On Some Erroneous Statements in the Paper "Optimality Conditions for Extended Ky Fan Inequality with Cone and Affine Constraints and Their Applications" by A. Capătă

SHORT COMMUNICATION

R.I. Boţ[1] and E.R. Csetnek[2]

Communicated by Igor Konnov

[1]Corresponding author. Department of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: radu.bot@mathematik.tu-chemnitz.de

[2]Department of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: robert.csetnek@mathematik.tu-chemnitz.de

Abstract. In this note we show that the main result of the paper [1] due to A. Capătă and, consequently, all its particular cases are false.

Key Words. Lagrange duality, quasi-relative interior, separation theorem, Ky Fan inequality

AMS Classification. 46N10, 90C25

1 The Incorrect Statements

The following statement, concerning a convex optimization problem (P) with geometric and cone constraints and its Lagrange dual problem (D), given in Adela Capătă's paper [1] captured our attention:

"The next corollary is an improvement of Corollary 4.1 of [18], where the authors considered a superfluous condition in order to prove the above mentioned corollary.

Corollary 5.1 Let $\operatorname{cl}(K-K)=Z$, and let $x \in S$ such that $g(x) \in -\operatorname{qri} K$. If $a \in A$ is a solution of (P) then there exists $\lambda \in K^*$ a weak solution of the dual problem (D) and the value of (P) equals the value of (D)."

The paper [18], to which the author of [1] refers, is our manuscript [R.I. Boţ, E.R. Csetnek, A. Moldovan: *Revisiting some duality theorems via the quasirelative interior in convex optimization*, Journal of Optimization Theory and Applications 139(1), 67–84, 2008], quoted in this note as reference [2].

Corollary 5.1 in [1] is obtained by successively particularizing the main result of this paper. In the following we show that **[1, Corollary 5.1] is false**, which has as a consequence the fact that the main result of the paper in discussion, as well as all its particular instances provide incorrect statements. For the beginning let us recall the framework considered in [1, Corollary 5.1].

Let X be a real linear space, $S \subseteq X$ a nonempty and convex set, $F : S \to \mathbb{R}$ a convex function on S, Z a separated locally convex space partially ordered by a nonempty convex cone K and $g : S \to Z$ a K-convex function on S, that is

$$g(\alpha s_1 + (1-\alpha)s_2) - \alpha g(s_1) - (1-\alpha)g(s_2) \in -K \ \forall \alpha \in]0,1[\ \forall s_1, s_2 \in S.$$

Corollary 5.1 in [1] concerns the primal optimization problem

$$(P) \qquad \inf_{x \in A} F(x),$$

where $A = \{x \in S : g(x) \in -K\}$, and its Lagrange dual problem

$$(D) \qquad \sup_{\lambda \in K^*} \inf_{x \in S} \{F(x) + \lambda(g(x))\}.$$

Here $K^* := \{\lambda \in Z^* : \lambda(k) \geq 0 \ \forall k \in K\}$ denotes the dual cone of K, Z^* being the topological dual space of Z. Let us also mention that, by a "weak solution of the dual problem (D)" the author simply means an optimal solution of the Lagrange dual to (P).

Further, let us recall that the quasi-relative interior of a convex set $M \subseteq Z$ is (see [3])

$$\operatorname{qri} M := \{z \in M : \operatorname{cl}\big(\operatorname{cone}(M - z)\big) \text{ is a linear subspace of } Z\},$$

where "cl" and "cone" denote the closure and the conic hull of a set, respectively. For properties and characterizations of this generalized interiority notion we invite the reader to consult [2-5] and the references therein.

Denoting by $v(P)$ the optimal objective value for (P), assumed to be finite, the conic extension for (P) is the set

$$\begin{aligned} \mathcal{E}_{v(P)} &:= \{(v(P) - F(x) - \alpha, -g(x) - y) : x \in S, \alpha \geq 0, y \in K\} \\ &= (v(P), 0) - (F, g)(S) - \mathbb{R}_+ \times K. \end{aligned}$$

The condition we gave in [2, Corollary 4.1], erroneously characterized by A. Capătă as "superfluous", reads $0 \notin \operatorname{qri} \mathcal{E}_{v(P)}$. The question, whenever one can omit it, was already addressed in [2], where one can also find an example showing that [2, Corollary 4.1] in the absence of this condition fails to be true. A careful reading of our paper would have made the author of [1] clear that her Corollary 5.1 is an incorrect result.

We present below the example in question below, concomitantly providing evidence for the fact that **[1, Corollary 5.1] is false**.

Example 1.1 (see also [2, Example 3.2]) Let be $X = S = Z = \ell^2$, where ℓ^2 is the real Hilbert space of real sequences $(x_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < +\infty$, equipped with the norm $\|\cdot\| : \ell^2 \to \mathbb{R}$, $\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}$ for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$. Take $K := \ell^2_+ = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 : x_n \geq 0 \ \forall n \in \mathbb{N}\}$ and define $F : \ell^2 \to \mathbb{R}$, $F(x) = \langle c, x\rangle_{\ell^2\times\ell^2}$, where $c = (c_n)_{n\in\mathbb{N}}, c_n = (1/n)$ for all $n \in \mathbb{N}$ and $g : \ell^2 \to \ell^2, g(x) = -Bx$, where $(Bx)_n = (1/2^n)x_n$ for all $n \in \mathbb{N}$. Then $A = \{x \in \ell^2 : Bx \in \ell^2_+\} = \ell^2_+$. It holds $\operatorname{cl}(\ell^2_+ - \ell^2_+) = \ell^2$ and $\operatorname{qri} \ell^2_+ = \{x = (x_n)_{n\in\mathbb{N}} \in \ell^2 : x_n > 0 \ \forall n \in \mathbb{N}\} \neq \emptyset$ (cf. [3]), while one can easily find an $\overline{x} \in \ell^2$ with $g(\overline{x}) \in -\operatorname{qri} \ell^2_+$. We also have that

$$v(P) = \inf_{x\in A} \langle c, x\rangle = 0$$

and $x = 0$ is an optimal solution of the primal problem. On the other hand, for $\lambda \in K^* = \ell^2_+$, it holds

$$\inf_{x\in S}\{F(x) + \lambda(g(x))\} = \inf_{x\in\ell^2}\{\langle c, x\rangle + \langle \lambda, g(x)\rangle\}$$

$$= \inf_{x=(x_n)_{n\in\mathbb{N}}\in\ell^2} \left(\sum_{n=1}^{\infty} \frac{1}{n} x_n - \sum_{n=1}^{\infty} \lambda_n \frac{1}{2^n} x_n \right) = \inf_{(x_n)_{n\in\mathbb{N}}\in\ell^2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{\lambda_n}{2^n} \right) x_n$$

$$= \begin{cases} 0, & \text{if } \lambda_n = \frac{2^n}{n} \ \forall n \in \mathbb{N}, \\ -\infty, & \text{otherwise.} \end{cases}$$

Since $(2^n/n)_{n\in\mathbb{N}}$ does not belong to ℓ^2, thus neither to ℓ^2_+, it follows that the optimal objective value of the dual problem is $v(D) = -\infty$.

Hence, all the hypotheses of [1, Corollary 5.1] are fulfilled, however its conclusion is wrong.

The article [1] mainly deals with the extended Ky Fan inequality:

$$(EKF) \qquad \text{find } \overline{a} \in A \text{ such that } f(\overline{a}, b) \notin -\operatorname{int} C \ \forall b \in A,$$

where X is a real linear space, Y, Z and W are real separated locally convex spaces, Y is partially ordered by a solid (that is, with nonempty interior), pointed and convex cone C, Z is partially ordered by a nonempty convex cone K, $S \subseteq X$ is a nonempty convex set, $f : S \times S \to Y$ is a C-convex function in its second variable fulfilling $f(x, x) = 0$ for all $x \in S$, $g : S \to Z$ is a K-convex function, $h : S \to W$ is an affine function, $A = \{x \in S : g(x) \in -K \text{ and } h(x) = 0\}$ and the following condition is fulfilled

$$\text{for all } (z^*, w^*) \in K^* \times W^* \setminus \{0, 0\} \text{ there is } x \in S$$

$$\text{such that } z^*(g(x)) + w^*(h(x)) < 0.$$

The main result of this paper reads:

“**Theorem 3.1** Let $\operatorname{qri}((g,h)(S)+K\times\{0\})\neq\emptyset$. A point $a\in A$ is a solution of (EKF) if and only if there exists $(y^*,z^*,w^*)\in C^*\setminus\{0\}\times K^*\times W^*$ such that

$$z^*(g(a))=0$$

and

$$0=y^*(f(a,a))+z^*(g(a))+w^*(h(a))=\min_{x\in S}\{y^*(f(a,x))+z^*(g(x))+w^*(h(x))\}.”$$

Since [1, Corollary 5.1] has been obtained by successively particularizing it, Example 1.1 leads to the conclusion that **this theorem, but also all its particular instances [1, Corollary 3.1, Theorem 4.1, Theorem 4.2, Theorem 5.1, Theorem 5.2 and Corollary 5.1] are false**.

2 Where Does the Error Come From?

In this section, we indicate the source of errors for [1, Theorem 3.1] and, from here, for the other results in [1] listed above. The proof of [1, Theorem 3.1] relies on the following separation statement given in [1, Corollary 2.1]:

“**Corollary 2.1** Let M be a nonempty convex subset of Y and $y_0\in Y$ such that $\operatorname{qri}M\neq\emptyset$ and $y_0\notin\operatorname{qri}M$. Then there exists $y^*\in Y^*\setminus\{0\}$ such that $y^*(y)\geq y^*(y_0)$ for all $y\in M$.”

Indeed, according to the proof of [1, Theorem 3.1], the author first proves, for a a solution of (EKF), that the set

$$M := \{(y,z,w) \in Y \times Z \times W : \exists x \in S, y \in f(a,x) + \operatorname{int} C, z \in g(x) + K, w = h(x)\}$$

has an nonempty quasi-relative interior and that $y_0 := (0,0,0) \notin \operatorname{qri} M$ and then applies the above corollary in this special setting. As it follows from the next example, **the separation statement [1, Corollary 2.1] is false**.

Example 2.1 (see also [6, Remark 2]) Let Y be an infinite dimensional normed space and $f : Y \to \mathbb{R}$ a linear and discontinuous functional. Define $M = \{y \in Y : f(y) = 1\}$, which is an affine and dense subset of Y. Take $y_0 := 0$. One can see that $\operatorname{qri} M = M \neq \emptyset$ and $y_0 \notin M = \operatorname{qri} M$. Hence all the hypotheses of [1, Corollary 2.1] are fulfilled. However, y_0 and M cannot be separated. Indeed, if there would exists $y^* \in Y^* \setminus \{0\}$ such that $y^*(y) \geq y^*(y_0)$ for all $y \in M$, then by employing the fact that M is dense in Y, one would obtain $y^* = 0$, which is a contradiction.

In [5, Corollary 2.1], in order to obtain a valid separation result, one has to assume that $y_0 \in M \setminus \operatorname{qri} M$, whereby the condition $\operatorname{qri} M \neq \emptyset$ needs not necessarily be fulfilled. Other separation theorems involving the quasi-relative interior of a convex set can be found in [2, 4, 5] and the references therein.

3 Concluding Remarks

In this note, we have shown that **the main result and all its particular cases of the paper [1] due to A. Capătă are false**, by sustaining our claims with examples and pointing out the flaw in the proof of the main result [1, Theorem 3.1]. Let us also mention that in [1] there is a considerable number of inaccuracies, inconsistencies and false assertions, however, we omit mentioning them, as this would exceed our intentions. All these aspects make us question the quality of the reviewing process of [1]. We stress that fact that, when a paper emphasizes a possible improvement of some results of another work, then the reviewers have to verify carefully at least this statement. This was for [1], definitively, not the case. Otherwise, a short look in [2] would have been enough to come to the conclusion that, due to [2, Example 3.2], the statement in [1, Corollary 5.1] is false.

References

1. Capătă, A.: Optimality conditions for extended Ky Fan inequality with cone and affine constraints and their applications, J. Optim. Theory Appl., published "Online First" on 10 September 2011

2. Boţ, R.I., Csetnek, E.R., Moldovan, A.: Revisiting some duality theorems via the quasirelative interior in convex optimization, J. Optim. Theory Appl. 139, 67–84 (2008)

3. Borwein, J.M., Lewis, A.S.: Partially finite convex programming, part I: Quasi relative interiors and duality theory, Math. Program. 57, 15–48 (1992)

4. Boţ, R.I., Csetnek, E.R.: Regularity conditions via generalized interiority notions in convex optimization: new achievements and their relation to some classical statements, Optimization DOI: 10.1080/02331934.2010.505649, Preprint arXiv:0906.0453v1, posted 2 June (2009)

5. Boţ, R.I., Csetnek, E.R., Wanka, G.: Regularity conditions via quasi-relative interior in convex programming, SIAM J. Optim. 19, 217–233 (2008)

6. Daniele, P., Giuffrè, S., Idone, G., Maugeri, A.: Infinite dimensional duality and applications, Math. Ann. 339, 221–239 (2007)