The Rockafellar Conjecture and type (FPV)

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Abstract

In this paper, using a technique of Verona and Verona, we show that a result announced recently by Eberhard and Wenczel implies the truth of the Rockafellar conjecture. We then show that there is a gap in the logic of the Eberhard–Wenczel result, which we tried unsuccessfully to close. We also discuss briefly the connection with maximally monotone multifunctions of type (FPV).

One of the fundamental results in the theory of monotone operators, which was proved by Rockafellar in [4, Theorem 1, pp. 76–83], is that if X is a reflexive Banach space, $S\colon X \rightrightarrows X^*$ and $T\colon X \rightrightarrows X^*$ are maximally monotone and $\mathrm{int}D(S) \cap D(T) \neq \emptyset$ then the Minkowski sum $S + T$ is maximally monotone. (As usual, "int" stands for "interior" and "$D(\cdot)$" stands for "domain of".) We will describe as the *Rockafellar Conjecture* the statement that this result is true if X is not assumed to be reflexive. Over the years, many people have tried unsuccessfully to prove or disprove the Rockafellar Conjecture. So, for the rest of this paper, we assume that X is a real, possibly nonreflexive, Banach space.

It is in this context that one must consider the assertion of Eberhard and Wenczel in [1, Theorem 36] (modified according to (24)), which we state formally as Conjecture 1:

Conjecture 1. *If $S\colon X \rightrightarrows X^*$ and $T\colon X \rightrightarrows X^*$ are maximally monotone, $D(S) \cap D(T)$ is bounded and $\mathrm{int}D(S) \cap D(T) \neq \emptyset$ then $S + T$ is maximally monotone.*

We will prove in Theorem 2 that Conjecture 1 implies the truth of the Rockafellar Conjecture. Our argument is based on an argument of Verona and Verona (see the preprint [8]). Their argument actually establishes a stronger result – we give a self contained but less technical proof of a weaker result here, which is adequate for our purposes.

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Theorem 2. *If Conjecture 1 is true then the Rockafellar Conjecture is true.*

Proof. Let us assume that Conjecture 1 is true, $S\colon X \rightrightarrows X^*$ and $T\colon X \rightrightarrows X^*$ are maximally monotone, and $\mathrm{int} D(S) \cap D(T) \neq \emptyset$. Write $U := S + T$. Let $(x, x^*) \in X \times X^*$ and (as usual, "$G(\cdot)$" stands for "graph of")

$$(u, u^*) \in G(U) \quad \Longrightarrow \quad \langle u - x, u^* - x^* \rangle \geq 0. \tag{1}$$

We will prove that

$$(x, x^*) \in G(U). \tag{2}$$

This establishes that $U = S + T$ is maximally monotone, and hence the truth of the Rockafellar Conjecture. Let B be any closed ball so large that $\mathrm{int} B \ni x$ and $\mathrm{int} B \cap \mathrm{int} D(S) \cap D(T) \neq \emptyset$. Let $N_B\colon X \rightrightarrows X^*$ be the normal cone operator of B. We now prove that

$$(u, v^*) \in G(U + N_B) \quad \Longrightarrow \quad \langle u - x, v^* - x^* \rangle \geq 0. \tag{3}$$

To this end, let $(u, v^*) \in G(U + N_B)$. Then we can choose $(u, u^*) \in G(U)$ and $w^* \in N_B u$ such that $v^* = u^* + w^*$. From (1), $\langle u - x, u^* - x^* \rangle \geq 0$ and, since $w^* \in N_B u$ and $x \in B$, the definition of normal cone implies that $\langle u - x, w^* \rangle \geq 0$. Thus $\langle u - x, v^* - x^* \rangle = \langle u - x, u^* - x^* \rangle + \langle u - x, w^* \rangle \geq 0$, which gives (3). We next prove that

$$U + N_B \text{ is maximally monotone.} \tag{4}$$

Since N_B is the subdifferential of the indicator function of B, N_B is maximally monotone. Now $D(N_B) \cap D(T) = B \cap D(T) \subset B$, which is bounded, and $\mathrm{int} D(N_B) \cap D(T) = \mathrm{int} B \cap D(T) \supset \mathrm{int} D(S) \cap \mathrm{int} B \cap D(T) \neq \emptyset$. Thus, from Conjecture 1 with S replaced by N_B, $N_B + T$ is maximally monotone. Next,

$$D(S) \cap D(N_B + T) = D(S) \cap D(N_B) \cap D(T) = D(S) \cap B \cap D(T) \subset B,$$

which is bounded, and

$$\mathrm{int} D(S) \cap D(N_B + T) = \mathrm{int} D(S) \cap B \cap D(T) \supset \mathrm{int} D(S) \cap \mathrm{int} B \cap D(T) \neq \emptyset.$$

From Conjecture 1 again with T replaced by $N_B + T$, $S + (N_B + T)$ is maximally monotone. Since $U + N_B = S + (N_B + T)$, this establishes (4). (4) and (3) now imply that $(x, x^*) \in G(U + N_B)$. Thus we can choose $y^* \in Ux$ and $z^* \in N_B x$ such that $x^* = y^* + z^*$. Since $x \in \mathrm{int} B$, $z^* = 0$, thus $x^* = y^*$ and $(x, x^*) = (x, y^*) \in G(U)$. This completes the proof of (2), and hence the truth of the Rockafellar Conjecture. $\square$

Theorem 2 encouraged us to examine the proof of Conjecture 1 in [1, Theorem 36] with great care. Unfortunately, when we did this, we found a gap in the logic, which we now explain: [1, Eqn. (11)] requires that $(y^*, y) \in \partial_\varepsilon h(x, x^*)$, where "$\partial_\varepsilon$" is the usual ε–subdifferential. However, when [1, Eqn. (11)] is used in [1, Proposition 21(3)], what is assumed is that $(y^*, y) \in \partial_\varepsilon \mathcal{F}_{M_h}(x, x^*)$. So the conditions do not match up. We attempted to remedy this situation. The analysis below shows that our attempt failed.

The following definition is made in [1], where $\mathcal{PC}(X \times X^*)$ stands for the set of proper, convex real functions on $X \times X^*$, $\dagger(x, x^*) := (x^*, x)$, "$\mathcal{F}_T$" is the Fitzpatrick function of T, * stands for Fenchel conjugate, and the ordering is pointwise on $X \times X^*$:

$$bR(T) := \{h \in \mathcal{PC}(X \times X^*) \colon\ {\mathcal{F}_T}^{*\dagger} \geq h \geq \mathcal{F}_T,\ h^{*\dagger} \geq h \geq \langle \cdot, \cdot \rangle\}.$$

Lemma 3. *Let $h \in bR(T)$. Then ${\mathcal{F}_{M_h}}^* \geq h^*$ on $X^* \times X$.*

Proof. Let $(z, z^*) \in X \times X^*$. Let $(y, y^*) \in M_h$: then we have $h(y, y^*) = \langle y, y^* \rangle$. From [5, Lemma 19.12, p. 82], (which is cited in [1, Lemma 5]),

$$h^*(y^*, y) = h^{*\dagger}(y, y^*) = \langle y, y^* \rangle.$$

Now let $(z^*, z) \in X^* \times X$. From the definition of h^*,

$$\langle y, z^* \rangle + \langle z, y^* \rangle - h(z, z^*) \leq h^*(y^*, y),$$

and so

$$\langle y, z^* \rangle + \langle z, y^* \rangle - \langle y, y^* \rangle = \langle y, z^* \rangle + \langle z, y^* \rangle - h^*(y^*, y) \leq h(z, z^*).$$

Taking the supremum over $(y, y^*) \in M_h$, $\mathcal{F}_{M_h}(z, z^*) \leq h(z, z^*)$. Thus

$$\mathcal{F}_{M_h} \leq h.$$

The result follows by taking conjugates. □

The hypotheses for Lemma 4 below are satisfied in [1, Proposition 21(3)]:

Lemma 4. *Let $h \in bR(T)$, $\mathcal{F}_{M_h}(x, x^*) = \langle x, x^* \rangle$, $\gamma := h(x, x^*) - \langle x, x^* \rangle$ and $\varepsilon > 0$. Then $\partial_\varepsilon \mathcal{F}_{M_h}(x, x^*) \subset \partial_{\varepsilon+\gamma} h(x, x^*)$.*

Proof. Let $(y^*, y) \in \partial_\varepsilon \mathcal{F}_{M_h}(x, x^*)$. Then, from Lemma 3,

$$\begin{aligned}\langle x, y^* \rangle + \langle y, x^* \rangle + \varepsilon \geq \mathcal{F}_{M_h}(x, x^*) + {\mathcal{F}_{M_h}}^*(y^*, y) &= \langle x, x^* \rangle + {\mathcal{F}_{M_h}}^*(y^*, y) \\ &\geq \langle x, x^* \rangle + h^*(y^*, y) = h(x, x^*) - \gamma + h^*(y^*, y).\end{aligned}$$

Thus $(y^*, y) \in \partial_{\varepsilon+\gamma} h(x, x^*)$. This completes the proof of Lemma 4. □

Discussion. Assuming that $(y^*, y) \in \partial_\varepsilon h(x, x^*)$, the first line of [1, Eqn. (11)] said:

$$\mathcal{F}_{M_h}(y, y^*) \leq \langle y, x^* \rangle + \langle x, y^* \rangle - \langle x, x^* \rangle + \varepsilon - \gamma. \tag{5}$$

Noting that $\delta := \varepsilon - \langle y - x, y^* - x^* \rangle$, this implies that

$$\mathcal{F}_{M_h}(y, y^*) - \langle y, y^* \rangle \leq \langle y, x^* \rangle + \langle x, y^* \rangle - \langle x, x^* \rangle - \langle y, y^* \rangle + \varepsilon - \gamma = \delta - \gamma.$$

If we only know that $(y^*, y) \in \partial_\varepsilon \mathcal{F}_{M_h}(x, x^*)$, then Lemma 4 enables us to replace ε by $\varepsilon + \gamma$ in (5), which now reads

$$\mathcal{F}_{M_h}(y, y^*) \leq \langle y, x^* \rangle + \langle x, y^* \rangle - \langle x, x^* \rangle + \varepsilon,$$

from which we derive, as above,

$$\mathcal{F}_{M_h}(y,y^*) - \langle y,y^*\rangle \le \langle y,x^*\rangle + \langle x,y^*\rangle - \langle x,x^*\rangle - \langle y,y^*\rangle + \varepsilon = \delta.$$

So, instead of [1, Eqn. (13)], we get $-\delta \le \delta$, that is to say, $\delta \ge 0$, from which it follows that $\langle y-x, y^*-x^*\rangle \le \varepsilon$. Since $\mathcal{F}_{M_h}(x,x^*) = \langle x,x^*\rangle$, we have $(x,x^*) \in (M_h)^\mu$. If we impose the additional condition that $(y^*,y) \in {M_h}^\dagger$ then $\langle y-x, y^*-x^*\rangle \ge 0$. So altogether we have $0 \le \langle y-x, y^*-x^*\rangle \le \varepsilon$. Since we could have $\langle y-x, y^*-x^*\rangle = 0$, this does not lead to a contradiction, because the connection between γ and ε has been lost.

Conclusion. We do not see a way of fixing the "gap" in the proof of [1, Proposition 21].

Comment on [1, Proposition 20.1($\Longleftarrow$)]. Constantin Zălinescu (personal communication) has made us aware of the following potential problem with the proof of [1, Proposition 20.1($\Longleftarrow$)]. The following simple example shows that [1, Eq. (9)] is incorrect, though it is still possible that [1, Proposition 20.1($\Longleftarrow$)] is correct, because M_h is a very special kind of monotone set.

Example 5. Let $x \in X$, $x^* \in X^*$ and $\langle x,x^*\rangle > 0$. Let M be the monotone subset $\{(0,0),(2x,2x^*)\}$ of $X \times X^*$. Since $(0,0) \in M$, for all $(y,y^*) \in X \times X^*$, $\mathcal{F}_M(y,y^*) \ge \langle 0,y^*\rangle + \langle y,0\rangle - \langle 0,0\rangle = 0$. On the other hand,

$$\mathcal{F}_M(x,x^*) = [\langle 0,x^*\rangle + \langle x,0\rangle - \langle 0,0\rangle] \vee [\langle 2x,x^*\rangle + \langle x,2x^*\rangle - \langle 2x,2x^*\rangle] = 0.$$

Thus (x,x^*) is a global minimizer for $\mathcal{F}_M$, from which $(0,0) \in \partial\mathcal{F}_M(x,x^*)$. Since $(0,0) \in M^\dagger$, $(0,0) \in \partial\mathcal{F}_M(x,x^*) \cap M^\dagger$. Clearly $(x,x^*) \in M^\mu$ also, but $\langle 0-x, 0-x^*\rangle > 0$, which contradicts [1, Eq. (9)].

Type (FPV)

We say a few words about *type (FPV)*, since it appears in a title role in [1]. Maximally monotone multifunction of *type (FPV)* were introduced (under the name *maximal monotone locally*) by Fitzpatrick–Phelps in [3, p. 65] and Verona–Verona in [6, p. 268]. Furthermore, it was proved by Fitzpatrick–Phelps in [2] and Verona–Verona in [7] that if the Rockafellar Conjecture is true then every maximally monotone multifunction is of type (FPV). [1, Theorem 38] asserts that every maximally monotone multifunction is of type (FPV). Since the proof of this result uses the result of [1, Theorem 36], the gap in the proof of [1, Theorem 36] also leads to a gap in the proof of [1, Theorem 38]. Of course, it is still possible that [1, Theorem 38] is true, even if [1, Theorem 36] is not.

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