On the Relations Between Different Dual Problems in Convex Mathematical Programming

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Abstract. Using the Fenchel-Rockafellar approach for the convex mathematical programming problem with inequality constraints different dual optimization problems by means of distinct perturbations of the primal problem are derived and studied. The classical Lagrange dual problem is one of those dual problems obtained by the perturbation of the right hand side of the inequality constraints.
For the various dual problems equality/inequality relations between the optimum values are verified under appropriate assumptions. Moreover, the duality relations to the primal problem are considered, in particular strong duality. Using the dual problems some optimality conditions are established.
The results are illustrated by some examples.
The application and usefulness for the construction of general multiobjective dual problems to the general multiobjective convex optimization problem is mentioned.

1 The constrained optimization problem and its conjugate duals

Consider the mathematical programming problem (called primal problem)

$$(P) \qquad \inf_{x\in G} f(x), \qquad G = \{x \in X : g(x) \leqq 0\},$$

with $X \subseteq \mathbb{R}^n$, $G \neq \emptyset$, $f : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $dom f = X$ (after redefining of an original $f : \mathbb{R}^n \to \mathbb{R}$), $g = (g_1, \dots, g_k)^T : \mathbb{R}^n \to \mathbb{R}^k$, where $g(x) \leqq 0$ means $g_i(x) \leq 0, i = 1, \dots, k$.

Applying the well-known so-called Fenchel-Rockafellar approach of defining a dual programming problem to a given primal problem we consider different perturbations of the primal problem (P) by introducing respective perturbation functions Φ_i :

$$(i) \qquad \Phi_1(x,q) = \begin{cases} f(x) & \text{, if } x \in X,\ g(x) \leqq q, \\ +\infty & \text{, otherwise,} \end{cases}$$

$$(ii) \qquad \Phi_2(x,p) = \begin{cases} f(x+p) & \text{, if } x \in X,\ g(x) \leqq 0, \\ +\infty & \text{, otherwise,} \end{cases}$$

$$(iii)\ \Phi_3(x,p,q) = \begin{cases} f(x+p) & \text{, if } x \in X,\ g(x) \leqq q, \\ +\infty & \text{, otherwise,} \end{cases}$$

$p \in \mathbb{R}^n$, $q \in \mathbb{R}^k$ are the respective perturbation variables.
For (i), (ii) and (iii) we consider the corresponding perturbed problems to (P) $\inf\limits_{x\in\mathbb{R}^n} \Phi_1(x,q)$, $\inf\limits_{x\in\mathbb{R}^n} \Phi_2(x,p)$ and $\inf\limits_{x\in\mathbb{R}^n} \Phi_3(x,p,q)$.

Setting the perturbation variables p and q equal to the zero vector in all three cases one gets (P). Now, dual problems may be defined by the conjugates Φ_i^* of the perturbation functions Φ_i as follows

$$(P_1^*) \quad \sup_{q^*\in\mathbb{R}^k} \{-\Phi_1^*(0,q^*)\},$$

$$(P_2^*) \quad \sup_{p^*\in\mathbb{R}^n} \{-\Phi_2^*(0,p^*)\},$$

$$(P_3^*) \quad \sup_{\substack{p^*\in\mathbb{R}^n \\ q^*\in\mathbb{R}^k}} \{-\Phi_3^*(0,p^*,q^*)\}$$

(recall the definition of the conjugate function $f^* : \mathbb{R}^n \to \bar{\mathbb{R}}$ to a function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ by $f^*(x^*) = \sup\limits_{x\in\mathbb{R}^n} \{x^{*T}x - f(x)\}$).

Calculating the respective conjugate function Φ_i^* we obtain the following three dual problems (P_i^*), $i = 1,2,3$, to (P):

$$(P_1^*) \quad \sup_{q^*\geqq 0} \inf_{x\in X} \left\{ f(x) + q^{*T} g(x) \right\},$$

$$(P_2^*) \quad \sup_{p^*\in\mathbb{R}^n} \left\{ -f^*(p^*) + \inf_{x\in G} p^{*T} x \right\},$$

$$(P_3^*) \quad \sup_{\substack{p^*\in\mathbb{R}^n \\ q^*\geqq 0}} \left\{ -f^*(p^*) + \inf_{x\in X} [p^{*T}x + q^{*T}g(x)] \right\}.$$

We observe that (P_1^*) indeed is the well-known and classical Lagrange dual problem with the Lagrangian $L(x,q^*) = f(x) + q^{*T}g(x), q^* \geqq 0$. By construction (cf.[1]) there is $\sup(P_i^*) \leq \inf(P)$ (weak duality).

2 The relations between the optimum values of the duals

Proposition 1 *It holds*

$$(i) \quad \sup(P_1^*) \geq \sup(P_3^*),$$

$$(ii) \quad \sup(P_2^*) \geq \sup(P_3^*).$$

Proof: (i) Let $q^* \geqq 0$ and $p^* \in \mathbb{R}^n$ be fixed. By the definition of the conjugate function we have for each $x \in X$ $f(x) \geqq p^{*T}x - f^*(p^*)$ (Young inequality). Adding to both sides $q^{*T}g(x)$ we obtain for each $x \in X$ $f(x) + q^{*T}g(x) \geqq -f^*(p^*) + p^{*T}x + q^{*T}g(x)$. This means that for each $q^* \geqq 0$ and $p^* \in \mathbb{R}^n$ holds

$$\inf_{x\in X}[f(x) + q^{*T}g(x)] \geqq -f^*(p^*) + \inf_{x\in X}[p^{*T}x + q^{*T}g(x)]. \tag{1}$$

We now may calculate the supremum over all $p^* \in \mathbb{R}^n$ and $q^* \geqq 0$ which implies the assertion.
(ii) Let $p^* \in \mathbb{R}^n$ be fixed.
For each $q^* \geqq 0$ the following inequalities are true

$$\inf_{x\in X}[p^{*T}x + q^{*T}g(x)] \leqq \inf_{x\in G}[p^{*T}x + q^{*T}g(x)] \leqq \inf_{x\in G} p^{*T}x.$$

This implies for each $p^* \in \mathbb{R}^n$

$$\sup_{q^*\geqq 0} \inf_{x\in X}[p^{*T}x + q^{*T}g(x)] \leqq \inf_{x\in G} p^{*T}x. \tag{1'}$$

Adding $-f^*(p^*)$ and computing the supremum over $p^* \in \mathbb{R}^n$ yields the wanted result. ■

The following examples show that the inequalities in Proposition 1 may be fulfilled strictly.

Example 1: Let be $X = [0, +\infty]$, $f : \mathbb{R} \to \bar{\mathbb{R}}$, $f(x) = \begin{cases} -x^2 & \text{if } x \in X, \\ +\infty & \text{, otherwise,} \end{cases}$
$g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 - 1$ and consider (P) for those dates.
A straightforward calculation shows that the supremum of the Lagrange dual is $\sup(P_1^*) = -1$. On the other hand for the dual (P_3^*) we have $\sup(P_3^*) = -\infty$.

Example 2: Let now (P) be defined by $X = [0, +\infty)$, $f : \mathbb{R} \to \bar{\mathbb{R}}$, $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 1 - x^2$, $f(x) = \begin{cases} x & \text{if } x \in X, \\ +\infty & \text{, otherwise.} \end{cases}$ Then follows $sup(P_2^*) = 1 > \sup(P_3^*) = 0$.

But, under additional assumptions, e.g. convexity and regularity conditions, one can show that the inequalities in Proposition 1 are fulfilled as equalities.

Theorem 1 *Assume that $X \neq \emptyset$ is a convex set and f and g_i, $i = 1, \ldots, k$, are convex functions. Then it holds* $\sup(P_1^*) = \sup(P_3^*)$.

Theorem 1 results immediately from the following strong duality assertion.

Proposition 2 *Under the assumptions of Theorem 1 there is for all*
$q^* \geqq 0$

$$\inf_{x \in X} [f(x) + q^{*T} g(x)] = \sup_{p^* \in \mathbb{R}^n} \left\{ -f^*(p^*) + \inf_{x \in X} [p^{*T} x + q^{*T} g(x)] \right\}.$$

If $\inf_{x \in X} [f(x) + q^{*T} g(x)]$ is finite then the supremum problem on the right hand side has a solution p_0^* such that "sup" may be substituted by "max".

Proof: Let $q^* \in \mathbb{R}^k$, $q^* \geqq 0$, be fixed and define $\alpha := \inf_{x \in X} [f(x) + q^{*T} g(x)]$.

Because of $X \neq \emptyset$ follows $\alpha \in [-\infty, +\infty)$. (1) implies after calculation of the supremum over $p^* \in \mathbb{R}^n$

$$\alpha \geqq \sup_{p^* \in \mathbb{R}^n} \left\{ -f^*(p^*) + \inf_{x \in X} [p^{*T} x + q^{*T} g(x)] \right\}. \tag{2}$$

For $\alpha = -\infty$ the statement is trivial.
For $\alpha > -\infty$ we consider the sets

$$C := \operatorname{epi} f = \{(x, \mu) : x \in X,\ \mu \in \mathbb{R},\ f(x) \leqq \mu\} \subset \mathbb{R}^{n+1},$$

$$D := \left\{(x, \mu) : x \in X,\ \mu \in \mathbb{R},\ \mu + q^{*T} g(x) \leqq \alpha\right\} \subset \mathbb{R}^{n+1}.$$

Obviously, $C, D \neq \emptyset$, are convex sets and $(\operatorname{rint} C) \cap D = \emptyset$ (rint C denotes the relative interior of C) ($(\operatorname{rint} C) \cap D \neq \emptyset$ causes a contradiction to the definition of α).
Therefore also $(\operatorname{rint} C) \cap (\operatorname{rint} D) = \emptyset$ meaning C and D are properly separable (cf. [3]), i.e. there exist $(p^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R}$, $(p^*, \mu^*) \neq (0, 0)$, and $\alpha^* \in \mathbb{R}$ such

that for the separating hyperplane $H = \{(x,\mu) : p^{*T}x + \mu^*\mu = \alpha^*\}$ holds

$$(a) \qquad \inf\{p^{*T}c + \mu^*\mu : (x,\mu) \in D\} \geqq \alpha^* \geqq \sup\{p^{*T}x + \mu^*\mu : (x,\mu) \in C\}, \tag{3}$$

$$(b) \qquad \sup\{p^{*T}x + \mu^*\mu : (x,\mu) \in D\} > \inf\{p^{*T}x + \mu^*\mu : (x,\mu) \in C\}. \tag{4}$$

Assuming $\mu^* = 0$ allows to conclude a contradiction to (4) because one can see easily that in this case $p^{*T}x = \alpha^*$ for all $x \in X$.
Also $\mu^* > 0$ is not possible since then follows from (3) with letting converge $\mu \to +\infty$ that the supremum on the right hand side of (3) is $+\infty$, therefore $\alpha^* = +\infty$, contradicting $\alpha^* \in \mathbb{R}$ (finite). Therefore must be $\mu^* < 0$.
We may divide (3) by $(-\mu^*)$ having

$$\inf\{p_0^{*T}x - \mu : (x,\mu) \in D\} \geqq \alpha_0^* \geqq \sup\{p_0^{*T}x - \mu : (x,\mu) \in C\},$$

where $\alpha_0^* = \alpha^*/(-\mu^*)$ and $p_0^* = p^*/(-\mu^*)$. Hence,

$$p_0^{*T}x - \mu \leqq \alpha_0^* \ \forall (x,\mu) \in C, \tag{5}$$

$$\alpha_0^{*T} \leqq p_0^{*T}x - \mu \ \forall (x,\mu) \in D. \tag{6}$$

(5) yields

$$p_0^{*T}x - f(x) \leqq \alpha_0^* \ \forall x \in X \tag{7}$$

as a consequence of $(x, f(x)) \in C$. Calculating the supremum over $x \in X$ of the left hand side of (7) generates the conjugate $f^*(p_0^*)$, i.e.

$$f^*(p_0^*) \leqq \alpha_0^*. \tag{8}$$

Furthermore, there is $(x, \alpha - q^{*T}g(x)) \in D \ \forall x \in X$ (cf. definition of D). Then by (6) $\alpha_0^* \leqq p_0^{*T}x - (\alpha - q^{*T}g(x)) \ \forall x \in X$, i.e. $\alpha_0^* + \alpha \leqq \inf\limits_{x \in X}[p_0^{*T}x + q^{*T}g(x)]$, which gives with (8)

$$\alpha \leqq -f^*(p_0^*) + \inf_{x \in X}[p_0^{*T}x + q^{*T}g(x)].$$

Together with (2) it results

$$\alpha = \sup_{p^* \in \mathbb{R}^n} \left\{ -f^*(p^*) + \inf_{x \in X} [p^{*T}x + q^{*T}g(x)] \right\}.$$

Finally, by the definition of α this means $\inf_{x \in X}[f(x) + q^{*T}g(x)] = -f^*(p_0^*)$ $+ \inf_{x \in X}[p_0^{*T}x + q^{*T}g(x)] = \max_{p^* \in \mathbb{R}^n} \left\{ -f^*(p^*) + \inf_{x \in X}[p^{*T}x + q^{*T}g(x)] \right\}$. ■

In order to have $\sup(P_2^*) = \sup(P_3^*)$ we need a regularity condition (constraint qualification (CQ)). Therefore consider for $g(x) = (g_1(x), \ldots, g_k(x))^T$ the sets $L = \{i \in \{1, \ldots, k\} : g_i \text{ is an affine function}\}$,
$N = \{i \in \{1, \ldots, k\} : g_i \text{ is not an affine function}\}$.

CQ: There exists an element $x' \in \operatorname{rint} X$ such that $g_i(x') < 0$ for $i \in N$ and $g_i(x') \leqq 0$ for $i \in L$.

Theorem 2 *Assume that X is a convex set, g_i, $i = 1, \ldots, k$, are convex functions such that $G = \{x \in X : g(x) \leqq 0\} \neq \emptyset$ and (CQ) is fulfilled. Then it holds*

$$\sup(P_2^*) = \sup(P_3^*).$$

Proof: For $p^* \in \mathbb{R}^n$ fixed we first show that

$$\sup_{q^* \geqq 0} \inf_{x \in X} [p^{*T}x + q^{*T}g(x)] = \inf_{x \in G} p^{*T}x. \tag{9}$$

Let be $\beta := \inf_{x \in G} p^{*T}x$. Because $G \neq \emptyset$ follows $\beta \in [-\infty, \infty)$.
If $\beta = -\infty$ then by (1') follows

$$\sup_{q^* \geqq 0} \inf_{x \in X} [p^{*T}x + q^{*T}g(x)] = -\infty = \inf_{x \in G} p^{*T}x.$$

Let now be $\beta \in (-\infty, +\infty)$.
If g_L and g_N denote the vector functions with the affine components and the non-affine components of g, respectively, then the system of inequalities $p^{*T}x - \beta < 0$, $g_N(x) \leqq 0$, $g_L(x) \leqq 0$ has no solution in X. Using Theorem 2. 104 in [2] (characterizing the solvability of inequality systems of the above type with (CQ) fulfilled) we observe the existence of $u^* > 0$, $q^* = (v^*, w^*) \in \mathbb{R}_+^k$ such that for each $x \in X$

$$u^*(p^{*T}x - \beta) + q^{*T}g(x) = u^*(p^{*T}x - \beta) + v^{*T}g_N(x) + w^{*T}g_L(x) \geqq 0.$$

Dividing that inequality by u^* we obtain with $q_0^* = q^*/u^*$

$$p^{*T}x - \beta + q_0^{*T}g(x) \geqq 0 \ \forall x \in X$$

and, equivalently,

$$\inf_{x \in X} [p^{*T}x + q_0^{*T}g(x)] \geqq \beta.$$

This last inequality and (1') imply (9). By adding $-f^*(p^*)$ to (9) and calculating the supremum subject to $p^* \in \mathbb{R}^n$ one has $\sup(P_2^*) = \sup(P_3^*)$. ■

3 The strong duality and optimality conditions

Using Theorem 1 and Theorem 2 and verifying that $\inf(P) = \sup(P_1^*)$, which can be done applying the solvability condition of Theorem 2. 104 in [2] to the system $f(x) - \inf(P) < 0$, $g(x) \leqq 0$ in analogous manner as within the proof of Theorem 2, the following strong duality assertion may be derived.

Theorem 3 *Under the assumptions of both Theorem 1, 2 and if* $\inf(P)$ *is finite strong duality holds and* $(P_i^*), i = 1, 2, 3,$ *have solutions*

$$\inf(P) = \max(P_1^*) = \max(P_2^*) = \max(P_3^*).$$

By means of that strong duality one can use each of the different three duals to conclude optimality conditions. We want to restrict ourselves to present the optimality conditions arising from the strong duality between (P) and (P_3^*).

Theorem 4 (a) *Let the assumptions of Theorem 3 be fulfilled and let* x_0 *be a solution of* (P). *Then there exists a tupel* $(p_0^*, q_0^*) \in \mathbb{R}^n \times \mathbb{R}^k, q_0^* \geqq 0,$ *such that the following optimality conditions are satisfied*

$$(i) \qquad f(x_0) + f^*(p_0^*) = p_0^{*T}x_0,$$

$$(ii) \qquad q_0^{*T}g(x_0) = 0,$$

$$(iii) \qquad p_0^{*T}(x - x_0) \geq q_0^{*T}(g(x) - g(x_0)) \ \forall x \in X.$$

(b) *Let* $x_0 \in G$ *and* $(p_0^*, q_0^*) \in \mathbb{R}^n \times \mathbb{R}^k, q_0^* \geqq 0,$ *satisfying (i), (ii) and (iii). Then* x_0 *and* (p_0^*, q_0^*) *turn out to be solutions of* (P) *and* (P_3^*), *respectively, and strong duality holds.*

Proof: **(a)** Let x_0 be a solution of (P). By Theorem 3 there exists a solution to (P_3^*) $(p_0^*, q_0^*) \in \mathbb{R}^n \times \mathbb{R}^k$, $q_0^* \geqq 0$, such that

$$f(x_0) = -f^*(p_0^*) + \inf_{x \in X} [p_0^{*T} x + q_0^{*T} g(x)].$$

By adding the term $p_0^{*T} x_0 + q_0^{*T} g(x_0)$ we obtain

$$\begin{aligned} & f(x_0) + f^*(p_0^*) - p_0^{*T} x_0 + p_0^{*T} x_0 + q_0^{*T} g(x_0) \\ & - \inf_{x \in X} [p_0^{*T} x + q_0^{*T} g(x)] - q_0^{*T} g(x_0) = 0. \end{aligned} \tag{10}$$

On the other hand the following inequalities hold $f(x_0) + f^*(p_0^*) - p_0^{*T} x_0 \geqq 0$ (Young inequality), $p_0^{*T} x_0 + q_0^{*T} g(x_0) - \inf_{x \in X} [p_0^{*T} x + q_0^{*T} g(x)] \geqq 0$, $-q_0^{*T} g(x_0) \geqq 0$. By (10) all these inequalities have to be equations which means that (i), (ii) and (iii) are fulfilled.

(b) All calculations done within part **(a)** of the proof may be carried out in the inverse direction starting from (i), (ii) and (iii). Then x_0 solves (P) and (p_0^*, q_0^*) solves (P_3^*) and strong duality holds. ■

Finally, we would like to remark that, in particular, the strong duality between (P) and (P_3^*) may be applied for the construction of a dual problem in multiobjective convex optimization (cf. [4]).

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