An upper estimate for the Clarke subdifferential of an infimal value function proved via the Mordukhovich subdifferential

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Abstract. The aim of this note is to give an alternative proof for a recent result due to Dorsch, Jongen and Shikhman, which provides an upper estimate for the Clarke subdifferential of an infimal value function. We show the validity of this result under a weaker condition than the one assumed in the mentioned paper, while the use of the Mordukhovich subdifferential, as an intermediate step, will considerably shorten its proof.

Key Words. infimal value function, Clarke subdifferential, Mordukhovich subdifferential

AMS subject classification. 49J52, 90C26, 90C56

1 Motivation and preliminary results

The setting we work within in this article, which is in fact the one considered in [2], is the following. Let $g_0, ..., g_s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be real-valued and continuously differentiable functions and $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be defined as

$$\varphi(x) := \inf_{y \in \mathbb{R}^m} \max_{0 \le k \le s} g_k(x, y).$$

The topological structure of the upper-level set

$$M^{\max} := \{x \in \mathbb{R}^n : \varphi(x) \ge 0\}$$

of the *infimal value function* φ is of particular interest in the study of generalized semi-infinite optimization problems (cf. [3]).

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Further, let be

$$\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \sigma(x,y) := \max_{0 \le k \le s} g_k(x,y),$$

the set-valued operator

$$M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m, M(x) := \{y \in \mathbb{R}^m : \sigma(x,y) = \varphi(x)\}$$

and for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ the following set of indices

$$K(x,y) := \{k \in \{0,...,s\} : g_k(x,y) = \sigma(x,y)\}.$$

For $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ we consider (cf. [3]) the convex and compact set

$$V(x,y) := \left\{ \sum_{k \in K(x,y)} \mu_k D_x g_k(x,y) \;\middle|\; \begin{array}{r} \sum\limits_{k \in K(x,y)} \mu_k D_y g_k(x,y) = 0, \\ \sum\limits_{k \in K(x,y)} \mu_k = 1, \\ \mu_k \ge 0 \ \forall k \in K(x,y) \end{array} \right\}.$$

Here, for a function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $D_x g$ and $D_y g$ denote the *gradients* of g with respect to the variables x, respectively, y. Further, let be

$$V : \mathbb{R}^n \rightrightarrows \mathbb{R}^m, V(x) := \bigcup_{y \in M(x)} V(x,y).$$

The following condition was introduced in [2].

Compactness Condition CC. *One says that the Compactness Condition CC is fulfilled, if for all sequences $(x_k, y_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ with*

$$\left\{ \begin{array}{ll} \bullet & x_k \to x \in \mathbb{R}^n \ (k \to \infty) \\ \bullet & \text{either } \sigma(x_k, y_k) \to a \ (k \to \infty) \text{ and } a \le \varphi(x) \\ & \text{or } \sigma(x_k, y_k) \to -\infty \ (k \to \infty) \end{array} \right.$$

the sequence $(y_k)_{k \in \mathbb{N}}$ contains a convergent subsequence.

One of the main results of [2] is represented by the following upper estimate for the *Clarke subdifferential* of the function φ.

Theorem 1 *Let the Compactness Condition CC be fulfilled and let $\bar{x} \in \mathbb{R}^n$. Then it holds*

$$\partial^C \varphi(\bar{x}) \subseteq \operatorname{conv} V(\bar{x}). \tag{1}$$

In the above result $\partial^C \varphi(\bar{x})$ denotes the Clarke subdifferential of φ at $\bar{x}$, while $\operatorname{conv} V(\bar{x})$ is the convex hull of the set $V(\bar{x})$.

The proof given in [2] for this result is quite involved and makes use of some characterizations of the Clarke subdifferential from [1]. We will give in this note an alternative proof for the above inclusion under a weaker assumption than the Compactness Condition CC, by using as intermediate tool the *Mordukhovich subdifferential*. This proof will allow us to point out which are the difficulties one has to face when trying to discuss the situation when the inclusion in Theorem 1 becomes an equality.

The condition which will turn out to be sufficient for (1) was given in [2], too, and has the following formulation.

Condition C*. *One says that Condition C* is fulfilled if*

- $(C1)$ *for all $x \in \mathbb{R}^n$ and sequences $(y_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^m$ with $\sigma(x, y_k) \to \varphi(x)$ $(k \to \infty)$ there exists a convergent subsequence of $(y_k)_{k\in\mathbb{N}}$ and*
- $(C2)$ *the mapping $x \rightrightarrows M(x)$ is locally bounded, i.e., for all $\bar{x} \in \mathbb{R}^n$ there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ such that $\bigcup_{x\in U_{\bar{x}}} M(x)$ is bounded.*

According to [2, Lemma 2.1], Condition C* guarantees the following local description of φ:

for every $\bar{x} \in \mathbb{R}^n$ there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ and a compact set $W \subseteq \mathbb{R}^m$ such that

$$\varphi(x) = \min_{y\in W} \sigma(x, y) \ \forall x \in U_{\bar{x}}. \tag{2}$$

As proved in [2], Condition C* is implied by the Compactness Condition CC and the two conditions are not equivalent. However, according to [2], Condition C* is not stable with respect to C^0-perturbations of the functions involved, which is not the case for the Compactness Condition CC. Nevertheless, for guaranteeing (1), one does not necessarily have to assume that the latter is fulfilled, as we will prove in the next section. To this end, we need several notions and results, which we introduce in the following.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function with a nonempty *effective domain* $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The *epigraph* of f is the set $\operatorname{epi} f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$. We say that f is *lower semicontinuous around* $\bar{x} \in \operatorname{dom} f$ if there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ such that f is lower semicontinuous at x for all $x \in U_{\bar{x}}$. We say that f is *locally Lipschitzian around* $\bar{x} \in \operatorname{dom} f$ if there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ and a real number $L > 0$ such that $|f(x) - f(y)| \leq L\|x - y\|$ for all $x, y \in U_{\bar{x}}$.

For $\varepsilon \geq 0$ the *Fréchet ε-subdifferential* (or the *analytic ε-subdifferential*) of f at $\bar{x} \in \operatorname{dom} f$ is defined by

$$\partial^F_\varepsilon f(\bar{x}) := \left\{ x^* \in X^* : \liminf_{\|h\|\to 0} \frac{f(\bar{x} + h) - f(\bar{x}) - \langle x^*, h\rangle}{\|h\|} \geq -\varepsilon \right\},$$

while for $\overline{x} \notin \operatorname{dom} f$ we set $\partial^F_\varepsilon f(\overline{x}) := \emptyset$. Further, $\partial^F f(\bar{x}) := \partial^F_0 f(\bar{x})$ denotes the classical *Fréchet subdifferential* of f at $\bar{x}$.

For a given function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, which is locally Lipschitzian around $\bar{x} \in \operatorname{dom} f$, $f^\circ(\bar{x}; \cdot) : \mathbb{R}^n \to \mathbb{R}$, defined by

$$f^\circ(\bar{x}; d) = \limsup_{\substack{x \to \bar{x} \\ t \downarrow 0}} \frac{f(x + td) - f(x)}{t},$$

denotes its *Clarke generalized derivative*, while

$$\partial^C f(\bar{x}) := \{v \in \mathbb{R}^n : v^T d \leq f^\circ(\bar{x}; d) \ \forall d \in \mathbb{R}^n\}$$

stays for the *Clarke subdifferential* (cf. [1]) of f at $\bar{x}$.

For a set $S \subseteq \mathbb{R}^n$ we denote by $\delta_S : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$,

$$\delta_S(x) = \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{otherwise,} \end{cases}$$

the *indicator function* of S and by $\operatorname{dist}_S : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $\operatorname{dist}_S(x) := \inf_{s \in S} \|x - s\|$, the *distance function* to S. For $\bar{x} \in S$ the *Clarke normal cone* to S at $\bar{x}$ is defined as

$$N_C(S; \bar{x}) := \operatorname{cl}\left(\bigcup_{\lambda > 0} \lambda \partial^C \operatorname{dist}_S(\bar{x}) \right),$$

while for $\bar{x} \notin S$ one takes $N_C(S; \bar{x}) := \emptyset$.

We can now define the *Clarke subdifferential* of an arbitrary function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $\bar{x} \in \operatorname{dom} f$, as being

$$\partial^C f(\bar{x}) := \{v \in \mathbb{R}^n : (v, -1) \in N_C(\operatorname{epi} f; (\bar{x}, f(\bar{x})))\},$$

while for $\bar{x} \notin \operatorname{dom} f$ we take $\partial^C f(\bar{x}) := \emptyset$.

Having a set-valued operator $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ we define its *graph* as being

$$\operatorname{gph} F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in F(x)\},$$

while its *domain* is the set

$$\operatorname{dom} F := \{x \in \mathbb{R}^n : F(x) \neq \emptyset\}.$$

For $\bar{x} \in \operatorname{dom} F$ we say that F is *closed-graph at* $\bar{x}$ if $\bar{y} \in F(\bar{x})$ whenever $x_k \to \bar{x}$ $(k \to \infty)$ and $y_k \to \bar{y}$ $(k \to \infty)$ with $y_k \in F(x_k)$ for all $k \geq 1$. Further, one says that F is *inner semicompact* at $\bar{x}$ (cf. [5]) if for every sequence $x_k \to \bar{x}$ $(k \to \infty)$ there is a sequence $y_k \in F(x_k)$ for all $k \geq 1$ that contains a convergent subsequence as $k \to \infty$. We also recall here the *Painlevé-Kuratowski upper/outer limit* of F at a point $\bar{x} \in \mathbb{R}^n$, which is the set

$$\operatorname*{Limsup}_{x \to \bar{x}} F(x) := \{\bar{y} \in \mathbb{R}^m : \ \exists \text{ sequences } x_k \to \bar{x} \ (k \to \infty) \text{ and } y_k \to \bar{y} \ (k \to \infty) \text{ with } y_k \in F(x_k) \text{ for all } k \geq 1\}.$$

By means of the Painlevé-Kuratowski upper/outer limit we can introduce a further subdifferential notion.

For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $\bar{x} \in \operatorname{dom} f$ we denote by

$$\partial^M f(\bar{x}) := \operatorname*{Limsup}_{\substack{x \xrightarrow{f} \bar{x} \\ \varepsilon \downarrow 0}} \partial^F_\varepsilon f(x) \tag{3}$$

the *basic/limiting/Mordukhovich subdifferential* of f at $\bar{x}$ (cf. [5, 6]) and we put $\partial^M f(\bar{x}) := \emptyset$ when $\bar{x} \notin \operatorname{dom} f$. Here the symbol $x \xrightarrow{f} \bar{x}$ means that $x \to \bar{x}$ with $f(x) \to f(\bar{x})$. Whenever f is lower semicontinuous around $\bar{x} \in \operatorname{dom} f$ one can equivalently put $\varepsilon = 0$ in (3). Further, one has for all $\bar{x} \in \mathbb{R}^n$ the following relation between the three subdifferential notions introduced above

$$\partial^F f(\bar{x}) \subseteq \partial^M f(\bar{x}) \subseteq \partial^C f(\bar{x}),$$

the inclusions being in general strict.

Given a set $S \subseteq X$ and $\bar{x} \in S$ the *basic/limiting/Mordukhovich normal cone* to S at $\bar{x}$ is defined as

$$N_M(S; \bar{x}) := \partial^M \delta_S(\bar{x}),$$

while for $\bar{x} \notin S$ one takes $N_M(S; \bar{x}) := \emptyset$. Finally, for a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $\bar{x} \in \operatorname{dom} f$ we denote by

$$\partial^\infty f(\bar{x}) := \{v \in \mathbb{R}^n : (v, 0) \in N_M(\operatorname{epi} f; (\bar{x}, f(\bar{x})))\} \tag{4}$$

the *singular subdifferential* of f at $\bar{x}$ (cf. [5]) and we put $\partial^\infty f(\bar{x}) := \emptyset$ for $\bar{x} \notin \operatorname{dom} f$. For $\bar{x} \in \operatorname{dom} f$ one always has that $0 \in \partial^\infty f(\bar{x})$. If f is locally Lipschitzian around $\bar{x} \in \operatorname{dom} f$, then it holds $\partial^\infty f(\bar{x}) = \{0\}$ (cf. [5, Corollary 1.81]).

By making use of the last two subdifferentials introduced in this first section, one can deliver the following useful characterization of the Clarke subdifferential of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $\bar{x} \in \operatorname{dom} f$ (see [5, Theorem 3.57 (b)]), which closes this section.

Theorem 2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous around $\bar{x} \in \operatorname{dom} f$. Then it holds*

$$\partial^C f(\bar{x}) = \operatorname{cl} \operatorname{conv} \left[\partial^M f(\bar{x}) + \partial^\infty f(\bar{x}) \right]. \tag{5}$$

2 The main result

The main result of this note states the existence of the upper estimate in (1) under the weak assumption Condition C*. The proof we give for it considerably shortens the one given for the same result in [2], one of its essential ingredients being the following result taken from [5, Theorem 1.108].

Theorem 3 *Let $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a given function, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ a set-valued operator, the infimal value function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, $\varphi(x) = \inf_{y \in G(x)} \sigma(x, y)$, and the argminimum mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m, M(x) = \{y \in G(x) : \sigma(x, y) = \varphi(x)\}$. For $\bar{x} \in \mathbb{R}^n$ we assume that $\varphi(\bar{x}) \in \mathbb{R}$, $M(\bar{x}) \neq \emptyset$, M is inner semicompact at $\bar{x}$, G is closed-graph at $\bar{x}$ and σ is lower semicontinuous on* $\operatorname{gph} G$. *Then one has*

$$\partial^M \varphi(\bar{x}) \subseteq \left\{ v \in \mathbb{R}^n : (v, 0) \in \bigcup_{y \in M(\bar{x})} \partial^M (\sigma + \delta_{\operatorname{gph} G})(\bar{x}, y) \right\} \tag{6}$$

and

$$\partial^\infty \varphi(\bar{x}) \subseteq \left\{ v \in \mathbb{R}^n : (v, 0) \in \bigcup_{y \in M(\bar{x})} \partial^\infty (\sigma + \delta_{\operatorname{gph} G})(\bar{x}, y) \right\}. \tag{7}$$

The main result follows.

Theorem 4 *Let the Condition C^* be fulfilled and let $\bar{x} \in \mathbb{R}^n$. Then it holds*

$$\partial^C \varphi(\bar{x}) \subseteq \operatorname{conv} V(\bar{x}). \tag{8}$$

Proof. We notice first that σ is a continuous function. Due to the local description of φ, there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ and a compact set $W \subseteq \mathbb{R}^m$ such that (cf. (2))

$$\varphi(x) = \min_{y \in W} \sigma(x, y) \ \forall x \in U_{\bar{x}},$$

which implies that $\varphi(\bar{x}) = \min_{y \in W} \sigma(\bar{x}, y)$. Thus there exists $\bar{y} \in W$ with $\varphi(\bar{x}) = \sigma(\bar{x}, \bar{y})$. Consequently, $\varphi(\bar{x}) \in \mathbb{R}$ and, as $\bar{x}$ was arbitrarily chosen, one has $\operatorname{dom} \varphi = \mathbb{R}^n$. Next we show that, for $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ taken such that $\operatorname{gph} G = \mathbb{R}^n \times \mathbb{R}^m$, the hypotheses of Theorem 3 are fulfilled. Obviously, G is closed-graph at $\bar{x}$ and σ is lower semicontinuous on $\operatorname{gph} G$. Further, $\bar{y} \in M(\bar{x})$, where $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m, M(x) = \{y \in \mathbb{R}^m : \sigma(x, y) = \varphi(x)\}$. It remains to prove that M is inner semicompact at $\bar{x}$.

Indeed, consider an arbitrary sequence x_k converging to $\bar{x}$ when $k \to \infty$. Without loss of generality we can assume that $x_k \in U_{\bar{x}}$ for all $k \geq 1$. Thus there exists $y_k \in W$ such that $\varphi(x_k) = \sigma(x_k, y_k)$ or, equivalenlty, $y_k \in M(x_k)$ for all $k \geq 1$. Since W is compact, y_k contains a convergent subsequence $(y_{k_l})_{l \in \mathbb{N}}$ as $l \to \infty$ and this provides the inner semicompactness of M at $\bar{x}$. Now, according to Theorem 3, one has

$$\partial^M \varphi(\bar{x}) \subseteq \bigcup_{y \in M(\bar{x})} \left\{ v \in \mathbb{R}^n : (v, 0) \in \partial^M \sigma(\bar{x}, y) \right\} \tag{9}$$

and

$$\partial^\infty \varphi(\bar{x}) \subseteq \bigcup_{y \in M(\bar{x})} \{v \in \mathbb{R}^n : (v, 0) \in \partial^\infty \sigma(\bar{x}, y)\}. \tag{10}$$

The functions $g_k, k = 0, ..., s$, are continuously differentiable, thus locally Lipschitzian around any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. This means that σ is locally Lipschitz around any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, too, as it is a maximum of a family of locally Lipschitz functions. Consequently, $\partial^\infty \sigma(x, y) = \{(0, 0)\}$ for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and, taking into account (10), it holds $\partial^\infty \varphi(\bar{x}) = \{0\}$.

The next step of the proof concerns the lower semicontinuity of the function φ around $\bar{x}$. More precisely, we will show that φ is lower semicontinuous on $U_{\bar{x}}$. To this end, take an arbitrary $\bar{z} \in U_{\bar{x}}$. Assume to the contrary that there exists $a \in \mathbb{R}$ such that $\liminf_{z \to \bar{z}} \varphi(z) < a < \varphi(\bar{z})$. Thus there exists a sequence $(z_k)_{k \in \mathbb{N}}$ in $U_{\bar{x}}$ such that $z_k \to \bar{z}$ when $k \to \infty$ and $\varphi(z_k) < a$ for all $k \geq 1$. Due to the local description of φ, for all $k \geq 1$ there exists $u_k \in W$ such that $\varphi(z_k) = \sigma(z_k, u_k)$. The sequence $(u_k)_{k \in \mathbb{N}}$ contains a convergent subsequence $(u_{k_l})_{l \in \mathbb{N}}$ which converges to an element $\bar{u} \in W$ as $l \to \infty$. Thus $\sigma(z_{k_l}, u_{k_l}) < a$ for all $l \geq 1$ and due to the continuity of σ one has $\sigma(\bar{z}, \bar{u}) \leq a$. On the other hand, we have $\varphi(\bar{z}) \leq \sigma(\bar{z}, \bar{u})$, which implies that $\varphi(\bar{z}) \leq a$, furnishing the desired contradiction.

Consequently, we can apply Theorem 2 and, by combining it with relation (9), we get

$$\begin{aligned}
\partial^C \varphi(\bar{x}) &= \operatorname{cl} \operatorname{conv}[\partial^M \varphi(\bar{x}) + \partial^\infty \varphi(\bar{x})] = \operatorname{cl} \operatorname{conv} \partial^M \varphi(\bar{x}) \\
&\subseteq \operatorname{cl} \operatorname{conv} \bigcup_{y \in M(\bar{x})} \left\{v \in \mathbb{R}^n : (v, 0) \in \partial^M \sigma(\bar{x}, y)\right\} \\
&\subseteq \operatorname{cl} \operatorname{conv} \bigcup_{y \in M(\bar{x})} \left\{v \in \mathbb{R}^n : (v, 0) \in \partial^C \sigma(\bar{x}, y)\right\}.
\end{aligned}$$

Using Proposition 2.3.12 in [1] one rapidly can notice that

$$\left\{v \in \mathbb{R}^n : (v, 0) \in \partial^C \sigma(\bar{x}, y)\right\} = V(\bar{x}, y),$$

which implies that

$$\partial^C \varphi(\bar{x}) \subseteq \operatorname{cl} \operatorname{conv} V(\bar{x}).$$

Condition C* also guarantees that $M(\bar{x})$ is compact and thus $V(\bar{x})$ is a compact set as a compact union of compact sets. This means that $\operatorname{cl} \operatorname{conv} V(\bar{x}) = \operatorname{conv} V(\bar{x})$ and, consequently,

$$\partial^C \varphi(\bar{x}) \subseteq \operatorname{conv} V(\bar{x}).$$

■

Remark 1 Assuming that Condition C* is fulfilled, from the proof of Theorem 4 we get as a by-product, since φ is lower semicontinuous at $\bar{x}$ and $\partial^\infty\varphi(\bar{x}) = \{0\}$, the fact that the infimal value function φ is locally Lipschitzian around $\bar{x}$ (see [5, Theorem 3.52]).

As already noticed in [2], answering questions like whether the inclusion in (1) is in fact an equality or under which sufficient conditions it becomes one, seems to be very challenging. As follows from the proof of Theorem 4, exact formulae (this means, not only upper estimates) for the subdifferential of an infimal value function could be very useful with this respect. Unfortunately, these formulae are available in the literature on nonsmooth analysis only for very special situations.

We discuss one of them in the following, namely, in the case when $s = 0$, which means that $\sigma = g_0$ is a continuously differentiable function. Thus $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ becomes $\varphi(x) = \inf_{y\in\mathbb{R}^m} g_0(x, y)$ and for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ one has $K(x, y) = \{0\}$ and

$$V(x, y) = \{D_x g_0(x, y) \mid D_y g_0(x, y) = 0\}.$$

The treatment of this particular situation relies on some results given in [7], which we recall as follows. Let $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a given function, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ a set-valued operator, the infimal value function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, $\varphi(x) = \inf_{y\in G(x)} \sigma(x, y)$, and the argminimum mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $M(x) = \{y \in G(x) : \sigma(x, y) = \varphi(x)\}$.

We assume that the infimal value function φ is finite at some $\bar{x} \in \operatorname{dom} M$, that σ is differentiable at $(\bar{x}, \bar{y})$ for some $\bar{y} \in M(\bar{x})$ and that the solution mapping $M : \operatorname{dom} G \rightrightarrows \mathbb{R}^m$ admits a *local upper Lipschitzian selection* at $(\bar{x}, \bar{y})$. This means that there exists a single-valued mapping $h : \operatorname{dom} G \to \mathbb{R}^m$, which is locally upper Lipschitzian at $\bar{x}$ satisfying $h(\bar{x}) = \bar{y}$ and $h(x) \in M(x)$ for all $x \in \operatorname{dom} G$ in an open neighborhood of $\bar{x}$. According to [8], the single-valued mapping $h : \operatorname{dom} G \to \mathbb{R}^m$ is *locally upper Lipschitzian* at $\bar{x}$ if there exists an open neighborhood $U_{\bar{x}} \subseteq \mathbb{R}^n$ of $\bar{x}$ and a number $L \geq 0$ such that

$$\|h(x) - h(\bar{x})\| \leq L\|x - \bar{x}\| \text{ whenever } x \in U_{\bar{x}} \cap \operatorname{dom} G.$$

Then, due to [7, Theorem 2], it holds

$$\partial^F\varphi(\bar{x}) = D_x\sigma(\bar{x}, \bar{y}) + D^{F*}G(\bar{x}, \bar{y})(D_y\sigma(\bar{x}, \bar{y})),$$

where

$$D^{F*}G(\bar{x}, \bar{y})(w) := \{v \in \mathbb{R}^n : (v, -w) \in N_F((\bar{x}, \bar{y}); \operatorname{gph} G)\}$$

is the *Fréchet coderivative* of G at $(\bar{x}, \bar{y})$ and

$$N_F((\bar{x}, \bar{y}); \operatorname{gph} G) := \partial^F\delta_{\operatorname{gph} G}(\bar{x}, \bar{y})$$

denotes the *Fréchet normal cone* to $\operatorname{gph} G$ at $(\bar{x}, \bar{y})$. The concept of *coderivative* has been introduced by Mordukhovich in [4].

We can prove the following result.

Theorem 5 *Consider the continuously differentiable function $g_0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, the infimal value function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$, $\varphi(x) = \inf_{y\in\mathbb{R}^m} g_0(x, y)$, the argminimum mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m, M(x) = \{y \in \mathbb{R}^m : g_0(x, y) = \varphi(x)\}$, $\bar{x} \in \mathbb{R}^n$ and assume that Condition C^* is fulfilled. If for each $y \in M(\bar{x})$ with $D_y g_0(\bar{x}, y) = 0$ the mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ admits a local upper Lipschitzian selection at $(\bar{x}, y)$, then*

$$\partial^C \varphi(\bar{x}) = \operatorname{conv} V(\bar{x}).$$

Proof. By taking $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ such that $\operatorname{gph} G = \mathbb{R}^n \times \mathbb{R}^m$ we are in the setting considered in [7] and described above. As shown in the proof of Theorem 4, since Condition C^* is fulfilled, the function φ is finite at $\bar{x}$ and $M(\bar{x}) \neq \emptyset$. More than that, for every $(x, y) \in \operatorname{gph} G$ one has $N_F((x, y); \operatorname{gph} G) = \{(0, 0)\}$ and, consequently, $D^{F*}G(x, y)(w) = \{0\}$ if $w = 0$, being equal to the empty set, otherwise.

Let be an arbitrary $y \in M(\bar{x})$ such that $D_y g_0(\bar{x}, y) = 0$. Since $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ admits a local upper Lipschitzian selection at $(\bar{x}, y)$, by [7, Theorem 2], it holds $\partial^F \varphi(\bar{x}) = \{D_x \sigma(\bar{x}, y)\}$. Thus $V(\bar{x}, y) = \partial^F \varphi(\bar{x}) \subseteq \partial^C \varphi(\bar{x})$, which means that $V(\bar{x}) \subseteq \partial^C \varphi(\bar{x})$. Using the fact that $\partial^C \varphi(\bar{x})$ is a convex set, it yields $\operatorname{conv} V(\bar{x}) \subseteq \partial^C \varphi(\bar{x})$, which, combined with Theorem 4, provides the desired conclusion. ■

3 The convex case

In the last section of the paper we discuss another particular situation for which the inclusion in (1) becomes an equality, provided that Condition C* is fulfilled. To this end we assume that the functions $g_0, ..., g_s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are *convex* and differentiable (not necessarily continuously differentiable). Thus the function $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \sigma(x, y) = \max_{0\le k\le s} g_k(x, y)$, is convex, too, which at its turn furnishes the convexity of the infimal value function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$, $\varphi(x) = \inf_{y\in\mathbb{R}^m} \sigma(x, y)$.

One of the intermediate tools which we will use in this section is the convex subdifferential of a function. Having a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and an element $\bar{x} \in \operatorname{dom} f$ the *convex subdifferential* of f at $\bar{x}$ is the set

$$\partial f(\bar{x}) := \{v \in \mathbb{R}^n : f(x) - f(\bar{x}) \ge v^T(x - \bar{x}) \ \forall x \in \mathbb{R}^n\},$$

while for $\bar{x} \notin \operatorname{dom} f$ we put $\partial f(\bar{x}) := \emptyset$.

According to [9, Theorem 2.6.1 (ii)], for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ it holds

$$(v, 0) \in \partial\sigma(x, y) \Leftrightarrow \varphi(x) = \sigma(x, y) \text{ and } v \in \partial\varphi(x). \tag{11}$$

On the other hand, according to [9, Corollary 2.8.11], for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ we have

$$\partial\sigma(x, y) = \left\{ \sum_{k\in K(x,y)} \mu_k (D_x g_k(x, y), D_y g_k(x, y)) \;\middle|\; \begin{array}{c} \sum_{k\in K(x,y)} \mu_k = 1, \\ \mu_k \ge 0 \ \forall k \in K(x, y) \end{array} \right\}, \tag{12}$$

which means that

$$(v,0) \in \partial\sigma(x,y) \Leftrightarrow v \in V(x,y). \tag{13}$$

Working in the convex setting, one gets equality in (1), without any supplementary assumption, excepting Condition C*.

Theorem 6 *Let the Condition C* be fulfilled and let $\bar{x} \in \mathbb{R}^n$. Then it holds*

$$\partial^C \varphi(\bar{x}) = V(\bar{x}) = \operatorname{conv} V(\bar{x}). \tag{14}$$

Proof. As we have already seen, Condition C* guarantees that $M(x) \neq \emptyset$ for all $x \in \mathbb{R}^n$, which implies that $\operatorname{dom}\varphi = \mathbb{R}^n$. From (11) it follows that

$$v \in \partial\varphi(\bar{x}) \text{ if and only if } \exists y \in M(\bar{x}) \text{ such that } (v,0) \in \partial\sigma(\bar{x},y),$$

which can be equivalently formulated as (cf. (13))

$$v \in \partial\varphi(\bar{x}) \Leftrightarrow v \in \bigcup_{y \in M(\bar{x})} V(\bar{x},y) = V(\bar{x}).$$

Further, since φ is a convex function, one gets

$$\partial^C \varphi(\bar{x}) = V(\bar{x}),$$

while (14) follows as a consequence of the convexity of the set $\partial^C \varphi(\bar{x})$. ∎

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