Duality for composed convex functions with applications in location theory

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Abstract. In this paper we consider, in a general normed space, the optimization problem with the objective function being a composite of a convex and componentwise increasing function with a vector convex function. Perturbing the primal problem, we obtain, by means of the Fenchel-Rockafellar approach, a dual problem for it. The existence of strong duality is proved and the optimality conditions are derived.

Using this general result, we introduce the dual problem and the optimality conditions for the single facility location problem in a general normed space in which the existing facilities are represented by sets of points.

The classical Weber problem and minmax problem with demand sets are studied as particular cases of this problem.

Keywords. convex optimization – conjugate duality – monotonic norms – location problems – optimality conditions

1 Introduction

This article is motivated by the work of Nickel, Puerto and Rodriguez-Chia ([9]). In [9] they introduced a single facility problem in a general normed space in which the existing facilities are represented by sets of points. For this problem the authors obtained a geometrical characterization of the set of optimal solutions.

The aim of our paper is to construct a dual problem for the optimization problem treated in [9] and for its particular instances, the Weber problem and the minmax problem with demand sets. On the other hand, we show how it is possible to derive the optimality conditions for these optimization problems, via strong duality.

In order to do this, we consider a more general optimization problem and, then, we particularize the results for the location problems in [9]. The optimization problem, from which we start, has as objective function a composite of a convex and componentwise increasing function with a vector convex function. Applying the Fenchel-Rockafellar duality approach and using some appropriate perturbations we construct a dual problem for it. The dual problem is formulated in terms of conjugate functions, and the existence of strong duality is proved. Afterwards, by means of strong duality, we derive the optimality conditions for the primal optimization problem.

In the past, optimization problems with the objective function being a composed convex function have been considered by different authors. We remind here the works [5] and [6], where form of the subdifferential of a composed convex function has been described, and, also, [3], [7] and [8], where some results with regard to duality has been given.

Recently, optimization problems of this type have found applications in goal programming problems [2] and average distance problems [10]. Concerning duality, Volle studied in [12] the same problem as a particular case of a D.C. programming problem. But, as well the dual problem introduced in [12] as the dual problems presented in [3], [7] and [8] are different from the dual proposed by us. A deeper investigation of the relations between these duals will be presented in a forthcoming paper.

2 The optimization problem with a composed convex function as objective function

Let $(X, \|\cdot\|)$ be a normed space, $g_i : X \to \mathbb{R}, i = 1, \ldots, m$, convex and continuous functions and $f : \mathbb{R}^m \to \mathbb{R}$ a convex and componentwise increasing function, i.e. for $y = (y_1, \ldots, y_m)^T, z = (z_1, \ldots, z_m)^T \in \mathbb{R}^m$,

$$y_i \geq z_i, i = 1, \ldots, m \Rightarrow f(y) \geq f(z).$$

The optimization problem which we consider here is the following one

$$(P) \quad \inf_{x \in X} f(g(x)),$$

where $g : X \to \mathbb{R}^m$, $g(x) = (g_1(x), \ldots, g_m(x))^T$.

In this first chapter we find out a dual problem to (P) and prove the existence of weak and strong duality. Moreover, by means of strong duality we derive the optimality conditions for (P).

The approach, we use to find a dual problem to (P), is the so-called Fenchel-Rockafellar approach and it was very well described in [4]. It offers the possibility to construct different dual problems to a primal optimization problem, by perturbing it in different ways (cf. [13], [14] and [15]).

In order to find a dual problem to (P), we consider the following perturbation function $\Psi : \underbrace{X \times \ldots \times X}_{m+1} \times \mathbb{R}^m \to \mathbb{R}$,

$$\Psi(x, q, d) = f((g_1(x + q_1), \ldots, g_m(x + q_m))^T + d),$$

where $q = (q_1, \ldots, q_m) \in X \times \ldots \times X$ and $d \in \mathbb{R}^m$ are the so-called perturbation variables.

Then the dual problem to (P), obtained by using the perturbation function Ψ, will be

$$(D) \quad \sup_{\substack{p_i \in X^*, i=1,\ldots,m, \\ \lambda \in \mathbb{R}^m}} \{-\Psi^*(0, p, \lambda)\},$$

where $\Psi^* : \underbrace{X^* \times \ldots \times X^*}_{m+1} \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is the conjugate function of Ψ. Here, $p_i, i = 1, ..., m$, and $\lambda \in \mathbb{R}^m$ are the dual variables.

We recall that for a function $h : Y \to \mathbb{R}$, Y being a Hausdorff locally convex vector space, its conjugate function $h^* : Y^* \to \mathbb{R} \cup \{+\infty\}$ has the form $h^*(y^*) = \sup_{y \in Y} \{\langle y^*, y\rangle - h(y)\}$. Y^* is the topological dual to Y.

The conjugate function of Ψ can be calculated by the following formula

$$\begin{aligned}\Psi^*(x^*, p, \lambda) = \sup_{\substack{q_i \in X, i=1,\ldots,m, \\ x \in X, d \in \mathbb{R}^m}} & \Big\{ \langle x^*, x\rangle + \sum_{i=1}^m \langle p_i, q_i\rangle + \langle \lambda, d\rangle \\ & -f((g_1(x + q_1), \ldots, g_m(x + q_m))^T + d) \Big\}.\end{aligned}$$

To find these expression, we introduce, at first, the new variable t instead of d and, then, the new variables r_i instead of q_i, by

$$t = d + (g_1(x + q_1), \ldots, g_m(x + q_m))^T \in \mathbb{R}^m$$

and

$$r_i = x + q_i \in X, i = 1, ..., m.$$

This implies

$$\begin{aligned}
\Psi^*(x^*, p, \lambda) &= \sup_{\substack{q_i \in X, i=1,\ldots,m,\\ x \in X, t \in \mathbb{R}^m}} \Big\{ \langle x^*, x\rangle + \sum_{i=1}^m \langle p_i, q_i\rangle \\
&\quad + \big\langle \lambda, t - (g_1(x+q_1), \ldots, g_m(x+q_m))^T \big\rangle - f(t) \Big\} \\
&= \sup_{\substack{r_i \in X, i=1,\ldots,m,\\ x \in X}} \Big\{ \langle x^*, x\rangle + \sum_{i=1}^m \langle p_i, r_i - x\rangle \\
&\quad - \big\langle \lambda, (g_1(r_1), \ldots, g_m(r_m))^T \big\rangle \Big\} + \sup_{t \in \mathbb{R}^m} \{\langle \lambda, t\rangle - f(t)\} \\
&= \sum_{i=1}^m \sup_{r_i \in X} \{\langle p_i, r_i\rangle - \lambda_i g_i(r_i)\} + \sup_{x \in X} \left\langle x^* - \sum_{i=1}^m p_i, x \right\rangle \\
&\quad + f^*(\lambda) \\
&= f^*(\lambda) + \sum_{i=1}^m (\lambda_i g_i)^*(p_i) + \sup_{x \in X} \left\langle x^* - \sum_{i=1}^m p_i, x \right\rangle.
\end{aligned}$$

We have now to consider $x^* = 0$ and, so, the dual problem of (P) has the following form

$$(D) \quad \sup_{\substack{\lambda \in \mathbb{R}^m, p_i \in X^*,\\ i=1,\ldots,m}} \left\{ -f^*(\lambda) - \sum_{i=1}^m (\lambda_i g_i)^*(p_i) + \inf_{x \in X} \left\langle \sum_{i=1}^m p_i, x \right\rangle \right\}.$$

In the objective function of (D), if $\sum\limits_{i=1}^m p_i \neq 0_{X^*}$, there exists $x_0 \in X, x_0 \neq 0_X$, such that $\left\langle \sum\limits_{i=1}^m p_i, x_0 \right\rangle < 0$. But, for all $\alpha > 0$, we have

$$\inf_{x \in X} \left\langle \sum_{i=1}^m p_i, x \right\rangle < \alpha \cdot \left\langle \sum_{i=1}^m p_i, x_0 \right\rangle,$$

and this means that, in this case, $\inf\limits_{x \in X} \left\langle \sum\limits_{i=1}^m p_i, x \right\rangle = -\infty$.

In conclusion, in order to have supremum in (D), we must consider $\sum\limits_{i=1}^m p_i = 0$.

By this, the dual problem of (P) will be

$$(D) \quad \sup_{\substack{\lambda \in \mathbb{R}^m, p_i \in X^*,\\ i=1,\ldots,m, \sum\limits_{i=1}^m p_i = 0}} \left\{ -f^*(\lambda) - \sum_{i=1}^m (\lambda_i g_i)^*(p_i) \right\}. \tag{1}$$

Let us point out that, by the Fenchel-Rockafellar approach, between (P) and (D) weak duality, i.e. $\inf(P) \geq \sup(D)$, always holds (cf. [4]).

But, we are interested in the existence of strong duality $\inf(P) = \sup(D)$. This can be shown, by proving that the problem (P) is stable (cf. [4]). Therefore, we show that the stability criterion described in Proposition III.2.3 in [4] is fulfilled. For the beginning, we need the following proposition.

Proposition 1. *The function* $\Psi : \underbrace{X \times \ldots \times X}_{m+1} \times \mathbb{R}^m \to \mathbb{R}$,

$$\Psi(x, q, d) = f((g_1(x + q_1), \ldots, g_m(x + q_m))^T + d)$$

is convex.

The convexity of Ψ follows from the convexity of the functions f and g and the fact that f is a componentwise increasing function.

Theorem 1 (strong duality for (P)). *If* $\inf(P) > -\infty$, *then the dual problem has a solution and strong duality holds, i.e.*

$$\inf(P) = \max(D).$$

Proof. By Proposition 1, we have that the perturbation function Ψ is convex. Moreover, $\inf(P)$ is a finite number and the function

$$(q_1, \ldots, q_m, d) \longrightarrow \Psi(0, q_1, \ldots, q_m, d)$$

is finite and continuous in $(\underbrace{0, ..., 0}_{m}, 0_{\mathbb{R}^m}) \in \underbrace{X \times \ldots \times X}_{m} \times \mathbb{R}^m$. This means that the stability criterion in Proposition III.2.3 in [4] is fulfilled, which implies that the problem (P) is stable. Finally, the Propositions IV.2.1 and IV.2.2 in [4] conduce us to the desired conclusions. $\square$

The structure of the problem (P) looks like a scalarization of a vector optimization problem by means of the monotonic function f. The results concerning duality for the problem (P) could be used to derive duality statements in the multiobjective optimization. But, this is the subject of some of our present research.

The last part of this section is devoted to the presentation of the optimality conditions for the primal problem (P). They are derived, by the use of the equality between the optimal values of the primal and dual problem.

Theorem 2 (optimality conditions for (P)).

(1) Let $\bar{x} \in X$ *be a solution to* (P). *Then there exist* $\bar{p}_i \in X^*, i = 1, ..., m$, *and* $\bar{\lambda} \in \mathbb{R}^m$, *such that* $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ *is a solution to* (D) *and the following optimality conditions are satisfied*

$$(i)\ f(g(\bar{x})) + f^*(\bar{\lambda}) = \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}),$$

$$(ii)\ \bar{\lambda}_i g_i(\bar{x}) + (\bar{\lambda}_i g_i)^*(\bar{p}_i) = \langle \bar{p}_i, \bar{x} \rangle, i = 1, \ldots, m,$$

$$(iii)\ \sum_{i=1}^m \bar{p}_i = 0.$$

(2) If $\bar{x} \in X$, $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ *is feasible to* (D) *and* (i)-(iii) *are fulfilled, then* $\bar{x}$ *is a solution to* (P), $(\bar{\lambda}, \bar{p}_1, \ldots, \bar{p}_m)$ *is a solution to* (D) *and strong duality holds*

$$f(g(\bar{x})) = -f^*(\bar{\lambda}) - \sum_{i=1}^m (\bar{\lambda}_i g_i)^*(\bar{p}_i).$$

Proof. (1) By Theorem 1, it follows that there exist $\bar{p}_i \in X^*, i = 1, ..., m$, and $\bar{\lambda} \in \mathbb{R}^m$, such that $(\bar{\lambda}, \bar{p}_1, \dots, \bar{p}_m)$ is a solution to (D) and $\inf(P) = \max(D)$. This means that $\sum\limits_{i=1}^{m} \bar{p}_i = 0$ and

$$f(g(\bar{x})) = -f^*(\bar{\lambda}) - \sum_{i=1}^{m} (\bar{\lambda}_i g_i)^*(\bar{p}_i). \tag{2}$$

The last equality is equivalent to

$$0 = f(g(\bar{x})) + f^*(\bar{\lambda}) - \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) + \sum_{i=1}^{m} [\bar{\lambda}_i g_i(\bar{x}) + (\bar{\lambda}_i g_i)^*(\bar{p}_i) - \langle \bar{p}_i, \bar{x} \rangle]. \tag{3}$$

From the definition of the conjugate functions we have that the following so-called Young-inequalities

$$f(g(\bar{x})) + f^*(\bar{\lambda}) \geq \left\langle \bar{\lambda}, g(\bar{x}) \right\rangle = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) \tag{4}$$

and

$$\bar{\lambda}_i g_i(\bar{x}) + (\bar{\lambda}_i g_i)^*(\bar{p}_i) \geq \langle \bar{p}_i, \bar{x} \rangle, i = 1, \dots, m, \tag{5}$$

are true. By (4) and (5), it follows that all the terms of the sum in (3) must be equal to zero. In conclusion, the equalities in (i) and (ii) must hold.

(2) All the calculations and transformations done within part (1) may be carried out in the inverse direction starting from the conditions (i), (ii) and (iii). Thus the equality (2) results, which is the strong duality, and shows that $\bar{x}$ solves (P) and $(\bar{\lambda}, \bar{p}_1, \dots, \bar{p}_m)$ solves (D). □

3 The case of monotonic norms

In this section we particularize the problem presented in the previous section. Therefore, let be $\Phi : \mathbb{R}^m \to \mathbb{R}$ a monotonic norm on $\mathbb{R}^m$. Recall that a norm Φ is said to be monotonic (cf. [1]), if

$$\forall\ u, v \in \mathbb{R}^m, |u_i| \leq |v_i|, i = 1, \dots, m \Rightarrow \Phi(u) \leq \Phi(v).$$

Let us introduce now the following primal problem

$$(P_\Phi) \quad \inf_{x \in X} \Phi^+(g(x)),$$

where $\Phi^+ : \mathbb{R}^m \to \mathbb{R}, \Phi^+(t) := \Phi(t^+)$, with $t^+ = (t_1^+, \dots, t_m^+)^T$ and $t_i^+ = \max\{0, t_i\}$, $i = 1, \dots, m$.

Proposition 2. *The function $\Phi^+ : \mathbb{R}^m \to \mathbb{R}$ is convex and componentwise increasing.*

Proof. First, let us point out that the function $(\cdot)^+ : \mathbb{R}^m \to \mathbb{R}^m_+$, defined by $(t)^+ = (t_1^+, ..., t_m^+)^T$, for $t \in \mathbb{R}^m$, is a convex functions. This means that, for $u, v \in \mathbb{R}^m$ and $\alpha \in [0, 1]$, it holds

$$(\alpha u + (1 - \alpha)v)^+ \leqq \alpha u^+ + (1 - \alpha)v^+.$$

Here, "$\leqq$" is the ordering induced on $\mathbb{R}^m$ by the cone of non-negative elements $\mathbb{R}^m_+$.

By the convexity and monotonicity of the norm Φ, we have then, for $u, v \in \mathbb{R}^m$ and $\alpha \in [0,1]$,

$$\begin{aligned}\Phi^+(\alpha u + (1-\alpha)v) &= \Phi((\alpha u + (1-\alpha)v)^+) \leq \Phi(\alpha u^+ + (1-\alpha)v^+) \\ &\leq \alpha\Phi(u^+) + (1-\alpha)\Phi(v^+) = \alpha\Phi^+(u) + (1-\alpha)\Phi^+(v).\end{aligned}$$

This means that the function Φ^+ is convex.

In order to prove that Φ^+ is componentwise increasing, let be $u, v \in \mathbb{R}^m$, such that $u_i \leq v_i, i = 1, ..., m$. We have then, $u_i^+ \leq v_i^+$, which implies that $|u_i^+| \leq |v_i^+|, i = 1, \ldots, m$. Φ being a monotonic norm, we have $\Phi(u^+) \leq \Phi(v^+)$, where $u^+ = (u_1^+, ..., u_m^+)^T, v^+ = (v_1^+, ..., v_m^+)^T$ or, equivalently, $\Phi^+(u) \leq \Phi^+(v)$.

In conclusion, the function Φ^+ is componentwise increasing. □

By the approach described in section 2, a dual problem to (P_Φ) is

$$(D_\Phi) \quad \sup_{\substack{\lambda \in \mathbb{R}^m, p_i \in X^*, \\ i=1,...,m, \sum\limits_{i=1}^m p_i = 0}} \left\{ -(\Phi^+)^*(\lambda) - \sum_{i=1}^m (\lambda_i g_i)^*(p_i) \right\}.$$

Proposition 3. *The conjugate function $(\Phi^+)^* : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ of Φ^+ verifies*

$$(\Phi^+)^*(\lambda) = \begin{cases} 0, \text{ if } \lambda \geqq 0 \text{ and } \Phi^0(\lambda) \leq 1, \\ +\infty, \text{ otherwise}, \end{cases}$$

where Φ^0 is the dual norm of Φ in $\mathbb{R}^m$.

Proof. Let be $\lambda \in \mathbb{R}^m$. For $t \in \mathbb{R}^m$, we have $|t_i| \geq |t_i^+|, i = 1, \ldots, m$, which implies that $\Phi(t) \geq \Phi(t^+)$ and

$$\Phi^*(\lambda) = \sup_{t \in \mathbb{R}^m} \{\langle \lambda, t\rangle - \Phi(t)\} \leq \sup_{t \in \mathbb{R}^m} \{\langle \lambda, t\rangle - \Phi^+(t)\} = (\Phi^+)^*(\lambda). \tag{6}$$

On the other hand, for the conjugate of the norm Φ we have the following formula (cf. [11])

$$\Phi^*(\lambda) = \sup_{t \in \mathbb{R}^m} \{\langle \lambda, t\rangle - \Phi(t)\} = \begin{cases} 0, \text{ if } \Phi^0(\lambda) \leq 1, \\ +\infty, \text{ otherwise}. \end{cases} \tag{7}$$

If $\Phi^0(\lambda) > 1$, by (6) and (7), we have $+\infty = \Phi^*(\lambda) \leq (\Phi^+)^*(\lambda)$. From here, $(\Phi^+)^*(\lambda) = +\infty$.

Let be now $\Phi^0(\lambda) \leq 1$. If there exists an $i_0 \in \{1, \ldots, m\}$, such that $\lambda_{i_0} < 0$, we have

$$\begin{aligned}(\Phi^+)^*(\lambda) &= \sup_{t \in \mathbb{R}^m} \{\langle \lambda, t\rangle - \Phi^+(t)\} = \sup_{t \in \mathbb{R}^m} \{\langle \lambda, t\rangle - \Phi(t^+)\} \\ &\geq \sup_{t_{i_0} < 0} \left\{\left\langle \lambda, (0, \ldots, t_{i_0}, \ldots, 0)^T \right\rangle - \Phi((0, \ldots, t_{i_0}, \ldots, 0)^+)\right\} \\ &= \sup_{t_{i_0} < 0} \lambda_{i_0} t_{i_0} = +\infty.\end{aligned}$$

Like in the previous case, $(\Phi^+)^*(\lambda) = +\infty$.

Finally, let be $\Phi^0(\lambda) \leq 1$ and $\lambda \geqq 0$. For every $t \in \mathbb{R}^m$, it holds then $\langle \lambda, t\rangle \leq \langle \lambda, t^+\rangle$ and $\langle \lambda, t^+\rangle \leq \Phi(t^+)$. By using this two inequalities, we obtain for the conjugate function of Φ^+

$$(\Phi^+)^*(\lambda) = \sup_{t \in \mathbb{R}^m} \{\langle \lambda, t\rangle - \Phi(t^+)\} \leq \sup_{t \in \mathbb{R}^m} \left\{\left\langle \lambda, t^+\right\rangle - \Phi(t^+)\right\} \leq 0.$$

But, by (6) and (7), it holds $(\Phi^+)^*(\lambda) \geq \Phi^*(\lambda) = 0$. So, we must have $(\Phi^+)^*(\lambda) = 0$, and the proposition is proved. □

By Proposition 3, the dual of (P_Φ) will have the following formulation

$$(D_\Phi) \quad \sup_{\substack{\lambda\in\mathbb{R}^m_+,p_i\in X^*,i=1,\dots,m,\\ \sum\limits_{i=1}^m p_i=0,\Phi^0(\lambda)\le 1}} \left\{-\sum_{i=1}^m (\lambda_i g_i)^*(p_i)\right\}.$$

In the objective function of the dual let us separate the terms for which $\lambda_i > 0$ from the terms for which $\lambda_i = 0$. The dual can be then written as

$$(D_\Phi) \quad \sup_{\substack{p_i\in X^*,i=1,\dots,m,\sum\limits_{i=1}^m p_i=0,\\ \Phi^0(\lambda)\le 1, I\subseteq\{1,\dots,m\},\\ \lambda_i>0(i\in I),\lambda_i=0(i\notin I)}} \left\{-\sum_{i\in I}(\lambda_i g_i)^*(p_i) - \sum_{i\notin I}(0)^*(p_i)\right\}. \tag{8}$$

For $i \notin I$, it holds

$$0^*(p_i) = \sup_{x\in X}\{\langle p_i, x\rangle - 0\} = \sup_{x\in X}\langle p_i, x\rangle = \begin{cases} 0, \text{ if } p_i = 0,\\ +\infty, \text{ otherwise,}\end{cases}$$

and this means that, in order to have supremum in (D_Φ), we must take $p_i = 0,\ \forall i \notin I$. The dual problem will be then

$$(D_\Phi) \quad \sup_{\substack{\Phi^0(\lambda)\le 1, I\subseteq\{1,\dots,m\},\\ \lambda_i>0(i\in I),\lambda_i=0(i\notin I),\\ p_i\in X^*,i\in I,\sum\limits_{i\in I} p_i=0}} \left\{-\sum_{i\in I}(\lambda_i g_i)^*(p_i)\right\}.$$

For $\lambda_i > 0, i \in I$, let us apply the following property of the conjugate functions $(\lambda_i g_i)^* = \lambda_i g_i^*\left(\frac{1}{\lambda_i}p_i\right),\ \forall i \in I$ (cf. [4]). Denoting $p_i := \frac{1}{\lambda_i}p_i$, we obtain, finally,

$$(D_\Phi) \quad \sup_{(I,\lambda,p)\in Y_\Phi} \left\{-\sum_{i\in I}\lambda_i g_i^*(p_i)\right\},$$

with

$$Y_\Phi = \Big\{(I,\lambda,p) : I \subseteq \{1,\dots,m\}, \lambda = (\lambda_1,\dots,\lambda_m)^T, p = (p_1,\dots,p_m),$$
$$\Phi^0(\lambda) \le 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i\in I}\lambda_i p_i = 0\Big\}.$$

In Proposition 2 we have shown that Φ^+ is a convex and componentwise increasing function. Moreover, one can observe that $\inf(P_\Phi)$ is finite, being greater or equal than zero. This last observation, together with Theorem 1, permits us to formulate the following strong duality theorem for the problems (P_Φ) and (D_Φ).

Theorem 3 (strong duality for (P_Φ)). *The dual problem (D_Φ) has a solution and strong duality holds, i.e.*

$$\inf(P_\Phi) = \max(D_\Phi).$$

As for the general problem (P), we can derive now the optimality conditions for (P_Φ).

Theorem 4 (optimality conditions for (P_Φ)).

(1) Let $\bar{x} \in X$ be a solution to (P_Φ). There exists then $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi$, solution to (D_Φ), such that the following optimality conditions are satisfied

$$(i)\ \bar{I} \subseteq \{1, \ldots, m\}, \bar{\lambda}_i > 0 (i \in \bar{I}), \bar{\lambda}_i = 0 (i \notin \bar{I}),$$

$$(ii)\ \Phi^0(\bar{\lambda}) \leq 1, \sum_{i \in \bar{I}} \bar{\lambda}_i \bar{p}_i = 0,$$

$$(iii)\ \Phi^+(g(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i g_i(\bar{x}),$$

$$(iv)\ g_i(\bar{x}) + g_i^*(\bar{p}_i) = \langle \bar{p}_i, \bar{x} \rangle, i \in \bar{I}.$$

(2) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi$ and (i)-(iv) are fulfilled, then $\bar{x}$ is a solution to (P_Φ), $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi$ is a solution to (D_Φ) and strong duality holds

$$\Phi^+(g(\bar{x})) = -\sum_{i \in \bar{I}} \bar{\lambda}_i g_i^*(\bar{p}_i).$$

Proof. (1) By Theorem 3, it follows that there exists $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi$, solution to (D_Φ), such that (i)-(ii) are fulfilled and

$$\Phi^+(g(\bar{x})) = -\sum_{i \in \bar{I}} \bar{\lambda}_i g_i^*(\bar{p}_i).$$

The last equality is equivalent to

$$0 = \Phi^+(g(\bar{x})) + (\Phi^+)^*(\bar{\lambda}) - \sum_{i \in \bar{I}} \bar{\lambda}_i g_i(\bar{x}) + \sum_{i \in \bar{I}} \bar{\lambda}_i [g_i(\bar{x}) + g_i^*(\bar{p}_i) - \langle \bar{p}_i, \bar{x} \rangle].$$

Using again the Young-inequalities

$$\Phi^+(g(\bar{x})) + (\Phi^+)^*(\bar{\lambda}) \geq \langle \bar{\lambda}, g(\bar{x}) \rangle = \sum_{i \in \bar{I}} \bar{\lambda}_i g_i(\bar{x}) \tag{9}$$

and

$$g_i(\bar{x}) + g_i^*(\bar{p}_i) \geq \langle \bar{p}_i, \bar{x} \rangle, i \in \bar{I}, \tag{10}$$

it follows that (9) and (10) turn over in equalities. This means that

$$\Phi^+(g(\bar{x})) + (\Phi^+)^*(\bar{\lambda}) = \sum_{i \in \bar{I}} \bar{\lambda}_i g_i(\bar{x}) \tag{11}$$

and

$$g_i(\bar{x}) + g_i^*(\bar{p}_i) = \langle \bar{p}_i, \bar{x} \rangle, i \in \bar{I}. \tag{12}$$

On the other hand, by Proposition 3, we have that $(\Phi^+)^*(\bar{\lambda}) = 0$, and, so, (11) conduces us to $\Phi^+(g(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i g_i(\bar{x})$. In conclusion, the relations (iii) and (iv) must also hold.

(2) All the calculations and transformations done within part (1) may be carried out in the inverse direction. □

Remark 1. In Theorem 4 we do not exclude the possibility that the set $\bar{I}$ should be empty. This would mean that, in the optimality conditions, $\bar{\lambda} = 0$ and, from (iii), $\Phi^+(g(\bar{x})) = 0$. But, this can be the case just if the following equivalent relations are true

$$\Phi(g(\bar{x})^+) = 0 \Leftrightarrow g^+(\bar{x}) = 0 \Leftrightarrow g_i^+(\bar{x}) = 0, i = 1, \ldots, m \Leftrightarrow g_i(\bar{x}) \leq 0, i = 1, \ldots, m.$$

4 The location model involving sets as existing facilities

After we have studied in the sections 2 and 3 the duality for two quite general optimization problems, we consider now the problem treated by Nickel, Puerto and Rodriguez-Chia in [9]. This problem is a single facility location problem in a general normed space in which the existing facilities are represented by sets.

Let be $\mathcal{A} = \{A_1, \ldots, A_m\}$ a family of convex sets in X, such that $\bigcap_{i=1}^{m} \overline{A}_i = \emptyset$. For $i = 1, ..., m$, we consider $g_i : X \to \mathbb{R}, g_i(x) = d_i(x, A_i)$, where

$$d_i(x, A_i) = \inf\{\gamma_i(x - a_i) : a_i \in A_i\}.$$

Here, γ_i is a continuous norm on X, for $i = 1, \ldots, m$. This means that the functions $g_i, i = 1, ..., m$, are convex and continuous on X.

Let be $d : X \to \mathbb{R}^m$ the vector function defined by

$$d(x) := (d_1(x, A_1), \ldots, d_m(x, A_m))^T.$$

The location problem with sets as existing facilities studied in [9] is

$$(P_{\Phi}(\mathcal{A})) \quad \inf_{x \in X} \Phi(d(x)).$$

But, because of

$$\Phi^+(d(x)) = \Phi(d^+(x)) = \Phi(d(x)), \ \forall x \in X,$$

we can write $(P_{\Phi}(\mathcal{A}))$ in the equivalent form

$$(P_{\Phi}(\mathcal{A})) \quad \inf_{x \in X} \Phi^+(d(x)).$$

This problem is a particular case of the problem studied in section 3. Therefore, the dual problem of $(P_{\Phi}(\mathcal{A}))$ is

$$(D_{\Phi}(\mathcal{A})) \quad \sup_{(I,\lambda,p) \in Y_{\Phi}(\mathcal{A})} \left\{ -\sum_{i \in I} \lambda_i d_i^*(p_i) \right\},$$

with

$$Y_{\Phi}(\mathcal{A}) = \Big\{ (I, \lambda, p) : I \subseteq \{1, \ldots, m\}, \lambda = (\lambda_1, \ldots, \lambda_m)^T, p = (p_1, \ldots, p_m),$$
$$\Phi^0(\lambda) \le 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i \in I} \lambda_i p_i = 0 \Big\}.$$

By the use of the Theorems 3 and 4 we can present for $(P_{\Phi}(\mathcal{A}))$ and $(D_{\Phi}(\mathcal{A}))$ the strong duality theorem and the optimality conditions.

Theorem 5 (strong duality for $(P_{\Phi}(\mathcal{A}))$). *The dual problem $(D_{\Phi}(\mathcal{A}))$ has a solution and strong duality holds, i.e.*

$$\inf(P_{\Phi}(\mathcal{A})) = \max(D_{\Phi}(\mathcal{A})).$$

Theorem 6 (optimality conditions for $(P_{\Phi}(\mathcal{A}))$).

(1) Let $\bar{x} \in X$ be a solution to $(P_{\Phi}(\mathcal{A}))$. There exists then $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_{\Phi}(\mathcal{A})$, solution to $(D_{\Phi}(\mathcal{A}))$, such that the following optimality conditions are satisfied

$$(i) \ \bar{I} \subseteq \{1, \ldots, m\}, \bar{I} \neq \emptyset, \bar{\lambda}_i > 0 (i \in \bar{I}), \bar{\lambda}_i = 0 (i \notin \bar{I}),$$

$$(ii) \ \Phi^0(\bar{\lambda}) = 1, \sum_{i \in \bar{I}} \bar{\lambda}_i \bar{p}_i = 0,$$

$$(iii) \ \Phi(d(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i d_i(\bar{x}, A_i),$$

$$(iv) \ \bar{x} \in \partial d_i^*(\bar{p}_i), i \in \bar{I}.$$

(2) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi(\mathcal{A})$ and (i) - (iv) are fulfilled, then $\bar{x}$ is a solution to $(P_\Phi(\mathcal{A}))$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi(\mathcal{A})$ is a solution to $(D_\Phi(\mathcal{A}))$ and strong duality holds

$$\Phi(d(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i d_i(\bar{x}, A_i) = -\sum_{i \in \bar{I}} \bar{\lambda}_i d_i^*(\bar{p}_i).$$

Proof. (1) By Theorem 5, it follows that there exists $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_\Phi(\mathcal{A})$, solution to $(D_\Phi(\mathcal{A}))$, such that

$$(i')\ \bar{I} \subseteq \{1, \ldots, m\}, \bar{\lambda}_i > 0 (i \in \bar{I}), \bar{\lambda}_i = 0 (i \notin \bar{I}),$$

$$(ii')\ \Phi^0(\bar{\lambda}) \leq 1, \sum_{i \in \bar{I}} \bar{\lambda}_i \bar{p}_i = 0,$$

$$(iii')\ \Phi^+(d(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i d_i(\bar{x}, A_i),$$

$$(iv')\ d_i(\bar{x}, A_i) + d_i^*(\bar{p}_i) = \langle \bar{p}_i, \bar{x} \rangle, i \in \bar{I}.$$

We prove now that $(\bar{I}, \bar{\lambda}, \bar{p})$ verifies the relations (i)-(iv). If $\bar{I}$ were empty, then by Remark 1, it would follow that

$$g_i(\bar{x}) = d_i(\bar{x}, A_i) = 0, i = 1, \ldots, m.$$

But, this would imply that $\bar{x} \in \bigcap\limits_{i=1}^{m} \overline{A}_i$, which would contradict the hypothesis that $\bigcap\limits_{i=1}^{m} \overline{A}_i = \emptyset$. By this, the relation (i) is proved.
From (iii'), we have that

$$\Phi^+(d(\bar{x})) = \Phi(d(\bar{x})) = \sum_{i \in \bar{I}} \bar{\lambda}_i d_i(\bar{x}, A_i), \tag{13}$$

and, so, (iii) is also proved.
From (iv'), we have that $\bar{p}_i \in \partial d_i(\bar{x}, A_i)$, for $i \in \bar{I}$ (cf. [4]). On the other hand, the distance function d_i, being convex and continuous, verifies (cf. [4] and [16])

$$\bar{p}_i \in \partial d_i(\bar{x}, A_i) \Leftrightarrow \bar{x} \in \partial d_i^*(\bar{p}_i),\ \forall i \in \bar{I},$$

which proves (iv).
In order to finish the proof, it remains us to show that $\Phi^0(\bar{\lambda}) = 1$. By the definition of the dual norm, we have

$$\Phi^0(\bar{\lambda}) = \sup_{\substack{\Phi(v) \leq 1, \\ v \in \mathbb{R}^m}} |\langle \bar{\lambda}, v \rangle|.$$

Because of $\bigcap\limits_{i=1}^{m} \overline{A}_i = \emptyset$, it holds $\Phi(d(\bar{x})) > 0$. Let be $\bar{v} = \frac{1}{\Phi(d(\bar{x}))} d(\bar{x}) \in \mathbb{R}^m$. Then we have $\Phi(\bar{v}) = 1$ and, by (iii) and (13),

$$\Phi^0(\bar{\lambda}) \geq \langle \bar{\lambda}, \bar{v} \rangle = \frac{\sum\limits_{i \in \bar{I}} \bar{\lambda}_i d_i(\bar{x}, A_i)}{\Phi(d(\bar{x}))} = 1.$$

This last inequality, together with (ii'), gives us $\Phi^0(\bar{\lambda}) = 1$.
(2) All the calculations and transformations done within part (1) may be carried out in the inverse direction. □

Remark 2. (a) Lemma 3.3 in [9], which characterizes the solutions of $(P_\Phi(\mathcal{A}))$, can be automatically obtained by means of the optimality conditions given in Theorem 6.

(b) In [9], the authors made the assumption, that the sets $A_i, i = 1, ..., m$, have to be compact. As one can see, in order to formulate the strong duality theorem and the optimality conditions for $(P_\Phi(\mathcal{A}))$, the compactness of the sets $A_i, i = 1, ..., m$, is not necessary.

In the last two sections of this paper we consider the Weber problem with infimal distances and the minmax problem with infimal distances with sets as existing facilities. One may notice that these problems may be related to the linear and Tchebycheff scalarization, respectively, of a multiobjective location problems. For the mentioned problems we formulate their duals and present the optimality conditions. Therefore, we write both problems, equivalently, as particular cases of the problem $(P_\Phi(\mathcal{A}))$.

5 The Weber problem with infimal distances

The Weber problem with infimal distances for the data $\mathcal{A}$ is

$$(P_W(\mathcal{A})) \quad \inf_{x\in X} \sum_{i=1}^{m} w_i d_i(x, A_i),$$

where $d_i(x, A_i) = \inf\limits_{a_i\in A_i} \gamma_i(x - a_i), i = 1, ..., m$, and $w_i > 0, i = 1, ..., m$, are positive weights.

We introduce now, for $i = 1, ..., m$, the continuous norms $\gamma_i' : X \to \mathbb{R}$, $\gamma_i' = w_i\gamma_i$ and the corresponding distance functions $d_i'(\cdot, A_i) : X \to \mathbb{R}$, $d_i'(x, A_i) = \inf\limits_{a_i\in A_i} \gamma_i'(x - a_i)$. This means that

$$d_i'(x, A_i) = \inf_{a_i\in A_i} \gamma_i'(x - a_i) = w_i d_i(x, A_i), i = 1, \ldots, m. \tag{14}$$

By (14), the primal problem $(P_W(\mathcal{A}))$ becomes

$$(P_W(\mathcal{A})) \quad \inf_{x\in X} \sum_{i=1}^{m} d_i'(x, A_i) = \inf_{x\in X} l_1(d'(x)),$$

where $d' : X \to \mathbb{R}^m$, $d'(x) = (d_1'(x, A_1), \ldots, d_m'(x, A_m))^T$ and $l_1 : \mathbb{R}^m \to \mathbb{R}, l_1(\lambda) = \sum\limits_{i=1}^{m} |\lambda_i|$. One may easy observe that the l_1-norm is a monotonic norm.

The dual problem of $(P_W(\mathcal{A}))$ will be then

$$(D_W(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_W(\mathcal{A})} \left\{ -\sum_{i\in I} \lambda_i (d_i')^*(p_i) \right\},$$

with

$$Y_W(\mathcal{A}) = \Big\{ (I, \lambda, p) : I \subseteq \{1, \ldots, m\}, \lambda = (\lambda_1, \ldots, \lambda_m)^T, p = (p_1, \ldots, p_m), \\ l_1^0(\lambda) \le 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i\in I} \lambda_i p_i = 0 \Big\}.$$

For $i = 1, \ldots, m$, we have that (cf. [4]) $(d_i')^*(p_i) = (w_i d_i)^*(p_i) = w_i d_i^* \left(\frac{1}{w_i} p_i \right)$. Moreover, the dual norm of the l_1- norm is $l_1^0(\lambda) = \max\limits_{i=1,\ldots,m} |\lambda_i|$. Denoting $p_i :=$

$\frac{1}{w_i}p_i, i = 1, \ldots, m$, we obtain the following formulation for the dual problem

$$(D_W(\mathcal{A})) \quad \sup_{(I,\lambda,p)\in Y_W(\mathcal{A})} \left\{ -\sum_{i\in I} \lambda_i w_i d_i^*(p_i) \right\},$$

with

$$Y_W(\mathcal{A}) = \Big\{ (I, \lambda, p) : I \subseteq \{1, \ldots, m\}, \lambda = (\lambda_1, \ldots, \lambda_m)^T, p = (p_1, \ldots, p_m), \max_{i\in I} \lambda_i \leq 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i\in I} \lambda_i w_i p_i = 0 \Big\}.$$

Let us give now the strong duality theorem and the optimality conditions for $(P_W(\mathcal{A}))$ and its dual $(D_W(\mathcal{A}))$.

Theorem 7 (strong duality for $(P_W(\mathcal{A}))$). *The dual problem $(D_W(\mathcal{A}))$ has a solution and strong duality holds, i.e.*

$$\inf(P_W(\mathcal{A})) = \max(D_W(\mathcal{A})).$$

Theorem 8 (optimality conditions for $(P_W(\mathcal{A}))$).

(1) Let $\bar{x} \in X$ be a solution to $(P_W(\mathcal{A}))$. There exists then $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_W(\mathcal{A})$, solution to $(D_W(\mathcal{A}))$, such that the following optimality conditions are satisfied

$$(i)\ \bar{I} \subseteq \{1, \ldots, m\}, \bar{I} \neq \emptyset, \bar{\lambda}_i = 1 (i \in \bar{I}), \bar{\lambda}_i = 0 (i \notin \bar{I}),$$

$$(ii)\ \sum_{i\in\bar{I}} w_i \bar{p}_i = 0,$$

$$(iii)\ \sum_{i=1}^{m} w_i d_i(\bar{x}, A_i) = \sum_{i\in\bar{I}} w_i d_i(\bar{x}, A_i),$$

$$(iv)\ \bar{x} \in \partial d_i^*(\bar{p}_i), i \in \bar{I}.$$

(2) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_W(\mathcal{A})$ and (i)-(iv) are fulfilled, then $\bar{x}$ is a solution to $(P_W(\mathcal{A}))$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_W(\mathcal{A})$ is a solution to $(D_W(\mathcal{A}))$ and strong duality holds

$$\sum_{i=1}^{m} w_i d_i(\bar{x}, A_i) = \sum_{i\in\bar{I}} w_i d_i(\bar{x}, A_i) = -\sum_{i\in\bar{I}} \bar{\lambda}_i w_i d_i^*(\bar{p}_i).$$

Proof. (1) By Theorem 7, it follows that there exists a triplet $(\bar{I}', \bar{\lambda}', \bar{p}')$, such that

$$(i')\ \bar{I}' \subseteq \{1, \ldots, m\}, \bar{I}' \neq \emptyset, \bar{\lambda}_i' > 0 (i \in \bar{I}'), \bar{\lambda}_i' = 0 (i \notin \bar{I}'),$$

$$(ii')\ l_1^0(\bar{\lambda}') = 1, \sum_{i\in\bar{I}'} \bar{\lambda}_i' \bar{p}_i' = 0,$$

$$(iii')\ l_1(d'(\bar{x})) = \sum_{i\in\bar{I}'} \bar{\lambda}_i' d_i'(\bar{x}, A_i),$$

$$(iv')\ \bar{x} \in \partial (d_i')^*(\bar{p}_i'), i \in \bar{I}'.$$

By (ii'), we have that $l_1^0(\bar{\lambda}) = \max_{i\in\bar{I}'} \bar{\lambda}_i' = 1$. From (iii'), it follows

$$l_1(d'(\bar{x})) = \sum_{i=1}^{m} d_i'(\bar{x}, A_i) = \sum_{i\in\bar{I}'} \bar{\lambda}_i' d_i'(\bar{x}, A_i) \leq \sum_{i\in\bar{I}'} d_i'(\bar{x}, A_i) \leq \sum_{i=1}^{m} d_i'(\bar{x}, A_i).$$

We must then have, for $i \in \bar{I}'$, $\bar{\lambda}'_i = 1$. Substituting in (ii'), it follows $\sum\limits_{i \in \bar{I}'} \bar{p}'_i = 0$. Considering $\bar{I} = \bar{I}'$, $\bar{\lambda} = \bar{\lambda}'$ and $\bar{p}_i = \frac{1}{w_i}\bar{p}'_i$, for $i = 1, \ldots, m$, the triplet $(\bar{I}, \bar{\lambda}, \bar{p})$ is feasible for $(D_W(\mathcal{A}))$. Moreover, it is obvious that $(i) - (iii)$ are verified. On the other hand, from (iv'), we have $\forall i \in \bar{I}$ (cf. [11]),

$$\bar{x} \in \partial(d'_i)^*(\bar{p}'_i) \Leftrightarrow \bar{p}'_i \in \partial d'_i(\bar{x}, A_i) \Leftrightarrow \bar{p}'_i \in \partial(w_i d_i)(\bar{x}, A_i) = w_i \partial d_i(\bar{x}, A_i),$$

which implies that $\bar{p}_i = \frac{1}{w_i} p'_i \in \partial d_i(\bar{x}, A_i)$, or, equivalently, $\bar{x} \in \partial d^*_i(\bar{p}_i)$.

(2) All the calculations and transformations done within part (1) may be carried out in the inverse direction. □

6 The minmax problem with infimal distances

The last optimization problem which we consider in this paper is the minmax problem with infimal distances for the data $\mathcal{A}$

$$(P_H(\mathcal{A})) \quad \inf_{x \in X} \max_{i=1,\ldots,m} w_i d_i(x, A_i),$$

where $d_i(x, A_i) = \inf\limits_{a_i \in A_i} \gamma_i(x - a_i), i = 1, ..., m$, and $w_i > 0, i = 1, ..., m$, are positive weights.

Like for the Weber problem studied above, let be, for $i = 1, ..., m$, the continuous norms $\gamma'_i : X \to \mathbb{R}$, $\gamma'_i = w_i \gamma_i$ and the corresponding distance functions $d'_i(\cdot, A_i) : X \to \mathbb{R}$, $d'_i(x, A_i) = \inf\limits_{a_i \in A_i} \gamma'_i(x - a_i)$.

This means that the equality in (14) is true and the primal problem $(P_H(\mathcal{A}))$ becomes

$$(P_H(\mathcal{A})) \quad \inf_{x \in X} \max_{i=1,\ldots,m} d'_i(x, A_i) = \inf_{x \in X} l_\infty(d'(x)),$$

where $d' : X \to \mathbb{R}^m$, $d'(x) = (d'_1(x, A_1), \ldots, d'_m(x, A_m))^T$ and $l_\infty : \mathbb{R}^m \to \mathbb{R}, l_\infty(\lambda) = \max\limits_{i=1,\ldots,m} |\lambda_i|$. The l_∞-norm is also a monotonic norm.

The dual problem of $(P_H(\mathcal{A}))$ will be then

$$(D_H(\mathcal{A})) \quad \sup_{(I,\lambda,p) \in Y_H(\mathcal{A})} \left\{ -\sum_{i \in I} \lambda_i (d'_i)^*(p_i) \right\},$$

with

$$Y_H(\mathcal{A}) = \Big\{ (I, \lambda, p) : I \subseteq \{1, \ldots, m\}, \lambda = (\lambda_1, \ldots, \lambda_m)^T, p = (p_1, \ldots, p_m), \\ l^0_\infty(\lambda) \leq 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i \in I} \lambda_i p_i = 0 \Big\}.$$

For $i = 1, \ldots, m$, we have again (cf. [4]) $(d'_i)^*(p_i) = (w_i d_i)^*(p_i) = w_i d^*_i \left(\frac{1}{w_i} p_i\right)$. The dual norm of the l_∞-norm is $l^0_\infty(\lambda) = \sum\limits_{i=1}^{m} |\lambda_i|$. Denoting $p_i := \frac{1}{w_i} p_i$, for $i = 1, \ldots, m$, we obtain

$$(D_H(\mathcal{A})) \quad \sup_{(I,\lambda,p) \in Y_H(\mathcal{A})} \left\{ -\sum_{i \in I} \lambda_i w_i d^*_i(p_i) \right\},$$

with

$$Y_H(\mathcal{A}) = \Big\{ (I, \lambda, p) : I \subseteq \{1, \ldots, m\}, \lambda = (\lambda_1, \ldots, \lambda_m)^T, p = (p_1, \ldots, p_m), \\ \sum_{i \in I} \lambda_i \leq 1, \lambda_i > 0 (i \in I), \lambda_i = 0 (i \notin I), \sum_{i \in I} \lambda_i w_i p_i = 0 \Big\}.$$

Like in the previous section, we state the strong duality theorem and formulate the optimality conditions.

Theorem 9 (strong duality for $(P_H(\mathcal{A}))$). *The dual problem $(D_H(\mathcal{A}))$ has a solution and strong duality holds, i.e.*

$$\inf(P_H(\mathcal{A})) = \max(D_H(\mathcal{A})).$$

Theorem 10 (optimality conditions for $(P_H(\mathcal{A}))$).

(1) Let $\bar{x} \in X$ be a solution to $(P_H(\mathcal{A}))$. There exists then $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_H(\mathcal{A})$, solution to $(D_H(\mathcal{A}))$, such that the following optimality conditions are satisfied

$$(i)\ \bar{I} \subseteq \{1, \ldots, m\}, \bar{I} \neq \emptyset, \bar{\lambda}_i > 0 (i \in \bar{I}), \bar{\lambda}_i = 0 (i \notin \bar{I}),$$

$$(ii)\ \sum_{i\in\bar{I}} \bar{\lambda}_i = 1, \sum_{i\in\bar{I}} w_i\bar{\lambda}_i\bar{p}_i = 0,$$

$$(iii)\ \max_{i=1,\ldots,m} w_i d_i(\bar{x}, A_i) = w_i d_i(\bar{x}, A_i),\ \forall i \in \bar{I},$$

$$(iv)\ \bar{x} \in \partial d_i^*(\bar{p}_i), i \in \bar{I}.$$

(2) If $\bar{x} \in X$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_H(\mathcal{A})$ and (i)-(iv) are fulfilled, then $\bar{x}$ is a solution to $(P_H(\mathcal{A}))$, $(\bar{I}, \bar{\lambda}, \bar{p}) \in Y_H(\mathcal{A})$ is a solution to $(D_H(\mathcal{A}))$ and strong duality holds

$$\max_{i=1,\ldots,m} w_i d_i(\bar{x}, A_i) = \sum_{i\in\bar{I}} \bar{\lambda}_i w_i d_i(\bar{x}, A_i) = -\sum_{i\in\bar{I}} \bar{\lambda}_i w_i d_i^*(\bar{p}_i).$$

Proof. (1) By Theorem 9, it follows that there exists a triplet $(\bar{I}', \bar{\lambda}', \bar{p}')$, such that

$$(i')\ \bar{I}' \subseteq \{1, \ldots, m\}, \bar{I}' \neq \emptyset, \bar{\lambda}_i' > 0 (i \in \bar{I}'), \bar{\lambda}_i' = 0 (i \notin \bar{I}'),$$

$$(ii')\ l_\infty^0(\bar{\lambda}') = 1, \sum_{i\in\bar{I}} \bar{\lambda}_i'\bar{p}_i' = 0,$$

$$(iii')\ l_\infty(d'(\bar{x})) = \sum_{i\in\bar{I}'} \bar{\lambda}_i' d_i'(\bar{x}, A_i),$$

$$(iv')\ \bar{x} \in \partial(d_i')^*(\bar{p}_i'), i \in \bar{I}'.$$

By (ii'), we have that $l_\infty^0(\bar{\lambda}') = \sum_{i\in\bar{I}'} \bar{\lambda}_i' = 1$. On the other hand, from (iii'), it follows

$$l_\infty(d'(x)) = \max_{i=1,\ldots,m} d_i'(x, A_i) = \sum_{i\in\bar{I}'} \bar{\lambda}_i' d_i'(\bar{x}, A_i) \leq \max_{i=1,\ldots,m} d_i'(\bar{x}, A_i).$$

For $i \in \bar{I}'$, we must then have $\max\limits_{i=1,\ldots,m} d_i'(\bar{x}, A_i) = d_i'(\bar{x}, A_i)$. From here, by (14), we have $\max\limits_{i=1,\ldots,m} w_i d_i(\bar{x}, A_i) = w_i d_i(\bar{x}, A_i),\ \forall i \in \bar{I}'$.

Considering $\bar{I} = \bar{I}'$, $\bar{\lambda} = \bar{\lambda}'$ and $\bar{p}_i = \frac{1}{w_i} p_i', i = 1, \ldots, m$, the triplet $(\bar{I}, \bar{\lambda}, \bar{p})$ is feasible for $(D_H(\mathcal{A}))$. So, it is obvious that $(i) - (iii)$ are verified. Relation (iv) can be obtained in the same way like in the proof of Theorem 8.

(2) All the calculations and transformations done within part (1) may be carried out in the inverse direction. □

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