Looking for appropriate qualification conditions for subdifferential formulae and dual representations for convex risk measures

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Abstract. A fruitful idea, when providing subdifferential formulae and dual representations for convex risk measures, is to make use of the conjugate duality theory in convex optimization. In this paper we underline the outstanding role played by the qualification conditions in the context of different problem formulations in this area. We show that not only the meanwhile classical generalized interiority point ones come here to bear, but also a recently introduced one formulated by means of the quasi-relative interior.

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1 Introduction

1.1 Preliminaries of convex analysis

Let $\mathcal{X}$ be a separated locally convex vector space and $\mathcal{X}^*$ its topological dual space. We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in \mathcal{X}^*$ at $x \in \mathcal{X}$.

For a subset C of $\mathcal{X}$ we denote by $\operatorname{co} C$, $\operatorname{cl} C$ and $\operatorname{int} C$ its *convex hull*, *closure* and *interior*, respectively. The set $\operatorname{cone} C := \cup_{\lambda \geq 0} \lambda C$ denotes the *cone generated by* C, while the *normal cone* of C at $x \in C$ is given by $N_C(x) = \{x^* \in \mathcal{X}^* : \langle x^*, y - x\rangle \leq 0 \ \forall y \in C\}$. When C is a convex and closed set, by $C_\infty := \{x \in \mathcal{X} : x + C \subseteq C\}$, which is in this case a convex closed cone, we denote the *asymptotic cone* of C.

The *indicator function* of a set $C \subseteq \mathcal{X}$, denoted by δ_C, is defined by $\delta_C : \mathcal{X} \to \overline{\mathbb{R}} :=$

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$\mathbb{R} \cup \{\pm\infty\}$,

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

For a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ we denote by $\operatorname{dom} f = \{x \in \mathcal{X} : f(x) < +\infty\}$ its *effective domain* and by $\operatorname{epi} f = \{(x, r) \in \mathcal{X} \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\operatorname{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in \mathcal{X}$. The *Fenchel-Moreau conjugate* of f is the function $f^* : \mathcal{X}^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in \mathcal{X}} \{\langle x^*, x\rangle - f(x)\} \, \forall x^* \in \mathcal{X}^*.$$

Similarly, when $\mathcal{X}^*$ is endowed with the weak* topology, the *biconjugate function* of f, $f^{**} : \mathcal{X} \to \overline{\mathbb{R}}$, is given by

$$f^{**}(x) = \sup_{x^* \in \mathcal{X}^*} \{\langle x^*, x\rangle - f^*(x^*)\} \, \forall x \in \mathcal{X}.$$

By the *Fenchel-Moreau Theorem*, whenever $f : \mathcal{X} \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function, one has $f = f^{**}$.

For $f : \mathcal{X} \to \overline{\mathbb{R}}$ an arbitrary function the set $\partial f(x) = \{x^* \in \mathcal{X}^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \ \forall y \in \mathcal{X}\}$, when $f(x) \in \mathbb{R}$, denotes the *subdifferential* of f at x, while if $f(x) \in \{\pm\infty\}$ we take by convention $\partial f(x) = \emptyset$.

Regarding a function and its conjugate we have the *Young-Fenchel inequality* $f^*(x^*) + f(x) \geq \langle x^*, x\rangle$ for all $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$. Moreover, for all $x \in \mathcal{X}, x^* \in \mathcal{X}^*$ one has

$$f^*(x^*) + f(x) = \langle x^*, x\rangle \Leftrightarrow x^* \in \partial f(x). \tag{1}$$

If $f : \mathcal{X} \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous, then by $f_\infty : \mathcal{X} \to \overline{\mathbb{R}}$ we denote the *recession function* of f, which is defined as being the function whose epigraph is $(\operatorname{epi} f)_\infty$. The recession function is in this setting a proper, sublinear and lower semicontinuous function, while for all $d \in \mathcal{X}$ one has

$$f_\infty(d) = \sup\{f(x + d) - f(x) : x \in \operatorname{dom} f\}$$

and (see, for instance, [30])

$$f_\infty(d) = \lim_{t \to +\infty} \frac{f(x + td) - f(x)}{t} = \sup_{t > 0} \frac{f(x + td) - f(x)}{t} \ \forall x \in \operatorname{dom} f. \tag{2}$$

Having $f_i : \mathcal{X} \to \overline{\mathbb{R}}, i = 1, ..., m$, given proper functions we denote by $f_1 \square ... \square f_m : \mathcal{X} \to \overline{\mathbb{R}}$, $f_1 \square ... \square f_m(x) = \inf\{\sum_{i=1}^m f_i(x_i) : \sum_{i=1}^m x_i = x\}$, for $x \in \mathcal{X}$, their *infimal convolution*.

In the formulation of the qualification conditions which we employ in the investigations made in this paper we will make use of several generalized interiority notions. For a convex set $C \subseteq \mathcal{X}$, we recall those interiority notions we need in the following:

- the *algebraic interior* or *core* of C (cf. [30]),
  $\operatorname{core} C = \{x \in C : \operatorname{cone}(C - x) = \mathcal{X}\}$;

- the *strong quasi-relative interior* of C (cf. [6, 30]),
  $\operatorname{sqri} C = \{x \in C : \operatorname{cone}(C - x) \text{ is a closed linear subspace of } \mathcal{X}\}$;

- the *quasi-relative interior* of C (cf. [7]),
  $\operatorname{qri} C = \{x \in C : \operatorname{cl}\operatorname{cone}(C - x) \text{ is a linear subspace of } \mathcal{X}\}$
- the *quasi interior* of C (cf. [21]),
  $\operatorname{qi} C = \{x \in C : \operatorname{cl}\operatorname{cone}(C - x) = \mathcal{X}\}$.

For the last two notions we have the following dual characterizations.

Proposition 1 *(cf. [7, 11]) Let C be a nonempty convex subset of $\mathcal{X}$ and $x \in C$. Then:*

(i) $x \in \operatorname{qri} C \iff N_C(x)$ *is a linear subspace of* $\mathcal{X}^*$*;*

(ii) $x \in \operatorname{qi} C \iff N_C(x) = \{0\}$.

For the generalized interiority notions from above the following relations of inclusion hold:

$$\operatorname{int} C \subseteq \operatorname{core} C \subseteq \begin{matrix} \operatorname{sqri} C \\ \\ \operatorname{qi} C \end{matrix} \subseteq \operatorname{qri} C \subseteq C,$$

all of them being in general strict. Between sqri and qi no relation of inclusion holds in general. For a comprehensive discussion, examples and counterexamples with this respect we refer to [9]. If $\mathcal{X}$ is a finite-dimensional space, then $\operatorname{qi} C = \operatorname{int} C = \operatorname{core} C$ (cf. [21]) and $\operatorname{qri} C = \operatorname{sqri} C = \operatorname{ri} C$ (cf. [7]), where $\operatorname{ri} C$ is the *relative interior* of C. In case $\operatorname{int} C \neq \emptyset$ all the generalized interiority notions collapse into the topological interior of the set C.

In the following we turn our attention to the *Lagrange duality* for the optimization problem with geometric and cone constraints

$$(P) \quad \inf_{\substack{x \in S \\ g(x) \in -K}} f(x).$$

Here $\mathcal{X}$ and $\mathcal{Z}$ are two separated locally convex spaces, the latter being partially ordered by the nonempty convex cone $K \subseteq \mathcal{Z}$, $S \subseteq \mathcal{X}$ is a nonempty set, $f : \mathcal{X} \to \overline{\mathbb{R}}$ is a proper function and $g : \mathcal{X} \to \mathcal{Z}$ is a vector function fulfilling $\operatorname{dom} f \cap S \cap g^{-1}(-K) \neq \emptyset$. We denote by $\geq_K$ the *partial ordering* induced by K on $\mathcal{Z}$, defined for $u, v \in \mathcal{Z}$ by $u \geq_K v$ whenever $u - v \in K$, and by $K^* = \{x^* \in \mathcal{X}^* : \langle x^*, x \rangle \geq 0 \ \forall x \in K\}$ the *dual cone* of K.

The K-*epigraph* of $g : \mathcal{X} \to \mathcal{Z}$ is the set $\operatorname{epi}_K g = \{(x, z) \in \mathcal{X} \times \mathcal{Z} : z \geq_K g(x)\}$. The vector function g is said to be K-*convex* if $\operatorname{epi}_K g$ is convex and K-*epi closed* if $\operatorname{epi}_K g$ is closed.

We further assume that S is a convex set, f is a convex function and g a K-convex vector function. The Lagrange dual problem associated to (P) is

$$(D) \quad \sup_{\lambda \in K^*} \inf_{x \in S} \{f(x) + \langle \lambda, g(x) \rangle\}.$$

By $v(P)$ and $v(D)$ we denote the optimal objective values of the primal and the dual problem, respectively. It is a known fact that between the primal and the dual problem *weak duality*, i.e. $v(P) \geq v(D)$, always holds. In order to guarantee strong duality, i.e. the situation when $v(P) = v(D)$ and (D) has an optimal solution, we additionally need to require the fulfillment of a so-called *qualification condition.* In the literature one can distinguish between two main classes of qualification conditions, the so-called *generalized* *interiority point* and *closedness-type* conditions, respectively. For an overview on the relations between these two classes we refer to [8].

Throughout this paper we deal with qualification conditions of the first type and discuss their applicability in the context of different topics involving convex risk measures. To this end we consider the *Slater constraint qualification*

$$(QC_1) \quad \exists x' \in \operatorname{dom} f \cap S \text{ such that } g(x') \in -\operatorname{int} K$$

as well as the generalized interiority point qualification conditions (cf. [8])

$$\begin{aligned}(QC_2) \qquad & \mathcal{X} \text{ and } \mathcal{Z} \text{ are Fréchet spaces, } S \text{ is closed, } f \text{ is lower semicontinuous,}\\ & g \text{ is } K\text{-epi closed and } 0 \in \operatorname{core}(g(\operatorname{dom} f \cap S) + K),\end{aligned}$$

and

$$\begin{aligned}(QC_3) \qquad & \mathcal{X} \text{ and } \mathcal{Z} \text{ are Fréchet spaces, } S \text{ is closed, } f \text{ is lower semicontinuous,}\\ & g \text{ is } K\text{-epi closed and } 0 \in \operatorname{sqri}(g(\operatorname{dom} f \cap S) + K).\end{aligned}$$

Assuming that $v(P) \in \mathbb{R}$, along the above qualification conditions, we consider also the following one introduced in [11] (see, also, [9,10]) and expressed by means of the quasi interior and quasi-relative interior

$$\begin{aligned}(QC_4) \qquad & \exists x' \in \operatorname{dom} f \cap S \text{ such that } g(x') \in -\operatorname{qri} K,\ \operatorname{cl}(K-K) = \mathcal{Z} \text{ and}\\ & (0,0) \notin \operatorname{qri}\left[\operatorname{co}\left(\mathcal{E}_{v(P)} \cup \{(0,0)\}\right)\right]\end{aligned}$$

where $\mathcal{E}_{v(P)} = \{(f(x) - v(P) + \epsilon, g(x) + z) : x \in \operatorname{dom} f \cap S, z \in K, \epsilon \geq 0\}$ is the set in analogy to the *the conic extension,* a notion used by Giannessi in the theory of *image spaces analysis* (see [20]). If $0 \in \operatorname{qi}[(g(\operatorname{dom} f \cap S) + K) - (g(\operatorname{dom} f \cap S) + K)]$, then $(0,0) \notin \operatorname{qri}\left[\operatorname{co}\left(\mathcal{E}_{v(P)} \cup \{(0,0)\}\right)\right]$ is equivalent to $(0,0) \notin \operatorname{qi}\left[\operatorname{co}\left(\mathcal{E}_{v(P)} \cup \{(0,0)\}\right)\right]$. On the other hand, whenever (P) has an optimal solution one has $\operatorname{co}\left(\mathcal{E}_{v(P)} \cup \{(0,0)\}\right) = \mathcal{E}_{v(P)}$. For further qualification conditions expressed by means of the quasi interior and quasi relative-interior we refer to [9,11]. Different to $(QC_i), i \in \{2,3\}$, these conditions have the remarkable property that they do not require the fulfillment of any topological assumption for the set S or for the functions f and g and they do not restrict the spaces $\mathcal{X}$ and $\mathcal{Z}$ to be Fréchet. More than that, they find applicability in situations where K is the ordering cone of a separable Banach space, like ℓ^p or L^p, $p \in [1,\infty)$ (see [9–11]). This is because of the fact that these ordering cones have nonempty quasi-relative interiors and quasi interiors, all the other interiority notions furnishing the empty set. The assumption that $v(P)$ is a real number is not restrictive at all, since, otherwise, namely, when $v(P) = -\infty$, strong duality is automatically fulfilled.

Remark 1 When $\mathcal{X}$ and $\mathcal{Z}$ are Fréchet spaces and f, g are proper, convex and lower semicontinuous functions we have the following relations between the above qualification conditions $(QC_1) \Rightarrow (QC_2) \Rightarrow (QC_3)$ and, whenever $v(P) \in \mathbb{R}$, $(QC_1) \Rightarrow (QC_2) \Rightarrow (QC_4)$. In general the conditions (QC_3) and (QC_4) cannot be compared, for more on this topic the reader being invited to consult [9].

Theorem 2 *Assume that $v(P) \in \mathbb{R}$. If one of the qualification conditions $(QC_i), i \in \{1,...,4\}$, is fulfilled, then $v(P) = v(D)$ and the dual problem has an optimal solution.*

Proof. Assuming one of the conditions $(QC_i), i \in \{1,2,3\}$, fulfilled, the statement follows from Theorem 3.4 in [8]. If (QC_4) holds, then the strong duality is a consequence of Theorem 4.1 in [10] (see also [8,11]). ■

1.2 Convex risk measures on L^p

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be an atomless probability space, where Ω denotes the space of future states ω, $\mathfrak{F}$ is a σ-*algebra* on Ω and $\mathbb{P}$ is a *probability measure* on $(\Omega, \mathfrak{F})$. For a measurable random variable $X : \Omega \to \mathbb{R} \cup \{+\infty\}$ the *expectation value* with respect to $\mathbb{P}$ is defined by $\mathbb{E}(X) := \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$. Whenever X takes the value $+\infty$ on a subset of positive measure we have $\mathbb{E}(X) = +\infty$. The *essential supremum* of X, which represents the smallest essential upper bound of the random variable, is $\operatorname{essup} X := \inf\{a \in \mathbb{R} : \mathbb{P}\,(\omega : X(\omega) > a) = 0\}$, while its *essential infimum* is defined by $\operatorname{essinf} X := -\operatorname{essup}(-X)$.

Further, for $p \in [1, \infty)$ let we consider the following space of random variables

$$L^p := L^p(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}) = \left\{ X : \Omega \to \mathbb{R} : X \text{ is measurable, } \int_\Omega |X(\omega)|^p d\mathbb{P}(\omega) < +\infty \right\}.$$

The space L^p equipped with the norm $\|X\|_p = (\mathbb{E}(|X|^p))^{\frac{1}{p}}$ is a Banach space. To complete the picture of L^p spaces, we introduce the space corresponding to the limiting value $p = \infty$, namely

$$L^\infty := L^\infty(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{R}) = \{X : \Omega \to \mathbb{R} : X \text{ is measurable, } \operatorname{essup}|X| < +\infty\},$$

which, being equipped with the norm $\|X\|_\infty = \operatorname{essup}|X|$, is a Banach space, too. For $p, q \in [1, \infty], p \geq q$, it holds $L^p \subseteq L^q$. We denote the topological dual space of L^p by $(L^p)^*$ and for $p \in [1, \infty)$ one has that $(L^p)^* = L^q$, where $q \in (1, \infty]$ fulfills $q = p/(p-1)$ (with the convention $1/0 = \infty$). In what concerns $(L^\infty)^*$, the topological dual space of L^∞, this can be identified with *ba,* the space of all bounded finitely additive measures on $(\Omega, \mathfrak{F})$ which are absolutely continuous with respect to $\mathbb{P}$ and it is usually much bigger than L^1. For the dual pairing $(X, X^*) \in (L^p, (L^p)^*)$ we shall write $\langle X^*, X\rangle = \mathbb{E}(X^* X)$ (even in the case $p = \infty$, by making an abuse of notation).

Equalities between random variables are to be interpreted in an *almost everywhere (a.e.)* way, while for $X, Y \in L^p$ we write $X \geq Y$ if and only if $X - Y \in L^p_+ := \{X \in L^p : X \geq 0 \text{ a.e.}\}$. We also write $X > Y$ if $X(\omega) > Y(\omega)$ for almost every $\omega \in \Omega$. Random variables $X : \Omega \to \mathbb{R}$ which take a constant value $c \in \mathbb{R}$, i.e $X = c$ a.e., will be identified with the real number c. Each random variable $X : \Omega \to \mathbb{R}$ can be represented as $X = X_+ - X_-$, where $X_+, X_- : \Omega \to \mathbb{R}$ are the random variables defined by $X_+(\omega) = \max\{0, X(\omega)\}$ and $X_-(\omega) = \max\{0, -X(\omega)\}$ for all $\omega \in \Omega$. The *characteristic function* of a set $G \in \mathfrak{F}$ is defined as being $\mathbf{1}_G : \Omega \to \mathbb{R}$,

$$\mathbf{1}_G(\omega) = \begin{cases} 1, & \text{if } \omega \in G, \\ 0, & \text{otherwise} \end{cases}$$

and, in view of the above notion, the expectation of a random variable X admits the equivalent representation $\mathbb{E}(X) = \langle \mathbf{1}_\Omega, X\rangle$, which will be used several times in this article.

The notions we introduce next will play a central role in what follows.

Definition 1 We call *risk function* a proper function $\rho : L^p \to \overline{\mathbb{R}}, p \in [1, \infty]$. The risk function ρ is said to be

(i) *convex*, if: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)\ \forall \lambda \in [0,1]\ \forall X, Y \in L^p$;

(ii) *positively homogeneous*, if: $\rho(0) = 0$ and $\rho(\lambda X) = \lambda\rho(X)\ \forall \lambda > 0\ \forall X \in L^p$;

(iii) *monotone*, if: $X \geq Y \Rightarrow \rho(X) \leq \rho(Y)\ \forall X, Y \in L^p$;

(iv) *cash-invariant*, if: $\rho(X + a) = \rho(X) - a\ \forall X \in L^p\ \forall a \in \mathbb{R}$;

(v) a *convex risk measure*, if: ρ is convex, monotone and cash-invariant;

(vi) a *coherent risk measure*, if: ρ is a positively homogeneous convex risk measure.

The first axiomatic way of defining risk measures has been given by Artzner, Delbaen, Eber and Heath in [1] and refers to *coherent risk measures*. Nevertheless, it has become a standard in modern risk management to assess the riskness of a portfolio by means of *convex risk measures*. The latter have been introduced by Föllmer and Schied in [18].

While the elements of $L^p, p \in [1, \infty]$ can be seen as describing future net worths, the value $\rho(X)$ can be understood as a capital requirement for X. Consequently, a convex risk measures guarantees that the capital requirement of the convex combination of two positions does not exceed the convex combination of the capital requirements of the positions taken separately. The monotonicity property says that if one has the certitude that Y will be smaller than X in (almost) every state of the world, than the capital requirement for Y should be greater than for X. Cash-invariance means that adding a constant amount of money a to X should reduce the capital requirement for X by a. For the economic interpretation of the other notions given in Definition 1 we refer to [1, 18, 19].

The reader can find examples of coherent and convex risk measures in [12, 14–18, 23–25, 27, 29], some of them being objects of the investigations we make in the forthcoming sections. The literature on convex risk measures has known in the last time a rapid growth, several aspects in connection with these notions being addressed. In this paper we will concentrate ourselves on providing subdifferential formulae and dual representations for different convex risk measures by making use of the conjugate duality theory in convex optimization.

One of the most challenging topics to be addressed in this field is the formulation of optimality conditions for portfolio optimization problems with a convex risk measure as objective function. Since for this class of functions differentiability is not necessarily guaranteed, one will be forced to make use of the convex subdifferential when characterizing optimality (see, for instance, [13]). This is why it is important to be in the possession of easily handleable formulae for the subdifferential of the risk measures which could come into consideration with this respect. Among the most relevant literature on this topic one has to mention [23, 24, 26, 27, 29].

Within a short time after the introduction of the convex risk measures, one could notice a intensification of the efforts to provide dual representations for these. With this respect, we refer the reader to [12, 14, 15, 22, 23, 26, 27, 29]. As it has been noticed, for instance, in [22, 23], the importance of having dual representations is given by the fact, that these can be used for deriving several properties for the risk measures in a very simple manner.

One of the major aims of the article is to emphasize the fact that different research aspects in connection with convex risk measures on L^p, $p \in [1, \infty]$, lead to some cone-constrained optimization problems having as ordering cone L^p_+, which has to be investigated from the point of view of the convex duality theory. As seen in Subsection 1.1, one needs to this end to have a qualification condition fulfilled. The Slater qualification condition is with this respect useful only when $p = \infty$, as in

$$\text{int}(L^\infty_+) = \{X \in L^\infty : \text{essinf}\, X > 0\}$$

is a nonempty set. On the other hand, for $p \in [1, \infty)$ one has (see [7])

$$\operatorname{int} L^p_+ = \operatorname{core} L^p_+ = \operatorname{sqri} L^p_+ = \emptyset,$$

which means that the Slater condition is in this situation unusable, while the conditions (QC_2) and (QC_3) could be employed only when the image of $\operatorname{dom} f \cap S$ through g is not a subset of the ordering cone. More appropriate for this situation will prove to be the qualification condition (QC_4), since one has that (see [7])

$$\operatorname{qi} L^p_+ = \operatorname{qri} L^p_+ = \{X \in L^p : X > 0\}.$$

1.3 Outline of the paper

In Section 2 we consider a generalized convex risk measure defined via a so-called *utility function* and associated with the Optimized Certainty Equivalent (OCE), a notion introduced and explored in [4, 5]. This convex risk measure is expressed as an infimal value function, thus we provide first of all a weak sufficient condition for the attainment of the infimum in its definition. Further, we give formulae for its conjugate function and its subdifferential. The generalized convex risk measure we consider has the advantage that, for some particular choices of the utility function, it leads to some well-known convex risk measures, for the conjugate and subdifferential of which we are consequently able to derive the corresponding formulae.

The results in the sections 3 and 4 are motivated the paper of Filipović and Kupper [16], where for a convex risk function the so-called *monotone cash-invariant hull* has been introduced, which is actually the greatest monotone and cash-invariant function majorized by the risk function. This function has been formulated by making use of the *infimal convolution.* In other words, the monotone cash-invariant hull at a given point is nothing else than the optimal objective value of a convex optimization problem. Having as a starting point this observation, we give a dual representation of the monotone and cash-invariant hull by employing the Lagrange duality theory along with a qualification condition, under the hypothesis that the risk function is *lower semicontinuous.* This guarantees the vanishing of the duality gap and, implicitly, the validity of the dual representation. The examples considered in [16] are discussed from this new point of view.

In the last section of the paper we deal with the same problem as in Section 3, but by considering this time a convex risk function which does not fulfill the lower semicontinuity assumption. For this function we can easily establish the *monotone hull* and we can also give a dual representation for it by making use of the quasi-relative interiority-type qualification condition (QC_4). We also refer to the limitations of this approach in the context of the determination of the *monotone cash-invariant hull* for the function in discussion.

2 Conjugate and subdifferential formulae for convex risk measures via Optimized Certainty Equivalent

In this section we will furnish first formulae for both conjugate and subdifferential of a generalized convex risk measure, associated with the Optimized Certainty Equivalent (OCE). The Optimized Certainty Equivalent was introduced by Ben-Tal and Teboulle in [4] by making use of a *concave utility function.* For properties of OCE and for relations with other certainty equivalent measures we refer to [4, 5]. For the investigations in this paper we adapt the definition of the Optimized Certainty Equivalent and the setting in which this has been introduced, by considering a *convex utility function*, as this better suits in the general framework of convex duality. We close the section by particularizing the general results to some convex risk measures widely used in the literature.

Let us start by fixing the framework in which we work throughout the section.

Assumption Let $u : \mathbb{R} \to \overline{\mathbb{R}}$ be a proper, convex, lower semicontinuous and nonincreasing function such that $u(0) = 0$ and $-1 \in \partial u(0)$.

Remark 2 The two conditions we impose on the utility function u are also known as *normalization conditions*. By exploiting the definition of the subdifferential, they can be equivalently written as $u(0) = 0$ and $u(t) + t \geq 0$ for all $t \in \mathbb{R}$.

Having as starting point the definition of the Optimized Certainty Equivalent given in [5] we define for $p \in [1, \infty]$ the following *generalized convex risk measure* $\rho_u : L^p \to \mathbb{R} \cup \{+\infty\}$

$$\rho_u(X) = \inf_{\lambda \in \mathbb{R}} \{\lambda + \mathbb{E}(u(X + \lambda))\}. \tag{3}$$

One can easily see that, due to the Assumption, $\rho_u(X) \geq -\mathbb{E}(X)$ for all $X \in L^p$ and that this function satisfies the properties required in the definition of a convex risk measure. Next we provide a formula for the conjugate of ρ_u.

Lemma 3 *The conjugate function of ρ_u, $\rho_u^* : (L^p)^* \to \overline{\mathbb{R}}$, is given by*

$$\rho_u^*(X^*) = \begin{cases} \mathbb{E}(u^*(X^*)), & \text{if } \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4}$$

Proof. By the definition of the conjugate function we get for all $X^* \in (L^p)^*$

$$\begin{aligned} \rho_u^*(X^*) &= \sup_{\substack{X \in L^p \\ \lambda \in \mathbb{R}}} \{\langle X^*, X\rangle - \lambda - \mathbb{E}(u(X + \lambda))\} = \sup_{\substack{R \in L^p \\ \lambda \in \mathbb{R}}} \{\langle X^*, R - \lambda\rangle - \lambda - \mathbb{E}(u(R))\} \\ &= \sup_{\lambda \in \mathbb{R}} \{-\lambda(\mathbb{E}(X^*) + 1)\} + \sup_{R \in L^p} \{\langle X^*, R\rangle - \mathbb{E}(u(R))\}. \end{aligned}$$

Using the interchangeability property of minimization and integration (see, for instance, [28, Theorem 14.60]) the second expression from above can be written as

$$\sup_{R \in L^p} \{\langle X^*, R\rangle - \mathbb{E}(u(R))\} = \mathbb{E}\left\{\sup_{r \in \mathbb{R}} (rX^* - u(r))\right\} = \mathbb{E}(u^*(X^*)).$$

On the other hand, since $\sup_{\lambda \in \mathbb{R}} \{-\lambda(\mathbb{E}(X^*) + 1)\} = \delta_{\{0\}}(\mathbb{E}(X^*) + 1)$, one obtains the desired conclusion. ■

Before providing a subdifferential formula for ρ_u, we deliver via Lagrange duality a sufficient condition the utility function u has to fulfill in order to guarantee the attainment of the infimum in the definition of $\rho_u(X)$ for all $X \in L^p$. According to [4, 5], for those $X \in L^p$ having as support a bounded and closed interval, the infimum in (3) is attained. But what we provide here, is a condition which ensures this fact independently from the choice of the random variable.

Let $X \in L^p$ be fixed. Consider the following primal optimization problem

$$\inf_{\substack{\Xi \in L^q \\ \mathbb{E}(\Xi)=-1}} \Big[\mathbb{E}(u^*(\Xi)) - \langle X, \Xi\rangle\Big], \tag{5}$$

where $q := \frac{p}{p-1}$, if $p \in [1,\infty)$, and $q := 1$, if $p = \infty$. The *Lagrange dual* optimization problem to (5) is given by

$$\sup_{\lambda\in\mathbb{R}} \inf_{\Xi\in L^q} \Big[\mathbb{E}(u^*(\Xi)) - \langle X, \Xi\rangle + \lambda(\mathbb{E}(\Xi)+1)\Big] = \sup_{\lambda\in\mathbb{R}} \Big[\lambda - \sup_{\Xi\in L^q} \Big(\langle X-\lambda, \Xi\rangle - \mathbb{E}(u^*(\Xi))\Big)\Big].$$

Again, via [28, Theorem 14.60], it holds

$$\sup_{\Xi\in L^q} \Big(\langle X-\lambda, \Xi\rangle - \mathbb{E}(u^*(\Xi))\Big) = \mathbb{E}\Big(\sup_{r\in\mathbb{R}}(r(X-\lambda) - u^*(r))\Big) = \mathbb{E}(u(X-\lambda))$$

and this leads to the following dual problem to (5)

$$\sup_{\lambda\in\mathbb{R}} \Big[-\lambda - \mathbb{E}(u(X+\lambda))\Big]. \tag{6}$$

Let us notice that the optimal objective value of the dual problem (6) is equal to $-\rho_u(X)$.

Theorem 4 *Assume that for the recession function of the utility function u fulfills the following condition*

$$\{d \in \mathbb{R} : u_\infty(d) = -d\} = \{0\}. \tag{7}$$

Then for all $X \in L^p$ there exists $\bar\lambda(X) \in \mathbb{R}$ such that $\rho_u(X) = \bar\lambda(X) + \mathbb{E}(u(X+\bar\lambda(X)))$.

Proof. We consider $X \in L^p$ fixed and prove that under condition (7) for the primal-dual pair (5)-(6) strong duality holds. This will guarantee among others the existence of an optimal solution $\bar\lambda(X)$ for the dual, which will prove the assertion.

Define $s : \mathbb{R} \to \overline{\mathbb{R}}$ as being $s(t) = u(t) + t$. Notice that s is proper, convex and lower semicontinuous, too, and for all $t^* \in \mathbb{R}$ it holds $s^*(t^*) = u^*(t^*-1)$, so $\operatorname{dom} s^* = \operatorname{dom} u^* + 1$. On the other hand, since $0 \in \operatorname{dom} s$, it holds (see (2))

$$s_\infty(d) = \sup_{t>0} \frac{s(td)-s(0)}{t} = \sup_{t>0} \frac{u(td)+td}{t} = \sup_{t>0} \frac{u(td)-u(0)}{t} + d = u_\infty(d) + d \ \forall d \in \mathbb{R}. \tag{8}$$

Thus condition (7) is nothing else than asking that $\{d \in \mathbb{R} : s_\infty(d) = 0\} = \{0\}$. On the other hand, from (8) it follows that $s_\infty(d) \geq 0$ for all $d \in \mathbb{R}$. By taking into account [2, Theorem 3.2.1.] we get that $0 \in \operatorname{ri}(\operatorname{dom} s^*) = \operatorname{ri}(\operatorname{dom} u^* + 1)$.

Further, we notice that, by taking $f : L^q \to \overline{\mathbb{R}}$, $f(\Xi) = \mathbb{E}(u^*(\Xi)) - \langle X, \Xi\rangle$ and $g : L^q \to \mathbb{R}$, $g(\Xi) = \mathbb{E}(\Xi) + 1$, which are both convex functions, the qualification condition (QC_3) is fulfilled. Indeed, f is lower semicontinuous, g is continuous and $0 \in \operatorname{ri}(\operatorname{dom} u^* + 1) = \operatorname{sqri}(\mathbb{E}(\operatorname{dom} u^*) + 1)$. Thus the existence of strong duality for (5)-(6) and, consequently, of an optimal solution for (6) is shown. ∎

Next we provide a formula for the subdifferential of the general convex risk measure ρ_u.

Theorem 5 *Assume that condition (7) is fulfilled. Let $X \in L^p$ and $\bar{\lambda}(X) \in \mathbb{R}$ be the element where the infimum in the definition of $\rho_u(X)$ is attained. Then it holds*

$$\partial\rho_u(X) = \{X^* \in (L^p)^* : X^*(\omega) \in \partial u(X(\omega)+\bar{\lambda}(X)) \text{ for almost every } \omega \in \Omega, \mathbb{E}(X^*) = -1\}. \tag{9}$$

Proof. We fix an $X \in L^p$ and let $\bar{\lambda}(X) \in \mathbb{R}$ be such that $\rho_u(X) = \bar{\lambda}(X)+\mathbb{E}(u(X+\bar{\lambda}(X)))$. Then, via (1),

$$X^* \in \partial\rho_u(X) \Leftrightarrow \rho_u^*(X^*) + \partial\rho_u(X) = \langle X^*, X\rangle$$

or, equivalently, (see Lemma 3)

$$\mathbb{E}(X^*) = -1 \text{ and } \mathbb{E}(u^*(X^*) + u(X + \bar{\lambda}(X)) - \langle X^*, X + \bar{\lambda}(X)\rangle) = 0.$$

On the other hand, by the Young-Fenchel inequality, it holds

$$u^*(X^*(\omega)) + u(X(\omega) + \bar{\lambda}(X)) - X^*(\omega)(X(\omega) + \bar{\lambda}(X)) \geq 0 \ \forall \omega \in \Omega,$$

which means that $\mathbb{E}(u^*(X^*) + u(X + \bar{\lambda}(X)) - \langle X^*, X + \bar{\lambda}(X)\rangle) = 0$ is nothing else than $u^*(X^*(\omega)) + u(X(\omega) + \bar{\lambda}(X)) - X^*(\omega)(X(\omega) + \bar{\lambda}(X)) = 0$ for almost every $\omega \in \Omega$. In conclusion, $X^* \in \partial\rho_u(X)$ if and only if

$$\mathbb{E}(X^*) = -1 \text{ and } X^*(\omega) \in \partial u(X(\omega) + \bar{\lambda}(X)) \text{ for almost every } \omega \in \Omega.$$

■

In the sequel we rediscover for particular choices of the utility function u several well-known convex risk measures and provide formulae for their conjugates and subdifferentials.

2.1 Conditional value-at-risk (CVaR)

For $\gamma_2 < -1 < \gamma_1 \leq 0$ we consider the utility function $u_1 : \mathbb{R} \to \mathbb{R}$ defined by

$$u_1(t) = \begin{cases} \gamma_2 t, & \text{if } t \leq 0, \\ \gamma_1 t, & \text{if } t > 0, \end{cases}$$

and notice that it satisfies all the requirements in the Assumption. This gives rise to the following convex risk measure $\rho_{u_1} : L^p \to \mathbb{R}$,

$$\rho_{u_1}(X) = \inf_{\lambda \in \mathbb{R}} \{\lambda + \gamma_1 \mathbb{E}(X + \lambda)_+ - \gamma_2 \mathbb{E}(X + \lambda)_-\}.$$

Since $u_1^* = \delta_{[\gamma_2,\gamma_1]}$, via Lemma 3 one gets for $\rho_{u_1}^* : (L^p)^* \to \overline{\mathbb{R}}$ the following expression

$$\rho_{u_1}^*(X^*) = \begin{cases} 0, & \text{if } \gamma_2 \leq X^* \leq \gamma_1, \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Noticing that for all $d \in \mathbb{R}$,

$$(u_1)_\infty(d) = \begin{cases} \gamma_2 d, & \text{if } d < 0, \\ 0, & \text{if } d = 0, \\ \gamma_1 d, & \text{if } d > 0, \end{cases}$$

one can easily see that condition (7) is satisfied. Thus for all $X \in L^p$ there exists $\bar{\lambda}(X) \in \mathbb{R}$ such that $\rho_{u_1}(X) = \bar{\lambda}(X) + \gamma_1 \mathbb{E}(X + \bar{\lambda}(X))_+ - \gamma_2 \mathbb{E}(X + \bar{\lambda}(X))_-$. Further, according to Theorem 5, we will make use of $\bar{\lambda}(X)$ when giving the formula for the subdifferential of ρ_{u_1} at X. Since

$$\partial u_1(t) = \begin{cases} \{\gamma_2\}, & \text{if } t < 0, \\ [\gamma_2, \gamma_1], & \text{if } t = 0, \\ \{\gamma_1\}, & \text{if } t > 0, \end{cases}$$

via (9) we obtain for all $X \in L^p$ the following formula

$$\partial \rho_{u_1}(X) = \left\{ X^* \in (L^p)^* : \mathbb{E}(X^*) = -1, \begin{array}{ll} X^*(\omega) = \gamma_2, & \text{if } X(\omega) < -\bar{\lambda}(X), \\ X^*(\omega) \in [\gamma_2, \gamma_1], & \text{if } X(\omega) = -\bar{\lambda}(X), \\ X^*(\omega) = \gamma_1, & \text{if } X(\omega) > -\bar{\lambda}(X) \end{array} \right\}. \quad (10)$$

When $\gamma_1 = 0$ and $\gamma_2 = -1/\beta$, where $\beta \in (0,1)$, the convex risk measure ρ_{u_1} turns out to be the classical so-called *conditional value-at-risk* (see, for instance, [24, 25]), $\mathrm{CVaR}_\beta : L^p \to \mathbb{R}$,

$$\mathrm{CVaR}_\beta(X) = \inf_{\lambda \in \mathbb{R}} \left\{ \lambda + \frac{1}{\beta} \mathbb{E}[(X + \lambda)_-] \right\}. \quad (11)$$

Thus, for all $X^* \in (L^p)^*$ its conjugate function $\mathrm{CVaR}_\beta^* : (L^p)^* \to \overline{\mathbb{R}}$ looks like

$$\mathrm{CVaR}_\beta^*(X^*) = \begin{cases} 0, & \text{if } -\frac{1}{\beta} \leq X^* \leq 0, \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

For all $X \in L^p$ the element where the infimum in the definition of $\mathrm{CVaR}_\beta(X)$ is attained, is the so-called *value-at-risk of X at level β*,

$$\mathrm{VaR}_\beta(X) = -\inf\{\alpha : \mathbb{P}(X \leq \alpha) > \beta\}.$$

This fact, along with (10), furnishes for all $X \in L^p$ the following formula for the subdifferential of the conditional value-at-risk

$$\partial\, \mathrm{CVaR}_\beta(X) = \left\{ X^* \in (L^p)^* : \mathbb{E}(X^*) = -1, \begin{array}{ll} X^*(\omega) = -1/\beta, & \text{if } X(\omega) < -\mathrm{VaR}_\beta(X), \\ X^*(\omega) \in [-1/\beta, 0], & \text{if } X(\omega) = -\mathrm{VaR}_\beta(X), \\ X^*(\omega) = 0, & \text{if } X(\omega) > -\mathrm{VaR}_\beta(X) \end{array} \right\}.$$

For alternative approaches for deriving the formula of the subdifferential of the conditional value-at-risk we refer to [26, 29].

2.2 Entropic risk measure

Consider the utility function $u_2 : \mathbb{R} \to \mathbb{R}$, $u_2(t) = \exp(-t) - 1$, which obviously fulfills the hypotheses in the Assumption. The convex risk measure we define via u_2 is $\rho_{u_2} : L^p \to \mathbb{R}$, $\rho_{u_2}(X) = \inf_{\lambda \in \mathbb{R}}\{\lambda + \mathbb{E}(\exp(-X - \lambda) - 1)\}$. With the convention $0 \ln(0) = 0$ we have for all $t^* \in \mathbb{R}$ that

$$u_2^*(t^*) = \begin{cases} -t^* \ln(-t^*) + t^* + 1, & \text{if } t^* \leq 0, \\ +\infty, & \text{if } t^* > 0, \end{cases}$$

and, so, from Lemma 3 it follows that for all $X^* \in (L^p)^*$ one has

$$\rho_{u_2}^*(X^*) = \begin{cases} -\mathbb{E}(X^* \ln(-X^*)), & \text{if } X^* < 0,\ \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $(u_2)_\infty = \delta_{[0,+\infty)}$, condition (7) is fulfilled and for all $X \in L^p$ there exists $\bar{\lambda}(X) \in \mathbb{R}$ such that the infimum in the definition of $\rho_{u_2}(X)$ is attained at this point. But in this special case one can easily see that $\bar{\lambda}(X) = \ln(\mathbb{E}(\exp(-X))$ and therefore the risk measure can be equivalently written as $\rho_{u_2}(X) = \ln(\mathbb{E}(\exp(-X)))$. This is the so-called *entropic risk measure* introduced and investigated in [3].

Noticing that $\partial u_2(t) = \{\nabla u_2(t)\} = \{-\exp(-t)\}$ for all $t \in \mathbb{R}$, the subdifferential of the entropic risk measure at $X \in L^p$ is $\partial \rho_{u_2}(X) = \{\nabla \rho_{u_2}(X)\} = \left\{\frac{-1}{\mathbb{E}(\exp(-X))}\exp(-X)\right\}$.

2.3 The worst-case risk measure

By taking as utility function $u_3 = \delta_{[0,+\infty)}$ one rediscovers under $\rho_{u_3} : L^p \to \mathbb{R} \cup \{+\infty\}$,

$$\rho_{u_3}(X) = \inf_{\substack{\lambda \in \mathbb{R} \\ X+\lambda \geq 0}} \lambda = -\operatorname{essinf} X, \tag{12}$$

the so-called *worst-case risk measure*. As $u_3^* = \delta_{(-\infty,0]}$, we have for all $X^* \in (L^p)^*$ that

$$\rho_{u_3}^*(X^*) = \begin{cases} 0, & \text{if } X^* \leq 0,\ \mathbb{E}(X^*) = -1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Noticing that $(u_3)_\infty = \delta_{[0,+\infty)}$, one can easily see that (7) is fulfilled, which means that for all $X \in L^p$ there exists $\bar{\lambda}(X) \in \mathbb{R}$ at which the infimum in (12) is attained. If $\operatorname{essinf} X = -\infty$, then one can take $\bar{\lambda}(X)$ arbitrarily in $\mathbb{R}$, while, when $\operatorname{essinf} X \in \mathbb{R}$, $\bar{\lambda}(X) = -\operatorname{essinf} X$. Since

$$\partial u_3(t) = \begin{cases} \emptyset, & \text{if } t < 0, \\ (-\infty, 0], & \text{if } t = 0, \\ \{0\}, & \text{if } t > 0, \end{cases}$$

we can provide via Theorem 5 the formula for the subdifferential of the worst-case risk measure. Indeed, for $X \in L^p$ with $\operatorname{essinf} X = -\infty$ one has $\partial \rho_{u_3}(X) = \emptyset$, while, if $\operatorname{essinf} X \in \mathbb{R}$, it holds

$$\partial \rho_{u_3}(X) = \left\{ X^* \in (L^p)^* : \mathbb{E}(X^*) = -1, \begin{array}{ll} X^*(\omega) \in (-\infty, 0], & \text{if } X(\omega) = \operatorname{essinf} X, \\ X^*(\omega) = 0, & \text{if } X(\omega) > \operatorname{essinf} X \end{array} \right\}.$$

3 Dual representations of monotone and cash-invariant hulls

Throughout the economical literature one finds a vast variety of risk functions, along the coherent and convex ones some very irregular ones, which are neither monotone nor cash-invariant being present, too. In order to overcome the lack of monotonicity or cash-invariance and to provide better tools for quantifying risk, Filipović and Kupper have proposed in [16] the notions of *monotone* and *cash-invariant hulls*, which are the greatest monotone and, respectively, cash-invariant functions majorized by the risk function in discussion. For the majority of the examples treated in [16] these hulls are not given in their initial formulation, but tacitly some dual representations of them are used.

In this section we show that these dual representations are nothing else than the dual problems of the primal optimization problems hidden in the definition of the monotone and cash-invariant hulls and formulate sufficient qualification conditions for the existence of strong duality. This is the premise for making the dual representations viable. Finally, we discuss the examples from [16] and show that for those particular situations the qualification conditions are automatically fulfilled, fact which permits the formulation of *refined* dual representations.

For the beginning we work in the general setting of a separated locally convex vector space $\mathcal{X}$ with $\mathcal{X}^*$ its topological dual space. Further, let $\mathcal{P}$ be a nonempty convex closed cone in $\mathcal{X}$, $\Pi \in \mathcal{X} \setminus \{0\}$ and $f : \mathcal{X} \to \overline{\mathbb{R}}$ a proper function. The following notions have been introduced in [16] having as a starting point the corresponding ones in the definition of a convex risk measure.

Definition 2 The function f is called:

(i) $\mathcal{P}$-*monotone*, if: $x \geq_{\mathcal{P}} y \Rightarrow f(x) \leq f(y)\ \forall x, y \in \mathcal{X}$;

(ii) Π-*invariant*, if: $f(x + a\Pi) = f(x) - a\ \forall x \in \mathcal{X}\ \forall a \in \mathbb{R}$.

If $\mathcal{X} = L^p$, $\mathcal{P} = L^p_+$ and $\Pi = 1$, then one rediscovers in the definition above the monotonicity and cash-invariance, respectively, as introduced in Definition 1.

Before introducing the following notions we consider the set $\mathcal{D} := \{x^* \in \mathcal{X}^* : \langle x^*, \Pi\rangle = -1\}$ and notice that for the conjugate of the indicator function of $\mathcal{D}$ we have (see, for instance, [16, Lemma 3.3]) for all $x \in \mathcal{X}$ that

$$\delta^*_{\mathcal{D}}(x) = \sup_{x^* \in \mathcal{D}} \langle x^*, x\rangle = \begin{cases} -a, & \text{if } \exists a \in \mathbb{R} \text{ such that } x = a\Pi, \\ +\infty, & \text{otherwise.} \end{cases}$$

This means that $\operatorname{dom} \delta^*_{\mathcal{D}} = \mathbb{R}\Pi := \cup_{a \in \mathbb{R}} a\Pi$.

Definition 3 For the given function f we call

(i) $\mathcal{P}$-*monotone hull* of f the function $f_{\mathcal{P}} : \mathcal{X} \to \overline{\mathbb{R}}$ defined as

$$f_{\mathcal{P}}(x) := f \Box \delta_{\mathcal{P}}(x) = \inf\{f(y) : y \in \mathcal{X}, x \geq_{\mathcal{P}} y\}$$

(ii) Π-*invariant hull* of f the function $f_{\Pi} : \mathcal{X} \to \overline{\mathbb{R}}$ defined as

$$f_{\Pi}(x) := f \Box \delta^*_{\mathcal{D}}(x) = \inf_{a \in \mathbb{R}} \{f(x - a\Pi) - a\}.$$

(iii) $\mathcal{P}$-*monotone* Π-*invariant hull* of f the function $f_{\mathcal{P},\Pi} : \mathcal{X} \to \overline{\mathbb{R}}$ defined as

$$f_{\mathcal{P},\Pi}(x) := f \Box \delta_{\mathcal{P}} \Box \delta^*_{\mathcal{D}}(x) = \inf\{f(y) - a : y \in \mathcal{X}, a \in \mathbb{R}, x \geq_{\mathcal{P}} y + a\Pi\}.$$

Obviously, $\operatorname{dom} f_{\mathcal{P}} = \operatorname{dom} f + \mathcal{P}$, $\operatorname{dom} f_{\Pi} = \operatorname{dom} f + \mathbb{R}\Pi$ and $\operatorname{dom} f_{\mathcal{P},\Pi} = \operatorname{dom} f + \mathcal{P} + \mathbb{R}\Pi$. Moreover, f is $\mathcal{P}$-monotone if and only if $f = f_{\mathcal{P}}$, while f is Π-invariant if and only if $f = f_{\Pi}$.

In the following we assume that f is a proper and convex function and provide a dual representation for $f_{\mathcal{P},\Pi}$ by making use of the convex duality theory. This approach is based on the observation that the value of the $\mathcal{P}$-monotone Π-invariant hull at a given point is nothing else than the optimal objective value of a convex optimization problem.

Theorem 6 *Let* $f : \mathcal{X} \to \overline{\mathbb{R}}$ *be a proper and convex function and* $x \in \operatorname{dom} f + \mathcal{P} + \mathbb{R}\Pi$. *If one of the following qualification conditions*

$$\exists (y', a') \in \operatorname{dom} f \times \mathbb{R} \text{ such that } y' + a'\Pi - x \in -\operatorname{int} \mathcal{P} \tag{13}$$

and

$$\mathcal{X} \text{ is a Fréchet space}, f \text{ is lower semicontinuous and } x \in \operatorname{sqri}(\operatorname{dom} f + \mathbb{R}\Pi + \mathcal{P}) \tag{14}$$

is fulfilled, then one has

$$f_{\mathcal{P},\Pi}(x) = \max_{\substack{x^* \in -\mathcal{P}^* \\ \langle x^*, \Pi \rangle = -1}} \{\langle x^*, x \rangle - f^*(x^*)\}, \tag{15}$$

where by the use of max *instead of* sup *we signalize the fact that the supremum is attained.*

Proof. For the beginning we would like to notice that $f_{\mathcal{P},\Pi}(x)$ is the optimal objective value of the convex optimization problem

$$\inf_{\substack{y \in \mathcal{X}, a \in \mathbb{R} \\ y + a\Pi - x \in -\mathcal{P}}} f(y) - a. \tag{16}$$

Its Lagrange dual problem looks like

$$\sup_{x^* \in \mathcal{P}^*} \inf_{y \in \mathcal{X}, a \in \mathbb{R}} \{f(y) - a + \langle x^*, y + a\Pi - x \rangle\}$$

or, equivalently,

$$\sup_{x^* \in \mathcal{P}^*} \left\{ \inf_{a \in \mathbb{R}} \{a(\langle x^*, \Pi \rangle - 1)\} + \inf_{y \in \mathcal{X}} \{\langle x^*, y \rangle + f(y)\} - \langle x^*, x \rangle \right\}.$$

Since $\inf_{a \in \mathbb{R}} \{a(\langle x^*, \Pi \rangle - 1)\} = -\delta_{\{0\}}(\langle x^*, \Pi \rangle - 1)$, the dual problem becomes

$$\sup_{\substack{x^* \in -\mathcal{P}^* \\ \langle x^*, \Pi \rangle = -1}} \{\langle x^*, x \rangle - f^*(x^*)\}. \tag{17}$$

Since $g : \mathcal{X} \times \mathbb{R} \to \mathcal{X}$, $g(y, a) = y + a\Pi - x$, is an affine and continuous function, condition (13) is nothing else than the Slater constraint qualification (QC_1), while (14) coincides with the qualification condition (QC_3). Consequently, according to Theorem 2, one has strong duality for the primal-dual pair (16)-(17). This means that

$$f_{\mathcal{P},\Pi}(x) = \max_{\substack{x^* \in -\mathcal{P}^* \\ \langle x^*, \Pi \rangle = -1}} \{\langle x^*, x \rangle - f^*(x^*)\}.$$

■

Remark 3 If f is $\mathcal{P}$-monotone, then $f = f \square \delta_{\mathcal{P}}$, which means that $f^* = f^* + \delta_{-\mathcal{P}^*}$. In this situation one would get for $f_{\mathcal{P},\Pi}(x) = f_\Pi(x)$ the following dual representation

$$\sup_{\langle x^*, \Pi \rangle = -1} \{\langle x^*, x \rangle - f^*(x^*)\}. \tag{18}$$

On the other hand, if f is Π-invariant, then $f = f \square \delta^*_{\mathcal{D}}$, which means that $f^* = f^* + \delta_{\mathcal{D}}$. In this situation one would get for $f_{\mathcal{P},\Pi}(x) = f_{\mathcal{P}}(x)$ the following dual representation

$$\sup_{x^* \in -\mathcal{P}^*} \{\langle x^*, x \rangle - f^*(x^*)\}. \tag{19}$$

In the following we investigate the verifiability of the qualification conditions (13) and (14) in the context of risk measure theory, namely by assuming that $\mathcal{X} = L^p$ and $\mathcal{P} = L^p_+$ for $p \in [1, \infty]$. Working in this framework, one can easily see that condition (13) could be a valuable one only when $p = \infty$, since for $p \in [1, \infty)$ the ordering cone L^p_+ has an empty interior; therefore we will concentrate ourselves in the latter situation on condition (14) and assume to this end that f *is lower semicontinuous.* A situation when f fails to have this topological property will be addressed in connection to qualification condition (QC_4) in the next section.

Corollary 7 *For* $p \in [1, \infty]$ *let* $f : L^p \to \overline{\mathbb{R}}$ *be a convex risk function. If one of the following conditions*
• *when* $p \in [1, \infty]$

$$f \text{ is lower semicontinuous and } -L^p_+ \subseteq \operatorname{dom} f; \tag{20}$$

• *when* $p = \infty$

$$\operatorname{essinf} \Pi \cdot \operatorname{essup} \Pi > 0; \tag{21}$$

is fulfilled, then one has for all $X \in L^p$ *that*

$$f_{\mathcal{P},\Pi}(X) = \max_{\substack{X^* \in -(L^p_+)^* \\ \mathbb{E}(X^*\Pi) = -1}} \{\mathbb{E}(X^*X) - f^*(X^*)\},$$

where by the use of max *instead of* sup *we signalize the fact that the supremum is attained.*

Proof. For $p \in [1, \infty]$, if $-L^p_+ \subseteq \operatorname{dom} f$, then one has $\operatorname{dom} f + \mathbb{R}\Pi + L^p_+ = L^p$, meaning that (14) is valid for all $X \in L^p$. The conclusion follows via Theorem 6.

Take now $p = \infty$ and assume that $\operatorname{essinf} \Pi \cdot \operatorname{essup} \Pi > 0$. Obviously, $\operatorname{dom} f + \mathbb{R}\Pi + L^\infty_+ \subseteq L^\infty$. As f is proper we can choose an element $Y \in \operatorname{dom} f$. Thus, for all $X \in L^\infty$ there exists $a \in \mathbb{R}$ such that $X - Y - a\Pi \in L^\infty_+$. This is because of the assumption made on Π, which actually means that either $\operatorname{essup} \Pi \geq \operatorname{essinf} \Pi > 0$ or $0 > \operatorname{essup} \Pi \geq \operatorname{essinf} \Pi$. Consequently, $\operatorname{dom} f + \mathbb{R}\Pi + L^\infty_+ = L^\infty$.

We consider an arbitrary element $X \in L^\infty = \operatorname{dom} f + \mathbb{R}\Pi + L^\infty_+$. Thus there exist $Y' \in \operatorname{dom} f$ and $a \in \mathbb{R}$ such that $X - Y' - a\Pi \in L^\infty_+$. Again, the condition (21) ensures the existence of $a' \in \mathbb{R}$ such that $X - Y' - a'\Pi \in \operatorname{int} L^\infty_+ = \{Z \in L^\infty : \operatorname{essinf} Z > 0\}$, meaning that (13) is fulfilled. The conclusion follows also in this case via Theorem 6. ■

Remark 4 One can notice that for $p = \infty$ the condition (21) in the theorem above is fulfilled when $\Pi \in L^\infty$ is a *constant numeraire.*

In the last part of this section we discuss the examples treated in [16] from this new perspective given by the duality theory, investigate the fulfillment of the conditions (20) and (21) and provide some refined dual representations for the risk functions in discussion. We will use the notion *monotone* for L^p_+*-monotone* and *cash-invariant* for 1*-invariant.* The same applies when we speak about the corresponding hulls.

Example 1 For $p \in [1, \infty)$ and $c > 0$ consider the L^p *deviation risk measure* $f : L^p \to \mathbb{R}$ defined by $f(X) = c\|X - \mathbb{E}(X)\|_p - \mathbb{E}(X)$. This is a convex, continuous and cash-invariant

($\Pi = 1$) risk function, but not monotone in general. For the conjugate formula of the L^p deviation risk measure we refer to [12]. This is for $X^* \in L^q$ given by

$$f^*(X^*) = \begin{cases} 0, & \text{if } \exists Y^* \in L^q \text{ such that } c(Y^* - \mathbb{E}(Y^*)) - 1 = X^*,\ \|Y^*\|_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

As $\operatorname{dom} f = L^p$, (20) is valid and thus the *monotone hull* of f looks for all $X \in L^p$ like (see also Remark 3)

$$f_{L^p_+,1}(X) = f_{L^p_+}(X) = \max_{\substack{\|Y^*\|_q \leq 1 \\ c(Y^* - \mathbb{E}(Y^*)) \leq 1}} c[\mathbb{E}(Y^*)\mathbb{E}(X) - \mathbb{E}(Y^*X)] - \mathbb{E}(X).$$

In this way we rediscover the formula given in [16, Subsection 5.1].

Example 2 Closely related to previous example we consider for $p \in [1, \infty)$ and $c > 1$ the *L^p semi-deviation risk measure* $f : L^p \to \mathbb{R}$ defined as $f(X) = c\|(X - \mathbb{E}(X))_-\|_p - \mathbb{E}(X)$. This is a convex, continuous and cash-invariant ($\Pi = 1$) risk function, but not monotone in general. For its conjugate function we have for $X^* \in L^q$ the following formula (see [12])

$$f^*(X^*) = \begin{cases} 0, & \text{if } \exists Y^* \in -L^q_+ \text{ such that } c(Y^* - \mathbb{E}(Y^*)) - 1 = X^*,\ \|Y^*\|_q \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

Consequently, since (20) is valid, the *monotone hull* of f is for all $X \in L^p$ given by (see also [16, Subsection 5.2])

$$f_{L^p_+,1}(X) = f_{L^p_+}(X) = \max_{\substack{Y^* \in L^q_+, \|Y^*\|_q \leq 1 \\ c(Y^* - \mathbb{E}(Y^*)) \leq 1}} c[\mathbb{E}(Y^*)\mathbb{E}(X) - \mathbb{E}(Y^*X)] - \mathbb{E}(X).$$

Example 3 For $p \in [1, \infty)$ and $c > 0$ consider the *mean-L^p risk measure* $f : L^p \to \mathbb{R}$ defined as $f(X) = c/p\mathbb{E}(|X|^p) - \mathbb{E}(X) = c/p\|X\|_p^p - \mathbb{E}(X)$, which is a convex and continuous risk function but neither monotone nor cash-invariant ($\Pi = 1$). Its conjugate function can be easily derived from [12] and for $X^* \in L^q$ it looks like

$$f^*(X^*) = \frac{p-1}{pc^{\frac{1}{p-1}}}\mathbb{E}(|X^* + 1|^q).$$

Again, $\operatorname{dom} f = L^p$, which means that the *monotone cash-invariant hull* of f has for all $X \in L^p$ the following formulation

$$f_{L^p_+,1}(X) = \max_{\substack{X^* \in -L^q_+ \\ \mathbb{E}(X^*) = -1}} \mathbb{E}\left[X^*X - \frac{1}{c^{q-1}q}|X^* + 1|^q\right].$$

Different to the approach in [16, Subsection 5.3], the use of the strong duality theory allows us to guarantee the attainment of the supremum in the formula above.

Example 4 For $p \in [1, \infty)$ and $c > 0$ consider now the *L^p semi-moment risk measure* $f : L^p \to \mathbb{R}$ defined as $f(X) = 1/c\mathbb{E}[(X_-)^p] = 1/c\|X_-\|_p^p$, which is a convex, continuous and monotone risk function, but not cash-invariant ($\Pi = 1$). Its conjugate function is for $X^* \in L^q$ given by (see [12])

$$f^*(X^*) = \begin{cases} \frac{p-1}{c} \left\| \frac{c}{p} X^* \right\|_q^q, & \text{if } X^* \in -L^q_+, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $\operatorname{dom} f = L^p$, the *cash-invariant hull* of f has for all $X \in L^p$ the following formulation (see also Remark 3)

$$f_{L^p_+,1}(X) = f_1(X) = \max_{\substack{X^* \in -L^q_+ \\ \mathbb{E}(X^*)=-1}} \left\{ \mathbb{E}(X^* X) - \frac{p-1}{c} \left\| \frac{c}{p} X^* \right\|_q^q \right\}.$$

Example 5 The next risk function we consider is the *exponential risk measure* defined for $p \in [1, \infty]$ as being $f : L^p \to \mathbb{R}$, $f(X) = \mathbb{E}(\exp(-X)) - 1$. This is a convex, continuous and monotone, but not cash-invariant ($\Pi = 1$) risk function. The conjugate function of f is for $X^* \in (L^p)^*$ given by

$$f^*(X^*) = \sup_{X \in L^p} \{\langle X^*, X \rangle - \mathbb{E}(\exp(-X)) + 1\},$$

which by the interchangeability of minimization and integration (see [28, Theorem 14.60]) becomes (we make use again of the convention $0 \ln(0) = 0$)

$$\mathbb{E}\left\{ \sup_{x \in \mathbb{R}} \{X^* x - \exp(-x) + 1\} \right\} = \begin{cases} \mathbb{E}[-X^* \ln(-X^*) + X^*] + 1, & \text{if } X^* \leq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Consequently, one obtains via Remark 3 for all $X \in L^p$ the following representation for the *cash-invariant hull* of f

$$f_{L^p_+,1}(X) = f_1(X) = \max_{\substack{X^* \in (L^p_+)^* \\ \mathbb{E}(X^*)=1}} \mathbb{E}[-X^* X - X^* \ln(X^*)].$$

Example 6 For $p = \infty$ the so-called *logarithmic risk measure* $f : L^\infty \to \overline{\mathbb{R}}$,

$$f(X) = \begin{cases} \mathbb{E}(-\ln(X)) - 1, & \text{if } X > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

is a convex, lower semicontinuous and monotone risk function which fails to be cash-invariant ($\Pi = 1$). Its conjugate function is given for $X^* \in (L^\infty)^*$ by

$$f^*(X^*) = \sup_{X > 0} \{\langle X^*, X \rangle + \mathbb{E}(\ln(X) + 1)\}$$

and can be further calculated by using [28, Theorem 14.60]. Indeed, one has

$$f^*(X^*) = \mathbb{E}\left\{ \sup_{x > 0} \{X^* x + \ln(x) + 1\} \right\} = \begin{cases} -\mathbb{E}(\ln(-X^*)), & \text{if } X^* < 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Before giving a dual representation for the cash-invariant hull of the logarithmic risk measure, one should notice that we are now in a situation where (20) fails, but (21) is valid. Consequently, the *cash-invariant hull* of f can be for all $X \in L^\infty$ given by

$$f_{L^\infty_+,1}(X) = f_1(X) = \max_{\substack{X^* \in (L^\infty)^*, X^* > 0 \\ \mathbb{E}(X^*)=1}} \mathbb{E}[-X^* X + \ln(X^*)].$$

Remark 5 The cash-invariant hull of a monotone and convex risk function has been object of investigation in the papers of Cheridito and Li [14, 15]. The two authors work there with risk measures on *Orlicz hearts*, which contain the L^p spaces, $p \in [1, \infty]$, as particular instances, for which they provide, among others, dual representations. The latter make use of a so-called *penalty function*, a notion which is strongly connected to the conjugate of the convex measure in discussion. Conditions for attainment of the infimum in the definition of the cash-invariant hull a risk function are given. For several classes of monotone and convex risk functions dual representations for their cash-invariant hulls are also discussed. It could be of particular interest to find out if the investigations done in this paper can be extended to the general setting considered in [14, 15].

4 The situation of missing lower semicontinuity

In the following we deal with the same problem of furnishing dual representations for the monotone and cash-invariant hull of a convex risk function by using the duality approach developed in Section 3, treating the particular case of a risk function which *fails to be lower semicontinuous.* We also discuss the difficulties which can arise when this topological assumption is missing.

For $p \in [1, \infty]$ consider $f : L^p \to \overline{\mathbb{R}}$ defined by

$$f(X) = \begin{cases} \|X - \mathbb{E}(X)\|_p, & \text{if } X_- \in L^\infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

This risk function is convex and fails to be lower semicontinuous for $p \in [1, \infty)$. One can easily verify that $\operatorname{dom} f = L^\infty + L^p_+$.

Like in the previous section we take as ordering cone L^p_+, but work with a *not necessarily constant numeraire* $\Pi \in L^p \setminus \{0\}$. Our goal is to furnish a dual representation for the *monotone* Π*-invariant hull* of f. To this end we will make use of the conjugate formula of $Y \mapsto \|Y - \mathbb{E}(Y)\|_p$, $p \in [1, \infty]$, which looks for $X^* \in (L^p)^*$ like (see [12, Fact 4.3])

$$(\|\cdot - \mathbb{E}(\cdot)\|_p)^*(X^*) = \begin{cases} 0, & \text{if } \exists Y^* \in (L^p)^*, \|Y^*\|_{(L^p)^*} \leq 1, \text{ such that } X^* = Y^* - \mathbb{E}(Y^*), \\ +\infty, & \text{otherwise.} \end{cases} \tag{22}$$

The case $p = \infty$. In this situation $\operatorname{dom} f = L^\infty$, f is a convex and *continuous* function and one can, consequently, use the qualification condition (20), which is obviously fulfilled. Thus for the monotone Π-invariant hull of f one can employ again formula (15). This means that, by taking into consideration (22), the *monotone* Π*-invariant hull* of f looks for all $X \in L^\infty$ like

$$f_{L^\infty_+, \Pi}(X) = \max_{\substack{\|Y^*\|_{(L^\infty)^*} \leq 1, \mathbb{E}(Y^*) - Y^* \in (L^\infty_+)^* \\ \mathbb{E}(Y^*\Pi) - \mathbb{E}(Y^*)\mathbb{E}(\Pi) + 1 = 0}} \mathbb{E}(Y^* X) - \mathbb{E}(Y^*)\mathbb{E}(X). \tag{23}$$

One can easily notice that if Π is a *constant numeraire*, then $f_{L^\infty_+, \Pi} \equiv -\infty$.

The case $p \in [1, \infty)$. In this second case we will proceed as follows: we first establish the monotone hull of f, along with a dual representation for it, then we discuss which are the difficulties that appear when trying to determine the dual representation of $f_{L^p_+, \Pi}$. Recall that $f_{L^p_+, \Pi}(X) = (f_{L^p_+})_\Pi(X)$ for all $X \in L^p$. In this setting we denote the dual space of L^p with L^q, $q = p/(p-1)$ (with the convention $1/0 = \infty$) and the same applies for the corresponding norm.

As $\operatorname{dom} f_{L^p_+} = \operatorname{dom} f + L^p_+ = L^\infty + L^p_+$, for every X outside this set one has $f_{L^p_+}(X) = +\infty$. For $X \in L^\infty + L^p_+$ we have

$$f_{L^p_+}(X) = \inf_{\substack{Y \in L^\infty + L^p_+ \\ Y - X \in -L^p_+}} \|Y - \mathbb{E}(Y)\|_p \tag{24}$$

and, obviously, $f_{L^p_+}(X) \geq 0$. On the other hand, since $X = Z + Y$ for $Z \in L^\infty$ and $Y \in L^P_+$, it holds $X \geq \operatorname{essinf} Z$, thus $\operatorname{essinf} Z$ is feasible for the optimization problem in the right-hand side of (24), which means that $f_{L^p_+}(X) = 0$. Consequently, $f_{L^p_+} = \delta_{L^\infty + L^p_+}$.

Before furnishing the monotone Π-invariant hull of f, let us shortly investigate how one could give dual representation for $f_{L^p_+}$. For $X \in L^\infty + L^p_+$ fixed one has to consider the convex optimization problem

$$\inf_{\substack{Y \in L^\infty + L^p_+ \\ Y - X \in -L^p_+}} \|Y - \mathbb{E}(Y)\|_p \tag{25}$$

and its *Lagrange dual problem* (notice that L^∞ is dense in L^p)

$$\sup_{X^* \in L^q_+} \inf_{Y \in L^\infty + L^p_+} \{\|Y - \mathbb{E}(Y)\|_p + \langle X^*, Y - X\rangle\} = \sup_{X^* \in L^q_+} \inf_{Y \in L^p} \{\|Y - \mathbb{E}(Y)\|_p + \langle X^*, Y - X\rangle\}$$

or, equivalently,

$$\sup_{X^* \in -L^q_+} \{\langle X^*, X\rangle - (\|\cdot - \mathbb{E}(\cdot)\|_p)^*(X^*)\} = \sup_{\|Y^*\|_q \leq 1, \mathbb{E}(Y^*) - Y^* \in L^q_+} \mathbb{E}(Y^* X) - \mathbb{E}(Y^*)\mathbb{E}(X). \tag{26}$$

In order to show that for the primal-dual pair (25)-(26) strong duality holds, one needs to use the quasi-relative interior-type condition (QC_4). Indeed, (25) is of the same type as the problem (P) from the preliminary section, when taking $\mathcal{X} = \mathcal{Z} = S = L^p$, $K = L^p_+$, $g : L^p \to L^p$, $g(Y) = Y - X$ and having as objective function $f : L^p \to \overline{\mathbb{R}}$,

$$f(Y) = \begin{cases} \|Y - \mathbb{E}(Y)\|_p, & \text{if } Y_- \in L^\infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $X \in L^\infty + L^p_+$, for $X' := X - 1 \in \operatorname{dom} f$ one has $g(X') \in -\operatorname{qri} L^p_+$ (see [7]), while obviously $\operatorname{cl}(L^p_+ - L^p_+) = L^p$. More than that, as $(g(\operatorname{dom} f \cap S) + L^p_+) - (g(\operatorname{dom} f \cap S) + L^p_+) = L^p$ and the optimal objective value of (25) is $f_{L^p_+}(X) = 0$, in order to show that (QC_4) is verified, it is enough to prove that $(0,0) \notin \operatorname{qi}(\mathcal{E}_{f_{L^p_+}(X)})$, where

$$\mathcal{E}_{f_{L^p_+}(X)} = \{(\|Y - \mathbb{E}(Y)\|_p + \epsilon, Y - X + Z) : Y \in L^\infty + L^p_+, Z \in L^p_+, \epsilon \geq 0\}.$$

Indeed, $(-1, 0) \in N_{\mathcal{E}_{f_{L^p_+}(X)}}(0,0)$ and via Proposition 1(ii) we get the desired conclusion. The qualification condition being verified it follows that

$$f_{L^p_+}(X) = \max_{\|Y^*\|_q \leq 1, \mathbb{E}(Y^*) - Y^* \in L^q_+} \mathbb{E}(Y^* X) - \mathbb{E}(Y^*)\mathbb{E}(X)$$

and so one obtains for the monotone hull of f for all $X \in L^p$ the following dual representation

$$f_{L^p_+}(X) = \begin{cases} \max\limits_{\|Y^*\|_q \le 1, \mathbb{E}(Y^*) - Y^* \in L^q_+} \mathbb{E}(Y^*X) - \mathbb{E}(Y^*)\mathbb{E}(X), & \text{if } X \in L^\infty + L^p_+, \\ +\infty, & \text{otherwise.} \end{cases}$$

The *monotone* Π-*invariant hull* of f is the Π-invariant hull of $f_{L^p_+}$ and for its derivation we use the direct formulation of the latter, $f_{L^p_+} = \delta_{L^\infty + L^p_+}$, as it is easier to handle with. For all $X \in L^P$ the monotone Π-invariant hull of f is

$$f_{L^p_+,\Pi}(X) = \inf_{a \in \mathbb{R}} \{f_{L^p_+}(X - a\Pi) - a\} = \inf_{\substack{(Y,a) \in (L^\infty + L^p_+) \times \mathbb{R} \\ Y + a\Pi - X = 0}} -a.$$

Since $f_{L^p_+,\Pi}$ is the optimal objective value of a convex optimization problem, it is natural to ask if a dual formulation for it, via the duality theory, can be provided. Unfortunately, *we are not always able to answer this question.* What we can say is, that for $X \in L^\infty + L^p_+ + \mathbb{R}\Pi = \operatorname{dom} f_{L^p_+,\Pi}$ it holds $f_{L^p_+,\Pi}(X) = +\infty$. For $X \notin L^\infty + L^p_+ + \mathbb{R}\Pi$ one get as *Lagrange dual problem* to

$$\inf_{\substack{(Y,a) \in (L^\infty + L^p_+) \times \mathbb{R} \\ Y + a\Pi - X = 0}} -a \tag{27}$$

the following optimization problem

$$\sup_{X^* \in L^q} \inf_{(Y,a) \in (L^\infty + L^p_+) \times \mathbb{R}} [-a + \langle X^*, Y + a\Pi - X \rangle],$$

which, since L^∞ is dense in L^p, is nothing else than

$$\sup_{X^* \in L^q} \left[-\langle X^*, X \rangle + \inf_{a \in \mathbb{R}} a(\langle X^*, \Pi \rangle - 1) + \inf_{Y \in L^p} \langle X^*, Y \rangle \right] = -\infty \tag{28}$$

Nevertheless, we cannot be sure that this is the value which $f_{L^p_+,\Pi}(X)$ takes, since no known qualification condition can be verified for (27)-(28). This applies as well as for the classical generalized interior ones ($L^\infty + L^p_+$ is not closed) as for the one of quasi-relative interior-type. This emphasizes the fact that one can have exceptional situations for which the approach we use is, unfortunately, not suitable.

Let us also mention that whenever $\Pi \in L^\infty$ (which includes the situation when Π is a *constant numeraire*), then for all $a \in \mathbb{R}$ there exists $Y \in L^\infty + L^p_+$ such that $X = a\Pi + Y$ and so $f_{L^p_+,\Pi}(X) = -\infty$. In this case we have for all $X \in L^p$

$$f_{L^p_+,\Pi}(X) = \begin{cases} -\infty, & \text{if } X \in L^\infty + L^p_+ + \mathbb{R}\Pi, \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark 6 The fact that $L^\infty + L^p_+$ is not closed does not make the applicability of the other main class of qualification conditions, the *closedness-type* ones, for the convex optimization problem in (27) possible, too.

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