient solution to $(P)$, such that $L\overline{x} = \overline{U}b + \overline{v}$.*

**Proof.** Let $\overline{d} := \overline{U}b + \overline{v}$ and suppose that $\overline{d} \notin L(\mathcal{A})$. Using Theorem 2 it follows easily that $\overline{d} \notin L(\mathcal{A}) + K$, too. Then Tuckey's separation theorem (see [2, Theorem 2.1.5]) guarantees the existence of $\gamma \in V^* \backslash \{0\}$ and $a \in \mathbb{R}$ such that

$$\langle \gamma, \overline{d} \rangle < a < \langle \gamma, Lx + k \rangle \ \forall x \in \mathcal{A} \ \forall k \in K. \tag{1}$$

Assuming that $\gamma \notin K^*$ would yield the existence of some $k \in K$ for which $\langle \gamma, k \rangle < 0$. Taking into account that $K$ is a cone, this implies a contradiction to (1), consequently $\gamma \in K^*$. Taking $k = 0$ in (1) it follows

$$\langle \gamma, \overline{d} \rangle < \langle \gamma, Lx \rangle \ \forall x \in \mathcal{A}. \tag{2}$$

On the other hand, one has $\langle \overline{\lambda}, \overline{d} \rangle = \langle \overline{\lambda}, \overline{U}b + \overline{v} \rangle = \langle \overline{U}^* \overline{\lambda}, b \rangle \leq \langle \overline{U}^* \overline{\lambda}, Ax \rangle$ for all $x \in \mathcal{A}$, so it holds

$$\langle \overline{\lambda}, Lx - \overline{d} \rangle \geq \langle (L - \overline{U} \circ A)^* \overline{\lambda}, x \rangle \geq 0 \ \forall x \in \mathcal{A}. \tag{3}$$

Now, taking $p := a - \langle \gamma, \overline{d} \rangle > 0$ it follows $\langle (r\overline{\lambda} + (1-r)\gamma), \overline{d} \rangle = a - p + r(\langle \overline{\lambda}, \overline{d} \rangle - a + p)$ for all $r \in \mathbb{R}$. Note that there exists an $\overline{r} \in (0,1)$ such that $\overline{r}(\langle \overline{\lambda}, \overline{d} \rangle - a + p) < p/2$ and $\overline{r}(\langle \overline{\lambda}, \overline{d} \rangle - a) > -p/2$, and let $\lambda := \overline{r}\overline{\lambda} + (1-\overline{r})\gamma$. It is clear that $\lambda \in K^{*0}$. By (2) and (3) it follows $r\langle \overline{\lambda}, \overline{d} \rangle + (1-r)a < \langle r\overline{\lambda} + (1-r)\gamma, Lx \rangle$ for all $x \in \mathcal{A}$ and all $r \in (0,1)$, consequently

$$\langle \lambda, \overline{d} \rangle = \overline{r}\langle \overline{\lambda}, \overline{d} \rangle + (1-\overline{r})\langle \gamma, \overline{d} \rangle = \overline{r}\langle \overline{\lambda}, \overline{d} \rangle + (1-\overline{r})(a-p)$$
$$< \frac{p}{2} + \overline{r}(a-p) + (1-\overline{r})(a-p) = a - \frac{p}{2} < \langle \lambda, Lx \rangle \ \forall x \in \mathcal{A}.$$

Moreover, there exists $\overline{k} \in K \backslash \{0\}$ such that $\langle \lambda, \overline{k} \rangle = 1$. Like in the proof of Theorem 3, the validity of $(RC)$ yields strong duality for the scalar optimization problem $\inf_{x \in \mathcal{A}} \langle \lambda, Lx \rangle$ and its Lagrange dual, i.e. there exists an $\overline{\eta} \in C^*$ with $L^*\lambda - A^*\overline{\eta} \in S^*$ for which $\inf_{x \in \mathcal{A}} \langle \lambda, Lx \rangle = \langle \overline{\eta}, b \rangle$.

Let $U \in \mathcal{L}(Z, V)$ be defined by $Uz := \langle \overline{\eta}, z \rangle \overline{k}$, $z \in Z$. Then $U^*\lambda = \overline{\eta} \in C^*$ and $(L - U \circ A)^*\lambda \in S^*$. Consequently, the hyperplane $\mathcal{H} := \{Ub + v : v \in V, \langle \lambda, v \rangle = 0\}$, which is nothing but the set $\{w \in V : \langle \lambda, w \rangle = \langle \lambda, Ub \rangle\}$, is contained in $h(\mathcal{B})$.

On the other hand, as $\langle \lambda, \overline{d} \rangle < \langle \overline{\eta}, b \rangle = \langle \lambda, Ub \rangle$, there exists a $\overline{k} \in K \backslash \{0\}$ such that $\langle \lambda, \overline{d} + \overline{k} \rangle = \langle \lambda, Ub \rangle$. Hence $\overline{d} + \overline{k} \in \mathcal{H} \subseteq h(\mathcal{B})$. Noting that $\overline{d} \leq_K \overline{d} + \overline{k}$, we have just arrived to a contradiction to the maximality of $\overline{d}$ to the set $h(\mathcal{B})$. Therefore our initial supposition is false, consequently $\overline{d} \in L(\mathcal{A})$. Then there exists $\overline{x} \in \mathcal{A}$ such that $L\overline{x} = \overline{d} = \overline{U}b + \overline{v}$. Using (3), it follows that $\overline{x}$ is a properly efficient solution to $(P)$. $\square$

*Remark 5.* From Theorem 1, Theorem 3 and Theorem 4 one can conclude that when $(RC)$ is fulfilled and $L(\mathcal{A}) + K$ is closed the following inclusion scheme holds

$$\mathrm{Max}(h^J(\mathcal{B}^J), K) \subseteq \mathrm{PMin}(L(\mathcal{A}), K) = \mathrm{Max}(h(\mathcal{B}), K) = \mathrm{Max}(h^L(\mathcal{B}^L), K) \tag{4}$$

and the inclusion becomes equality when $b \neq 0$. When $X = \mathbb{R}^n$, $S = \mathbb{R}^n_+$, $V = \mathbb{R}^k$, $Z = \mathbb{R}^m$ and $C = \{0\}$, taking into consideration Remark 3 and [1, Section 4], (4) is valid because $L(\mathcal{A}) + K$ is closed and there is no need to impose the fulfillment of any regularity condition since $(RC)$ can be replaced by a simple feasibility condition that is already satisfied.

*Remark 6.* In the framework of [1, Theorem 4] the set $L(\mathcal{A}) + K$ is closed. Thus, a natural question is under which conditions can the closedness of the mentioned set be guaranteed in more general settings. Sufficient conditions that ensure this can be found for instance in [8, Corollary 9.1.2]), [9, Theorem 1.1.8] and [10, Corollary 3.12].

## 4 Extending another vector dual

After successfully generalizing the vector dual from [1] to infinite-dimensional spaces, a natural challenge is to try doing the same with the vector dual to $(CP)$ from [5]. To this end, we propose the following vector dual to $(P)$

$$(D^H) \quad \operatorname*{Max}_{U \in \mathcal{B}^H} h^H(U),$$

where

$$\mathcal{B}^H = \big\{U \in \mathcal{L}(Z,V) : ((L - U \circ A)(S) + U(C)) \cap (-K) = \{0\}\big\}$$

and

$$h^H(U) = Ub + \operatorname{PMin}\big((L - U \circ A)(S) + U(C), K\big).$$

When $X = \mathbb{R}^n$, $S = \mathbb{R}^n_+$, $V = \mathbb{R}^k$, $Z = \mathbb{R}^m$ and $C = \{0\}$ this turns out to be exactly the vector dual problem proposed in [5] to $(CP)$, taking also in consideration that (see [1, Theorem 1]) in that framework the properly efficient solutions of the vector minimization problem in the objective function of $(D^H)$ coincide with the efficient ones of the same problem.

We begin with a result that establishes a connection between the feasible elements of $(D^J)$ and the one of $(D^H)$. Note that in the framework of [1] it is valid in both directions. A good way to achieve also here such an equivalence is by strongly separating the sets $(L - U \circ A)(S) + U(C)$ and $-K$. This could be done, under additional hypotheses, for instance by [7, Theorem 3.22], [5, Lemma 2.2] or [8, Theorem 11.4].

**Proposition 3.** *If $\lambda \in K^{*0}$ and $U \in \mathcal{L}(Z,V)$ fulfill $U^*\lambda \in C^*$ and $(L - U \circ A)^*\lambda \in S^*$, then $((L - U \circ A)(S) + U(C)) \cap (-K) = \{0\}$.*

**Proof.** Assume to the contrary that the conclusion is false. Then there exist $x \in S$ and $c \in C$ such that $0 \neq (L - U \circ A)x + Uc \in -K$. Consequently, $\langle \lambda, (L - U \circ A)x + Uc\rangle < 0$. But $\langle \lambda, (L - U \circ A)x + Uc\rangle = \langle (L - U \circ A)^*\lambda, x\rangle + \langle U^*\lambda, c\rangle$ and the hypotheses imply the nonnegativity of the both terms in the right-hand side of the last equality, so we reached the desired contradiction. $\square$

Let us see now where does the image set of this vector dual lie, in relation to the other vector duals considered in this paper.

**Proposition 4.** *It holds* $h^J(\mathcal{B}^J) \subseteq h^H(\mathcal{B}^H)$.

**Proof.** Let $d \in h^J(\mathcal{B}^J)$. Thus, there exists $(\lambda, U) \in \mathcal{B}^J$ such that $d = Ub$. By Proposition 3 we obtain immediately that $U \in \mathcal{B}^H$. Moreover, $(L - U \circ A)(0) + U(0) = 0$ and whenever $x \in S$ and $c \in C$ there holds $\langle \lambda, (L - U \circ A)x + Uc \rangle = \langle (L - U \circ A)^* \lambda, x \rangle + \langle U^* \lambda, c \rangle$ and this is nonnegative because $(\lambda, U) \in \mathcal{B}^J$. Consequently, $0 \in \mathrm{PMin}\big((L - U \circ A)(S) + U(C), K\big)$ and $d \in h^H(\mathcal{B}^H)$. $\square$

**Proposition 5.** *It holds* $h^H(\mathcal{B}^H) \subseteq h(\mathcal{B})$.

**Proof.** Let $d \in h^H(\mathcal{B}^H)$. Thus, there exists $U \in \mathcal{B}^H$ such that $d = Ub + v$, with $v \in \mathrm{PMin}\big((L - U \circ A)(S) + U(C), K\big)$. Then, there exist $\gamma \in K^{*0}$, $\overline{x} \in S$ and $\overline{c} \in C$ such that $v = (L - U \circ A)\overline{x} + U\overline{c}$ and

$$\langle \gamma, (L - U \circ A)\overline{x} + U\overline{c} \rangle \leq \langle \gamma, (L - U \circ A)x + Uc \rangle \ \forall x \in S \ \forall c \in C. \tag{5}$$

Taking in the right-hand side of (5) $c := \overline{c}$, it follows $\langle \gamma, (L - U \circ A)\overline{x} \rangle \leq \langle \gamma, (L - U \circ A)x \rangle$ for all $x \in S$. As $S$ is a cone, the existence of a point $\tilde{x} \in S$ for which $\langle \gamma, (L - U \circ A)\tilde{x} \rangle < 0$ would yield $\langle \gamma, (L - U \circ A)\overline{x} \rangle = -\infty$, that is impossible, so $\langle \gamma, (L - U \circ A)x \rangle \geq 0$ for all $x \in S$. Consequently, $(L - U \circ A)^* \gamma \in S^*$. As $0 \in S$, it follows also that $\langle \gamma, (L - U \circ A)\overline{x} \rangle \leq 0$, so $\langle \gamma, (L - U \circ A)\overline{x} \rangle = 0$.

Back to (5), taking now $x := \overline{x}$ one gets $\langle \gamma, U\overline{c} \rangle \leq \langle \gamma, Uc \rangle$ for all $c \in C$. Since $C$ is a cone, too, the same argumentation as above leads to $U^* \gamma \in C^*$ and $\langle \gamma, U\overline{c} \rangle = 0$. Consequently, $\langle \gamma, (L - U \circ A)\overline{x} + U\overline{c} \rangle = \langle \gamma, v \rangle = 0$, so $(\gamma, U, v) \in \mathcal{B}$ and $h(\gamma, U, v) = d$. Therefore $d \in h(\mathcal{B})$. $\square$

*Remark 7.* Employing Proposition 2, Proposition 4 and Proposition 5, one can arrange the image sets of the vector duals we treated in this paper in the following scheme

$$h^J(\mathcal{B}^J) \subseteq h^H(\mathcal{B}^H) \subseteq h(\mathcal{B}) \subseteq h^L(\mathcal{B}^L).$$

Examples showing that the just proven inclusions can be sometimes strict were given in [5, Section 4.3], [2, Example 5.5.1] and [1, Remark 5], respectively.

Let us investigate now the duality properties of $(D^H)$. First note that from Theorem 2 and Proposition 5 one can deduce the following weak duality statement for $(D^H)$.

**Corollary 1.** *There exist no* $x \in \mathcal{A}$, $U \in \mathcal{B}^H$ *and* $v \in \mathrm{PMin}\big((L - U \circ A)(S) + U(C), K\big)$ *such that* $Lx \leq_K Ub + v$.

Strong duality for $(D^H)$ can be proven under the same hypotheses as for $(D)$. Note that Remark 3 is valid in this case, too.

**Theorem 5.** *If $\overline{x}$ is a properly efficient solution to $(P)$ and $(RC)$ is fulfilled, there exists $\overline{U} \in \mathcal{B}^H$, an efficient solution to $(D^H)$, such that $L\overline{x} = \overline{U}b + \overline{v}$, where $\overline{v} \in \mathrm{PMin}\big((L - \overline{U} \circ A)(S) + U(C), K\big)$.*

**Proof.** Like in the proof of Theorem 3, the proper efficiency of $\overline{x}$ to $(P)$ delivers a $\overline{\lambda} \in K^{*0}$ and the fulfillment of $(RC)$ an $\overline{\eta} \in C^*$ such that $\langle \overline{\lambda}, L\overline{x} \rangle = \langle \overline{\eta}, b \rangle$ and $L^*\overline{\lambda} - A^*\overline{\eta} \in S^*$. As $\overline{\lambda} \in K^{*0}$, there exists $k \in K \backslash \{0\}$ such that $\langle \overline{\lambda}, k \rangle = 1$. Let $\overline{U} \in \mathcal{L}(Z, V)$ be defined by $\overline{U}z := \langle \overline{\eta}, z \rangle k$, $z \in Z$. Proposition 3 yields then $\overline{U} \in \mathcal{B}^H$.

Take now $\overline{v} := L\overline{x} - \overline{U}b$. Then it can be rewritten as $\overline{v} = (L - \overline{U} \circ A)\overline{x} + \overline{U}(A\overline{x} - b)$, so $\overline{v} \in (L - \overline{U} \circ A)(S) + \overline{U}(C)$. One has $\langle \overline{\lambda}, \overline{v} \rangle = \langle \overline{\lambda}, L\overline{x} - \overline{U}b \rangle = \langle \overline{\eta}, b \rangle - \langle \overline{\lambda}, \langle \overline{\eta}, b \rangle k \rangle = 0$ and $\langle \overline{\lambda}, (L - \overline{U} \circ A)x + \overline{U}c \rangle \geq 0$ for all $x \in S$ and $c \in C$. Consequently, $\overline{v} \in \mathrm{PMin}\big((L - \overline{U} \circ A)(S) + \overline{U}(C), K\big)$.

Assuming that $\overline{U}$ were not efficient to $(D^H)$, i.e. the existence of another feasible solution $U \in \mathcal{B}^H$ satisfying $\overline{U}b + \overline{v} \leq_K Ub + v$ for a $v \in \mathrm{PMin}\big((L - U \circ A)(S) + U(C), K\big)$, it follows $L\overline{x} \leq_K Ub + v$, which contradicts Corollary 1. Consequently, $\overline{U}$ is an efficient solution to $(D)$ for which $L\overline{x} = \overline{U}b + \overline{v}$. □

*Remark 8.* From Theorem 5 one can conclude that when $(RC)$ is fulfilled one has

$$\mathrm{PMin}(L(\mathcal{A}), K) \subseteq \mathrm{Max}(h^H(\mathcal{B}^H), K).$$

It remains an open challenge to find out under which conditions does this inclusion turn into an equality and also to compare $\mathrm{Max}(h^H(\mathcal{B}^H), K)$ with $\mathrm{Max}(h(\mathcal{B}), K)$ and $\mathrm{Max}(h^L(\mathcal{B}^L), K)$. Note that $\mathrm{Max}(h^H(\mathcal{B}^H), K)$ coincides with the equal sets from (4) in the framework of [1], i.e. when $X = \mathbb{R}^n$, $S = \mathbb{R}^n_+$, $V = \mathbb{R}^k$, $Z = \mathbb{R}^m$ and $C = \{0\}$.

# References


[1] R.I. Boţ, S.-M. Grad, G. Wanka, *Classical linear vector optimization duality revisited*, Optimization Letters, DOI: 10.1007/s11590-010-0263-1.

[2] R.I. Boţ, S.-M. Grad, G. Wanka, *Duality in vector optimization*, Springer-Verlag, Berlin-Heidelberg, 2009.

[3] R.I. Boţ, G. Wanka, *An analysis of some dual problems in multiobjective optimization (I)*, Optimization 53(3):281–300, 2004.

[4] A. Guerraggio, E. Molho, A. Zaffaroni, *On the notion of proper efficiency in vector optimization*, Journal of Optimization Theory and Applications 82(1):1–21, 1994.

[5] A.H. Hamel, F. Heyde, A. Löhne, C. Tammer, K. Winkler, *Closing the duality gap in linear vector optimization*, Journal of Convex Analysis 11(1):163–178, 2004.

[6] J. Jahn, *Duality in vector optimization*, Mathematical Programming 25(3):343–353, 1983.

[7] J. Jahn, *Vector optimization - theory, applications, and extensions*, Springer-Verlag, Berlin, 2004.

[8] R.T. Rockafellar, *Convex analysis*, Princeton University Press, Princeton, 1970.

[9] C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific, Singapore, 2002.

[10] C. Zălinescu, *Stability for a class of nonlinear optimization problems and applications*, in: Nonsmooth Optimization and Related Topics (Erice 1988), Plenum, New York, 437–458, 1988.