Multiobjective Duality for Convex-Linear Problems

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1. Introduction

We consider the following multiobjective optimization problem with convex objective functions and linear inequality constraints

$$(P) \quad \underset{x \in \mathcal{A}}{\mathrm{v}-\min} \quad F(x),$$

$$F(x) = (f_1(x), \ldots, f_m(x))^T,$$

$$\mathcal{A} = \left\{ x \in \mathbb{R}^n : x \underset{K_0}{\geqq} 0\,, Ax + b \underset{K_1}{\leqq} 0 \right\} .$$

$K_0 \subseteq \mathbb{R}^n$ and $K_1 \subseteq \mathbb{R}^l$ are assumed to be convex closed cones defining partial orderings according to $x_1 \underset{K_0}{\geqq} x_2$ if and only if $x_1 - x_2 \in K_0$ (analogously for K_1 instead of K_0).

The functions $f_i(x)$, mapping $i = 1, \ldots, m$, from $\mathbb{R}^n$ into $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ are convex. Moreover, let be the interior $\operatorname{int}\left(\bigcap\limits_{i=1}^{m} \operatorname{dom} f_i \right) \neq \emptyset$ and $f_i(x) > -\infty \, \forall x \in \mathbb{R}^n$, $i = 1, \ldots, m$, where $\operatorname{dom} f_i = \{x \in \mathbb{R}^n : f_i(x) < \infty\}$.

By A is denoted any real $l \times n$ matrix and $b \in \mathbb{R}^l$, $b \neq (0, \ldots, 0)^T$.

An element $x \in \mathcal{A}$ is called admissible for the problem (P) and the set $\mathcal{A}$ is the admissible domain.

The notation "$\mathrm{v}-\min$" refers to a vector minimum problem is investigated. This is a symbolic denotation requires to explain the considered notion of solutions. In this paper minimal and proper minimal solutions of the problem (P) are studied. We introduce the well-known solution concept of so-called efficient or Pareto-optimal solutions.

Definition 1

An element $\bar{x} \in \mathcal{A}$ is said to be efficient (or minimal or Pareto-minimal) if from

$$F(x) \underset{\mathbb{R}^m_+}{\leqq} F(\bar{x}) \quad \text{for} \quad x \in \mathcal{A} \quad \text{follows} \quad F(x) = F(\bar{x}).$$

Here $\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m : x_i \geqq 0, i = 1, \ldots, m\}$ denotes the ordering cone of the non-negative elements in $\mathbb{R}^m$.

For (P) we are concerned with a sharpened notation, the so-called proper efficiency.

Definition 2

An element $\bar{x} \in \mathcal{A}$ is said to be properly efficient or properly minimal if there exist positive numbers $\lambda_i, i = 1, \ldots, m$, such that $\sum\limits_{i=1}^{m} \lambda_i f_i(\bar{x}) \leqq \sum\limits_{i=1}^{m} \lambda_i f_i(x) \quad \forall x \in \mathcal{A}$.

Of course, a properly efficient element also turns out to be an efficient one (even if the functions f are not convex).

By this definition a properly efficient element $\bar{x} \in \mathcal{A}$ is a solution of the scalarized problem (P_λ) to (P)

$$(P_\lambda) \quad \inf_{x \in \mathcal{A}} \sum_{i=1}^{m} \lambda_i f_i(x)\,.$$

2. The dual of the scalarized problem

Our aim is to construct a multiobjective dual problem to (P). To do so we want to use a dual problem of the scalarized problem (P_λ).

But the usual Lagrangian dual problem

$$(P^*_{Lag}) \quad \sup_{p \underset{\mathbb{R}^\ell_+}{\geqq} 0} \quad \inf_{x \underset{K_0}{\geqq} 0} \quad L(x,p)$$

with the Lagrangian

$$L(x,p) = \sum_{i=1}^{m} \lambda_i f_i(x) + p^T(Ax + b)$$

is not a suitable dual problem for our purpose to construct a multiobjective dual problem to (P).

To overcome this situation we will derive another dual problem by means of the Fenchel-Rockafellar approach of establishing a dual problem using a perturbation of the primal problem (P_λ). This approach permits to form different dual problems to an original primal problem depending on the kind of perturbation.

We introduce the following perturbation function $\Phi(x,\varphi_1,\ldots,\varphi_m,\gamma)$

$$\Phi(x,\varphi_1,\ldots,\varphi_m,\gamma)=\begin{cases}\sum\limits_{i=1}^{m}\lambda_i f_i(x+\varphi_i)\,, & \text{if} \quad x \underset{K_0}{\geqq} 0, Ax+b \underset{K_1}{\leqq} \gamma\,,\\ \infty\,, & \text{otherwise}\,,\end{cases} \tag{1}$$

with the perturbation variables

$$\varphi_i\in\mathbb{R}^n,\ i=1,\ldots,m, \text{ and } \gamma\in\mathbb{R}^\ell\ .$$

So we have the perturbed optimization problem to (P_λ) $(\varphi=(\varphi_1,\ldots,\varphi_m))$

$$(P_{\lambda;\,\varphi,\gamma})\quad \inf_{x\in\mathbb{R}^n}\ \Phi(x,\varphi_1,\ldots,\varphi_m,\gamma)\,.$$

For $\varphi_i=(0,\ldots,0)^T,\gamma=(0,\ldots,0)^T$ (we agree to write $\varphi=0,\ \gamma=0$) we get $(P_{\lambda;0,0})=(P_\lambda)$. Then (cf. Ekeland, Temam Buch) a perturbed dual problem (P^*_λ) to (P_λ) may be defined by

$$(P^*_{\lambda;x^*})\ \sup_{\substack{\varphi_i^*\in\mathbb{R}^n,\\ i=1,\ldots,m,\\ \gamma^*\in\mathbb{R}^\ell}}\ \{-\Phi^*(x^*,\varphi_1^*,\ldots,\varphi_m^*,\gamma^*)\}$$

using the conjugate function Φ^* to Φ

$$\Phi^*(x^*,\ \varphi_1^*,\ldots,\varphi_m^*,\gamma^*)=\sup_{\substack{x,\varphi_i\in\mathbb{R}^n,\\ i=1,\ldots,m,\\ \gamma\in\mathbb{R}^\ell}}\left\{x^{*T}x+\sum_{i=1}^{m}\varphi_i^{*T}\varphi_i+\gamma^{*T}\gamma-\Phi(x,\varphi_1,\ldots,\varphi_m,\gamma)\right\}. \tag{2}$$

There is $x^*,\varphi_i^*\in\mathbb{R}^n$, $i=1,\ldots,m,\gamma^*\in\mathbb{R}^\ell$ and x^* represents the perturbation variable of the dual problem. For $x^*=(0,\ldots,0)^T$ (we write as usual $x^*=0$) the dual problem (P^*_λ) to (P_λ) is

$$(P^*_\lambda)\ \sup_{\substack{\varphi_i^*\in\mathbb{R}^n,\\ i=1,\ldots,m,\\ \gamma^*\in\mathbb{R}^\ell}}\ \{-\Phi^*(0,\varphi_1^*,\ldots,\varphi_m^*,\gamma^*\}\,.$$

To deduce (P^*_λ) we replace Φ in (2) by means of (1)

$$\Phi^*(x^*,\varphi_1^*,\ldots,\varphi_m^*,\gamma^*) = \sup_{\substack{x,\varphi_i\in\mathbb{R}^n,\\ i=1,\ldots,m,\\ Ax+b \underset{K_1}{\leqq} \gamma,\, x \underset{K_0}{\geqq} 0}} \left\{ x^{*T}x + \sum_{i=1}^m \varphi_i^{*T}\varphi_i + \gamma^{*T}\gamma - \sum_{i=1}^m \lambda_i f_i(x+\varphi_i)\right\}.$$

To calculate this expression we introduce new variables y_i instead of φ_i and z instead of γ by

$$y_i = x+\varphi_i\,,\ i=1,\ldots,m\,,\ z=\gamma-Ax-b\,.$$

This implies

$$\Phi^*(x^*,\varphi_1^*,\ldots,\varphi_m^*,\gamma^*)$$

$$= \sup_{\substack{y_i\in\mathbb{R}^n,\\ i=1,\ldots,m,\\ x \underset{K_0}{\geqq} 0\,,\ z \underset{K_1}{\geqq} 0}} \left\{ x^{*T}x + \sum_{i=1}^m \varphi_i^{*T}(y_i-x) + \gamma^{*T}(z+Ax+b) - \sum_{i=1}^m \lambda_i f_i(y_i)\right\}$$

$$= \sum_{i=1}^m \lambda_i \sup_{y_i\in\mathbb{R}^n} \left\{ \frac{1}{\lambda_i}\varphi_i^{*T}y_i - f_i(y_i)\right\} + \sup_{x \underset{K_0}{\geqq} 0} \left\{ (-\sum_{i=1}^m \varphi_i^* + x^* + A^T\gamma^*)^T x\right\} + \sup_{z \underset{K_1}{\geqq} 0} \gamma^{*T}z + \gamma^{*T}b\,.$$

We compute the different suprema and get

$$\sup_{y_i\in\mathbb{R}^n} \left\{ \frac{1}{\lambda_i}\varphi_i^{*T}y_i - f_i(y_i)\right\} = f_i^*\left(\frac{1}{\lambda_i}\varphi_i^*\right),$$

$$\sup_{x \underset{K_0}{\geqq} 0} \left\{ (-\sum_{i=1}^m \varphi_i^* + x^* + A^T\gamma^*)^T x\right\} = \begin{cases} 0\,, \text{if} -\sum_{i=1}^m \varphi_i^* + x^* + A^T\gamma^* \underset{K_0^*}{\leqq} 0\,, \\ \infty\,, \qquad \text{otherwise}\,, \end{cases}$$

$$\sup_{z \underset{K_1}{\geqq} 0} \gamma^{*T}z = \begin{cases} 0\,,\ \text{if}\ \gamma^* \underset{K_1^*}{\leqq} 0\,, \\ \infty\,,\ \text{otherwise}\,, \end{cases}$$

using the dual cones K_0^* and K_1^* to K_0 and K_1, respectively.

The dual cone $K^* \subseteq \mathbb{R}^k$ to the cone $K \subseteq \mathbb{R}^k$ is defined by $K^* = \{x^*\in\mathbb{R}^k : x^{*T}x \geqq 0$ for all $x\in K\}$.

Substituting $p_i^* = \frac{1}{\lambda_i}\varphi_i^*$ the perturbed dual problem $(P^*_{\lambda;x^*})$ is

$$(P^*_{\lambda;x^*}) \qquad \sup_{\substack{\gamma^* \underset{K_1^*}{\leqq} 0, \\ -\sum\limits_{i=1}^{m} \lambda_i p_i^* + A^T\gamma^* \underset{K_0^*}{\leqq} -x^*}} \left\{ -\sum_{i=1}^{m} \lambda_i f_i^*(p_i^*) - \gamma^{*T} b \right\} .$$

Setting $x^* = 0$ the dual problem (P^*_λ) to (P_λ) is

$$(P^*_\lambda) \qquad \sup_{(p_1^*, \ldots, p_m^*, \gamma^*) \in \mathcal{B}_\lambda} \left\{ -\sum_{i=1}^{m} \lambda_i f_i^*(p_i^*) - \gamma^{*T} b \right\} ,$$

where the set of constraints is given by

$$\mathcal{B}_\lambda = \left\{ (p_1^*, \ldots, p_m^*, \gamma^*) : \gamma^* \underset{K_1^*}{\leqq} 0 \, , \, -\sum_{i=1}^{m} \lambda_i p_i^* + A^T\gamma^* \underset{K_0^*}{\leqq} 0 \right\} .$$

Indeed, this is a dual problem that allows to have an idea how the multiobjective dual problem to (P) could look like.

Before intoducing the vector dual problem some properties of the above scalar dual problems (P_λ) and (P^*_λ) are mentioned.

First we point out that there is weak duality between (P_λ) and (P^*_λ) by construction (cf. Ekeland, Temam), i. e. $\sup(P^*_\lambda) \leqq \inf(P_\lambda)$.

But, we are interested in the existence of strong duality $\sup(P^*_\lambda) = \inf(P_\lambda)$ or even $\max(P^*_\lambda) = \min(P_\lambda)$ meaning the existence of solutions to the problems. One classical assumption assuring the strong duality is that a constraint qualification (Slater condition) is fulfilled. This means that there exists an admissible element $x' \in \mathcal{A}$ such that $f_i(x')$, $i = 1, \ldots, m$, is continuous (i.e. $x' \in \operatorname{int}\,(\bigcap\limits_{i=1}^{m} \operatorname{dom} f_i)$) fulfilling the inequality $Ax' + b \underset{K_1}{\leqq} 0$ (i.e. $Ax' + b \in -K_1$) in the strict sense $Ax' + b \in -\operatorname{int} K_1$, also described by $Ax' + b \underset{K_1}{<} 0$. Obviously, this implies that $\operatorname{int} K_1 \neq \emptyset$. This condition is sufficient for strong duality (cf. Ekeland/Temam) but, as well-known, not necessary. Thus, other types of constraint qualifications still exist. Moreover, according to the general duality theory the dual problem (P^*_λ) has a solution. Thus we can formulate the following strong duality theorem.

Theorem 1

Let there exists an element $x' \in \text{int}\,(\bigcap_{i=1}^{m} \text{dom}\ f_i)$ fulfilling $x' \underset{K_0}{\geqq} 0$ and the constraint qualification $Ax' + b \in -\ \text{int}\ K_1$. Then the dual problem (P_λ^*) has a solution and strong duality $\inf(P_\lambda) = \max(P_\lambda^*)$ holds.

Remark:

If we set $m = 1, f_1 = f,\ \lambda_1 = 1$ (the case of singleobjective optimization) and $K_0 = \mathbb{R}^n$, $K_1 = \mathbb{R}^\ell$ (meaning $\mathcal{A} = \mathbb{R}^n$) we obtain as primal problem

$$\inf_{x \in \mathbb{R}^n} f(x)$$

and the dual problem takes the form

$$\sup_{(p^*, \gamma^*) \in \mathcal{B}} \{-f^*(p^*) - \gamma^{*T} b\}\,,$$

where

$$\begin{aligned} \mathcal{B} &= \{(p^*,\, \gamma^*) : \gamma^* \underset{K_1^*}{\leqq} 0\,,\ -p^* + A^T\gamma^* \underset{K_0^*}{\leqq} 0\} \\ &= \{(p^*,\, \gamma^*) : \gamma^* = 0\,,\ -p^* + A^T\gamma^* = 0\} \\ &= \{(0,0)\} \end{aligned}$$

because $K_0^* = \{0\}$, $K_1^* = \{0\}$, i.e. $\sup_{(p^*, \gamma^*) \in \mathcal{B}} \{-f^*(p^*) - \gamma^{*T} b\} = -f^*(0)$.

This is the well-known trivial relation

$$-f^*(0) = \inf_{x \in \mathbb{R}^n} f(x) \quad \text{coming from} \quad f^*(0) = \sup_{x \in \mathbb{R}^n} \{0^T x - f(x)\} = -\inf_{x \in \mathbb{R}^n} f(x)\,.$$

For investigating later the multiobjective duality to (P) we need optimality conditions regarding to the scalar problem (P_λ) and its dual (P_λ^*). These are formulated in the following theorem.

Theorem 2

(a) Under the assumptions of Theorem 1 let $\bar{x}$ be a solution to (P_λ)

$$\sum_{i=1}^{m} \lambda_i f_i(\bar{x}) = \min_{x \in \mathcal{A}} \left\{ \sum_{i=1}^{m} \lambda_i f_i(x) \right\}\,.$$

Then a tupel $(\bar{p}_1^*, \ldots, \bar{p}_m^*, \bar{\gamma}^*)$, $\bar{p}_i^* \in \mathbb{R}^n$, $i = 1, \ldots, m$, $\bar{\gamma}^* \in \mathbb{R}^\ell$ exists fulfilling the inequalities

$$\bar{\gamma}^* \underset{K_1^*}{\leqq} 0\,, \; -\sum_{i=1}^m \lambda_i \bar{p}_i^* + A^T \bar{\gamma}^* \underset{K_0^*}{\leqq} 0$$

such that the following optimality conditions are satisfied

$$\begin{aligned}
&(i) \quad f_i^*(\bar{p}_i^*) + f_i(\bar{x}) = \bar{p}_i^{*T} \bar{x}\,,\; i = 1, \ldots, m,\\
&(ii) \quad \left(- \sum_{i=1}^m \lambda_i \bar{p}_i^* + A^T \bar{\gamma}^* \right)^T \bar{x} = 0,\\
&(iii) \quad \bar{\gamma}^{*T}(A\bar{x} + b) = 0.
\end{aligned}$$

(b) Let $\bar{x}$ be admissible to (P_λ) and $(\bar{p}_1^*, \ldots, \bar{p}_m^*, \bar{\gamma}^*)$ be admissible to (P_λ^*) satisfying $(i), (ii), (iii)$.

Then $\bar{x}$ and $(\bar{p}_1^*, \ldots, \bar{p}_m^*, \bar{\gamma}^*)$ turn out to be solutions to (P_λ) and (P_λ^*), respectively.

Remark:

(a) As well-known in convex optimization the optimality conditions are necessary and sufficient. But for the sufficiency the constraint qualification is not necessary.

(b) The conditions (ii) and (iii) have the well-known structure of so-called complementary slackness conditions.

(c) The tupel $(\bar{p}_1^*, \ldots, \bar{p}_m^*, \bar{\gamma}^*)$ in (a) (of Theorem 2) even represents a solution of the dual problem (P_λ^*) (cf. the proof).

(d) The condition (i) shows that the so-called Young inequality $f_i(x) + f_i^*(p_i^*) \geq p_i^{*T} x$ is fulfilled as equality. This means that $\bar{p}_i^*$ belongs to the subdifferential of f_i at $\bar{x}$, *i.e.* $\bar{p}_i^* \in \partial f_i(\bar{x})$ and vice versa, whence $\bar{x} \in \partial f_i^*(\bar{p}_i^*)$.

Therefore condition (ii) in case of $K_0 = \mathbb{R}_+^n$ and $\bar{x} \in \operatorname{int} \mathbb{R}_+^n$ may be written $-\sum_{i=1}^m \lambda_i \bar{p}_i^* + A^T \bar{\gamma}^* = 0$ and hence $A^T \bar{\gamma}^* \in \sum_{i=1}^m \lambda_i \partial f_i(\bar{x})$. Anyway, if a solution $(\bar{p}_1^*, \ldots, \bar{p}_m^*, \bar{\gamma}^*)$ to (P_λ^*) is known then the condition $(i), (ii)$ and (iii) permit to identify a solution to (P_λ).

Proof:

(a) Let $\bar{x}$ be a solution to (P_λ). Then because of theorem 1 (strong duality) a solution $(\bar{p}_1^*, \ldots, \bar{p}_m^*, \bar{\gamma}^*)$ to (P_λ^*) exists and the objective function values are equal.

This means

$$\sum_{i=1}^{m} \lambda_i f_i(\bar{x}) = -\sum_{i=1}^{m} \lambda_i f_i(\bar{p}_i^*) - \bar{\gamma}^{*T} b \,. \tag{3}$$

Adding $\sum_{i=1}^{m} \lambda_i \bar{p}_i^{*T} \bar{x} - \sum_{i=1}^{m} \lambda_i \bar{p}_i^{*T} \bar{x} + (A^T\bar{\gamma}^*)^T \bar{x} - (A^T\bar{\gamma}^*)^T \bar{x} = 0$ to (3) yields after some transformations

$$\begin{aligned} 0 = & \sum_{i=1}^{m} \lambda_i [f_i^*(\bar{p}_i^*) + f_i(\bar{x})] + \bar{\gamma}^{*T} b - \sum_{i=1}^{m} \lambda_i \bar{p}_i^{*T} \bar{x} + \sum_{i=1}^{m} \lambda_i \bar{p}_i^{*T} \bar{x} + (A^T\bar{\gamma}^*)^T \bar{x} - (A^T\bar{\gamma}^*)^T \bar{x} \\ = & -\sum_{i=1}^{m} \lambda_i [f_i^*(\bar{p}_i^*) - (\bar{p}_i^{*T} - f_i(\bar{x}))] + \left(-\sum_{i=1}^{m} \lambda_i \bar{p}_i^* + A^T \bar{\gamma}^* \right)^T (-\bar{x}) + \bar{\gamma}^{*T}(A\bar{x} + b) \,. \end{aligned} \tag{4}$$

Because of the definition of the conjugate function

$$f_i^*(\bar{p}_i^*) = \sup_{x \in \mathbb{R}^n} \{\bar{p}_i^{*T} x - f_i(x)\} \geqq \bar{p}_i^{*T} \bar{x} - f_i(\bar{x}) \qquad \text{follows}$$

$$f_i^*(\bar{p}_i^*) - (\bar{p}_i^{*T} \bar{x} - f_i(\bar{x})) \geqq 0 \,.$$

Further, because of $\quad \bar{x} \underset{K_0}{\geqq} 0 \quad$ and $\quad -\sum_{i=1}^{m} \lambda_i \bar{p}_i^* + A^T \bar{\gamma}^* \underset{K_0^*}{\leqq} 0 \quad$ it is

$$\left(-\sum_{i=1}^{m} \lambda_i \bar{p}_i^* + A^T \bar{\gamma}^* \right)^T (-\bar{x}) \geqq 0 \quad \text{and since} \qquad \bar{\gamma}^* \underset{K_1^*}{\leqq} 0 \text{ and } A\bar{x} + b \underset{K_1}{\leqq} 0$$

$$\text{follows } \bar{\gamma}^{*T}(A\bar{x} + b) \geqq 0 \,.$$

Now (4) implies that all those expressions must be equal to zero. This gives the optimality conditions (i), (ii) and (iii).

(b) All calculations and transformations done within part (a) may be carried out in the inverse direction starting from the conditions (i), (ii) and (iii). Thus the quation (3) results which is strong duality and shows $\bar{x}$ solves (P_λ) and $(\bar{p}_i^*, \ldots, \bar{p}_m^*, \bar{\gamma}^*)$ solves (P_λ^*) . □

3. The multiobjective dual problem

Now, with the above preparation we are able to formulate a multiobjective dual problem to (P).

First of all we introduce an usual definition of weak and strong duality in vector optimization.

Let be given two multiobjective optimization problems, a minimum problem

$$\underset{x \in \mathcal{A}}{\mathrm{v} - \min}\ F(x) \tag{5}$$

and a maximum one

$$\underset{y \in \mathcal{B}}{\mathrm{v} - \max}\ G(x) \tag{6}$$

where $F(x), G(y) \in \mathbb{R}^m$.

Definition 3

Between (5) and (6) there is weak duality if there is no $x \in \mathcal{A}$ and no $y \in \mathcal{B}$ fulfilling

$G(y) \underset{\mathbb{R}^m_+}{\geqq} F(x)$ and $G(y) \neq F(x)$.

Remark:

(a) Here the partial ordering in $\mathbb{R}^m$ given by $\mathbb{R}^m_+$ is considered. But of course it is possible to underlay another partial ordering in $\mathbb{R}^m$ (or in another objective space Z). Then the definition of efficient solutions has to be changed by substituting the corresponding partial ordering (ordering cone, respectively).

(b) Obviously, this definition represents a natural generalization of the so-called weak duality within the scalar mathematical programming theory as verified above for (P_λ) and (P^*_λ).

If under the supposition of weak duality there are elements x_0 and y_0 such that $F(x_0) = G(y_0)$, thus, as in scalar optimization, we call this strong duality. The elements x_0 and y_0 are then efficient to (5) and (6), respectively, as can be proved easily (cf. G"opfert/Nebse).

But this strong duality is connected with the point $(F(x_0)(= G(y_0))$ and thus with x_0 and y_0. So this strong duality is a local property. It may happen that for another efficient solution $x_1 \in \mathcal{A}$ there is no $y_1 \in \mathcal{B}$ realizing $F(x_1) = G(y_1)$.

Therefore, one normally is interested in such a global form of strong duality where to each properly efficient point $x \in \mathcal{A}$ of (5) there is a point $y \in \mathcal{B}$ (which then necessarily is efficient to (6)) with $F(x) = G(y)$ or vice versa.

We will later create this global form of strong duality for our original multiobjective problem (P).

Now a dual multiobjective optimization problem (P^*) to (P) is introduced by

$$(P^*) \quad \underset{(p^*,\delta^*)\in\mathcal{B}}{\text{v}-\max} \quad G(p^*,\delta^*)$$

with

$$G(p^*,\delta^*) = \begin{pmatrix} g_1(p^*,\delta^*) \\ \vdots \\ g_m(p^*,\delta^*) \end{pmatrix} = \begin{pmatrix} -f_1^*(p_1^*) - \delta_1^{*T} b \\ \vdots \\ f_m^*(p_m^*) - \delta_m^{*T} b \end{pmatrix}$$

and with the dual variables

$$p^* = \ (p_1, \ldots, p_m^*),\ p_i^* \in \mathbb{R}^n,\ \delta^* = (\delta_1^*, \ldots, \delta_m^*),\ \ \delta_i^* \in \mathbb{R}^\ell, i = 1, \ldots, m,$$

and with the set of constraints

$$\begin{aligned} \mathcal{B} = \ & \{(p^*,\delta^*) : \exists \lambda_i > 0, 1 = 1, \ldots, m, \qquad \text{such that} \\ & \sum_{i=1}^m \lambda_i \delta_i^* \underset{K_1^*}{\leqq} 0,\ \sum_{i=1}^m \lambda_i(-p_i^* + A^T \delta_i^*) \underset{K_0^*}{\leqq} 0\}. \end{aligned} \tag{7}$$

With the symbolic notation "v-max" we mean again (in an analogous manner to "v-min" for (P)) efficient solutions, but now in the sense of a maximum, therefore also called maximal (or Pareto-maximal) elements.

Definition 4 An element $(\bar{p}^*, \bar{\delta}^*) \in \mathcal{B}$ is said to be efficient or maximal (or Pareto-maximal) for (P^*) if from

$$G(p^*,\delta^*) \underset{\mathbb{R}_+^m}{\geqq} G(\bar{p}^*,\bar{\delta}^*) \text{ for } \quad (p^*,\delta^*) \in \mathcal{B}$$

follows $G(p^*,\delta^*) = G(\bar{p}^*,\bar{\delta}^*)$.

First, we will show that we are entitled to call (P^*) a dual problem to (P) because the weak duality property according to Definition 3 may be pointed out. Afterwards, strong duality will be established. This follows within the next section.

4. Weak and strong duality

The following theorem states the weak duality assertion (cf. Definition 3).

Theorem 3

There is no $x \in \mathcal{A}$ and no $(p^*, \delta^*) \in \mathcal{B}$ fulfilling $G(p^*, \delta^*) \underset{\mathbb{R}^m_+}{\geqq} F(x)$ and $G(p^*, \delta^*) \neq F(x)$.

For the proof we refer to [...].

The following theorem expresses the strong duality in the global sense observed in section 3.

Theorem 4

Assume the existence of an element $x' \in \text{int}\,(\bigcap\limits_{i=1}^{m} \text{dom}\, f_i)$ fulfilling $x' \underset{K_0}{\geqq} 0$ and $Ax' + b \in -\text{int}K_1$. Assume $b \neq (0, \ldots, 0)^T$. Let $\bar{x}$ be a properly efficient element to (P). Then an efficient solution $(\bar{p}^*, \bar{\delta}^*) \in \mathcal{B}$ to the dual problem (P^*) exists and the strong duality is true $F(\bar{x}) = G(\bar{p}^*, \bar{\delta}^*)$.

Proof:

Assume $\bar{x}$ to be properly efficient to (P). From Definition 2 follows the existence of a corresponding scalarizing vector $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \text{int}\,\mathbb{R}^m_+$ such that $\bar{x}$ solves (P_λ). Theorem 1 assures the existence of an element $(\bar{p}^*, \bar{\gamma}^*)$ to the dual problem (P^*_λ) and Theorem 2 and the attached remarks say that the optimality condition (i), (ii) and (iii) of Theorem 2 are satisfied.

Let us define the elements $\bar{\delta}^*_i, i = 1, \ldots, m$, by means of $\bar{x}$ and $(\bar{p}^*, \bar{\gamma}^*)$.

$$\bar{\delta}^*_i = \begin{cases} -\frac{\bar{p}_i^{*T}\bar{x}}{\bar{\gamma}^{*T}b}\bar{\gamma}^* \text{ if } \bar{\gamma}^{*T}b \neq 0\,, \\ \frac{1}{m\lambda_i}\bar{\gamma}^* - (\bar{p}_i^{*T}\bar{x})\tilde{\gamma}^* \quad \text{with} \quad \tilde{\gamma}^* \in \mathbb{R}^l : \tilde{\gamma}^{*T}b = 1 \text{ if } \bar{\gamma}^{*T}b = 0\,. \end{cases} \tag{8}$$

Of course such a $\tilde{\gamma}^*$ exists, e.g. $\tilde{\gamma}^* = \frac{b}{||b||^2}$ may be chosen with $||b||$ the Euclidean norm of $b \in \mathbb{R}^\ell$. No it is verified that $(\bar{p}^*, \bar{\delta}^*)$, $\bar{\delta}^* = (\bar{\delta}^*_1, \ldots, \bar{\delta}^*_m)$, is admissible to (P^*) and satisfies $F(\bar{x}) = G(\bar{p}^*, \bar{\delta}^*)$, which claims the strong duality and the efficiency of $(\bar{p}^*, \bar{\delta}^*)$ to (P^*) according to the remark attached to Definition 3.

Therefore $(\bar{p}^*, \bar{\delta}^*) \in \mathcal{B}$ will be proved. First let be $\bar{\gamma}^{*T}b \neq 0$. Then (9), (ii) and (iii) from

Theorem 2 imply

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i \bar{\delta}_i^* &= \sum_{i=1}^{m} \lambda_i \tfrac{1}{\bar{\gamma}^{*T} b} (-\bar{p}_i^{*T} \bar{x}) \bar{\gamma}^* \\
&= \tfrac{1}{\bar{\gamma}^{*T} b} \left(- \sum_{i=1}^{m} \lambda_i \, \bar{p}_i^{*T} \right)^T \bar{x} \bar{\gamma}^* \\
&= \tfrac{1}{\bar{\gamma}^{*T} b} \left(-A^T \bar{\gamma}^* \right)^T \bar{x} \bar{\gamma}^* \\
&= \tfrac{1}{\bar{\gamma}^{*T} b} \left(\bar{\gamma}^{*T} b \right) \bar{\gamma}^* = \bar{\gamma}^* \,.
\end{aligned}$$

For $\bar{\gamma}^{*T} b = 0$ we obtain in an analogous manner

$$\begin{aligned}
\sum_{i=1}^{m} \lambda_i \bar{\delta}_i^* &= \sum_{i=1}^{m} \lambda_i \tfrac{1}{m\lambda_i} \bar{\gamma}^* + \sum_{i=1}^{m} \lambda_i \, (-\bar{p}^{*T} \bar{x}) \tilde{\gamma}^* \\
&= \bar{\gamma}^* + \left(- \sum_{i=1}^{m} \lambda_i \, \bar{p}_i^* \right)^T \bar{x} \tilde{\gamma}^* \\
&= \bar{\gamma}^* + \left(-A^T \bar{\gamma}^* \right)^T \bar{x} \tilde{\gamma}^* \\
&= \bar{\gamma}^* + \left(\bar{\gamma}^{*T} b \right) \tilde{\gamma}^* = \bar{\gamma}^* \,.
\end{aligned}$$

¿From $(\bar{p}^*, \bar{\gamma}^*) \in \mathcal{B}_\lambda$ follows $\bar{\gamma}^* \underset{K_1^*}{\leqq} 0$ and therefore $\sum_{i=1}^{m} \lambda_i \bar{\delta}_i^* \underset{K_1^*}{\leqq} 0$ as well as $\sum_{i=1}^{m} \lambda_i \, (-\bar{p}_i^* + A^T \bar{\delta}_i^*) = \sum_{i=1}^{m} \lambda_i \, (-\bar{p}_i^*) + A^T \bar{\gamma}^* \underset{K_1^*}{\leqq} 0$. This means $(\bar{p}^*, \bar{\delta}^*) \in \mathcal{B}$, i.e. it is admissible to (P^*). Next, we demonstrate the equality of the values of the objective functions $F(\bar{x})$ and $G(\bar{p}^*, \bar{\delta}^*)$. Let us start again with the case $\bar{\gamma}^{*T} b \neq 0$. With (8) and (i) from Theorem 2 holds for $i = 1, \ldots, m$

$$\begin{aligned}
g_i(\bar{p}^*, \bar{\delta}^*) &= -f_i^*(\bar{p}^*) - \bar{\delta}_i^{*T} b \\
&= -f_i^*(\bar{p}_i^*) + \tfrac{1}{\bar{\gamma}^{*T} b} (\bar{p}_i^{*T} \bar{x})(\bar{\gamma}^{*T} b) \\
&= f_i(\bar{x}) - \bar{p}_i^{*T} \bar{x} + \bar{p}_i^{*T} \bar{x} = f_i(\bar{x}) \,.
\end{aligned}$$

In case of $\bar{\gamma}^{*T} b = 0$ may be calculated

$$\begin{aligned}
g_i(\bar{p}^*, \bar{\delta}^*) &= -f_i^*(\bar{p}_i^*) - \bar{\delta}_i^{*T} b \\
&= -f_i^*(\bar{p}_i^*) - \tfrac{1}{m\lambda_i} \bar{\gamma}^{*T} b + \bar{p}_i^{*T} \bar{x})(\tilde{\gamma}^{*T} b) \\
&= f_i(\bar{x}) - \bar{p}_i^{*T} x + \bar{p}_i^{*T} \bar{x} = f_i(\bar{x}) \,.
\end{aligned}$$

Alltogether, $(\bar{p}^*, \bar{\delta}^*)$ must be efficient to (P^*) and the proof is complete. □